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Hypercomplex analysis

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Contents

1. Quaternionic analysis

The division ring of quaternions is denoted by $\mathbb{H}$ and is equal to

$$\mathbb{H} = \{t + x\,\mathfrak{i} + y\,\mathfrak{j} + z\,\mathfrak{k} \mid t, x, y, z \in \mathbb{R}\}$$

where $\mathfrak{i}^2 = \mathfrak{j}^2 = \mathfrak{k}^2 = -1$ and $\mathfrak{i}\,\mathfrak{j} = \mathfrak{k}, \mathfrak{j}\,\mathfrak{k} = \mathfrak{i}, \mathfrak{k}\,\mathfrak{i} = \mathfrak{j}$. Quternionic analysis is then the study of functions $f\colon U \to V$ where $U, V \subseteq \mathbb{H}$. Quternions inherit the topological notions of continuity from Euclidean space since $\mathbb{H} \cong \mathbb{R}^4$ as $\mathbb{R}$-vector spaces. We can therefore begin our study of quaternionic analysis with the definition of a quaternionic derivative [3].

Definition 1.1. A function $f\colon U \to V$ where $U, V \subseteq \mathbb{H}$ has a *left quaternionic derivative* at a point $w_0 \in U$ if the limit

$$\lim_{h\to 0} h^{-1}(f(w_0 + h) - f(w_0)) \tag{1}$$

exists. Similarly we say that f has a *right quaternionic derivative* at a point $w_0 \in U$ if the limit

$$\lim_{h\to 0} (f(w_0 + h) - f(w_0))h^{-1} \tag{2}$$

exists.

The following examples show that this definition is in fact to restrictive to be of any practical use.

Example 1.2. Let us consider the function $f\colon \mathbb{H} \to \mathbb{H}$ defined as $f(w) = w^2$. We then calculate that

$$h^{-1}(f(w+h) - f(w)) = h^{-1}((w+h)^2 - w^2) = h^{-1}(wh + hw + h^2) = h^{-1}wh + w + h.$$

It is now easy to see that the limit (1) does not exist for our function f. We first choose $h = h_0$ where $h_0 \in \mathbb{R}$. Limit (1) then becomes

$$\lim_{h_0\to 0}(h_0^{-1}wh_0 + w + h_0) = \lim_{h_0\to 0}(2w + h_0) = 2w.$$

We now choose $h = h_0\mathfrak{i}$ where $h_0 \in \mathbb{R}$. Limit (1) then becomes

$$\lim_{h_0\to 0}((h_0\mathfrak{i})^{-1}wh_0\mathfrak{i} + w + h_0\mathfrak{i}) = \lim_{h_0\to 0}(\mathfrak{i}^{-1}w\mathfrak{i} + w + h_0\mathfrak{i}) = \mathfrak{i}^{-1}w\mathfrak{i} + w.$$

These two limits are of course not the same for a generic $w \in \mathbb{H}$.

An analogous calculation shows that limit (2) also does not exist. Therefore f has neither a left nor a right quaternionic derivative.

Example 1.3. Let us consider the function $f\colon \mathbb{H} \to \mathbb{H}$ defined as $f(w) = a + bw$ where $a, b \in \mathbb{H}$ fixed. We then calculate that

$$h^{-1}(f(w+h) - f(w)) = h^{-1}(a + b(w+h) - a - bw) = h^{-1}bh.$$

Thus limit (1) will not exits unless $b \in Z(\mathbb{H}) = \mathbb{R}$ where $Z(\mathbb{H})$ denotes the centre of $\mathbb{H}$. On the other hand, we calculate that

$$(f(w+h) - f(w))h^{-1} = (a + b(w+h) - a - bw)h^{-1} = b.$$

Thus limit (2) always exists. The function f thus has a right quaternionic derivative and also has a left quaternionic derivative in the special case when $b \in \mathbb{R}$.

Example 1.4. In analogy to the previous example one can show that $f\colon \mathbb{H} \to \mathbb{H}$ defined as $f(w) = a + wb$ where $a, b \in \mathbb{H}$ fixed, always has a left quaternionic derivative and also has a right quaternoinic derivative in the special case when $b \in \mathbb{R}$.

