Lieb-Thirring bounds for interacting Bose gases

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based on joint work with
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Outline of Talk

1. Introduction to quantum gases
2. Old and new Lieb-Thirring inequalities
3. Repulsion $\Rightarrow$ local exclusion principle
4. Local uncertainty principle
5. General Lieb-Thirring type inequalities
6. Generalizations to fractional operators, HLT, interpolation

The interacting quantum gas

N-particle Hamiltonian with repulsive pair interaction $W(\boldsymbol{x}) \geq 0$:

$$\hat{H}_N = \hat{T} + \hat{V} + \hat{W} = \sum_{j=1}^{N} (-\Delta_j + V(\boldsymbol{x}_j)) + \sum_{1 \leq j < k \leq N} W(\boldsymbol{x}_j - \boldsymbol{x}_k),$$

acting on normalized wave functions $\Psi \in L^2((\mathbb{R}^d)^N)$. $\frac{\hbar^2}{2m} = 1$.

Bosons: $\Psi \in \bigotimes_{\text{sym}}^N L^2(\mathbb{R}^d)$

Fermions: $\Psi \in \bigwedge^N L^2(\mathbb{R}^d)$

$\Leftarrow$ Pauli's exclusion principle: $\psi \wedge \psi = 0$, $\psi \in L^2(\mathbb{R}^d)$

Total energy in the state Ψ:

$$E[\Psi] = \langle \Psi, \hat{H}_N \Psi \rangle = T_\Psi + V_\Psi + W_\Psi$$

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Local particle density

The one-body density associated to Ψ:

$$\rho_\Psi(\boldsymbol{x}) := \sum_{j=1}^{N} \int_{\mathbb{R}^{d(N-1)}} |\Psi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{j-1}, \boldsymbol{x}, \boldsymbol{x}_{j+1}, \ldots, \boldsymbol{x}_N)|^2 \prod_{k \neq j} d\boldsymbol{x}_k$$

Normalized $\int_{\mathbb{R}^d} \rho_\Psi = N$,

$\int_Q \rho_\Psi =$ expected number of particles on $Q \subseteq \mathbb{R}^d$.

Aim: Replace functionals of $\Psi \in L^2(\mathbb{R}^{dN})$ (where $N \to \infty$) by functionals of $\rho_\Psi \in L^1(\mathbb{R}^d)$

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The non-interacting Bose gas

Know: $\Psi_0 = \otimes^N \psi_0$,

ψ_0 normalized ground state of $\hat{H}_1 = -\Delta_{\mathbb{R}^d} + V(\boldsymbol{x})$

$$E[\Psi_0] = N\langle \psi_0, \hat{H}_1 \psi_0 \rangle = N \int_{\mathbb{R}^d} \left(|\nabla \psi_0|^2 + V|\psi_0|^2 \right) d\boldsymbol{x},$$

$$\rho_{\Psi_0}(\boldsymbol{x}) = N|\psi_0(\boldsymbol{x})|^2$$

The dilute interacting Bose gas (3D)

Dilute limit $a\bar{\rho}^{1/3} \to 0$ while $N \to \infty$. Expect: $\Psi_0 \sim \psi^{\otimes N}$

Gross-Pitaevskii limit: $Na/L \sim const.$ $\Rightarrow$

$$E[\Psi_0] \to \mathcal{E}_{\mathsf{GP}}[\phi_0], \qquad \rho_{\Psi_0}(\boldsymbol{x}) \to |\phi_0(\boldsymbol{x})|^2,$$

$$\mathcal{E}_{\mathsf{GP}}[\phi] := \int_{\mathbb{R}^3} \left(|\nabla\phi|^2 + V|\phi|^2 + 4\pi a|\phi|^4\right) d\boldsymbol{x}, \quad \int_{\mathbb{R}^3} |\phi|^2 = N$$

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Rigorous treatments first by Dyson 1957 (hard-sphere & $V = 0$), more recently and generally by Lieb, Yngvason, Seiringer, ...

