Recent studies of anyons

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based on work in collaborations with
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Nicolas Rougerie, Jan Philip Solovej

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Outline of Talk

1. Recall 2D anyons — ideal or extended

2. Lower bounds $\leftarrow$ local exclusion principle

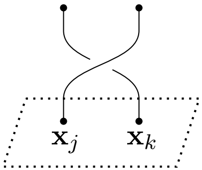
3. The extended anyon gas & Ideal anyons in a harmonic trap

4. Upper bounds $\leftarrow$ many-anyon trial states

Identical particles and statistics in 2D

Particle exchange in 2D: $\Psi \in L^2((\mathbb{R}^2)^N) \cong \bigotimes^N L^2(\mathbb{R}^2)$

$$\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_N) = e^{i\alpha\pi}\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_N)$$



$e^{i\alpha\pi} \in U(1)$ **any** phase

$\alpha = 0$: bosons

$\alpha = 1$: fermions

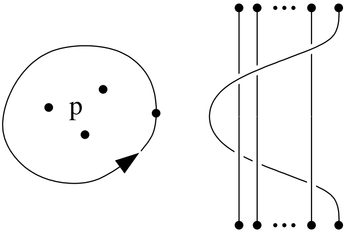
anyons: 'fractional'-statistics quasiparticles in confined systems — expected to arise in fractional quantum Hall systems

∼1970 Souriau, Streater & Wilde ... Leinaas & Myrheim '77; Goldin, Menikoff & Sharp '81; Wilczek '82 ...

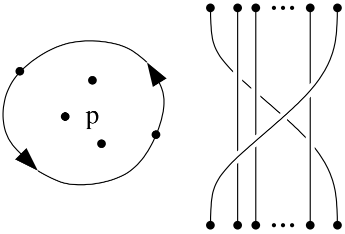
Reviews by Fröhlich '90, Wilczek '90, Lerda '92, Myrheim '99, Khare '05, Ouvry '07, Stern '08, ...

Past rigorous QM studies by Baker, Canright & Mulay '93, Dell'Antonio, Figari & Teta '97

Modelling anyons mathematically — anyon gauge



$e^{i2p\alpha\pi}$



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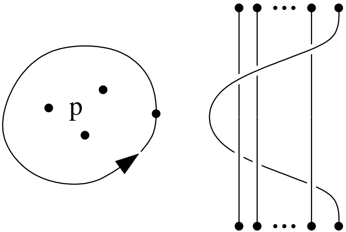

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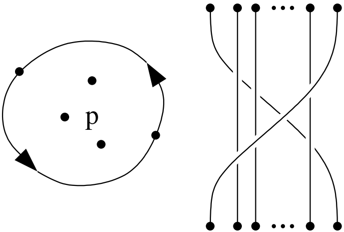

$e^{i(2p+1)\alpha\pi}$

Think: free kinetic energy $\hat{T}_0 = \frac{\hbar^2}{2m}\sum_{j=1}^{N}(-i\nabla_j)^2$ acting on multi-valued

$$\Psi_\alpha := U^{\alpha}\,\Psi_0, \qquad U := \prod_{j<k} e^{i\phi_{jk}} = \prod_{j<k} \frac{z_j - z_k}{|z_j - z_k|}.$$

Modelling anyons mathematically — magnetic gauge

Bosons ($\Psi \in L^2_{\mathrm{sym}}$) in $\mathbb{R}^2$ with Aharonov-Bohm magnetic interactions:

$$\hat{T}_\alpha := \frac{\hbar^2}{2m} \sum_{j=1}^{N} D_j^2, \quad D_j = -i\nabla_j + \alpha \mathbf{A}_j, \quad \mathbf{A}_j = \sum_{k \neq j} \frac{(\mathbf{x}_j - \mathbf{x}_k)^\perp}{|\mathbf{x}_j - \mathbf{x}_k|^2}$$