On has in fact the following theorem [4].

Theorem 1.5. *Let $f\colon U \to \mathbb{H}$ where $U \subseteq \mathbb{H}$ is a connected open set. Then f has a left quaternionic derivative at every point in U if and only if f has the form*

$$f(w) = a + wb$$

for some $a, b \in \mathbb{H}$.

These examples show that quaternionic polynomials in general are not differentiable. Suppose then that we would like to find a definition of analyticity such that quaternionic polynomials would become analytic. To do so we first note that natural monomial functions of a quaternionic variable of degree r are of the form

$$w \mapsto a_0 w a_1 \dots a_{r-1} w a_r$$

where $a_0, a_1, \dots, a_r \in \mathbb{H}$. The elementary monomials can be considered to be those with $a_0, a_1, \dots, a_r \in \{1, \mathfrak{i}, \mathfrak{j}, \mathfrak{k}\}$. However, if one writes $w = t + x\,\mathfrak{i} + y\,\mathfrak{j} + z\,\mathfrak{k}$ we then have

$$\begin{aligned} t &= \tfrac{1}{4}(w - \mathfrak{i}\,w\,\mathfrak{i} - \mathfrak{j}\,w\,\mathfrak{j} - \mathfrak{k}\,w\,\mathfrak{k}), \\ x &= \tfrac{1}{4\mathfrak{i}}(w - \mathfrak{i}\,w\,\mathfrak{i} + \mathfrak{j}\,w\,\mathfrak{j} + \mathfrak{k}\,w\,\mathfrak{k}), \\ y &= \tfrac{1}{4\mathfrak{j}}(w + \mathfrak{i}\,w\,\mathfrak{i} - \mathfrak{j}\,w\,\mathfrak{j} + \mathfrak{k}\,w\,\mathfrak{k}), \\ z &= \tfrac{1}{4\mathfrak{k}}(w + \mathfrak{i}\,w\,\mathfrak{i} + \mathfrak{j}\,w\,\mathfrak{j} - \mathfrak{k}\,w\,\mathfrak{k}). \end{aligned}$$

Thus a definition which would make all quaternionic polynomials analytic would also make all polynomials in four variables analytic (in a quaternionic sense), something which we do not want.

In search of a better definition of a derivative we recall from elementary complex analysis the Cauchy-Riemann operators (also called the Wirtinger derivatives) that are defined as

$$\partial = \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathfrak{i}\frac{\partial}{\partial y}\right) \quad \text{and} \quad \overline{\partial} = \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathfrak{i}\frac{\partial}{\partial y}\right)$$

where the $z = x + \mathfrak{i}\,y$ are the coordinates on $\mathbb{C}$. An analogous definition for $\mathbb{H}$ is the following [3].

Definition 1.6. A function $f\colon U \to V$ where $U, V \subseteq \mathbb{H}$ is called *left regular* on U if

$$D_L f = \frac{\partial f}{\partial t} + \mathfrak{i}\frac{\partial f}{\partial x} + \mathfrak{j}\frac{\partial f}{\partial y} + \mathfrak{k}\frac{\partial f}{\partial z} = 0. \tag{3}$$

Similarly f is called *right regular* on U if

$$D_R f = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\mathfrak{i} + \frac{\partial f}{\partial y}\mathfrak{j} + \frac{\partial f}{\partial z}\mathfrak{k} = 0. \tag{4}$$

A function f is called simply *regular* if it is both left- and right-regular.

The operators D_L and D_R are sometimes called the Fueter operators and equation (3) (or (4)) is sometimes called the left (or right) Cauchy-Riemann-Fueter equation.

We continue with an example the shows that we indeed have non-trivial functions that are left (or right) regular contrary to definition 1.1 where even certain linear functions fail to have a left (or right) quaternionic derivative.

