Recent studies of anyons

Douglas Lundholm
KTH Stockholm

based on work in collaborations with
Michele Correggi, Romain Duboscq, Simon Larson,
Nicolas Rougerie, Jan Philip Solovej

Happy 70th birthday, Barry Simon!
Fields Institute, Toronto

Outline of Talk

1. Fractional statistics in 2D and the emergence of anyons
2. An average-field theory for almost-bosonic anyons
3. Local exclusion principle and universal energy bounds
4. Anyons in a harmonic trap and many-anyon trial states

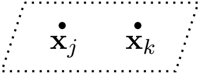
Identical particles and statistics in 2D

Particle exchange in 2D: $\Psi \in L^2((\mathbb{R}^2)^N) \cong \bigotimes^N L^2(\mathbb{R}^2)$

Identical particles and statistics in 2D

Particle exchange in 2D: $\Psi \in L^2((\mathbb{R}^2)^N) \cong \bigotimes^N L^2(\mathbb{R}^2)$

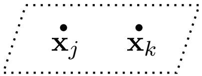
$$|\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_N)|^2 = |\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_N)|^2$$



Identical particles and statistics in 2D

Particle exchange in 2D: $\Psi \in L^2((\mathbb{R}^2)^N) \cong \bigotimes^N L^2(\mathbb{R}^2)$

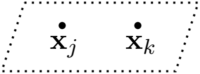
$$\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_N) = e^{i\alpha\pi} \Psi(\mathbf{x}_1, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_N)$$



Identical particles and statistics in 2D

Particle exchange in 2D: $\Psi \in L^2((\mathbb{R}^2)^N) \cong \bigotimes^N L^2(\mathbb{R}^2)$

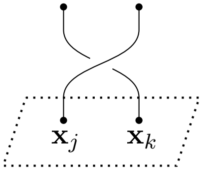
$$\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_N) = \pm\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_N)$$



Identical particles and statistics in 2D

Particle exchange in 2D: $\Psi \in L^2((\mathbb{R}^2)^N) \cong \bigotimes^N L^2(\mathbb{R}^2)$

$$\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_N) = e^{i\alpha\pi} \Psi(\mathbf{x}_1, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_N)$$



$e^{i\alpha\pi} \in U(1)$ **any** phase

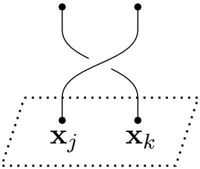
$\alpha = 0$: bosons

$\alpha = 1$: fermions

Identical particles and statistics in 2D

Particle exchange in 2D: $\Psi \in L^2((\mathbb{R}^2)^N) \cong \bigotimes^N L^2(\mathbb{R}^2)$

$$\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_N) = e^{i\alpha\pi}\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_N)$$



$e^{i\alpha\pi} \in U(1)$ **any** phase

$\alpha = 0$: bosons

$\alpha = 1$: fermions

anyons: 'fractional'-statistics quasiparticles in confined systems
— expected to arise in fractional quantum Hall systems

∼1970 Souriau, Streater & Wilde ... Leinaas & Myrheim '77; Goldin, Menikoff & Sharp '81; Wilczek '82 ...

Reviews by Fröhlich '90, Wilczek '90, Lerda '92, Myrheim '99, Khare '05, Ouvry '07, Stern '08, ...

Past rigorous QM studies by Baker, Canright & Mulay '93, Dell'Antonio, Figari & Teta '97

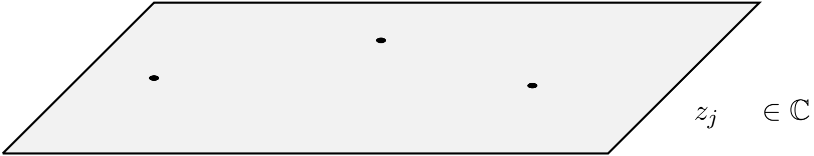
How to create an anyon in the lab?

How to create an anyon in the lab?

- Need several particles!
- Need 2D!

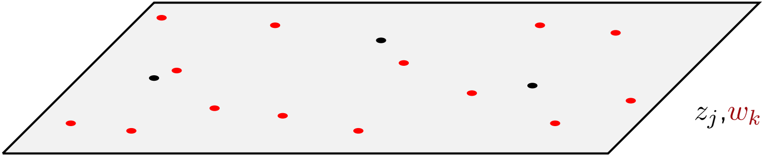
How to create anyons in the lab?

DL, Rougerie, Phys. Rev. Lett., 2016 — avoids usual Berry phase argument of Arovas, Schrieffer, Wilczek, 1984



How to create anyons in the lab?

DL, Rougerie, Phys. Rev. Lett., 2016 — avoids usual Berry phase argument of Arovas, Schrieffer, Wilczek, 1984

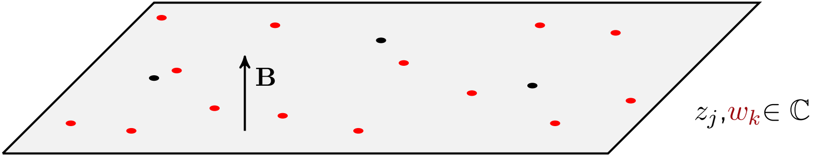


$z_j, w_k \in \mathbb{C}$

- Two species of particles in a plane (bosons or fermions)

How to create anyons in the lab?

