

DERIVED CATEGORY OF TORIC FIBRATIONS

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ABSTRACT. In this paper, we give a structure theorem for the derived category $D^b(X)$ of a toric fibration X over \mathbb{P}^n with fiber Y provided that Y has a full strongly exceptional collection of line bundles.

1. INTRODUCTION

Let X be a smooth projective variety defined over an algebraically closed field K of characteristic zero and let $D^b(X) = D^b(\mathcal{O}_X\text{-mod})$ be the derived category of bounded complexes of coherent sheaves of \mathcal{O}_X -modules. $D^b(X)$ is one of the most important algebraic invariants of a smooth projective variety X and, in spite of the increasing interest in understanding the structure of $D^b(X)$, very little progress has been achieved. An important approach to understand derived categories is to construct full strongly exceptional sequences.

Definition 1.1. A coherent sheaf E on a smooth projective variety X is called *exceptional* if it is simple and $\text{Ext}_{\mathcal{O}_X}^i(E, E) = 0$ for $i \neq 0$. An ordered collection (E_0, E_1, \dots, E_m) of coherent sheaves on X is an *exceptional collection* if each sheaf E_i is exceptional and $\text{Ext}_{\mathcal{O}_X}^i(E_k, E_j) = 0$ for $j < k$ and $i \geq 0$. An exceptional collection (E_0, E_1, \dots, E_m) is a *strongly exceptional collection* if in addition $\text{Ext}_{\mathcal{O}_X}^i(E_j, E_k) = 0$ for $i \geq 1$ and $j \leq k$. If an exceptional collection (E_0, E_1, \dots, E_m) of coherent sheaves on X generates $D^b(X)$, then it is called *full*.

The now classical result of Beilinson [2] states that $(\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n))$ is a full strongly exceptional collection on \mathbb{P}^n . The following problem can be considered by now a natural and important question in Algebraic Geometry.

Problem 1.2. Characterize smooth projective varieties which have a full strongly exceptional collection and investigate whether there is one consisting of line bundles.

Note that not all smooth projective varieties have a full strongly exceptional collection of coherent sheaves. Indeed, the existence of a full strongly exceptional collection (E_0, E_1, \dots, E_m) of coherent sheaves on a smooth projective variety X imposes a rather strong restriction on X , namely that the Grothendieck group $K_0(X) = K_0(\mathcal{O}_X\text{-mod})$ is isomorphic to \mathbb{Z}^{m+1} . In [4], [5], [6] and [7], we constructed full strongly exceptional collections of line bundles on smooth toric varieties with a splitting fan, on smooth complete toric varieties with small Picard number and on \mathbb{P}^d -fibrations on certain toric varieties.

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The goal of this paper is to construct a full strongly exceptional collection consisting of line bundles on any smooth toric Y -fibration over \mathbb{P}^n provided Y has a full strongly exceptional collection of line bundles (see Theorem 4.2).

In view of our results we make the following Conjecture:

Conjecture 1.3. *Let X be a smooth complete toric variety. Assume that X is an F -fibration over Y and that Y and F are smooth toric varieties with a full strongly exceptional collection of (line) bundles. Then X has a full strongly exceptional collection made up of (line) bundles.*

Next we outline the structure of this paper. In section 2, we recall the basic facts on toric varieties and toric fibrations as well as the results on the cohomology of line bundles on toric varieties and toric fibrations needed in the sequel. Section 3 is the heart of this paper. Indeed, in Theorem 4.2, we prove that if X is a Y -fibration over \mathbb{P}^n and Y has a full strongly exceptional collection of line bundles then X also has a full strongly exceptional collection consisting of line bundles. We divide the proof in two steps. First, we construct a strongly exceptional collection on X . To this end, we need to check the acyclicity of certain line bundles on X and this is achieved using the combinatorial properties of Y -fibrations over \mathbb{P}^n . Once we have constructed a full strongly exceptional collection on X , it suffices to show that it is full and this is done by induction on n . We end the paper with a short section where we give an account on the contributions to Conjecture 1.3.

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2. TORIC VARIETIES

In this section we will introduce the notation and facts on toric varieties and toric fibrations that we will use along this paper. We refer to [10] and [13] for more details.

2.1. Toric varieties. Let X be a smooth complete toric variety of dimension n over an algebraically closed field K of characteristic zero. X is defined by a fan $\Sigma := \Sigma_X$ of strongly convex polyhedral cones in $N \otimes_{\mathbb{Z}} \mathbb{R}$ where N is the lattice \mathbb{Z} . Let $M := \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ denote the dual lattice.

For any $0 \leq i \leq n$, let $\Sigma(i) := \{\sigma \in \Sigma \mid \dim(\sigma) = i\}$. In particular, associated to any 1-dimensional cone $\sigma \in \Sigma(1)$ there is a unique generator $v \in N$, called *ray generator*, such that $\sigma \cap N = \mathbb{Z}_{\geq 0} \cdot v$. Denote by $G_X = \{v_i \mid i \in J\}$ the set of ray generators of X . There is a one-to-one correspondence between such ray generators $\{v_i \mid i \in J\}$ and toric divisors $\{Z_i \mid i \in J\}$ on X .

If $|G_X| = l$ then the Picard number of X is $\rho(X) = l - n$ and the anticanonical divisor $-K_X$ is given by $-K_X = Z_1 + \cdots + Z_l$.

Now we introduce the notions of primitive collections and primitive relations due to V.V. Batyrev [1].

Definition 2.1. A set of toric divisors $\{Z_1, \dots, Z_k\}$ on X is called a *primitive set* if $Z_1 \cap \cdots \cap Z_k = \emptyset$ but $Z_1 \cap \cdots \cap \widehat{Z_j} \cap \cdots \cap Z_k \neq \emptyset$ for all j . Equivalently, if $\langle v_1, \dots, v_k \rangle \notin \Sigma$ but $\langle v_1, \dots, \widehat{v_j}, \dots, v_k \rangle \in \Sigma_X$ for all j . $\mathcal{P} = \{v_1, \dots, v_k\}$ is called a *primitive collection*.