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These are **ideal** anyons. One can also model ***R*-extended** anyons:

$$\mathbf{A}_j^R(\mathbf{x}_j) := \sum_{k \neq j} \frac{(\mathbf{x}_j - \mathbf{x}_k)^\perp}{|\mathbf{x}_j - \mathbf{x}_k|_R^2}, \qquad |\mathbf{x}|_R := \max\{|\mathbf{x}|, R\}$$

$$\Rightarrow \quad \mathrm{curl}\, \alpha \mathbf{A}_j^R = 2\pi\alpha \sum_{k \neq j} \frac{\mathbb{1}_{B_R(\mathbf{x}_k)}}{\pi R^2} \xrightarrow{R \to 0} 2\pi\alpha \sum_{k \neq j} \delta_{\mathbf{x}_k}$$

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We would like to understand the N-anyon ground state Ψ_0 and energy

$$E_0(N) := \inf\operatorname{spec}\hat{H}_N, \quad \hat{H}_N = \hat{T}_\alpha + \hat{V} = \sum_{j=1}^{N}\left(\frac{\hbar^2}{2m}D_j^2 + V(\mathbf{x}_j)\right)$$

Modelling anyons mathematically

Precise definition in magnetic gauge: (DL, Solovej, 2013/'14)

$$D\colon L^2_{\mathrm{sym}}(\mathbb{R}^{2N}) \to \mathscr{D}'(\mathbb{R}^{2N} \setminus \triangle; \mathbb{C}^{2N}), \qquad \int_{\mathbb{R}^{2N}} |D\Psi|^2 < \infty$$

$$\Psi \mapsto (D_j\Psi)_{j=1}^N = (-i\nabla_{\mathbf{x}_j}\Psi + \alpha \mathbf{A}_j\Psi)_{j=1}^N$$

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Def. / Theorem: $\hat{T}^{R\geq 0}_{\alpha\in\mathbb{R}} := \frac{\hbar^2}{2m}(D_{\text{min}})^* D_{\text{min}} = \frac{\hbar^2}{2m}(D_{\text{max}})^* D_{\text{max}}$

$$\begin{aligned}
\text{Dom}(\hat{T}^{R=0}_{\alpha=2n}) &= U^{-2n} H^2_{\text{sym}}(\mathbb{R}^{2N}) \\
\text{Dom}(\hat{T}^{R=0}_{\alpha=2n+1}) &= U^{-(2n+1)} H^2_{\text{asym}}(\mathbb{R}^{2N})
\end{aligned}$$

$$\text{Dom}(\hat{T}^{R>0}_{\alpha\in\mathbb{R}}) = \text{Dom}(\hat{T}^{R>0}_{0}) = H^2_{\text{sym}}(\mathbb{R}^{2N})$$

Modelling anyons mathematically

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Fermions in terms of bosons:

$$\Psi_{\alpha=1} = U^{-1}\Psi_{\mathrm{asym}} = \prod_{j<k} \frac{\bar{z}_j - \bar{z}_k}{|z_j - z_k|}\Psi_{\mathrm{asym}} \in L^2_{\mathrm{sym}}(\mathbb{R}^{2N})$$

Compare with the ideal Fermi gas in 2D

Know: $\Psi_0 = \bigwedge_{k=0}^{N-1} \varphi_k, \quad \varphi_k$ lowest states of $\hat{H}_1 = -\Delta_{\mathbb{R}^2} + V(\mathbf{x})$

Compare with the ideal Fermi gas in 2D

Know: $\Psi_0 = \bigwedge_{k=0}^{N-1} \varphi_k$, φ_k lowest states of $\hat{H}_1 = -\Delta_{\mathbb{R}^2} + V(\mathbf{x})$

The free Fermi gas in a box $Q \subset \mathbb{R}^2$:

$$E_0(N) = \sum_{k=0}^{N-1} \lambda_k \sim 2\pi \, (\underbrace{N/|Q|}_{\bar{\varrho}})^2 |Q|,$$

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$\Rightarrow$ Thomas–Fermi approximation: (Thomas, Fermi, 1927 — precursor to modern DFT)

$$\langle \Psi_0, (\hat{T}_{\alpha=1} + \hat{V})\Psi_0 \rangle \approx \int_{\mathbb{R}^2} \Big(2\pi \varrho_{\Psi_0}(\mathbf{x})^2 + V(\mathbf{x})\varrho_{\Psi_0}(\mathbf{x}) \Big) d\mathbf{x}$$

