1+1 Dimensional Quantum Field Theories of Anyons

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Based on joint work with P. Moosavi

Introduction

Bosons, Fermions and Anyons

Bosons & Fermions: N-identical-particle Schrödinger wave function $\Psi : (\mathbb{R}^d)^N \to \mathbb{C}$ in spatial dimension $d \geq 3$ is **symmetric** or **anti-symmetric** under permutations $\sigma \in S_N$

$$\Psi(x_1, \ldots, x_N) = (\pm)^\sigma \Psi(x_{\sigma_1}, \ldots, x_{\sigma_N})$$

For local relativistic QFT, fiels commute or anti-commute (**Spin-Statistics theorem**), see [Fierz '39] [Pauli '40]

$$\psi(x)\psi(x') = (\pm)\psi(x')\psi(x) \ , \qquad x \neq x'$$

Anyons: In $d = 2$, continuous exchange transformation allows for representations of the **Braid group**, more formally $\pi_1(X_2^N) = B_N$, for instance

$$\Psi(x_1, x_2) = \mathrm{e}^{\mathrm{i}\alpha\pi} \Psi(x_2, x_1) \ , \qquad \alpha \in [0, 2)$$

[Streater, Wilde '70] [Leinaas, Myrheim '77] [Goldin, Menikoff, Sharp '81] [Wilczek '82]

• Anyonic statistics also studied in **one spatial dimension**, [Klaiber '68] [Fröhlich, Marchetti '89] [Polychronakos '98]

Boson-Anyon Correspondence

Setting:

• Let $\alpha > 0$ be the statistics parameter

• We restrict ourselves on the interval $[-L/2, L/2]$ (IR cut-off). Observables satisfy periodic b.c. (momenta $p \in (2\pi/L)\mathbb{Z}$)

• Introduce **bosonic Hilbert space** with generated by the operators (acting on the vacuum $|\Psi_0\rangle$)

$$\{\rho_r(p)\}_{p\in 2\pi/L\mathbb{Z}, r=\pm}\ , \qquad \{R_r\}_{r=\pm}$$

satisfying $\rho_r(p)^\dagger = \rho_r(-p)$, $R_r^\dagger = R_r^{-1}$

$$\big[\rho_r(p), \rho_{r'}(-p')\big] = r\frac{Lp}{2\pi}\delta_{r,r'}\delta_{p,p'} \qquad \rho_r(p)|\Psi_0\rangle = 0\ , \qquad \forall p \geq 0$$

$$\big[\rho_r(0), R_{r'}\big] = r\sqrt{\alpha}\delta_{r,r'}R_{r'}\ , \qquad \langle\Psi_0|R_+^{q_+}R_-^{q_-}|\Psi_0\rangle = \delta_{q_+,0}\delta_{q_-,0}$$

• $\{\rho_r(p)\}_{p\neq 0}$ prop to creation and annihilation operators

• $\rho_r(0)$ are charge operators and R_r are raising/lowering unitaries.

Fields: Let $\epsilon > 0$ (UV cut-off), introduce regularized fields [Carey, Langmann '99]

$$\psi_r^-(x;\epsilon) = L^{-\alpha/2} {}^{\times}_{\times} R_r^{-r} \exp\left(\mathrm{i} r\sqrt{\alpha}\frac{2\pi}{L}\left[x\rho_r(0) + \sum_{p\neq 0}\frac{1}{\mathrm{i}p}\rho_r(p)\mathrm{e}^{\mathrm{i}px-\epsilon|p|/2}\right]\right) {}^{\times}_{\times}$$

- Operator-valued distribution as $\epsilon \to 0^+$
- **Vertex Operators** on a Hilbert space [Klaiber '67] [Carey, Langmann '99]
- More general operators, e.g., composite anyons, inhomogeneous setting [Moosavi et al. '17]

Exchange Relations: Equal time, $x \neq x'$

$$\psi_r^q(x;\epsilon)\psi_r^{q'}(x';\epsilon') = \mathrm{e}^{-\mathrm{i}r\pi qq'\alpha\,\mathrm{sgn}(x-x';\epsilon+\epsilon')}\psi_r^{q'}(x';\epsilon')\psi_r^q(x;\epsilon)$$

$$\psi_r^q(x;\epsilon)\psi_{-r}^{q'}(x';\epsilon') = \mathrm{e}^{-\mathrm{i}r\pi qq'\alpha}\psi_{-r}^{q'}(x';\epsilon')\psi_r^q(x;\epsilon)$$