If $\mathcal{P} = \{v_1, \dots, v_k\}$ is a primitive collection, the element $v := v_1 + \dots + v_k$ lies in the relative interior of a unique cone of Σ_X , say the cone generated by v'_1, \dots, v'_s and $v_1 + \dots + v_k = a_1 v'_1 + \dots + a_s v'_s$ with $a_i > 0$ is referred to as a *primitive relation* $r(\mathcal{P})$ associated to \mathcal{P} .

2.2. Toric fiber-fibrations.

Definition 2.2. A *Toric Fiber-bundle* is given by (X, π, F, Y) where X is a toric variety and π a surjective morphism over a normal variety Y such that every fiber is isomorphic to F .

Observe that with these assumptions Y and F are necessarily toric and the map π is equivariant. This implies that Σ_F can be seen as a subfan of Σ_X . The morphism π is in fact induced by an injective map of fans

$$i : \Sigma_F \hookrightarrow \Sigma_X.$$

Let $\dim(Y) = m$, $\dim(F) = r$. The morphism π being locally trivial translates to the fact that for each $\sigma \in \Sigma_Y(m)$, $\pi^{-1}(U_\sigma) \cong U_\sigma \times F$, where U_σ is the associated affine patch. This in turn implies that every $\sigma \in \Sigma_X(n)$ can be written as $\sigma = \nu + \tau$, where $\nu \in \Sigma_F(r)$ and $\tau \cap \Sigma_F = \emptyset$.

This property characterizes in fact every fiber bundle:

Proposition 2.3 (see for instance [9]; Theorem 6.7 or [10]; Section 2.4 for more details). *Let X be an n -dimensional toric variety. X has the structure of a toric fiber bundle if and only if there is a linear subspace $H \subset N_{\mathbb{R}}$ of dimension r such that for every n -dimensional cone $\sigma \in \Sigma_X$, we have $\sigma = \nu + \tau$, with $\nu \in \Sigma_X$, $\nu \subset H$, $\dim \nu = r$ and $\tau \cap H = \{0\}$.*

In fact:

- (1) The set $\Sigma_F = \{\sigma \in \Sigma_X | \sigma \subset H\}$ is the fan of a smooth, complete, r -dimensional toric variety F .
- (2) Denote by H^\perp the complementary space of H in $N_{\mathbb{R}}$ such that $N = (N \cap H) \oplus (N \cap H^\perp)$ and let $\pi : N_{\mathbb{R}} \rightarrow H^\perp$ be the projection. Then the set

$$\Sigma_Z = \{\pi(\sigma) | \sigma \in \Sigma_X\}$$

is the fan of a complete, smooth $(n - r)$ -dimensional toric variety Z .

- (3) The projection π induces an equivariant morphism $\bar{\pi} : X \rightarrow Z$ such that, for every affine invariant open subset $U \subset Z$, $\bar{\pi}^{-1}(U) \cong F \times U$ as toric varieties over U .

We will use the name F -fibration for a fiber bundle with fiber F .

Example 2.4. (1) Let F and Z be smooth projective toric varieties. Then $X \cong F \times Z$ is the trivial fibration over Z with fiber F .

(2) Let $X = \mathbb{P}(\mathcal{E})$ be a smooth projective toric variety which is the projectivization of a rank r vector bundle \mathcal{E} on a smooth projective toric variety Z . Then, X is a \mathbb{P}^{r-1} -fibration over Z .

Let X be an F -fibration over Z as in Proposition 2.3. Then $G_F = G_X \cap H$ and the primitive collections of Σ_X contained in H are exactly the primitive collections of Σ_F . These primitive collections have the same primitive relations in Σ_X and in Σ_F . On the other hand, the projection $\pi : N_{\mathbb{R}} \rightarrow H^\perp$ induces a bijection between $G_X \setminus G_F$ and G_Z .

Under this identification, the primitive collections of Σ_X not contained in H are exactly the primitive collections of Σ_Z . If we denote by \bar{x} the image of $x \in N$ by π , then a primitive collection in Σ_Z of the form

$$\bar{x}_1 + \cdots + \bar{x}_h - (a_1\bar{y}_1 + \cdots + a_k\bar{y}_k) = 0$$

lifts to a primitive relation in Σ_X of the form

$$x_1 + \cdots + x_h - (a_1y_1 + \cdots + a_ky_k + b_1z_1 + \cdots + b_lz_l) = 0,$$

with $l \geq 0$ and $z_i \in \Sigma_F$ for $1 \leq i \leq l$. In particular, the fibration is trivial, namely $X \cong F \times Z$, if and only if all primitive relations in Σ_Z remain unchanged when lifted to Σ_X .

For any smooth projective toric variety X , we denote by $P_X(t)$ its Poincaré polynomial. It is well known that the topological Euler characteristic of X , $\chi(X)$ verifies

$$\chi(X) = P_X(-1)$$

and $\chi(X)$ coincides with the number of maximal cones of X , that is, with the rank of the Grothendieck group $K_0(X)$ of X . On the other hand, if X is an F -fibration over Z we have ([10];Pg. 92-93):

$$P_X(t) = P_F(t) \cdot P_Z(t).$$

Thus putting altogether we deduce that:

$$(2.1) \quad \text{rank}(K_0(X)) = P_X(-1) = P_F(-1) \cdot P_Z(-1) = \text{rank}(K_0(F)) \cdot \text{rank}(K_0(Z)).$$

2.3. The cohomology of line bundles. The next section relies on some facts concerning the cohomology of line bundles on toric varieties and in particular on toric fibrations, which are recalled in this section.

