Physical applications of Clifford algebras

Point particle mechanics

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I will talk about applications of Clifford algebras in classical physics, that is relativistic and non-relativistic mechanics. I will formulate the action principle in this language and derive the Euler-Lagrange equations accordingly. I will furthermore try to be a little bit bold and "derive"the form of the action from some basic principles, this will however not be totally mathematically satisfactory. I will use the equations, derived for general dimensions, to show the corresponding equation in $(1,3)$ dimensions. Finally I may give a small introduction to classical field theory (EM etc.) in this language.

I. CLASSICAL MECHANICS

What I call classical mechanics is basically point particle mechanics, that is, the assumption that elementary entities from which everything is built is made from particles which are zero dimensional points. In course of *time* these particles trace out a curve which we want to describe. We could thus take it as a principle that everything is built from one dimensional curves and not points. The second and more important assumption is the **principle of least action**, this is no more complicated then saying that the universe is stable in some sort of minima.

II. CURVES IN $\mathbf{R}^{r,s}$

We now try to put these assumptions into formulas. We will assume that the point particles trace out a curve in the real space $\mathbf{R}^{r,s}$. Mathematically a curve is a mapping from a compact interval $I \subset \mathbf{R}$ to the space $\mathbf{R}^{r,s}$. We will take this interval to be fixed but in the end our result will not depend on it. This we could also take as a basic assumption and indeed we will, but not so explicitly. We will actually make a stronger assumption. The set of curves $x : I \to \mathbf{R}^{r,s}$ which are C^k will be denoted by $\mathscr{C}^k(\mathbf{R}^{r,s})$ or simply $\mathscr{C}^k$ when the space is known from context. We can introduce pointwise summation of curves and multiplication by constant

$$(x+y)(\lambda) = x(\lambda) + y(\lambda), \quad (ax)(\lambda) = ax(\lambda), \quad \text{where } x, y \in \mathscr{C}^k,\ a \in \mathbf{R} \text{ and } \lambda \in I.$$

Clearly the set $\mathscr{C}^k$ is a vector space. There is a subspace of $\mathscr{C}^k$ which includes all curves which start and end at the $0 \in \mathbf{R}^{r,s}$, denote this subset by $\mathscr{H}^k(\mathbf{R}^{r,s})$ or $\mathscr{H}^k$. The factor space $\mathscr{C}^k/\mathscr{H}^k$ includes equivelance classes of curves between two points, and infact the equivelance class only depends of the endpoints but includes all curves between these points.

From what follows we will only try to describe a single particle or curve. According to our assumption, the universe or the system is in some sort of minima, thus we need a functional to minimize. This functional is called an action and is denoted by S. We have however not specified what exactly S is, physically S should only depend on a path taken by the particle, or a curve and possibly the space itself. We could also let S depend on the interval I but this seems unnatural since I is only used to parametrize the curve. So

$$\textbf{Action:} \quad S : \mathscr{C}^k \to \mathbf{R},$$

we want S to map to $\mathbf{R}$ to make our life as simple as possible, but there is also a physical reason for this (symmetry).

III. THE ACTION

We now try to motivate how S should look like, S should be smooth in some sense. That is, it should not vary to much if we change the curve by a small ammount. It should also have some additive property, that is, if we take a curve x and split it up to x_1, x_2 using some reparametrization then $S[x] = S[x_1] + S[x_2]$. This can be motivated as follows, if x is a curve that minimizes S then x_1 and x_2 should also minimize S. All this indicates that we should write S as a line integral in $\mathbf{R}^{r,s}$.