Example 1.7. Consider the function $f\colon \mathbb{H} \to \mathbb{H}$ defined by $f(w) = w$. Then f is neither left nor right regular since

$$(D_L f)(w) = -2 \quad \text{and} \quad (D_R f)(w) = -2.$$

Example 1.8. Consider the function $f\colon \mathbb{H} \to \mathbb{H}$ defined by

$$f(w) = \frac{\overline{w}}{|w|^4} = \frac{t - x\,\mathfrak{i} - y\,\mathfrak{j} - z\,\mathfrak{k}}{(t^2 + x^2 + y^2 + z^2)^2}.$$

Then f is both left and right regular on $\mathbb{H}$ to show this we first calculate that

$$\begin{aligned}
\frac{\partial f}{\partial t}(w) &= \frac{(t^2 + x^2 + y^2 + z^2) - 4t(t - x\,\mathfrak{i} - y\,\mathfrak{j} - z\,\mathfrak{k})}{(t^2 + x^2 + y^2 + z^2)^3}, \\
\frac{\partial f}{\partial x}(w) &= \frac{-\mathfrak{i}(t^2 + x^2 + y^2 + z^2) - 4x(t - x\,\mathfrak{i} - y\,\mathfrak{j} - z\,\mathfrak{k})}{(t^2 + x^2 + y^2 + z^2)^3}, \\
\frac{\partial f}{\partial y}(w) &= \frac{-\mathfrak{j}(t^2 + x^2 + y^2 + z^2) - 4y(t - x\,\mathfrak{i} - y\,\mathfrak{j} - z\,\mathfrak{k})}{(t^2 + x^2 + y^2 + z^2)^3}, \\
\frac{\partial f}{\partial z}(w) &= \frac{-\mathfrak{k}(t^2 + x^2 + y^2 + z^2) - 4z(t - x\,\mathfrak{i} - y\,\mathfrak{j} - z\,\mathfrak{k})}{(t^2 + x^2 + y^2 + z^2)^3}.
\end{aligned}$$

It is now trivial to check that

$$(D_L f)(w) = (D_R f)(w) = 0$$

for all $w \in \mathbb{H} \setminus \{0\}$.

Definition 1.9. The *conjugate Fueter operators* are defined as

$$\overline{D}_L = \frac{\partial}{\partial t} - \mathfrak{i}\frac{\partial}{\partial x} - \mathfrak{j}\frac{\partial}{\partial y} - \mathfrak{k}\frac{\partial}{\partial z} \tag{5}$$

and

$$\overline{D}_R = \frac{\partial}{\partial t} - \frac{\partial}{\partial x}\mathfrak{i} - \frac{\partial}{\partial y}\mathfrak{j} - \frac{\partial}{\partial z}\mathfrak{k}. \tag{6}$$

We recall now that the Cauchy-Riemann operators factorize the Laplacian on $\mathbb{R}^2$ in the sense that $\partial\overline{\partial} = \overline{\partial}\partial = \frac{1}{4}\triangle$. Similarly the Fueter operators and their conjugates factorize the Laplacian on $\mathbb{R}^4$ in the sense that

$$D_L \overline{D}_L = \overline{D}_L D_L = D_R \overline{D}_R = \overline{D}_R D_R = \triangle.$$

One also has a quaternionic version of Cauchy's theorem as follows [4].

Theorem 1.10. *Suppose f is regular in an open set U. Let w_0 be a point in U, and let C be a rectifiable 3-chain which is homologous, in the singular homology of $U \setminus \{w_0\}$, to a differentiable 3-chain whose image is ∂B for some ball $B \subseteq U$. Then*

$$\frac{1}{2\pi^2} \int_C \frac{(w - w_0)^{-1}}{|w - w_0|^2} Dw f(w) = n f(w_0)$$

where n is the wrapping number of C about w_0.

The details are available in [4]. On can then also generalize other theorems form complex analysis that depend only on Cauchy's formula, such as the maximum principle, Morera's theorem and Liouville's theorem [4].

2. Dirac operators

Dirac operator are generalized Cauchy-Riemann operators. More precisely, we have the following construction [3].