- Interpolation between α even (bosons) and odd (fermions). Special case $\alpha = 1$ are "standard" Dirac Fermions.
- $\mathrm{sgn}(x;\epsilon)$ is a smooth regularization of the sign function, also found in [Fröhlich, Marchetti '89] and peculiar of one dimension

Why correspondence?:

• Generalization boson-fermion correspondence (Hilbert spaces) [Carey, Hurst '85]

• Boson operators are the **anyon currents** (anyon normal-ordering)

Anyon Normal-ordering: $\psi_r^{\pm}(x)$ operator-valued distribution, $\psi_r^+(x)\psi_r^-(x)$ meaningless

• Point-splitting/Operator product expansions

$$N[\psi_r^+(x)\partial_x^n\psi_r^+(x)] = \lim_{\epsilon\to 0}\lim_{x'\to x} N_\epsilon[\psi_r^+(x;\epsilon)\partial_{x'}^n\psi_r^+(x';\epsilon)]$$

$$\boxed{\rho_r(x) = N[\psi_r^+(x)\psi_r^-(x)]}$$

• Description of the models simpler in terms of the boson operators (**bosonisation**). Models exactly solvable via **Bogoliubov transformation**.

Remarks: Anomalous commutators (Schwinger terms)

$$\Big[N[\psi_r^+(x)\psi_r^-(x)], N[\psi_r^+(x)\psi_r^-(x)]\Big] \neq 0$$

QFTs of Anyons

Motivations and Goals

Question: Anyonic statistics interpolates between bosons and fermions: are physical properties of fermionic QFTs sensitive to the statistics?

Goals:

- Makes sense of some one dimensional QFTs for anyons
  - i) Free relativistic anyons
  - ii) Current-current interaction [Luttinger '63][Mattis, Lieb ' 65] / [Thirring '58][Klaiber '68]
  - iii) $U(1)$-gauge field interaction [Schwinger '62][Lowenstein, Swieca '71]
- Study condensation via ODLRO [Penrose, Onsager '56] [Yang '62]. ODLRO studied in the anyonic Lieb-Liniger model [Colcelli, Trombettoni '18 -'20]

Remark: Luttinger and Schwinger models **exactly solvable**, expect the anyonic extensions to be likewise (exact computation correlation functions)

Free Relativistic Anyons

Hamiltonian: Relativistic (Dirac) anyons with "Fermi velocity" v_0 formally descr. by

$$H_0 := \sum_{r=\pm} \int_{-L/2}^{L/2} \mathrm{d}x N[\psi_r^+(x)(-\mathrm{i}rv_0\partial_x)\psi_r^-(x)]$$

which is made precise in **bosonised form** as

$$H_0 = \sum_{r=\pm} \left(\frac{\pi}{L}\rho_r(0)^2 + \sum_{p>0} \frac{2\pi}{L}\rho_r(-rp)\rho_r(rp) \right)$$

Exactly Solvable: Free boson Hamiltonian with GS $|\Psi_0\rangle$ (α independent)

- Excitations and excitation energies explicit
- Correlation functions, e.g.,

$$\lim_{L\to\infty} \lim_{\epsilon\to 0^+} \langle\Psi_0|\psi_r^+(x,t;\epsilon)\psi_{r'}^-(x',t';\epsilon)|\Psi_0\rangle$$

$$= \frac{\delta_{r,r'}}{(2\pi)^\alpha} \left(\frac{\mathrm{i}r}{x - x' - rv_0(t-t') + \mathrm{i}r0^+} \right)^\alpha$$

Anyonic Luttinger Model

Hamiltonian: Adding current-current interaction $H_{\rm AL} := H_0 + H_{\rm int} - E_{\rm AL}$

$$H_{\rm int} := \sum_{r,r'=\pm} \frac{\pi v_0 \lambda}{L} \int_{-L/2}^{L/2} \mathrm{d}x \, N[\psi_r^+(x)\psi_r^-(x)] N[\psi_r^+(x)\psi_{r'}^-(x)]$$

which, by rearranging the potential and by **bosonisation** is made precise as

$$H_{\rm int} := v_0 \lambda \sum_{r=\pm} \Bigg(\left[\rho_r(0)\rho_{-r}(0) + \rho_r(0)^2\right] + \sum_{p>0} \frac{2\pi}{L} \left[\rho_r(-p)\rho_{-r}(p) + \rho_r(-p)\rho_r(p)\right] \Bigg)$$