Definition 2.5. Let X be a smooth complete toric variety. A line bundle \mathcal{L} on X is said to be *acyclic* if $H^i(X, \mathcal{L}) = 0$ for every $i \geq 1$.

Now we are going to provide a description of the cohomology of line bundles \mathcal{L} on an n -dimensional smooth complete toric variety X . To this end we need to fix some notation.

Notation 2.6. Let Z_1, \dots, Z_s be the toric divisors on X . For every $\mathbf{r} = (r_i)_{i=1}^s \in \mathbb{Z}^s$ we denote by $\text{Supp}(\mathbf{r})$ the simplicial complex on s vertices, which encodes all the cones σ of Σ_X for which all i with v_i in σ satisfy $r_i \geq 0$. As usual $H_q(\text{Supp}(\mathbf{r}), K)$ denotes the q -th reduced homology group of the simplicial complex $\text{Supp}(\mathbf{r})$. Given a line bundle \mathcal{L} on X , we will say that $\mathbf{r} = (r_i)_{i=1}^s \in \mathbb{Z}^s$ represents \mathcal{L} whenever $\mathcal{L} \cong \mathcal{O}_X(r_1Z_1 + r_2Z_2 + \cdots + r_sZ_s)$.

Proposition 2.7. *With the above notation, it is*

$$H^p(X, \mathcal{L}) \cong \bigoplus_{\mathbf{r}} H_{\text{rank } N - p}(\text{Supp}(\mathbf{r}), K)$$

where the sum is taken over all the representations $\mathbf{r} = (r_i)_{i=1}^s \in \mathbb{Z}^s$ of \mathcal{L} .

Proof. See [8]; Proposition 4.1. \square

The following is a useful consequence of the proposition above:

Corollary 2.8. *With the above notation, $H^0(X, \mathcal{L})$ is determined only by $\mathbf{r} = (r_i)_{i=1}^s \in \mathbb{Z}^s$ such that $\text{Supp}(\mathbf{r})$ is the entire fan Σ_X . Equivalently $H^0(X, \mathcal{L})$ is determined only by $\mathbf{r} = (r_i)_{i=1}^s \in \mathbb{Z}^s$ such that $\mathcal{L} \cong \mathcal{O}_X(r_1 Z_1 + r_2 Z_2 + \cdots + r_s Z_s)$ with all $r_i \geq 0$. On the other hand, $H^{\dim X}(X, \mathcal{L})$ is determined only by $\mathbf{r} = (r_i)_{i=1}^s \in \mathbb{Z}^s$ such that $\text{Supp}(\mathbf{r})$ is empty, i.e. by $\mathbf{r} = (r_i)_{i=1}^s \in \mathbb{Z}^s$ such that $\mathcal{L} \cong \mathcal{O}_X(r_1 Z_1 + r_2 Z_2 + \cdots + r_s Z_s)$ with all $r_i \leq -1$.*

We will end this section with the following lemma which is a direct consequence of well known facts concerning homology of simplicial complexes.

Lemma 2.9. *Assume that $\text{Supp}(\mathbf{r}) = \text{Supp}(\mathbf{r}_1) \otimes \text{Supp}(\mathbf{r}_2)$. Then the following Künneth formula holds:*

$$H_i(\text{Supp}(\mathbf{r}), K) \cong \bigoplus_{p+q=i} H_p(\text{Supp}(\mathbf{r}_1), K) \cdot H_q(\text{Supp}(\mathbf{r}_2), K).$$

3. DERIVED CATEGORY OF TORIC FIBRATIONS OVER \mathbb{P}^n

Let $\varphi : X \rightarrow \mathbb{P}^n$ be a smooth toric Y -fibration over \mathbb{P}^n . As already observed Y is a smooth toric variety. Let $\dim(Y) = d$. As in the previous sections we denote by $\Sigma_Y \subset \mathbb{Z}^d$ the fan of Y and we will use the notation $\Sigma_n \subset \mathbb{Z}^n$ for the fan of \mathbb{P}^n .

This section contains the main new results of this work. Its goal is to give a structure theorem for the derived category $D^b(X)$ of a toric fibration X over \mathbb{P}^n . This will be achieved by constructing a full strongly exceptional collection of line bundles on X being X a toric fibration over \mathbb{P}^n with fibers a toric variety Y whose derived category $D^b(Y)$ has an orthogonal basis consisting of line bundles.

We start by recalling the notions of exceptional sheaves, exceptional collections of sheaves, strongly exceptional collections of sheaves and full strongly exceptional collections of sheaves as well as the facts on derived categories needed in the sequel.

Definition 3.1. Let X be a smooth projective variety.

(i) A coherent sheaf E on X is *exceptional* if $\text{Hom}(E, E) = K$ and $\text{Ext}_{\mathcal{O}_X}^i(E, E) = 0$ for $i > 0$,

(ii) An ordered collection (E_0, E_1, \dots, E_m) of coherent sheaves on X is an *exceptional collection* if each sheaf E_i is exceptional and $\text{Ext}_{\mathcal{O}_X}^i(E_k, E_j) = 0$ for $j < k$ and $i \geq 0$.

(iii) An exceptional collection (E_0, E_1, \dots, E_m) is a *strongly exceptional collection* if in addition $\text{Ext}_{\mathcal{O}_X}^i(E_j, E_k) = 0$ for $i \geq 1$ and $j \leq k$.

(iv) An ordered collection (E_0, E_1, \dots, E_m) of coherent sheaves on X is a *full (strongly) exceptional collection* if it is a (strongly) exceptional collection and E_0, E_1, \dots, E_m generate the bounded derived category $D^b(X)$.

The following proposition characterizes when an exceptional collection is full.