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The non-interacting Fermi gas (3D)

Know: $\Psi_0 = \bigwedge_{k=0}^{N-1} \psi_k$, $\quad \psi_k$ lowest states of $\hat{H}_1 = -\Delta_{\mathbb{R}^d} + V(\boldsymbol{x})$

The free Fermi gas in a box $Q \subset \mathbb{R}^3$:

$$E_0 = \sum_{k=0}^{N-1} \lambda_k \sim C_{\mathsf{TF}} \, (\underbrace{N/|Q|}_{\bar{\rho}})^{5/3} |Q|, \quad C_{\mathsf{TF}} = \frac{3}{5}(6\pi^2)^{2/3}$$

$\Rightarrow$ Thomas-Fermi approximation: (Thomas, Fermi, 1927)

$$T_{\Psi_0} + V_{\Psi_0} \approx \int_{\mathbb{R}^3} \Big(C_{\mathsf{TF}} \, \rho_{\Psi_0}(\boldsymbol{x})^{5/3} + V(\boldsymbol{x}) \rho_{\Psi_0}(\boldsymbol{x}) \Big) \, d\boldsymbol{x}$$

(Precursor to modern density functional theory, DFT)

Pauli repulsion and Lieb-Thirring inequalities

Pauli exclusion: say $q \in \mathbb{N}$ particles allowed in each one-particle state of $\hat{H}_1 = -\Delta_{\mathbb{R}^d} + V(\boldsymbol{x})$

$\Rightarrow$ Lieb-Thirring inequality: (Lieb, Thirring, 1975)

$$\hat{H}_{N,\text{Pauli}} = \hat{T} + \hat{V} = \sum_{j=1}^{N} (-\Delta_j + V(\boldsymbol{x}_j))$$

$$\geq -q \sum_{k=0}^{\infty} |\lambda_k| \geq -q\, C_d \int_{\mathbb{R}^d} |V_-(\boldsymbol{x})|^{1+\frac{d}{2}}\, d\boldsymbol{x}$$

$\Leftrightarrow$ kinetic energy inequality: (cp. Thomas-Fermi. New approach due to Rumin, 2011)

$$T_\Psi = \int_{\mathbb{R}^{dN}} \sum_{j=1}^{N} |\nabla_j \Psi|^2\, dx \geq \frac{C'_d}{q^{2/d}} \int_{\mathbb{R}^d} \rho_\Psi(\boldsymbol{x})^{1+\frac{2}{d}}\, d\boldsymbol{x}$$

Bosons: $q = N \to \infty$ $\Rightarrow$ trivial bounds

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The uncertainty principle and LT

For fermions, $\Psi \in \bigwedge^N L^2(\mathbb{R}^d)$, $\|\Psi\| = 1$:

$$T_\Psi = \int_{\mathbb{R}^{dN}} \sum_{j=1}^{N} |\nabla_j \Psi|^2 \, dx \;\geq\; \underbrace{C_d'}_{\leq C_{\mathsf{TF}}} \int_{\mathbb{R}^d} \rho_\Psi(\boldsymbol{x})^{1+\frac{2}{d}} \, d\boldsymbol{x}$$

This can also be interpreted as a many-particle generalization of the Gagliardo-Nirenberg-Sobolev inequality

$$\int_{\mathbb{R}^d} |\nabla u|^2 dx \left(\int_{\mathbb{R}^d} |u|^2 dx \right)^{2/d} \geq C \int_{\mathbb{R}^d} |u|^{2(1+2/d)} \, dx,$$

which is a quantitative formulation of the uncertainty principle.

LT bounds for generalized particle statistics

DL, Solovej, 2011-2013

Abelian **anyons** in 2D, with interchange phase $e^{\alpha\pi i} \in \mathrm{U}(1)$:

$$T_\Psi^{(\alpha)} := \int_{\mathbb{R}^{2N}} \sum_{j=1}^{N} \left|(-i\nabla_j + \boldsymbol{A}_j^{(\alpha)})\Psi\right|^2 dx \;\geq\; C\, C_\alpha^2 \int_{\mathbb{R}^2} \rho_\Psi^2 \, d\boldsymbol{x},$$

$$C_\alpha := \inf_{p,q\in\mathbb{Z}} |(2p+1)\alpha - 2q| \;=\; \begin{cases} 1/\nu \text{ if } \alpha = \mu/\nu \in \mathbb{Q},\ \mu \text{ odd} \\ 0, \text{ otherwise} \end{cases}$$

Intermediate statistics particles in 1D, modeled as bosons with a pair interaction $W(x) = \eta\delta(x)$ or $\alpha(\alpha-1)/|x|^2$: (cp. Lieb-Liniger / Calogero-Sutherland)

$$T_\Psi + W_\Psi \;\geq\; C \int_{\mathbb{R}} e(c/\rho_\Psi(x))\rho_\Psi(x)^3 \, dx$$

$e(c/\rho) \sim$ local bound for the two-particle energy.