Remarks:

- This is **local** version, requires **multiplicative renormalization**
- Physical properties captured by

Renormalized velocity	$v := v_0\sqrt{1+2\lambda}$
Luttinger parameter	$K := 1/\sqrt{1+2\lambda}$

Anyonic Schwinger Model

Gauge Field:

- Electrodynamics with $F_{\mu\mu'} = \epsilon_{\mu\mu'} E/c$
- Quantization of E and A (temporal or Weyl gauge)

Hamiltonian: Gauge covariant derivative $\mathrm{i}\partial_x \to \mathrm{i}\partial_x - eA(x)$

$$H_{\mathrm{AS}} := \frac{1}{2}\int_{-L/2}^{L/2} \mathrm{d}x\, E(x)^2 + \sum_{r=\pm}\int_{-L/2}^{L/2} \mathrm{d}x\, \widetilde{N}\left[\psi_r^+(x) r v_0(-\mathrm{i}\partial_x + eA(x))\psi_r^+(x)\right] - E_{\mathrm{AS}}$$

- **Gauge invariance**, Gauge invariant normal-ordering

Gauss' Law: Non-dynamical constraint $\partial_x E = \varrho_{\mathrm{tot}}$

$$\widehat{G}(p) := -\mathrm{i}p\widehat{E}(p) + \frac{e}{\sqrt{\alpha}}\big(\rho_+(p) + \rho_-(p)\big) = \widehat{\varrho}_{\mathrm{ext}}(p)$$

- G generator (small) gauge transformations

Physical Applications

What is Off-Diagonal Long-Range Order?

Penrose-Onsager Criterion: For a homogeneous system, condensation (BEC, Superconductivity) when

$$\gamma^{(1)}(x,x') = \begin{pmatrix} \langle \psi_+^+(x)\psi_+^-(x)\rangle & \langle \psi_+^+(x)\psi_-^-(x)\rangle \\ \langle \psi_-^+(x)\psi_+^-(x)\rangle & \langle \psi_-^+(x)\psi_-^-(x)\rangle \end{pmatrix}$$

has largest eigenvalue $\lambda_1 \sim O(Volume)$. Via Fourier transform

$$\lambda_1 = \overline{n}_{k_{\min}} \qquad \overline{n}_k = \sum_{r=\pm} \int_{-L/2}^{L/2} \mathrm{d}x\, \tilde{\gamma}^{(1)}_{r,r}(x,0)\, \mathrm{e}^{-\mathrm{i}krx}$$

- ODLRO: Condensation $\leftrightarrow$ **decay/integrability** of $\gamma^{(1)}(x,0)$

Mesoscopic Condensation:

$\overline{n}_{k_{\min}} \sim L^{\mathcal{C}}$ **Condensation parameter** $\mathcal{C}$ close to 1

For **finite but large systems**, e.g. cold atoms, experimentally observable phenomenon!

Solution of the AL Model

Exact Solution:

- Make non-local with approximate delta function $\delta_a \to \delta$

$$H_{\text{int}}(a) := \sum_{r=\pm} \frac{1}{L} \int_{-L/2}^{L/2} \mathrm{d}x \, \mathrm{d}x' \, N[\psi_r^+(x)\psi_r^-(x)]\delta_a(x-x')N[\psi_{r'}^+(x')\psi_{r'}^-(x')]$$

- Hamiltonian diagonalized via **Bogoliubov transformation** $S_{\text{AL}}(a)$ and GS $|\Psi_{\text{AL}}(a)\rangle = \mathrm{e}^{-\mathrm{i}S_{\text{AL}}(a)}|\Psi_0\rangle$
- Singular local limit via **multiplicative renormalization**

$$G^{\text{AL}}_{r,r'}(x-x';t-t';\epsilon) := \lim_{a\to 0} Z^{-2}_{a;\epsilon}\langle\Psi_{\text{AL}}(a)|\,\psi_r^+(x,t;\epsilon)\psi_{r'}^-(x',t';\epsilon)\,|\Psi_{\text{AL}}(a)\rangle$$