Proposition 3.2. *(see [3]) Let X be a smooth projective variety. Assume that (E_1, \dots, E_r) is an exceptional collection in $D^b(X)$. Then, the following are equivalent:*

- (a) (E_1, \dots, E_r) is full, i.e. $\langle E_1, \dots, E_r \rangle = D^b(X)$;
- (b) $0 = {}^\perp \langle E_1, \dots, E_r \rangle := \{F \in D^b(X) \mid \text{Ext}^\bullet(F, E_i) = 0 \quad \forall i\}$;
- (c) $0 = \langle E_1, \dots, E_r \rangle^\perp := \{F \in D^b(X) \mid \text{Ext}^\bullet(E_i, F) = 0 \quad \forall i\}$; and
- (d) $0 = {}^\perp \langle E_1, \dots, E_k \rangle \cap \langle E_{k+1}, \dots, E_r \rangle^\perp$ for all k .

Remark 3.3. As mentioned in the Introduction, the existence of a full strongly exceptional collection (E_0, E_1, \dots, E_m) of coherent sheaves on a smooth projective variety X imposes a rather strong restriction on X , namely that the Grothendieck group $K_0(X) = K_0(\mathcal{O}_X\text{-mod})$ is isomorphic to \mathbb{Z}^{m+1} .

Let us illustrate the above definitions with examples:

Example 3.4. (1) Suppose that the vectors v_0, \dots, v_n generate a lattice N of rank n and $v_0 + \dots + v_n = 0$. Let Σ be the fan whose rational cones are generated by any proper subsets of the vectors v_0, \dots, v_n . It is well known that the toric variety associated to the fan Σ is \mathbb{P}^n . Denote by Z the toric divisor associated to v_0 . Then the collection of line bundles $(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(Z), \dots, \mathcal{O}_{\mathbb{P}^n}(nZ))$ is a full strongly exceptional collection of line bundles on \mathbb{P}^n .

(2) Let X_1 and X_2 be two smooth projective varieties and let $(F_0^i, F_1^i, \dots, F_{n_i}^i)$ be a full strongly exceptional collection of locally free sheaves on X_i , $i = 1, 2$. Then,

$$(F_0^1 \boxtimes F_0^2, F_1^1 \boxtimes F_0^2, \dots, F_{n_1}^1 \boxtimes F_0^2, \dots, F_0^1 \boxtimes F_{n_2}^2, F_1^1 \boxtimes F_{n_2}^2, \dots, F_{n_1}^1 \boxtimes F_{n_2}^2)$$

is a full strongly exceptional collection of locally free sheaves on $X_1 \times X_2$ (see [4]; Proposition 4.16). In particular, if Z_n is a toric divisor that generates $\text{Pic}(\mathbb{P}^n)$ and Z_m is a toric divisor that generates $\text{Pic}(\mathbb{P}^m)$, then the collection of line bundles

$$(\mathcal{O}_{\mathbb{P}^n} \boxtimes \mathcal{O}_{\mathbb{P}^m}, \mathcal{O}_{\mathbb{P}^n}(Z_n) \boxtimes \mathcal{O}_{\mathbb{P}^m}, \dots, \mathcal{O}_{\mathbb{P}^n}(nZ_n) \boxtimes \mathcal{O}_{\mathbb{P}^m}, \dots, \mathcal{O}_{\mathbb{P}^n} \boxtimes \mathcal{O}_{\mathbb{P}^m}(mZ_m), \\ \mathcal{O}_{\mathbb{P}^n}(Z_n) \boxtimes \mathcal{O}_{\mathbb{P}^m}(mZ_m), \dots, \mathcal{O}_{\mathbb{P}^n}(nZ_n) \boxtimes \mathcal{O}_{\mathbb{P}^m}(mZ_m))$$

is a full strongly exceptional collection of line bundles on $\mathbb{P}^n \times \mathbb{P}^m$.

(3) Let X be a smooth complete toric variety which is the projectivization of a rank r vector bundle \mathcal{E} over a smooth complete toric variety Y which has a full strongly exceptional collection of locally free sheaves. Then, X also has a full strongly exceptional collection of locally free sheaves (See [4]; Proposition 4.9).

As we said in the introduction $D^b(X)$ is an important algebraic invariant of a smooth projective variety but very little is known about the structure of $D^b(X)$. In particular whether $D^b(X)$ is freely and finitely generated and, hence, we are lead to consider the following problem

Problem 3.5. Characterize smooth projective varieties X which have a full strongly exceptional collection of coherent sheaves and, even more, if there is one made up of line bundles.

This problem is far from being solved and in this paper we will restrict our attention to the particular case of toric varieties since they admit a combinatorial description which allows many invariants to be expressed in terms of combinatorial data.

In [12], Kawamata proved that the derived category of a smooth complete toric variety has a full exceptional collection of objects. In his collection the objects are sheaves rather

than line bundles and the collection is only exceptional and not strongly exceptional. In the toric context, there are a lot of contributions to the above problem. For instance, it turns out that a full strongly exceptional collection made up of line bundles exists on projective spaces ([2]), multiprojective spaces ([4]; Proposition 4.16), smooth complete toric varieties with Picard number ≤ 2 ([4]; Corollary 4.13) and smooth complete toric varieties with a splitting fan ([4]; theorem 4.12). Nevertheless some restrictions are required because in [11], Hille and Perling constructed an example of smooth non Fano toric surface which does not have a full strongly exceptional collection made up of line bundles. Based on these results, the following quite natural conjecture was made (see [6]; Conjecture 3.3 and [7]; Conjecture 1.2):

Conjecture 3.6. *Every smooth complete Fano toric variety has a full strongly exceptional collection of line bundles.*

There is some numerical evidence towards the above conjecture (see, for instance, [6] and [7]). So far only partial results are known.

4. THE MAIN RESULT

The main result of this article provides some more evidence towards this Conjecture.