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LT bounds for repulsive Bose gases

DL, Portmann, Solovej, 2014

More generally, replace Pauli repulsion by $W \geq 0$.
Examples of new energy inequalities:

For the hard-sphere gas, $W = W_a^{\mathsf{hs}}$ with diameter $a > 0$, in 3D:

$$T_\Psi + W_\Psi \;\geq\; C \int_{\mathbb{R}^3} \min\left\{ a\rho_\Psi(\boldsymbol{x})^2, \rho_\Psi(\boldsymbol{x})^{5/3} \right\} d\boldsymbol{x}$$

cp. $E[\Psi_0]/\mathrm{Vol} \to 4\pi a \rho^2$ as $a\rho^{1/3} \to 0$

For hard disks, $W = W_a^{\mathsf{hd}}$, $a > 0$, in 2D:

$$T_\Psi + W_\Psi \;\geq\; C \int_{\mathbb{R}^2} \frac{\rho_\Psi(\boldsymbol{x})^2}{2 + \left(-\ln(a\rho_\Psi(\boldsymbol{x})^{1/2}/2)\right)_+} d\boldsymbol{x}$$

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Main idea: Local exclusion principle

Consider a d-dimensional box Q, and the local energy $(T+W)^Q_\Psi :=$

$$\sum_{j=1}^N \int_{\mathbb{R}^{dN}} \chi_Q(\boldsymbol{x}_j) \left(|\nabla_j \Psi|^2 + \frac{1}{2} \sum_{k \neq j} W(\boldsymbol{x}_j - \boldsymbol{x}_k) |\psi|^2 \right) dx.$$

If $W \geq 0$ then

$$(T+W)^Q_\Psi \geq \sum_{n=0}^N E_n \, p_n(Q),$$

where $E_n(|Q|; W)$ is the g.s. energy for n particles on Q with Neumann b.c., and $p_n(Q)$ the n-particle probability distribution,

$$\sum_{n=0}^N p_n(Q) = 1, \qquad \sum_{n=0}^N n \, p_n(Q) = \int_Q \rho_\Psi.$$

Local exclusion for fermions

cp. Dyson, Lenard, 1967

Let $\psi \in \bigwedge^n L^2(\mathbb{R}^d)$ be a wave function of n fermions and let Q be a d-cube. Then

$$\int_{Q^n} \sum_{j=1}^{n} |\nabla_j \psi|^2 \, dx \;\geq\; (n-1) \frac{\pi^2}{|Q|^{2/d}} \int_{Q^n} |\psi|^2 \, dx,$$

hence $E_n \geq (n-1)_+ \pi^2 / |Q|^{2/d}$.
It follows that

$$(T + \text{`}W_{\text{Pauli}}\text{'})_\Psi^Q \geq \frac{\pi^2}{|Q|^{2/d}} \left(\int_Q \rho_\Psi(\boldsymbol{x}) \, d\boldsymbol{x} - 1 \right)_+ .$$

Similarly for $W(x) \sim |\boldsymbol{x}|^{-2}$: $\quad W_\Psi^Q \geq \frac{C}{|Q|^{2/d}} \left((\int_Q \rho_\Psi)^2 - \int_Q \rho_\Psi \right)_+ .$

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Local uncertainty principle

We combine exclusion with uncertainty:

Lemma (Local uncertainty principle)

Let Ψ *be an* N*-particle wave function on* $\mathbb{R}^d$*, and* Q *a* d*-cube with volume* $|Q|$*. Then*

$$T_\Psi^Q \;\geq\; c_1 \frac{\int_Q \rho_\Psi^{1+2/d}\, d\boldsymbol{x}}{(\int_Q \rho_\Psi\, d\boldsymbol{x})^{2/d}} - c_2 \frac{\int_Q \rho_\Psi\, d\boldsymbol{x}}{|Q|^{2/d}},$$

where the constants $c_1, c_2 > 0$ *only depend on* d*.*

Idea of proof: $\int_{\mathbb{R}^d} |\nabla \sqrt{\rho_\Psi}|^2$ and Poincaré-Sobolev inequality on Q

General Lieb-Thirring type inequalities

General assumptions on W:

Assumption 1 (Local exclusion)