Two-point Correlation Function:

$$\lim_{L\to\infty} \lim_{\epsilon\to 0^+} G^{\text{AL}}_{r,r'}(x;t;\epsilon)$$

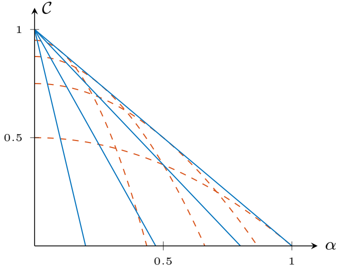
$$= \frac{\delta_{r,r'}}{(2\pi)^\alpha}\Big(\frac{\mathrm{i}}{x-vt+\mathrm{i}r0^+}\Big)^{2\alpha\Delta_r^+}\Big(\frac{-\mathrm{i}}{x+vt-\mathrm{i}r0^+}\Big)^{2\alpha\Delta_r^-} \qquad \Delta_r^\pm := \frac{(K\pm r)^2}{8K}$$

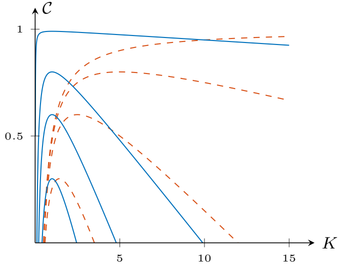
Results: Mesoscopic Condensation in the AL Model

Mesoscopic Condensation:

- Algebraic behaviour with decay $\sim |x|^{-\alpha(K^2+1)/2K}$, $\overline{n}_k$ decreasing

$$\mathcal{C}_{\mathrm{AL}} = 1 - \alpha \frac{K^2+1}{2K}$$





Comparison: **Anyonic Luttinger Model** vs Anyonic Lieb-Liniger Model

Solution of the AS Model

Gauge Transformations: Smooth periodic maps $[-L/2, L/2] \to \mathrm{U}(1)$:

$$x \mapsto \mathrm{e}^{\mathrm{i}\Lambda(x)}, \quad \Lambda(x) = 2\pi w \frac{x}{L} + \Lambda_{\mathrm{small}}(x) \ , \qquad w \in \mathbb{Z}$$

where $\Lambda_{\mathrm{small}}(x)$ is a periodic real function

Action on operators:

$$\psi_r^-(x) \to \mathrm{e}^{\mathrm{i}\alpha\Lambda(x)}\psi_r^-(x) \qquad \left(\psi_r^-(x+L) = \mathrm{e}^{\mathrm{i}\pi\alpha Q}\psi_r^-(x)\mathrm{e}^{\mathrm{i}\pi\alpha Q}\right)$$

$$A(x) \to A(x) - \frac{\alpha}{2\pi e}\partial_x\Lambda(x) \qquad E(x) \to E(x)$$

Anomaly: because $\left[\rho_r(p), \rho_r(-p)\right] = rLp/2\pi$

$$\rho_r(p) \to \rho_r(p) + \frac{r\sqrt{\alpha}}{2\pi}\widehat{\partial\Lambda}(p)$$

Gauge invariant currents

$$\widetilde{\rho}_r(p) := \rho_r(p) + r\frac{e}{\sqrt{\alpha}}\widehat{A}(p)$$

Representation:

- On physical Hilbert space $\mathcal{H}_{\rm phys}$ only **gauge invariant** quantities (up to $U(1)$)
- Gauss' law generator is **gauge invariant**

$$\widehat{G}(p) \equiv -\mathrm{i}p\widehat{E}(p) + \frac{e}{\sqrt{\alpha}}\left(\widetilde{\rho}_+(p) + \widetilde{\rho}_-(p)\right)$$

Gauss' law is **exact** $\Rightarrow U(1)$ representation of (small) gauge transformations

- Chiral current $\rho_+ - \rho_-$ **not** gauge invariant, that is, chiral symmetry is **broken**

$$\mathcal{U}_{\rm large}|\Psi_{AS}(\theta)\rangle = \mathrm{e}^{-\mathrm{i}\theta}|\Psi_{AS}(\theta)\rangle \ , \qquad \theta \in [0, 2\pi)$$

$$\mathcal{H}_{\rm phys} = \int^{\oplus} \mathcal{H}_{\rm phys}(\theta)$$

- Wilson lines: $w(x) = 2\pi e \partial_x^{-1} A(x) = \lim_{\epsilon \to 0^+} w_\epsilon(x)$

$$\psi_r^-(x;\epsilon) \to \mathrm{e}^{\mathrm{i}w_\epsilon(x)}\psi_r^-(x;\epsilon) =: \widetilde{\psi}_r^-(x;\epsilon)$$

$$G^{\rm AS}_{\theta;r,r'}(x,x';t,t') := \langle\Psi_{AS}(\theta)|\,\widetilde{\psi}_r^+(x,t;\epsilon)\widetilde{\psi}_r^-(x',t';\epsilon)\,|\Psi_{AS}(\theta)\rangle$$