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The Lieb–Thirring inequality: (Lieb, Thirring, 1975)

$$\langle \Psi, (\hat{T}_{\alpha=1} + \hat{V})\Psi \rangle \geq \int_{\mathbb{R}^2} \Big(C_{\mathsf{LT}} \, \varrho_{\Psi}(\mathbf{x})^2 + V(\mathbf{x})\varrho_{\Psi}(\mathbf{x}) \Big) d\mathbf{x}$$

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The Lieb–Thirring inequality: (Lieb, Thirring, 1975) ν part.s in each state

$$\langle \Psi, (\hat{T}_{\alpha=1} + \hat{V})\Psi\rangle \geq \int_{\mathbb{R}^2} \Big(C_{\mathsf{LT}}\,\nu^{-1}\varrho_{\Psi}(\mathbf{x})^2 + V(\mathbf{x})\varrho_{\Psi}(\mathbf{x})\Big)d\mathbf{x}$$

Average-field approximation

(see e.g. Wilczek 1990 review)

For anyons one may consider an **average-field** approximation

$$\langle \Psi_0, (\hat{T}_\alpha + \hat{V})\Psi_0 \rangle \approx \inf_{\substack{\varrho \geq 0 \\ \int \varrho = N}} \int_{\mathbb{R}^2} \Big(2\pi|\alpha|\varrho(\mathbf{x})^2 + V(\mathbf{x})\varrho(\mathbf{x}) \Big) d\mathbf{x},$$

where $B = \operatorname{curl} \alpha \mathbf{A}_j \approx 2\pi\alpha\varrho$ with LLL energy/particle $\sim |B|$.

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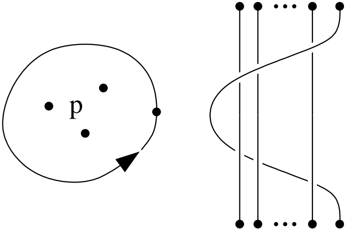
A particular **almost-bosonic** limit $\alpha = \beta/N$ leads to

$$\mathcal{E}^{\mathrm{af}}[u] := \int_{\mathbb{R}^2} \left(\left| \left(-i\nabla + \beta \mathbf{A}[|u|^2]\right) u \right|^2 + V|u|^2 \right), \quad u \in H^1(\mathbb{R}^2)$$

where $\operatorname{curl} \mathbf{A}[|u|^2] = 2\pi|u|^2$ and β the only parameter. DL, Rougerie, 2015

Universal bounds: A local exclusion principle for anyons



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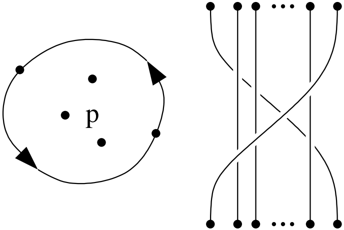
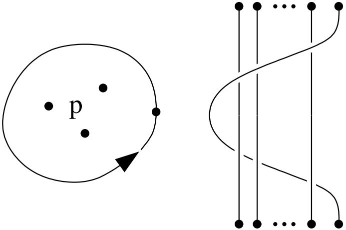


$e^{i(2p+1)\alpha\pi}$

Recall: 2-particle exchange phase $(2p+1)\alpha$ times π.
But anyons can also have pairwise relative angular momenta $\pm 2q$.

Universal bounds: A local exclusion principle for anyons



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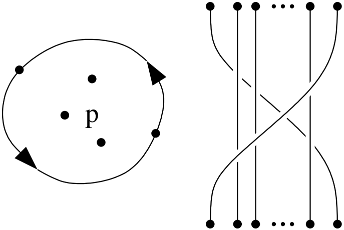

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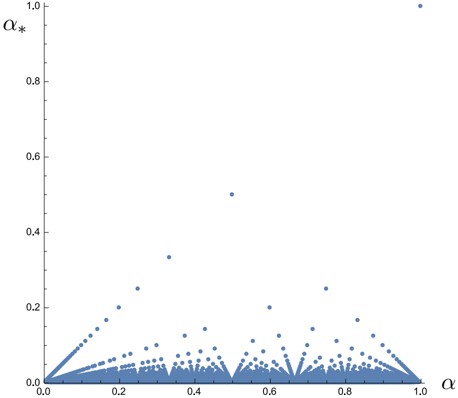
Recall: 2-particle exchange phase $(2p+1)\alpha$ times π.
But anyons can also have pairwise relative angular momenta $\pm 2q$.
$\Rightarrow$ effective **statistical repulsion** DL, Solovej, 2013

$$V_{\mathrm{stat}}(r) = |(2p+1)\alpha - 2q|^2 \frac{1}{r^2} \geq \frac{\alpha_N^2}{r^2}$$