Given W, there exists a function $e(\gamma)$ with

$$\gamma(|Q|) := \tau|Q|^{(2-\alpha)/d}, \qquad \alpha, \tau > 0,$$

where $e(\gamma)$ is monotone increasing and concave in γ with $e(0) = 0$, such that for any finite cube Q, any $N \geq 1$ and all normalized $\Psi \in H^1(\mathbb{R}^{dN})$ the local energy satisfies

$$(T+W)^Q_\Psi \geq \frac{1}{2} \frac{e(\gamma(|Q|))}{|Q|^{2/d}} \left(\int_Q \rho_\Psi \; - 1 \right)_+ ,$$

General Lieb-Thirring type inequalities

General assumptions on W:

Assumption 2 (Local uncertainty)

Given W, there exist $\alpha > 0$ and constants $S_1, S_2 > 0$ such that for any finite cube Q, any $N \geq 1$ and all normalized $\Psi \in H^1(\mathbb{R}^{dN})$ we have

$$(T+W)_\Psi^Q \geq \begin{cases} S_1 \dfrac{\int_Q \rho_\Psi^{1+2/d}}{(\int_Q \rho_\Psi)^{2/d}} - S_2 \dfrac{\int_Q \rho_\Psi}{|Q|^{2/d}}, & \text{for } 0 < \alpha \leq 2, \\ S_1 \dfrac{\left(\int_Q \rho_\Psi^{1+\alpha/d}\right)^{2/\alpha}}{(\int_Q \rho_\Psi)^{2/\alpha+2/d-1}} - S_2 \dfrac{\int_Q \rho_\Psi}{|Q|^{2/d}}, & \text{for } \alpha > 2. \end{cases}$$

General Lieb-Thirring type inequalities

We also need a boundedness assumption on $e(\gamma)$,

$$\underline{e}_K(\gamma) := \min\{e(\gamma), K\}, \quad K > 0,$$

(arbitrarily strong exclusion cannot be matched by uncertainty)

Theorem (Lieb-Thirring inequality)

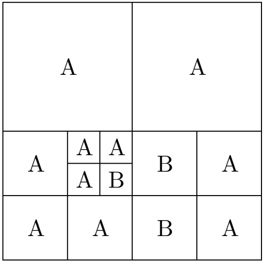
Let W satisfy Assumption 1 & 2 with an $\alpha > 0$ and e replaced by $\underline{e}_K$. Then there exists an explicit constant $C_{d,\alpha,K} > 0$, such that for any $N \geq 1$ and all normalized $\Psi \in H^1(\mathbb{R}^{dN})$, the total energy satisfies the bound

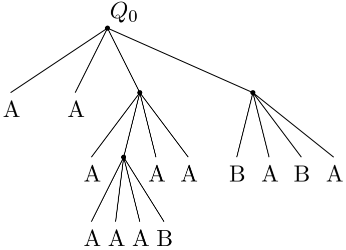
$$T_\Psi + W_\Psi \;\geq\; C_{d,\alpha,K} \int_{\mathbb{R}^d} \underline{e}_K(\gamma(2/\rho_\Psi(\boldsymbol{x})))\, \rho_\Psi(\boldsymbol{x})^{1+2/d}\, d\boldsymbol{x}.$$

DL, Portmann, Solovej, 2014

Proof uses a splitting algorithm (Covering lemma)

DL, Solovej, 2011. Convenient reformulation in DL, Nam, Portmann, 2015





Split a cube $Q_0 \subset \mathbb{R}^d$ recursively until each sub-cube contains ≈ 2 particles (B) or < 2 particles (A). Apply local uncertainty on every cube with non-constant density. Apply local exclusion on B-cubes, which also cover for A-cubes with $\sim$constant density.

Generalizations to fractional operators

DL, Nam, Portmann, 2015

Fractional kin. en. & homogeneous interaction, $d \geq 1$, $s > 0$:

$$\left\langle \Psi, \left(\sum_{j=1}^{N} (-\Delta_j)^s + \sum_{j<k} \frac{1}{|\boldsymbol{x}_j - \boldsymbol{x}_k|^{2s}} \right) \Psi \right\rangle \geq C \int_{\mathbb{R}^d} \rho_\Psi^{1+\frac{2s}{d}} \, d\boldsymbol{x}$$