To write down the integral we consider S to be a mapping from $\mathbf{R}^{r,s}$ to the Clifford algebra $\mathscr{G}^{r,s}$. Then the integral will only depend on functions which map from the space to the Clifford algebra and the boundary of the integral is the curve itself. We have seen that the integral can either depend on direction or not, that is, for line integrals we have two measures $\mathrm{d}\sigma$ and

$|\mathrm{d}\sigma|$ where $|\cdot|$ denotes the scalar length. We can now write

$$S[x] = \int_x F|\mathrm{d}\sigma| + G(\mathrm{d}\sigma),$$

where $F : \mathbf{R}^{r,s} \to \mathcal{G}^{r,s}$ and G is a Clifford algebra valued one-form. Since S must be real for any curve we have that F is scalar valued but we can write

$$G(\mathrm{d}\sigma) = A * \mathrm{d}\sigma$$

and we can take A to be a vector, since G is linear in $\mathrm{d}\sigma$.

We have seen that the most general action takes the form

$$S[x] = \int_x F|\mathrm{d}\sigma| + A * \mathrm{d}\sigma$$

where F and A depend on position. We say the particle is free if the action and therefore the physics of the system does not depend on either position nor direction. that would meen that A would have to vanish in this limit and F become a constant,

$$S_{\text{free}}[x] = \int_x m|\mathrm{d}\sigma|$$

where m is a constant called mass. Physicist usually think of the non-free case to be artificially created or controlled, so we could allways by hand turn off the external fields as they are called, then the action reads

$$S[x] = \int_x (m+\theta)|\mathrm{d}\sigma| + A * \mathrm{d}\sigma$$

where θ and A are the external fields. To actually calculate this integral we would have to introduce some parametrization $\lambda \in I$ (or any other interval)

$$S[x] = \int_I \Big((m + \theta(x(\lambda)))|\dot{x}(\lambda)| + A(x(\lambda)) * \dot{x}(\lambda) \Big) \mathrm{d}\lambda.$$

To write it more familiarly write $v = \dot{x}$, then

$$S[x] = \int_I \Big((m+\theta)|v| + A * v \Big) \mathrm{d}\lambda.$$

IV. FUNCTIONAL DERIVATIVES

To derive the equations of motion the easiest way is to review functional derivatives. Let V be a vector space over $\mathbf{R}$ equipped with a non-degenerate quadratic form Q or $\langle \cdot, \cdot \rangle$. Let $F : V \to \mathbf{R}$ be a functional. A functional derivative is a linear functional

$$\frac{\delta F}{\delta u}[v] = \frac{\mathrm{d}}{\mathrm{d}\epsilon} F[v + \epsilon u] \Big|_{\epsilon=0}$$

where $v \in V$ but $u \in H$ some subspace of V which satisfies some test function space like properties. I'm not going into details here, you of course need the limit to exist and hence some restrictions on the vector space (continuity, see distribution theory). If V is finite dimensional V is canonically isomorphic to the dual space V^* by the quadratic form. This enables us to define the functional "gradient"δF as follows

$$\frac{\delta F}{\delta u}[v] = \langle u, \delta F[v] \rangle.$$

This is even done for infinite dimensional spaces though it is not strictly true (see distribution theory). If F is such that $(\delta F/\delta u)[v] = 0$ for all $u \in H$ then $\delta F[v] = 0$.

V. EULER-LAGRANGE EQUATIONS AND SYMMETRIES

The vector space in question is $\mathscr{C}^k(\mathbf{R}^{r,s})$ and the subspace $\mathscr{H}^k$ functions as a test function space (compact support, this is however satisfied for all curves in $\mathscr{C}^k$). The inner product is given by

$$\langle x,y\rangle = \int_I x(\lambda)*y(\lambda)\mathrm{d}\lambda, \quad x,y\in\mathscr{C}^k.$$

Let $F:\mathscr{C}^k\to\mathbf{R}$ be given by

$$F[\gamma] = \int_I G(x(\lambda),\dot{x}(\lambda))\mathrm{d}\lambda, \quad \gamma\in\mathscr{C}^k.$$