Universal bounds: A local exclusion principle for anyons



$$\alpha_N := \min_{p\in\{0,1,\ldots,N-2\},\, q\in\mathbb{Z}} |(2p+1)\alpha - 2q|$$

$$\xrightarrow[N\to\infty]{} \alpha_* := \begin{cases} \frac{1}{\nu}, & \text{if } \alpha = \frac{\mu}{\nu} \text{ is a reduced fraction with } \mu \text{ odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Lieb–Thirring inequalities for anyons

Theorem ([DL-Solovej '13] LT inequality for ideal anyons)

Let Ψ be an N-anyon wave function on $\mathbb{R}^2$ with any $\alpha \in \mathbb{R}$. Then

$$\langle \Psi, \hat{T}_\alpha \Psi \rangle \ \geq \ C\, \alpha_N^2 \int_{\mathbb{R}^2} \varrho_\Psi(\mathbf{x})^2 \, d\mathbf{x},$$

for a constant $C > 0$,

So for $\alpha = \mu/\nu$ with **odd** μ and $\nu \geq 1$,

$$\langle \Psi, \hat{H}_N \Psi \rangle \geq \int_{\mathbb{R}^2} \Big(C\nu^{-2} \varrho_\Psi(\mathbf{x})^2 + V(\mathbf{x}) \varrho_\Psi(\mathbf{x}) \Big) d\mathbf{x}$$

Lieb–Thirring inequalities for anyons

DL, Solovej, 2013; LT with general local exclusion developed by DL, Nam, Portmann, Solovej, 2013-'15

Theorem ([Larson-DL '16] LT inequality for ideal anyons)

Let Ψ be an N-anyon wave function on $\mathbb{R}^2$ with any $\alpha \in \mathbb{R}$. Then

$$\langle \Psi, \hat{T}_\alpha \Psi \rangle \;\geq\; C\,(j'_{\alpha_N})^2 \int_{\mathbb{R}^2} \varrho_\Psi(\mathbf{x})^2\, d\mathbf{x},$$

for a constant $C > 0$, where $j'_\nu \geq \sqrt{2\nu}$ is first zero of J'_ν Bessel.

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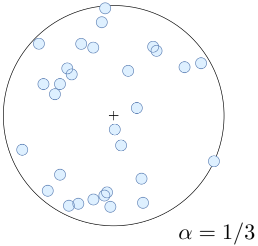
Extended case

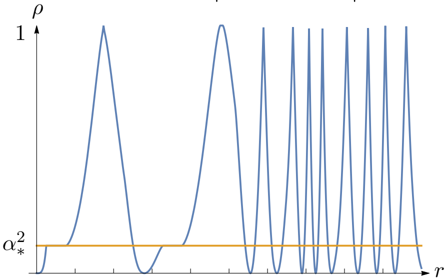
We use a magnetic Hardy inequality **with symmetry**

(cf. Laptev, Weidl, 1998; Hoffmann-Ostenhof^2, Laptev, Tidblom, 2008; Balinsky...)

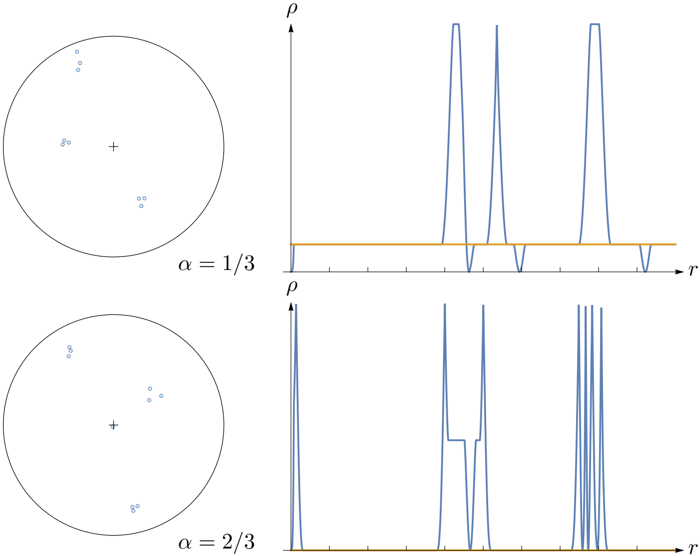
to consider the enclosed flux inside a two-particle exchange loop, subtracted with arbitrary pairwise angular momenta. Unwanted oscillation can be controlled by smearing (but analysis is tricky!)