Consider $\sigma = \langle \bar{v}_1, \dots, \bar{v}_d \rangle$, a maximal cone of Y . Since Y is a smooth toric variety, we can take

$$(4.1) \quad e_1 = \bar{v}_1, \dots, e_d = \bar{v}_d$$

to be a \mathbb{Z} -basis of \mathbb{Z}^d . Let $G_Y = \{\bar{v}_1, \dots, \bar{v}_d, \bar{v}_{d+1}, \dots, \bar{v}_r\}$ be the set of ray generators of Y expressed in this \mathbb{Z} -basis and denote by \tilde{Z}_i the corresponding toric divisors. We fix

$$\langle \tilde{Z}_{d+1}, \tilde{Z}_{d+2}, \dots, \tilde{Z}_r \rangle$$

to be a basis of $\text{Pic}(Y)$. Then, in this basis:

$$(4.2) \quad \tilde{Z}_i = \sum_{j=d+1}^r \gamma_j^i \tilde{Z}_j, \quad 1 \leq i \leq d$$

for some $\gamma_j^i \in \mathbb{Z}$. We will denote by $r(\mathcal{P}_1), \dots, r(\mathcal{P}_l)$ the primitive relations of Y associated to all primitive collections $\mathcal{P}_1, \dots, \mathcal{P}_l$ in Σ_Y .

Let e_{d+1}, \dots, e_{d+n} be a \mathbb{Z} -basis of \mathbb{Z}^n , and consider

$$(4.3) \quad \bar{v}_{r+i} = e_{d+i}, \quad \text{for } 1 \leq i \leq n \quad \text{and} \quad \bar{v}_{r+n+1} = -e_{d+1} - \dots - e_{d+n};$$

so that $G_{\mathbb{P}^n} = \langle \bar{v}_{r+1}, \dots, \bar{v}_{r+n}, \bar{v}_{r+n+1} \rangle$ is the set of ray generators of \mathbb{P}^n . We will denote by \bar{Z}_{r+i} , $1 \leq i \leq n+1$, the toric divisors associated to the ray generators \bar{v}_{r+i} , $1 \leq i \leq n+1$. Let $\bar{Z} := \bar{Z}_{r+n+1}$ be a basis of $\text{Pic}(\mathbb{P}^n)$. On \mathbb{P}^n , we have a unique primitive collection $\mathcal{P} = \{\bar{v}_{r+n+1}, \bar{v}_{r+1}, \dots, \bar{v}_{r+n}\}$ with primitive relation $r(\mathcal{P})$ given by

$$\bar{v}_{r+n+1} + \bar{v}_{r+1} + \dots + \bar{v}_{r+n} = 0.$$

The lattice morphism $\tilde{\varphi} : \mathbb{Z}^d \rightarrow \mathbb{Z}^{d+n} \oplus \mathbb{Z}$ is defined by

$$\tilde{\varphi}(\bar{v}_i) = (\bar{v}_i, 0) =: v_i \quad 1 \leq i \leq r,$$

and the projection $\pi : \mathbb{Z}^{n+d} \rightarrow \mathbb{Z}^n$ is defined by

$$\pi(v_{r+i}) = \bar{v}_{r+i} \quad 1 \leq i \leq n$$

$$\pi(v_{r+n+1}) = \pi(a_1, \dots, a_d, \bar{v}_{r+n+1}) = \bar{v}_{r+n+1}$$

where

$$v_{r+i} := (0, \bar{v}_{r+i}), \text{ for } 1 \leq i \leq n$$

$$v_{r+n+1} = (a_1, \dots, a_d, \bar{v}_{r+n+1}) \text{ for some } a_i \geq 0.$$

Under this identification, X has ray generators $G_X = \langle v_1, \dots, v_{r+n}, v_{r+n+1} \rangle$ with

$$(4.4) \quad v_{r+n+1} = -v_{r+1} - \dots - v_{r+n} + \sum_{i=1}^d a_i v_i, \quad a_i \geq 0.$$

The primitive relations of X are

$$r(\mathcal{P}_1), \dots, r(\mathcal{P}_l), \quad \text{and} \quad v_{r+n+1} + v_{r+1} + \dots + v_{r+n} = \sum_{i=1}^d a_i v_i.$$

We will denote by Z_1, \dots, Z_{r+n}, Z the toric divisors associated to the ray generators v_i , $1 \leq i \leq r+n$ and v_{r+n+1} respectively. Note that as a basis of $\text{Pic}(X)$ we can take

$$(4.5) \quad \text{Pic}(X) = \langle Z_{d+1}, \dots, Z_r, Z \rangle.$$

According to (4.2), in this basis the following linear equivalences hold:

$$(4.6) \quad \begin{aligned} Z_i &= \sum_{j=d+1}^r \gamma_j^i Z_j - a_i Z & 1 \leq i \leq d \\ Z_j &= Z & r+1 \leq j \leq r+n. \end{aligned}$$

Let $\mathcal{L} = \mathcal{O}_X(\alpha_{d+1} Z_{d+1} + \dots + \alpha_r Z_r + \alpha Z)$. If $\mathbf{r} = (c_1, \dots, c_r, b_1, \dots, b_n, b)$ is another representation of \mathcal{L} , i.e. $\mathcal{L} \cong \mathcal{O}_X(\sum_{i=1}^r c_i Z_i + \sum_{i=1}^n b_i Z_{r+i} + bZ)$, then using the relations (4.6), we have that:

$$\alpha_{d+1} Z_{d+1} + \dots + \alpha_r Z_r + \alpha Z = \sum_{i=1}^d c_i \left[\sum_{j=d+1}^r \gamma_j^i Z_j - a_i Z \right] + \sum_{i=d+1}^r c_i Z_i + \left(\left(\sum_{i=1}^n b_i \right) + b \right) Z$$

and thus, the following equations must hold:

$$(4.7) \quad \begin{aligned} \alpha_{d+1} &= \sum_{i=1}^d c_i \gamma_{d+1}^i + c_{d+1} \\ &\vdots \\ \alpha_k &= \sum_{i=1}^d c_i \gamma_k^i + c_k \\ &\vdots \\ \alpha_r &= \sum_{i=1}^d c_i \gamma_r^i + c_r \\ \alpha &= -\sum_{i=1}^d c_i a_i + \sum_{i=1}^n b_i + b. \end{aligned}$$