We now calculate the functional derivative of F, let $y\in\mathscr{H}^k$, then

$$\begin{aligned}\frac{\delta F}{\delta y}[x] &= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\int_I G(x+\epsilon y,\dot{x}+\epsilon\dot{y})\mathrm{d}\lambda\Big|_{\epsilon=0}\\ &= \int_I \Big\{y*\nabla+\dot{y}*\nabla_v\Big\}G\mathrm{d}\lambda\\ &= \int_I y*\Big\{\nabla G-\frac{\mathrm{d}}{\mathrm{d}\lambda}\nabla_v G\Big\}\mathrm{d}\lambda+\Big[y*\nabla_{\dot{x}}G\Big]_I.\end{aligned}$$

The last term is zero since y is zero on the boundary of I. Here ∇_v is the derivative with respect to second argument of G. So we have the desired form

$$\delta F[x] = \nabla_x G-\frac{\mathrm{d}}{\mathrm{d}t}\nabla_v G.$$

If we set $\delta F[x]=0$ we get the Euler-Lagrange equations.

Now let $x\in\mathscr{C}^k$ be such that the Euler-Lagrange equations are solved for $F[x]$. Let U be a subspace of $\mathscr{C}^k$ such that $F[x+u]=F[x]$ for any $u\in U$. Then

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}F[x+\epsilon u]\Big|_{\epsilon=0}=0$$

for any $u\in U$. By our above calculation we have that

$$0=\frac{\mathrm{d}}{\mathrm{d}\epsilon}F[x+\epsilon u]\Big|_{\epsilon=0}=\langle\delta F[x],u\rangle+\Big[u*\nabla_{\dot{x}}G\Big]_I=\Big[u*\nabla_{\dot{x}}G\Big]_I,$$

since $\delta F[x]=0$. Lets take two important examples, if U is the subspace of all constant curves, then for all constants $a\in\mathbf{R}^{r,s}$

$$0=\Big[a*\nabla_v G\Big]_I=a*\Big[\nabla_v G\Big]_I,$$

which shows that the momentum $p=\nabla_v G$ is conserved. We have shown that translational invariance leads to momentum conservation, but what about rotationl invariance? Let F be such that $F[RxR^\dagger]=F[x]$ for all rotors R. The rotors can be written for ($\mathbf{R}^{r,s}=\mathbf{R}^{0,n},\mathbf{R}^{n,0},\mathbf{R}^{n-1,1}$ or $\mathbf{R}^{1,n-1}$)

$$R=\pm e^{\theta B/2}, \quad \text{where } \theta\in\mathbf{R}, \text{ and } B \text{ is a unit bivector.}$$

We can now calculate

$$0=\frac{\mathrm{d}}{\mathrm{d}\epsilon}F\big[e^{\epsilon B/2}xe^{-\epsilon B/2}]\Big|_{\epsilon=0}=\frac{\mathrm{d}}{\mathrm{d}\epsilon}F[x+\epsilon(Bx-xB)/2+O(\epsilon^2)]\Big|_{\epsilon=0}.$$

Since the higher order terms don't matter our subspace of vectors is $\{B\rfloor x:\forall B \text{ unit bivectors}\}$. We can further calculate

$$0=\Big[(B\rfloor x)*p\Big]_I=-\Big[p*(x\llcorner B)\Big]_I=-\Big[(p\wedge x)*B\Big]_I=B*\Big[x\wedge p\Big]_I$$

which shows that the angular momentum $L=x\wedge p$ is conserved even though the momentum might not be conserved.

VI. EQUATIONS OF MOTION

Now we write down the equations of motion for the action above, remember that

$$S[x]=\int_I L(x,v)\mathrm{d}\lambda$$

where $L=(m+\theta)|v|+A*v$. Even though S may not be translational invariant it is common to write $p=\nabla_v L=(m+\theta)u+A$ where $u=v/|v|$ (since $2|v|\nabla_v|v|=\nabla_v v^2=2v$). From this we get the result

$$(p-A)^2=(m+\theta)^2.$$

The time derivative of the momentum is

$$\dot{p}=(v*\nabla)\theta\, u+(v*\nabla)A+(m+\theta)\dot{u}.$$

The Euler-Lagrange equations give that this equals $\nabla(\theta|v|+A*v)$ so

$$\frac{m+\theta}{|v|}\dot{u}=\nabla\theta-u(u*\nabla)\theta+\nabla(A*u)-(u*\nabla)A,$$

which simplifies to

$$\frac{m+\theta}{|v|}\dot{u}=\nabla\theta-u(u*\nabla)\theta+(\nabla\wedge A)\rfloor u.$$