$$V_{\text{stat}}(r) = \rho(r)\frac{1}{r^2}, \qquad \rho(r) = \min_{q\in\mathbb{Z}}\left|\frac{\Phi(r)}{2\pi} - 2q\right|^2$$





Extended case (clustering)



Universal bounds for the extended anyon gas

Consider ground-state energy on a box $Q \subset \mathbb{R}^2$:

$$E_0(N, Q, \alpha, R) := \inf\Big\{\langle \Psi, \hat{T}_\alpha^R \Psi \rangle \,:\, \Psi \in L_c^2(Q^N)\,,\, \|\Psi\| = 1\Big\}$$

In the thermodynamic limit, $N, |Q| \to \infty$ with $\bar{\varrho} = N/|Q|$ fixed, for dimensional reasons,

$$\frac{E_0(N, Q, \alpha, R)}{N} \to e(\alpha, \gamma)\bar{\varrho}, \qquad \gamma := R\sqrt{\bar{\varrho}}.$$

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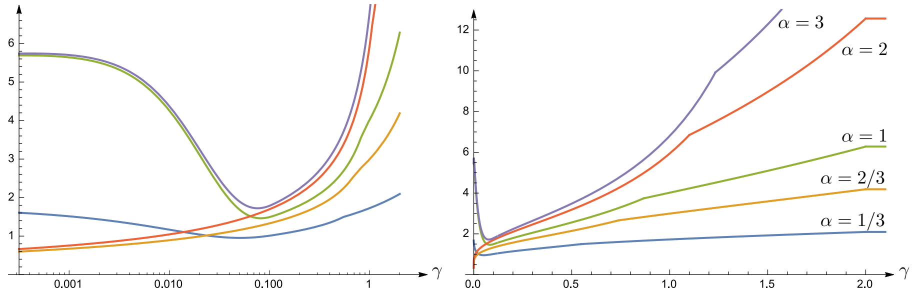
We define (with Dirichlet b.c.)

$$e(\alpha, \gamma) := \liminf_{\substack{N, |Q| \to \infty \\ N/|Q| = \bar{\varrho}}} \frac{E_0(N, Q, \alpha, R)}{\bar{\varrho} N}.$$

Universal bounds for the extended anyon gas



Theorem ([Larson-DL'16] Bounds for the extended anyon gas)

Up to some universal constant $C > 0$,

$$e(\alpha,\gamma) \gtrsim \begin{cases} \frac{2\pi}{|\ln\gamma|} + \pi(j'_{\alpha_*})^2 \geq 2\pi\alpha_*, & \gamma \to 0, \ \alpha \neq 0 \\ 2\pi|\alpha|, & \gamma \gtrsim 1. \end{cases}$$

Ideal anyons in a harmonic trap

Harmonic oscillator Hamiltonian:

$$\hat{H}_N = \hat{T}_\alpha + \hat{V} = \sum_{j=1}^{N} \left(\frac{1}{2m} (-i\nabla_j + \alpha \mathbf{A}_j)^2 + \frac{m\omega^2}{2} |\mathbf{x}_j|^2 \right).$$

Ideal anyons in a harmonic trap

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Rigorous bounds for the ground-state energy $E_0(N)$:

$$\hat{H}_N\big|_{\text{ang.mom.}=L} \;\geq\; \omega \left(N + \left| L + \alpha \tfrac{N(N-1)}{2} \right| \right)$$ (Chitra, Sen, 1992)