Let $Cl = Cl(V,q)$ be the Clifford algebra associated with a real non-degenerated quadratic vector space (V,q). The space $\mathcal{C}^\infty(U,Cl)$ of smooth Cl-valued functions on an open set $U \subseteq V$ is a Cl-module under pointwise multiplication. By identifying V with a subspace of Cl we can regard any $X \in V$ as an element in Cl and hence a multiplier on $\mathcal{C}^\infty(U,Cl)$. On the other hand, each $X \in V$ gives rise to a vector field ∂_X acting on $\mathcal{C}^\infty(U,Cl)$. The action of this vector field is usually defined as

$$\partial_X f(v) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(v + t\,X), \quad v \in U.$$

It then holds that

$$\partial_{\alpha X+\beta Y} = \alpha\partial_X + \beta\partial_Y, \quad \alpha,\beta \in \mathbb{R},\ X,Y \in V$$

and

$$\partial_X(af+bg) = a\partial_X f + b\partial_X g, \quad a,b \in \mathbb{R},\ f,g \in \mathcal{C}^\infty(U,Cl),\ X \in V.$$

Let now $\{e_i\}_{i=1}^n$ be a normalized basis for V and denote by ∂_i the vector field corresponding to e_i. We then have the following definition [3].

Definition 2.1. The *Dirac operator* D associated with the real non-degenerated quadratic vector space (V,q) is the first order differential operator

$$D = \sum_{i=1}^n q(e_i)e_i\partial_i \tag{7}$$

acting on $\mathcal{C}^\infty(U,Cl)$. The coefficients of D are the generators of $Cl(V,q)$ acting by pointwise multiplication on $\mathcal{C}^\infty(U,Cl)$. The *Laplace operator* $\triangle$ is the second order constant-coefficient differential operator

$$\triangle = \sum_{i=1}^n q(e_i)\partial_i^2 \tag{8}$$

acting on $\mathcal{C}^\infty(U,Cl)$.

It follows immediately from the definition of D that

$$D^2 = \left(\sum_{i=1}^n q(e_1)e_1\partial_i\right)^2 = \sum_{i=1}^n q(e_i)^2e_i^2\partial_i^2 = \sum_{i=1}^n q(e_i)\partial_i^2 = \triangle. \tag{9}$$

Although the definition of both D and $\triangle$ depends on a choice of a basis of Cl the following proposition shows the definitions are in fact independent of the particular choice of a basis [3].

Proposition 2.2. *Let $\{e_i\}_{i=1}^n$ and $\{f_i\}_{i=1}^n$ be two normalized bases of V such that $q(e_i) = q(f_i)$ for all $i = 1,2,\ldots,n$ and let ∂_i be the vector field associated to e_i and $\widetilde{\partial}_i$ the vector field associated to f_i. Then*

$$\sum_{i=1}^n q(e_i)e_i\partial_i = \sum_{i=1}^n q(f_i)f_i\widetilde{\partial}_i, \quad \sum_{i=1}^n q(e_i)\partial_i^2 = \sum_{i=1}^n q(f_i)\widetilde{\partial}_i^2.$$

Proof. Let $A \in O(V,q)$ be such that $f_i = Ae_i$ for all i. Let $A = \{a_{j,k}\}_{j,k=1}^n$. Then

$$f_i = \sum_{j=1}^n a_{i,j} e_j, \quad \sum_{i=1}^n q(f_i) a_{i,j} a_{i,k} = q(e_j)\delta_{j,k} \quad \text{and} \quad \widetilde{\partial}_i = \sum_{k=1}^n a_{i,k}\partial_k$$

where the middle identity follows from the fact that $A \in O(V,q)$. Thus

$$\sum_{i=1}^n q(f_i) f_i \widetilde{\partial}_i = \sum_{j,k=1}^n \left(\sum_{i=1}^n q(f_i) a_{i,j} a_{i,k} \right) e_j \partial_k = \sum_{j=1}^n q(e_j) e_j \partial_j$$

and

$$\sum_{i=1} q(f_i) \widetilde{\partial}_i^2 = \sum_{j,k=1}^n \left(\sum_{i=1}^n q(f_i) a_{i,j} a_{i,k} \right) \partial_j \partial_k = \sum_{j=1}^n q(e_j) \partial_j^2.$$

This completes the proof. □

Let now $\{e_{\vec{i}}\}_{\vec{i}}$ be a basis for Cl where the multi-index $\vec{i}$ satisfies $0 \leq |\vec{i}| < n$ and $\vec{i} = (i_1, i_2, \ldots, i_k)$, $k = |\vec{i}|$. We then have the following proposition [3].