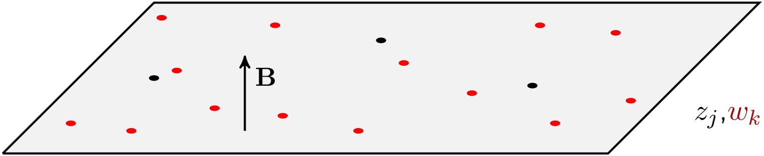
DL, Rougerie, Phys. Rev. Lett., 2016 — avoids usual Berry phase argument of Arovas, Schrieffer, Wilczek, 1984



- Two species of particles in a plane (bosons or fermions)
- Strong perpendicular magnetic field $B \Rightarrow$ LLL

How to create anyons in the lab?

DL, Rougerie, Phys. Rev. Lett., 2016 — avoids usual Berry phase argument of Arovas, Schrieffer, Wilczek, 1984



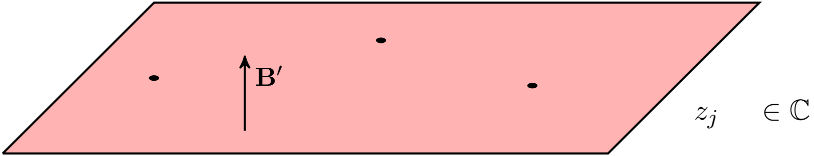
$z_j, w_k \in \mathbb{C}$

- Two species of particles in a plane (bosons or fermions)
- Strong perpendicular magnetic field B $\Rightarrow$ LLL
- Strong repulsion between particles $\Rightarrow$ Laughlin state

$$\Psi(\mathrm{z},\mathrm{w}) = \Phi(\mathrm{z})c(\mathrm{z}) \prod_{j,k}(z_j - w_k) \prod_{i<k}(w_i - w_k)^n \, e^{-B|\mathrm{w}|^2/4}$$

How to create anyons in the lab?

DL, Rougerie, Phys. Rev. Lett., 2016 — avoids usual Berry phase argument of Arovas, Schrieffer, Wilczek, 1984

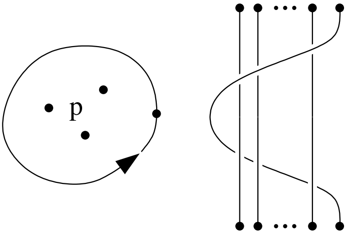


- Two species of particles in a plane (bosons or fermions)
- Strong perpendicular magnetic field $B \;\Rightarrow\;$ LLL
- Strong repulsion between particles $\;\Rightarrow\;$ Laughlin state

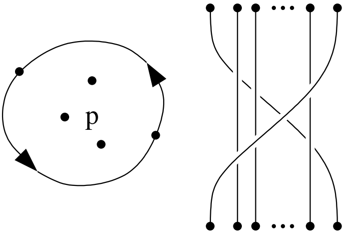
$$\Psi(\mathrm{z},\mathrm{w}) = \Phi(\mathrm{z})c(\mathrm{z}) \prod_{j,k}(z_j - w_k) \prod_{i<k}(w_i - w_k)^n \, e^{-B|\mathrm{w}|^2/4}$$

$\Rightarrow$ Effective Hamiltonian for Φ with a reduced magnetic field and $\alpha = \alpha_0 - 1/n$

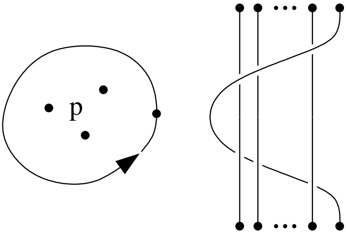
Modelling anyons mathematically — anyon gauge



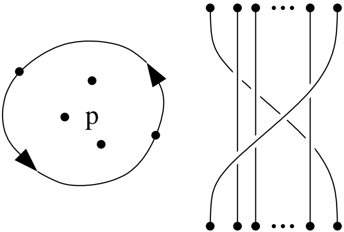
$e^{i2p\alpha\pi}$



$e^{i(2p+1)\alpha\pi}$



$e^{i2p\alpha\pi}$



$e^{i(2p+1)\alpha\pi}$

Think: free kinetic energy $\hat{T}_0 = \frac{\hbar^2}{2m}\sum_{j=1}^{N}(-i\nabla_j)^2$ acting on multi-valued

$$\Psi_\alpha := U^\alpha \, \Psi_0, \qquad U := \prod_{j<k} e^{i\phi_{jk}} = \prod_{j<k} \frac{z_j - z_k}{|z_j - z_k|}.$$