Ideal anyons in a harmonic trap

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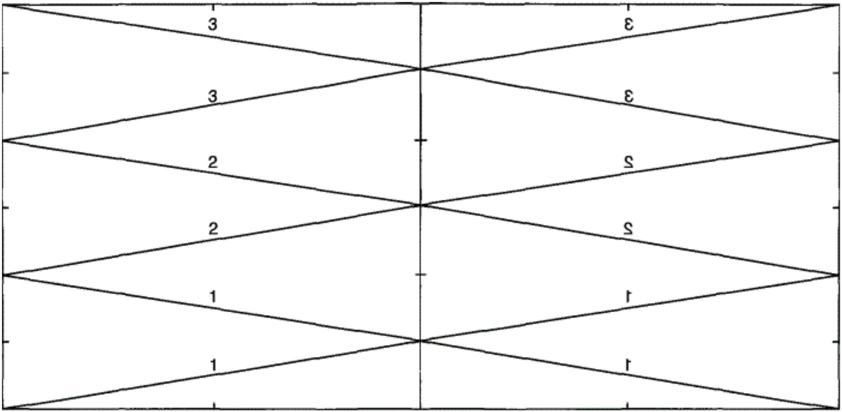
$$C_1\, j'_{\alpha_N} \leq E_0(N)/(\omega N^{\frac{3}{2}}) \leq C_2 \quad \forall \alpha, N$$ (DL, Solovej, 2013; Larson, DL, 2016)

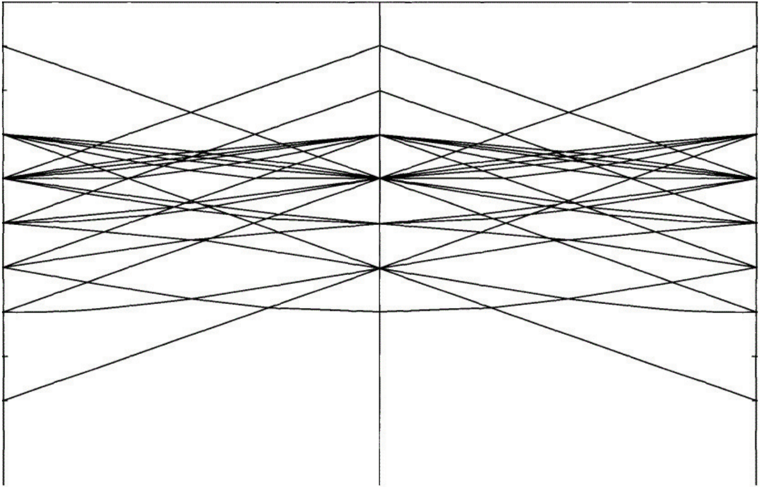
cp. with fermions in 2D: $E_0(N) \sim \frac{\sqrt{8}}{3} \omega N^{\frac{3}{2}}$ as $N \to \infty$

Anyons in a harmonic trap — exact spectrum



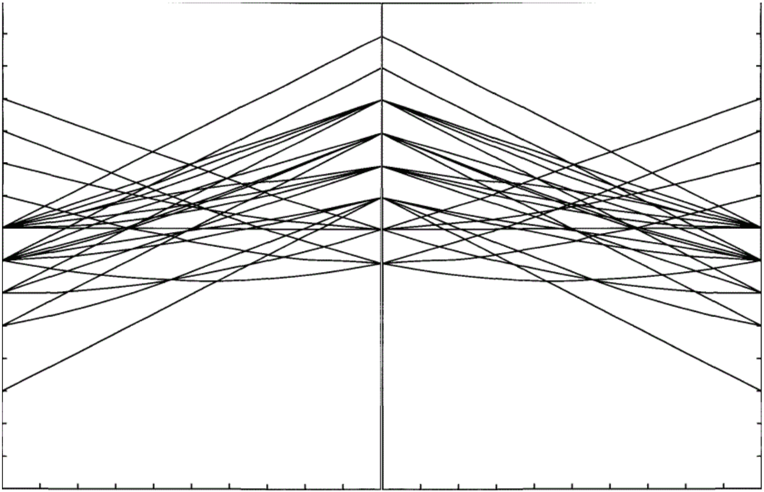
Exact $N = 2$ spectrum: Leinaas, Myrheim, 1977