Proposition 2.3. *If $f = \sum_{\vec{i}} f_{\vec{i}} e_{\vec{i}}$ is a solution of $Df = 0$ in $\mathcal{C}^\infty(U, Cl)$ with each $f_{\vec{i}}$ real-valued then $\triangle f_{\vec{i}} = 0$.*

Proof. Using (9) we get that

$$0 = D0 = D^2 f = \triangle f = \sum_{\vec{i}} (\triangle f_{\vec{i}}) e_{\vec{i}}.$$

Since $\{e_{\vec{i}}\}_{\vec{i}}$ be a basis for Cl it must hold that $\triangle f_{\vec{i}} = 0$. □

We now consider how the Dirac and Laplace operator act from the left or the right. Since the Laplace operator has scalar coefficients we have that

$$\triangle f = f \triangle .$$

The Dirac operator on the other hand has coefficients that are non-scalar elements of Cl. Therefore we have to distinguish between left and right action. Suppose we have a function $f \in \mathcal{C}^\infty(U, Cl)$. As in Proposition 2.3 we can then write

$$f(x) = \sum_{\vec{i}} f_{\vec{i}}(x) e_{\vec{i}}, \quad x \in U,$$

where the "coordinate" functions are scalar valued. We then have

$$Df = \sum_{j=1}^n \sum_{\vec{i}} q(e_j) e_j e_{\vec{i}} \partial_j f_{\vec{i}} \quad \text{and} \quad fD = \sum_{j=1}^n \sum_{\vec{i}} q(e_j) e_{\vec{i}} e_j \partial_j f_{\vec{i}}.$$

We thus have the following definition [1].

Definition 2.4. A function $f \in \mathcal{C}^\infty(U, Cl)$ is called *left-monogenic* if $Df = 0$ and *right-monogenic* if $fD = 0$. A function is called *monogenic* if it is both left- and right-monogenic.

Sometimes the terms used are left- and right- Clifford analytic [3].

We finish by returning to the case of quaternions.

Example 2.5. In Example 1.8 we showed that the function $f(w) = \frac{\overline{w}}{|w|^4}$ is regular. By proposition 2.3 its coordinate functions are harmonic. We will now check this via a direct computation. Let

$$f_1(t,x,y,z) = \frac{t}{(t^2+x^2+y^2+z^2)^2}.$$

We then have

$$\begin{aligned}
\frac{\partial^2 f_1}{\partial t^2}(t,x,y,z) &= \frac{\partial}{\partial t}\left(\frac{-3t^2+x^2+y^2+z^2}{(t^2+x^2+y^2+z^2)^3}\right) = \frac{-12t(-t^2+x^2+y^2+z^2)}{(t^2+x^2+y^2+z^2)^4},\\
\frac{\partial^2 f_1}{\partial x^2}(t,x,y,z) &= \frac{\partial}{\partial x}\left(\frac{-4tx}{(t^2+x^2+y^2+z^2)^3}\right) = \frac{-4t(t^2-5x^2+y^2+z^2)}{(t^2+x^2+y^2+z^2)^4},\\
\frac{\partial^2 f_1}{\partial y^2}(t,x,y,z) &= \frac{\partial}{\partial y}\left(\frac{-4ty}{(t^2+x^2+y^2+z^2)^3}\right) = \frac{-4t(t^2+x^2-5y^2+z^2)}{(t^2+x^2+y^2+z^2)^4},\\
\frac{\partial^2 f_1}{\partial z^2}(t,x,y,z) &= \frac{\partial}{\partial z}\left(\frac{-4tz}{(t^2+x^2+y^2+z^2)^3}\right) = \frac{-4t(t^2+x^2+y^2-5z^2)}{(t^2+x^2+y^2+z^2)^4}.
\end{aligned}$$

It is now trivial to see that we indeed have $\triangle f_1 = 0$. Similar calculations can be done for the other coordinate functions.

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