VII. $\mathbf{R}^{1,3}$ BASES AND TIMES

Consider from now on $\mathbf{R}^{1,3}$ where the first coordinate is interpreted as time. Since the action is reparametrization invariant the equations are also. This means that we can choose the parameter λ appropriate to our calculation. We will only talk about the two most important choices, if λ is such that $|v|=1$ always then $\tau=\lambda$ is called proper time by physicist. We will return to this later, but lets consider more classical choice.

Introduce a basis $\{e_i\}$ of our space such that a curve $x(\lambda)$ can be written $x=x^i(\lambda)e_i$ and then choose $\lambda=x^0=t$. This is our natural conception of time. Notice however that we don't need a complete basis to make this choice. Take any timelike ($w^2>0$) unit vector w and use it as the first basis vector, then $t=x*w$ and we can write

$$x=xww=(x*w+x\wedge w)w.$$

Remember that the even subalgebra of $\mathcal{G}^{1,3}$ is isomorphic to $\mathcal{G}^3$ and since for any vector $x\in\mathcal{G}^{1,3}$, $x\wedge w$ is a bivector or zero. This means that we can consider $\mathbf{x}=x\wedge w$ to be a vector in $\mathbf{R}^3$ that is the spatial coordinates. Note also with this choice of λ we get

$$\dot{x}w=\dot{x}*w+\dot{x}\wedge w=\frac{\mathrm{d}t}{\mathrm{d}t}+\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}=1+\mathbf{v}.$$

Furthermore

$$|v|=\sqrt{vv}=\sqrt{vwwv}=\sqrt{(1+\mathbf{v})(1-\mathbf{v})}=\sqrt{1-\mathbf{v}^2}=\gamma^{-1}.$$

We define $pw=E+\mathbf{p}$, $Aw=\phi+\mathbf{A}$ and $w\nabla=\partial_t+\boldsymbol{\nabla}$. Then we can find the equations for E and $\mathbf{p}$,

$$pw=E+\mathbf{p}=(m+\theta)\gamma(1+\mathbf{v})+\phi+\mathbf{A}$$

and

$$\frac{\mathrm{d}(E+\mathbf{p})}{\mathrm{d}t}=\nabla(\theta|v|+A*v)w=(\partial_t-\boldsymbol{\nabla})(\theta\gamma^{-1}+\phi-\mathbf{A}*\mathbf{v}).$$

Since

$$A*v=\frac{1}{2}(Av+vA)=\frac{1}{2}(Awwv+vwwA)=\frac{1}{2}\big((\phi+\mathbf{A})(1-\mathbf{v})+(1+\mathbf{v})(\phi-\mathbf{A})\big)=\phi-\frac{1}{2}(\mathbf{A}\mathbf{v}+\mathbf{v}\mathbf{A})=\phi-\mathbf{A}*\mathbf{v}.$$

Define $M = m + \theta$ and seperate the equations into scalar part and vector part,

$$\dot{E} = \frac{\mathrm{d}(M\gamma + \phi)}{\mathrm{d}t} = \partial_t(\theta\gamma^{-1} + \phi - \mathbf{A}*\mathbf{v}), \quad \dot{\mathbf{p}} = \frac{\mathrm{d}(M\gamma\mathbf{v} + \mathbf{A})}{\mathrm{d}t} = -\boldsymbol{\nabla}(\theta\gamma^{-1} + \phi - \mathbf{A}*\mathbf{v}).$$