Anyons in a harmonic trap — exact spectrum



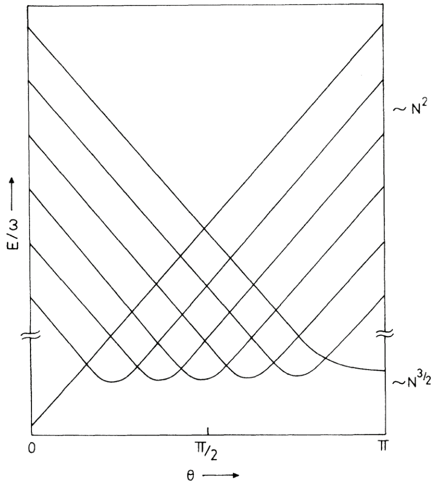
Numerical $N = 3$ spectrum: Murthy, Law, Brack, Bhaduri, 1991; Sporre, Verbaarschot, Zahed, 1991

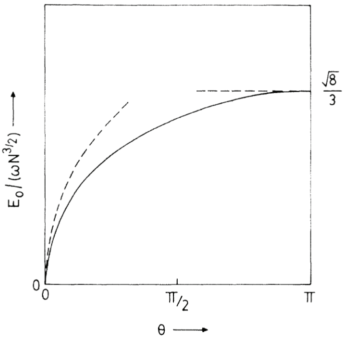
Anyons in a harmonic trap — exact spectrum



Numerical $N = 4$ spectrum: Sporre, Verbaarschot, Zahed, 1992

Anyons in a harmonic trap — qualitative spectrum

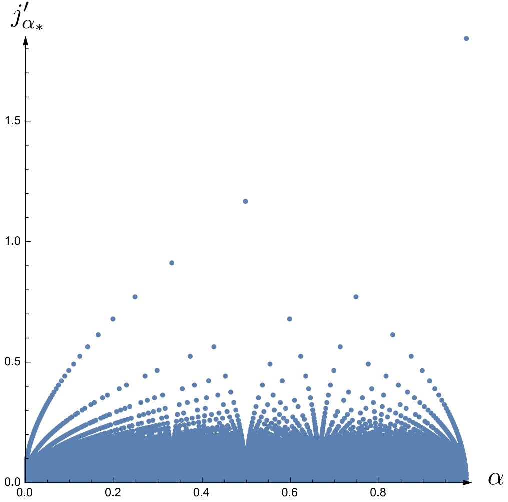




Schematic $N \to \infty$ spectrum: Chitra, Sen, 1992 $(\theta = \alpha\pi)$

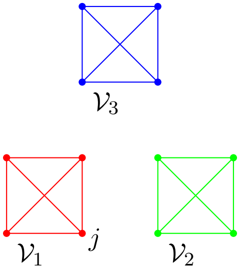
Anyons in a harmonic trap — current lower bound

$$\frac{E_0(N)}{\omega N^{\frac{3}{2}}} \gtrsim$$



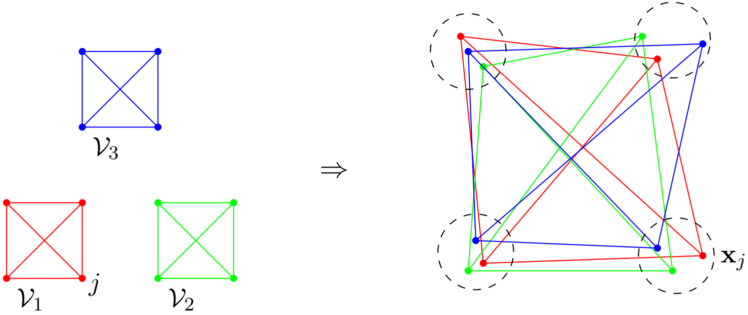
Rigorous lower bound: DL, Solovej, 2013/'14, improved in Larson, DL, 2016

Upper bounds: many-anyon trial states



$N = \nu K$ particles arranged into ν complete graphs $(\mathcal{V}_q, \mathcal{E}_q)$