Remark 4.1. (1) We observe that if $\mathbf{r} = (c_1, \dots, c_r, b_1, \dots, b_n, b) \in \mathbb{Z}^{r+n+1}$ is another representation of $\mathcal{L} = \mathcal{O}_X(\alpha_{d+1}Z_{d+1} + \dots + \alpha_r Z_r + \alpha Z)$ then $\mathbf{r}_1 = (c_1, \dots, c_r) \in \mathbb{Z}^r$ is a representation of $\mathcal{L}' = \mathcal{O}_Y(\alpha_{d+1}\tilde{Z}_{d+1} + \dots + \alpha_r \tilde{Z}_r)$ and $\mathbf{r}_2 = (b_1, \dots, b_n, b) \in \mathbb{Z}^{n+1}$ is a representation of $\mathcal{L}'' = \mathcal{O}_{\mathbb{P}^n}((\alpha + \sum_{i=1}^d c_i a_i)\bar{Z})$.

(2) In the above basis and by abuse of notation if there is no confusion, we will denote by the same letter D the divisor that in Y is expressed as

$$D = \sum_{i=d+1}^r d_i \tilde{Z}_i$$

and the divisor that in X is expressed as

$$D = \sum_{i=d+1}^r d_i Z_i.$$

We are now ready to state our main result.

Theorem 4.2. *Let X be a smooth toric Y -fibration over \mathbb{P}^n . Assume that $\mathcal{F} = (\mathcal{O}_Y, \mathcal{O}_Y(D_1), \dots, \mathcal{O}_Y(D_s))$ is a full strongly exceptional collection of line bundles on Y . Then, the ordered collection \mathcal{C} of line bundles on X :*

$$\begin{aligned} \mathcal{C} = & (\mathcal{O}_X, \mathcal{O}_X(Z), \mathcal{O}_X(2Z), \dots, \mathcal{O}_X(nZ), \\ & \mathcal{O}_X(D_1), \mathcal{O}_X(D_1 + Z), \mathcal{O}_X(D_1 + 2Z), \dots, \mathcal{O}_X(D_1 + nZ), \\ & \vdots \\ & \mathcal{O}_X(D_s), \mathcal{O}_X(D_s + Z), \mathcal{O}_X(D_s + 2Z), \dots, \mathcal{O}_X(D_s + nZ)) \end{aligned}$$

is a full strongly exceptional collection of line bundles in $D^b(X)$.

Proof. First of all we will prove that \mathcal{C} is strongly exceptional. To this end, given any pair of line bundles $\mathcal{L}_i, \mathcal{L}_j$ of the ordered collection \mathcal{C} we have to check that

$$\text{Ext}^k(\mathcal{L}_i, \mathcal{L}_j) = H^k(X, \mathcal{L}_j \otimes \mathcal{L}_i^\vee) = 0 \quad \text{for any } i, j \text{ and } k > 0$$

and

$$\text{Ext}^k(\mathcal{L}_i, \mathcal{L}_j) = H^k(X, \mathcal{L}_j \otimes \mathcal{L}_i^\vee) = 0 \quad \text{for } i > j \text{ and } k \geq 0.$$

Since we are dealing with line bundles and we already know that $H^i(\mathcal{O}_X) = 0$ for $i \geq 1$, it is enough to see that:

- (a) $\mathcal{O}_X(D_i - D_j + kZ)$ is acyclic for $i \geq j$ and $-n \leq k \leq n$.
- (b) $H^l(\mathcal{O}_X(D_j - D_i + kZ)) = 0$ for $l \geq 0$, $i > j$ and $-n \leq k \leq n$.
- (c) $H^0(\mathcal{O}_X(-kZ)) = 0$ for $1 \leq k \leq n$.
- (d) $H^l(\mathcal{O}_X(-D_i - kZ)) = 0$ for $l \geq 0$, $i \geq 1$ and $0 \leq k \leq n$.
- (e) $\mathcal{O}_X(D_i + kZ)$ is acyclic for $i \geq 1$ and $0 \leq k \leq n$.

(a) Assume that $\mathcal{O}_X(D_i - D_j + kZ)$ is not acyclic. By Proposition 2.7, this means that there exists some $m < \dim X$ and a representation $\mathbf{r} = (c_1, \dots, c_r, b_1, \dots, b_n, b) \in \mathbb{Z}^{r+n+1}$ of $\mathcal{O}_X(D_i - D_j + kZ)$ such that $H_m(\text{Supp}(\mathbf{r}), K) \neq 0$. On the other hand, by Remark 4.1;

(1) together with Proposition 2.3 and the definition of the simplicial complex $\text{Supp}(\mathbf{r})$, we obtain

$$\text{Supp}(\mathbf{r}) = \text{Supp}(\mathbf{r}_1) \otimes \text{Supp}(\mathbf{r}_2)$$

where $\mathbf{r}_1 = (c_1, \dots, c_r) \in \mathbb{Z}^r$ is a representation of $\mathcal{O}_Y(D_i - D_j)$ and $\mathbf{r}_2 = (b_1, \dots, b_n, b) \in \mathbb{Z}^{n+1}$ is a representation of $\mathcal{O}_{\mathbb{P}^n}((k + \sum_{i=1}^d c_i a_i) \overline{Z})$. Therefore, by Lemma 2.9, we have

$$0 \neq H_m(\text{Supp}(\mathbf{r}), K) \cong \bigoplus_{p+q=m} H_p(\text{Supp}(\mathbf{r}_1), K) \cdot H_q(\text{Supp}(\mathbf{r}_2), K).$$