Results: Screening in the AS Model

Mass Generation: Spectrum $E_{\mathrm{AS},\mathbf{n}} = \sum_p \omega(p)n(p) + \sum_{p\neq 0} \frac{v_0^2|\hat{G}(p)|^2}{2L\omega(p)^2}$

$$\omega(p) := v_0\sqrt{m^2v_0^2 + p^2} \qquad m := \frac{e}{\sqrt{\pi\alpha}v_0^{3/2}}$$

Screening: Study the "particle-antiparticle" potential

$$|\Psi_{\mathrm{AS}}(\theta; d)\rangle : \qquad \varrho_{ext}(x) = \mathsf{e}\big(\delta(x - d/2) + \delta(x + d/2)\big)$$

$$V_{\mathrm{AS}}(d) := \langle\Psi_{\mathrm{AS}}(\theta; d)|H_{\mathrm{AS}}|\Psi_{\mathrm{AS}}(\theta; d)\rangle - \langle\Psi_{\mathrm{AS}}(\theta)|H_{\mathrm{AS}c}|\Psi_{\mathrm{AS}}(\theta)\rangle$$

• In the **vacuum**, $V_{\mathrm{vacuum}}(d)$ is the Coulomb potential, which grows linearly $V_{\mathrm{vacuum}}(d) \sim d$

• If charges are screened, they do not interact at large distance and $V(d)$ saturates

$$\boxed{V_{\mathrm{AS}}(d) = \frac{\mathsf{e}^2}{2mv_0}\left(1 - \mathrm{e}^{-dmv_0}\right)}$$

Two-point Correlation Function:

$$\lim_{L\to\infty} G^{\mathrm{AS}}_{\theta;r,r'}(x,0;t,0) = \lim_{L\to\infty} \mathrm{e}^{\mathrm{i}\frac{(r-r')\theta}{2}} \tilde{G}_{0;r,r'}(x;t) R_{r,r'}(t) \mathrm{e}^{\alpha K_{r,r'}(x;t)}$$

Properties:

- $\tilde{G}_{0;r,r}(x;0)$ free anyon two-point correlation function. $\tilde{G}_{0;r,-r}(x;t) = 1$
- **No ODLRO & asymptotic freedom**:

  If $m = 0$ $K_{r,r'}(x;0) = 0$, otherwise for $|x|$ large

  $$K_{r,r'}(x;0) \lesssim -mv_0|x|$$

  Also, $\lim_{x\to 0} K_{r,r'}(x;0) = 0$ for $m \geq 0$.
- **Chiral condensate & revival**

  If $m > 0$ $R_{r,r'}(0)$ uniformly bounded, $R_{r,r'}(2\pi j/mv_0^2) \sim_j R_{r,-r}(0)$ for $j \in \mathbb{Z}$ and

  $$\lim_{L\to\infty} R_{r,-r}(0) = \left(\frac{mv_0\mathrm{e}^{\gamma}}{4\pi}\right)^{\alpha}$$

Furthermore, we have **suppression** [Kasher, Kogut, Susskind '73 - '74]

$$|R_{r,r'}(2\pi j/mv_0^2)| \leq \mathrm{e}^{-CtL^2} \qquad t \in [-\pi,\pi]/mv_0^2$$

Conclusions

- Given an introduction to anyons in one dimension via the **boson-anyon correspondence** and we have
- Constructed one dimensional interacting QFTs of anyons and controlled their correlation functions
- Showed that ODLRO possible in the weaker form of **mesoscopic condensation** in the AL model
- Showed that the AS model has a **mass generation** and hence exhibits **screening** regardless of the statistics parameter

Perspectives:

- How does the boson-anyon correspondence compare with with other approaches to anyons?
- Study other statistics-dependent properties, i.e., stability of matter
- Is it possible to interpret anyons as "creation/annihilation operators" on a suitable Hilbert space?

Thank you!