Special case $d = 3$, $s = 1/2$: N equally charged relativistic particles with Coulomb interaction,

$$\left\langle \Psi, \left(\sum_{j=1}^{N} \sqrt{-\Delta_j} + \sum_{j<k} \frac{1}{|\boldsymbol{x}_j - \boldsymbol{x}_k|} \right) \Psi \right\rangle \geq C \int_{\mathbb{R}^3} \rho_\Psi^{4/3} \, d\boldsymbol{x}$$

Generalizations to fractional operators

Recall Hardy's inequality: $(-\Delta)^s - \frac{\mathcal{C}_{d,s}}{|\boldsymbol{x}|^{2s}} \geq 0$ on $L^2(\mathbb{R}^d)$,

$d \geq 1$, $0 < s < d/2$. Hardy-Lieb-Thirring inequality:

(cp. Ekholm, Frank, Lieb, Seiringer)

$$\left\langle \Psi, \left(\sum_{j=1}^{N} \left((-\Delta_j)^s - \frac{\mathcal{C}_{d,s}}{|\boldsymbol{x}_j|^{2s}} \right) + \sum_{j<k} \frac{1}{|\boldsymbol{x}_j - \boldsymbol{x}_k|^{2s}} \right) \Psi \right\rangle \geq C \int_{\mathbb{R}^d} \rho_\Psi^{1+\frac{2s}{d}} \, d\boldsymbol{x}$$

Special case $d = 3$, $s = 1/2$: N equally charged relativistic particles with Coulomb interaction and a static 'nucleus' at $\boldsymbol{x} = 0$,

$$\hat{H}_N = \sum_{j=1}^{N} \left(\sqrt{-\Delta_j} - \frac{2/\pi}{|\boldsymbol{x}_j|} \right) + \sum_{j<k} \frac{1}{|\boldsymbol{x}_j - \boldsymbol{x}_k|}$$

One-body interpolation inequalities

Taking $\Psi = u^{\otimes N}$ in our Hardy-Lieb-Thirring inequality $\Rightarrow$

$$\left\langle u, \left((-\Delta)^s - \frac{\mathcal{C}_{d,s}}{|x|^{2s}}\right)u\right\rangle^{1-\frac{2s}{d}} \left(\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u(x)|^2|u(y)|^2}{|x-y|^{2s}}\,dxdy\right)^{\frac{2s}{d}} \geq C\int_{\mathbb{R}^d} |u(x)|^{2(1+2s/d)}\,dx,$$

for $0 < s < d/2$. Such an inequality, *without the Hardy term*, was recently proved by Bellazzini, Frank, Ozawa, Visciglia.

Theorem

For $0 < s < d/2$ and $s \leq 1$ this inequality is **equivalent** *to HLT.*

Idea of proof: Use Hoffmann-Ostenhof and Lieb-Oxford inequalities

An isoperimetric inequality

Our approach to proving LT inequalities can also be applied to prove other interpolation inequalities, for example:

Theorem (Isoperimetric inequality with non-local term)

For any $d \geq 2$ and $1/2 \leq s < d/2$ there exists a constant $C > 0$ depending only on d and s, such that for all functions $u \in W^{1,2s}(\mathbb{R}^d)$ we have

$$\left(\int_{\mathbb{R}^d} |\nabla u|^{2s} dx\right)^{1-\frac{2s}{d}} \left(\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(x)|^{2s}|u(y)|^{2s}}{|x-y|^{2s}} dx dy\right)^{\frac{2s}{d}} \geq C \int_{\mathbb{R}^d} |u|^{2s(1+2s/d)}\, dx.$$

References

D. L., J.P. Solovej, *Hardy and Lieb-Thirring inequalities for anyons*, Commun. Math. Phys. 322 (2013) 883, `arXiv:1108.5129`

D. L., J.P. Solovej, *Local exclusion principle for identical particles obeying intermediate and fractional statistics*, Phys. Rev. A 88 (2013) 062106, `arXiv:1205.2520`

D. L., J.P. Solovej, *Local exclusion and Lieb-Thirring inequalities for intermediate and fractional statistics*, Ann. Henri Poincaré 15 (2014) 1061, `arXiv:1301.3436`

D. L., F. Portmann, J.P. Solovej, *Lieb-Thirring bounds for interacting Bose gases*, `arXiv:1402.4463`, to appear in CMP.

D. L., P. T. Nam, F. Portmann, *Fractional Hardy-Lieb-Thirring and related inequalities for interacting systems*, `arXiv:1501.04570`