Hence, there exist integers p_0 and q_0 such that $p_0 + q_0 = m$,

$$H_{p_0}(\text{Supp}(\mathbf{r}_1), K) \neq 0 \quad \text{and} \quad H_{q_0}(\text{Supp}(\mathbf{r}_2), K) \neq 0.$$

Since \mathcal{F} is a strongly exceptional collection of line bundles on Y , $\mathbf{r}_1 = (c_1, \dots, c_r) \in \mathbb{Z}^r$ is a representation of an acyclic line bundle $\mathcal{O}_Y(D_i - D_j)$ on Y which forces to have $p_0 = \dim Y$. Thus, by Corollary 2.8, $\text{Supp}(\mathbf{r}_1)$ is the entire fan Σ_Y and, in particular, $c_i \geq 0$ for all $1 \leq i \leq r$. Since $\mathbf{r}_2 = (b_1, \dots, b_n, b) \in \mathbb{Z}^{n+1}$ is a representation of $\mathcal{O}_{\mathbb{P}^n}((k + \sum_{i=1}^d c_i a_i) \overline{Z})$ with $-n \leq k \leq n$, $c_i a_i \geq 0$ for $1 \leq i \leq d$ and $H_{q_0}(\text{Supp}(\mathbf{r}_2), K) \neq 0$, we must have $q_0 = n$. Applying again Corollary 2.8, we get that $\text{Supp}(\mathbf{r}_2)$ is the entire fan $\Sigma_{\mathbb{P}^n}$ and in particular we have $b_i \geq 0$ for all $1 \leq i \leq n$ and $b \geq 0$. Therefore, putting altogether we have that $\mathbf{r} = (c_1, \dots, c_r, b_1, \dots, b_n, b) \in \mathbb{Z}^{r+n+1}$ is a representation with all the coefficients positive and, by Corollary 2.8, this implies that it only contributes to $H^0(\mathcal{O}_X(D_i - D_j + kZ))$ which contradicts the fact that $m < \dim X$.

(b) Assume that there exists $l \geq 0$ such that $H^l(\mathcal{O}_X(D_j - D_i + kZ)) \neq 0$. By Proposition 2.7, this implies the existence of an integer m , $0 \leq m \leq \dim X$ and a representation $\mathbf{r} = (c_1, \dots, c_r, b_1, \dots, b_n, b) \in \mathbb{Z}^{n+r+1}$ of $\mathcal{O}_X(D_j - D_i + kZ)$ such that $H_m(\text{Supp}(\mathbf{r}), K) \neq 0$. On the other hand, by Remark 4.1; (1) together with Proposition 2.3 and the definition of the simplicial complex $\text{Supp}(\mathbf{r})$, we obtain

$$\text{Supp}(\mathbf{r}) = \text{Supp}(\mathbf{r}_1) \otimes \text{Supp}(\mathbf{r}_2)$$

where $\mathbf{r}_1 = (c_1, \dots, c_r) \in \mathbb{Z}^r$ is a representation of $\mathcal{O}_Y(D_j - D_i)$ and $\mathbf{r}_2 = (b_1, \dots, b_n, b) \in \mathbb{Z}^{n+1}$ is a representation of $\mathcal{O}_{\mathbb{P}^n}((k + \sum_{i=1}^d c_i a_i) \overline{Z})$. Applying Lemma 2.9, we get

$$0 \neq H_m(\text{Supp}(\mathbf{r}), K) \cong \bigoplus_{p+q=m} H_p(\text{Supp}(\mathbf{r}_1), K) \cdot H_q(\text{Supp}(\mathbf{r}_2), K).$$

Hence, there exist integers p_0 and q_0 such that $p_0 + q_0 = m$,

$$(4.8) \quad H_{p_0}(\text{Supp}(\mathbf{r}_1), K) \neq 0 \quad \text{and} \quad H_{q_0}(\text{Supp}(\mathbf{r}_2), K) \neq 0.$$

By hypothesis \mathcal{E} is a strongly exceptional collection on Y . Therefore, $\mathbf{r}_1 = (c_1, \dots, c_r) \in \mathbb{Z}^r$ is a representation of an acyclic line bundle $\mathcal{O}_Y(D_j - D_i)$ on Y without global sections. Thus, $H_p(\text{Supp}(\mathbf{r}_1), K) = 0$ for all $p \geq 0$ which contradicts (4.8).

(c) It easily follows from that fact that Z is an effective toric divisor.

(d) The proof of (d) is analogous to the proof of (b) and we leave it to the reader.

(e) The proof of (e) is analogous to the proof of (a) and we leave it to the reader.

Putting altogether we conclude that \mathcal{C} is a strong exceptional collection of line bundles on X .

Let us now see that \mathcal{C} is also full. To this end, we will proceed by induction on n . For $n = 0$ we have a Y -fibration over a point. So the result follows by assumption on Y . Fix $n > 0$. We already know that \mathcal{C} is an exceptional collection. So, by Proposition 3.2, it is enough to prove that for any $F \in \mathcal{C}^\perp$, $F = 0$.

Denote by $X_n = X$ the Y -fibration over \mathbb{P}^n given in the statement. According to the previous notation, it has ray generators

$$v_1, \dots, v_{r+n}, \quad \text{and} \quad v_{r+n+1} = -v_{r+1} - \dots - v_{r+n} + \sum_{i=1}^d a_i v_i, \quad a_i \geq 0.$$

Notice that the toric divisor Z_{r+n} on X_n can be naturally identified with the Y -fibration X_{n-1} over \mathbb{P}^{n-1} which has as ray generators

$$v_1, \dots, v_{r+n-1}, \quad \text{and} \quad v_{r+n} = -v_{r+1} - \dots - v_{r+n-1} + \sum_{i=1}^d a_i v_i, \quad a_i \geq 0.$$

Moreover, we have the short exact sequence

$$(4.9) \quad 0 \rightarrow \mathcal{O}_{X_n}(-Z_{r+n}) \cong \mathcal{O}_{X_n}(-Z) \rightarrow \mathcal{O}_{X_n} \rightarrow i_* \mathcal{O}_{X_{n-1}} \rightarrow 0$$

being $i : X_{n-1} \hookrightarrow X_n$ the natural inclusion.