The first equation becomes

$$\frac{\mathrm{d}(M\gamma)}{\mathrm{d}t} = \gamma^{-1}\partial_t\theta - \mathbf{v}*(\partial_t\mathbf{A} + \boldsymbol{\nabla}\phi) = \gamma^{-1}\partial_t\theta + \mathbf{v}*\mathbf{E},$$

where we defined the electric field $\mathbf{E} = -\partial_t\mathbf{A} - \boldsymbol{\nabla}\phi$. The second equation is then

$$\frac{\mathrm{d}(M\gamma\mathbf{v})}{\mathrm{d}t} = -\gamma^{-1}\boldsymbol{\nabla}\theta - \boldsymbol{\nabla}\phi - \partial_t\mathbf{A} - \mathbf{v}*\boldsymbol{\nabla}\mathbf{A} + \boldsymbol{\nabla}(\mathbf{A}*\mathbf{v}) = -\gamma^{-1}\boldsymbol{\nabla}\theta + \mathbf{E} + (\boldsymbol{\nabla}\wedge\mathbf{A}) \lrcorner\, \mathbf{v} = -\gamma^{-1}\boldsymbol{\nabla}\theta + \mathbf{E} + \mathbf{B} \lrcorner\, \mathbf{v}$$

which defines the bivector $\mathbf{B}$. We can put this into one big equation

$$M\gamma\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\gamma^{-1}(\boldsymbol{\nabla}\theta + \mathbf{v}\partial_t\theta) + \mathbf{E} + \mathbf{B} \lrcorner\, \mathbf{v} - \mathbf{v}(\mathbf{v}*\mathbf{E}).$$

If $|\mathbf{v}| \ll 1$ then

$$M\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\boldsymbol{\nabla}\theta - \mathbf{v}\partial_t\theta + \mathbf{E} + \mathbf{B} \lrcorner\, \mathbf{v} + O(\mathbf{v}^2).$$

If we furthermore make θ time-independant and $\theta \ll m$ so that $M = m + O(\theta/m)$ then we get (surpressing $O(\cdots)$)

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\boldsymbol{\nabla}\theta + \mathbf{E} + \mathbf{B} \lrcorner\, \mathbf{v}$$

which is the equation for classical particle in electromagnetic and (scalar) gravitational field.

VIII. SOLUTIONS USING ROTORS

We return now to the proper time formalism and put $\theta = 0$

$$m\dot{v} = F \lrcorner\, v$$

where $F = \nabla \wedge A$ and $\dot{v} = \frac{\mathrm{d}v}{\mathrm{d}\tau}$. Since v is timelike and "future pointing"for all τ and also normalized we can think of v as τ dependant rotation of some fixed vector. We introduce a fixed timelike, "future pointing"vector v_0 such that $v = Rv_0R^\dagger$ where R is τ dependant rotor. Then

$$\dot{v} = \dot{R}v_0R^\dagger + Rv_0\dot{R}^\dagger,$$

but from linearity of the † follows that for any $\mathcal{G}$ valued function F

$$\frac{\mathrm{d}}{\mathrm{d}\tau}F^\dagger = \left(\frac{\mathrm{d}F}{\mathrm{d}\tau}\right)^\dagger$$

since B is a sum of commuting bivectors. Now $RR^\dagger = 1$ and

$$0 = \frac{\mathrm{d}}{\mathrm{d}\tau}(RR^\dagger) = \dot{R}R^\dagger + R\dot{R}^\dagger$$

thus (since $\dot{R}R^\dagger = \dot{B}$ is a bivector)

$$\dot{v} = \dot{R}R^\dagger Rv_0R^\dagger + Rv_0R^\dagger R\dot{R}^\dagger = \dot{R}R^\dagger v - v\dot{R}R^\dagger = 2(\dot{R}R^\dagger) \lrcorner\, v.$$

We then finally get the equation

$$2m(\dot{R}R^\dagger) \lrcorner\, v = F \lrcorner\, v$$

which would follow if

$$\dot{R} = \frac{1}{2m}FR.$$

This equation can be much easier to solve then the original one. For instance take F to be constant, then

$$R = e^{\tau F/2m}$$

with the initial conditions $v(\tau = 0) = v_0$.