Upper bounds: many-anyon trial states



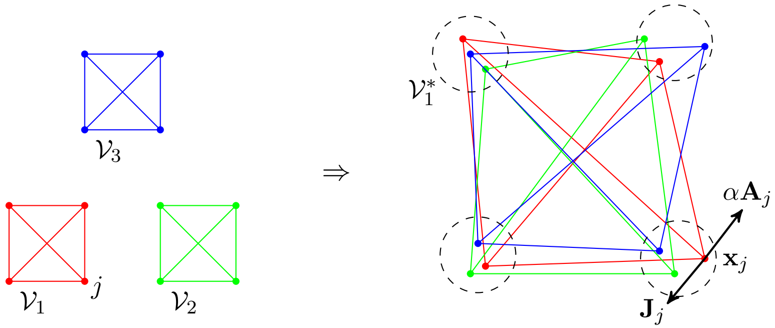
$N = \nu K$ particles arranged into ν complete graphs $(\mathcal{V}_q, \mathcal{E}_q)$

$\alpha = \frac{\mu}{\nu}$ **even**:

$$\psi_\alpha(\mathrm{z}) := \prod_{j<k} |z_{jk}|^{-\alpha} \, \mathcal{S} \left[\prod_{q=1}^{\nu} \prod_{(j,k)\in\mathcal{E}_q} (\bar{z}_{jk})^{\mu} \right] \prod_{k=1}^{N} \varphi_0(z_k)$$

(cf. Moore–Read (Pfaffian), Read–Rezayi)

Upper bounds: many-anyon trial states



$N = \nu K$ particles arranged into ν complete graphs $(\mathcal{V}_q, \mathcal{E}_q)$
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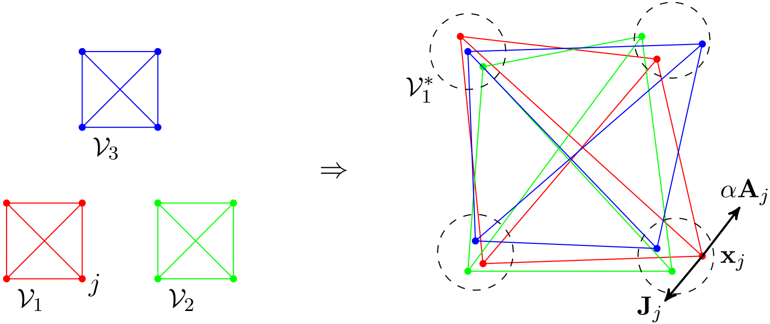
Upper bounds: many-anyon trial states



$N = \nu K$ particles arranged into ν complete graphs $(\mathcal{V}_q, \mathcal{E}_q)$

$\alpha = \frac{\mu}{\nu}$ **odd**:

$$\psi_\alpha(\mathrm{z}) := \prod_{j<k} |z_{jk}|^{-\alpha} \, \mathcal{S} \left[\prod_{q=1}^{\nu} \prod_{(j,k)\in\mathcal{E}_q} (\bar{z}_{jk})^{\mu} \bigwedge_{k=0}^{K-1} \varphi_k \left(z_{j\in\mathcal{V}_q} \right) \right]$$

(cf. Moore–Read (Pfaffian), Read–Rezayi)

Upper bounds: many-anyon trial states

R-extended case: Replace $\prod_{j<k} |z_{jk}|^{-\alpha}$ with $e^{-\alpha \sum_{j<k} w_R(\mathbf{x}_j - \mathbf{x}_k)}$.

Proposition: For the free gas on a box $Q \subset \mathbb{R}^2$, α **even**

$$\boxed{\hat{T}_\alpha^R \psi_\alpha = \alpha W_R \psi_\alpha},$$

$$W_R(\mathrm{x}) := \sum_{j \neq k=1}^{N} \Delta w_R(\mathbf{x}_j - \mathbf{x}_k) = 2\pi \sum_{j \neq k=1}^{N} \frac{\mathbb{1}_{B_R(0)}}{\pi R^2}(\mathbf{x}_j - \mathbf{x}_k).$$

Proposition: For $\Psi = \Phi \psi_\alpha$, $\Phi \in H_0^1(Q^N; \mathbb{R})$,

$$\langle \Psi, \hat{T}_\alpha^R \Psi \rangle \leq C \int_{Q^N} \left(\sum_{j=1}^{N} |\nabla_j \Phi|^2 + \alpha W_R |\Phi|^2 \right) |\psi_\alpha|^2 \, d\mathrm{x}.$$

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