Claim: For any $F \in \mathcal{C}^\perp$, $i^* F = 0$.

Proof of the Claim: Let $F \in \mathcal{C}^\perp$. This means that, for $0 \leq i \leq s$, $0 \leq k \leq n$, we have:

$$(4.10) \quad 0 = \text{Ext}^\bullet(\mathcal{O}_{X_n}(D_i + kZ), F) = H^\bullet(F(-D_i - kZ))$$

where we have set $D_0 := 0$. Twisting by $F(-D_i - jZ)$, $0 \leq i \leq s$ and $1 \leq j \leq n-1$, the short exact sequence (4.9) and taking cohomology we get the long exact sequence

$$\dots \rightarrow H^l(F(-D_i - jZ)) \rightarrow H^l(F(-D_i - jZ) \otimes i_* \mathcal{O}_{X_{n-1}}) \rightarrow H^{l+1}(F(-D_i - (j+1)Z)) \rightarrow \dots$$

Using the equalities (4.10), we deduce that for $0 \leq i \leq s$ and $1 \leq j \leq n-1$

$$0 = H^\bullet(X_{n-1}, F(-D_i - jZ) \otimes i_* \mathcal{O}_{X_{n-1}}) = \text{Ext}^\bullet(\mathcal{O}_{X_{n-1}}(D_i + jZ), i^* F),$$

namely that $i^* F \in \mathcal{C}_{n-1}^\perp$ where

$$\begin{aligned} \mathcal{C}_{n-1} := & (\mathcal{O}_{X_{n-1}}, \mathcal{O}_{X_{n-1}}(Z), \mathcal{O}_{X_{n-1}}(2Z), \dots, \mathcal{O}_{X_{n-1}}((n-1)Z), \\ & \mathcal{O}_{X_{n-1}}(D_1), \mathcal{O}_{X_{n-1}}(D_1 + Z), \mathcal{O}_{X_{n-1}}(D_1 + 2Z), \dots, \mathcal{O}_{X_{n-1}}(D_1 + (n-1)Z), \\ & \vdots \\ & \mathcal{O}_{X_{n-1}}(D_s), \mathcal{O}_{X_{n-1}}(D_s + Z), \mathcal{O}_{X_{n-1}}(D_s + 2Z), \dots, \mathcal{O}_{X_{n-1}}(D_s + (n-1)Z)). \end{aligned}$$

Since we have already seen that \mathcal{C} is an exceptional collection, the same is true for \mathcal{C}_{n-1} and by hypothesis of induction \mathcal{C}_{n-1} is full, i.e. it generates $D^b(X_{n-1})$. Therefore, applying Proposition 3.2, we get $i^* F = 0$ which finishes the proof of the Claim.

It follows from the Claim that for any $F \in \mathcal{C}^\perp$ and $i : X_{n-1} \hookrightarrow X_n$, $i^* F = 0$. Thus $\text{Im}(i) \cap \text{Supp}(F) = 0$, i.e. $\text{Supp}(F)$ does not meet X_{n-1} . Note that $X_{n-1} = \varphi^*(H)$ where

$H \in \mathcal{O}_{\mathbb{P}^n}(1)$ and $\varphi : X_n \rightarrow \mathbb{P}^n$ is the smooth toric fibration of the statement. Therefore, moving the toric divisor $X_{n-1} \subset X_n$ we cover the whole variety $X_n = X$ and we conclude that $\text{Supp}(F) = 0$.

Summing up we have seen that for any $F \in \mathcal{C}^\perp$, $F = 0$. hence, applying once more Proposition 3.2, we obtain that \mathcal{C} is also full which proves what we want. \square

Remark 4.3. (1) We want to point out that with the above notation we have that X is a Fano toric variety if and only if Y is a Fano toric variety and $\sum_i a_i \leq n$.

(2) Note that taking $a_1 = \dots = a_d = 0$, then $X \cong Y \times \mathbb{P}^n$ and we recover the collection given in Example 3.4; (2).

5. FINAL COMMENTS

The hypothesis "Fano" in Conjecture 3.6 can certainly be relaxed. In fact, in [4]; Theorem 4.12, full strongly exceptional collections of line bundles on families of smooth complete toric varieties none of which is entirely Fano were constructed.

This article provides more examples of Fano and non Fano toric varieties with a full strongly exceptional sequence. As previously observed Y is Fano only when $\sum_1^d a_i < n+1$. Notice that because $a_i \geq 0$ the fibrations will be Fano only in a finite number of cases.

Our proof uses cohomology vanishing and induction over \mathbb{P}^n . Even though these techniques are not extendable to a different toric basis we do believe that the result should hold true in a more general context.

Theorem 4.2 suggests - and proves in many cases - the following conjecture:

Conjecture 5.1. *Let X be a smooth complete toric variety. Assume that X is a F -fibration over Y and that Y and F are smooth toric varieties with a full strongly exceptional collection of (line) bundles. Then, X has a full strongly exceptional collection consisting of (line) bundles.*

The following result gives support to the above conjecture:

- The above Conjecture is true if X is the trivial fibration over Y with fiber F , i.e. $X \cong Y \times F$ (See [4]; Proposition 4.16).
- The above Conjecture is true if $Y \cong \mathbb{P}^r$ for some integer r (See Theorem 4.2).
- The above Conjecture is true if $F \cong \mathbb{P}^m$ for some integer m (See [5]; Main Theorem).

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