Modelling anyons mathematically — magnetic gauge

Bosons ($\Psi \in L^2_{\mathrm{sym}}$) in $\mathbb{R}^2$ with Aharonov-Bohm magnetic interactions:

$$\hat{T}_\alpha := \frac{\hbar^2}{2m} \sum_{j=1}^{N} D_j^2, \quad D_j = -i\nabla_j + \alpha \mathbf{A}_j, \quad \mathbf{A}_j = \sum_{k \neq j} \frac{(\mathbf{x}_j - \mathbf{x}_k)^\perp}{|\mathbf{x}_j - \mathbf{x}_k|^2}$$

Modelling anyons mathematically — magnetic gauge

Bosons ($\Psi \in L^2_{\mathrm{sym}}$) in $\mathbb{R}^2$ with Aharonov-Bohm magnetic interactions:

$$\hat{T}_\alpha := \frac{\hbar^2}{2m} \sum_{j=1}^N D_j^2, \quad D_j = -i\nabla_j + \alpha \mathbf{A}_j, \quad \mathbf{A}_j = \sum_{k \neq j} \frac{(\mathbf{x}_j - \mathbf{x}_k)^\perp}{|\mathbf{x}_j - \mathbf{x}_k|^2}$$

These are **ideal** anyons. One can also model R-**extended** anyons:

$$\mathbf{A}_j(\mathbf{x}_j) := \sum_{k \neq j} \frac{(\mathbf{x}_j - \mathbf{x}_k)^\perp}{|\mathbf{x}_j - \mathbf{x}_k|_R^2}, \qquad |\mathbf{x}|_R := \max\{|\mathbf{x}|, R\}$$

$$\Rightarrow \quad \operatorname{curl} \alpha \mathbf{A}_j = 2\pi\alpha \sum_{k \neq j} \frac{\mathbb{1}_{B_R(\mathbf{x}_k)}}{\pi R^2} \xrightarrow{R \to 0} 2\pi\alpha \sum_{k \neq j} \delta_{\mathbf{x}_k}$$

Modelling anyons mathematically — magnetic gauge

Bosons ($\Psi \in L^2_{\text{sym}}$) in $\mathbb{R}^2$ with Aharonov-Bohm magnetic interactions:

$$\hat{T}_\alpha := \frac{\hbar^2}{2m}\sum_{j=1}^{N} D_j^2, \quad D_j = -i\nabla_j + \alpha \mathbf{A}_j, \quad \mathbf{A}_j = \sum_{k\neq j} \frac{(\mathbf{x}_j - \mathbf{x}_k)^\perp}{|\mathbf{x}_j - \mathbf{x}_k|^2}$$

These are **ideal** anyons. One can also model R-**extended** anyons:

$$\mathbf{A}_j(\mathbf{x}_j) := \sum_{k\neq j} \frac{(\mathbf{x}_j - \mathbf{x}_k)^\perp}{|\mathbf{x}_j - \mathbf{x}_k|_R^2}, \qquad |\mathbf{x}|_R := \max\{|\mathbf{x}|, R\}$$

$$\Rightarrow \quad \operatorname{curl} \alpha \mathbf{A}_j = 2\pi\alpha \sum_{k\neq j} \frac{\mathbb{1}_{B_R(\mathbf{x}_k)}}{\pi R^2} \xrightarrow{R\to 0} 2\pi\alpha \sum_{k\neq j} \delta_{\mathbf{x}_k}$$

We would like to understand the N-anyon ground state Ψ_0 and energy

$$E_0(N) := \inf \operatorname{spec} \hat{H}_N, \quad \hat{H}_N = \hat{T}_\alpha + \hat{V} = \sum_{j=1}^{N}\left(\frac{\hbar^2}{2m} D_j^2 + V(\mathbf{x}_j)\right)$$

Compare with the ideal Fermi gas in 2D

Know: $\Psi_0 = \bigwedge_{k=0}^{N-1} \varphi_k, \quad \varphi_k$ lowest states of $\hat{H}_1 = -\Delta_{\mathbb{R}^2} + V(\mathbf{x})$

Compare with the ideal Fermi gas in 2D

Know: $\Psi_0 = \bigwedge_{k=0}^{N-1} \varphi_k, \quad \varphi_k$ lowest states of $\hat{H}_1 = -\Delta_{\mathbb{R}^2} + V(\mathbf{x})$

The free Fermi gas in a box $Q \subset \mathbb{R}^2$:

$$E_0(N) = \sum_{k=0}^{N-1} \lambda_k \sim 2\pi \, (\underbrace{N/|Q|}_{\bar{\varrho}})^2 |Q|,$$

Compare with the ideal Fermi gas in 2D

Know: $\Psi_0 = \bigwedge_{k=0}^{N-1} \varphi_k$, φ_k lowest states of $\hat{H}_1 = -\Delta_{\mathbb{R}^2} + V(\mathbf{x})$

The free Fermi gas in a box $Q \subset \mathbb{R}^2$:

$$E_0(N) = \sum_{k=0}^{N-1} \lambda_k \sim 2\pi \,(\underbrace{N/|Q|}_{\bar{\varrho}})^2 |Q|,$$

$\Rightarrow$ Thomas–Fermi approximation: (Thomas, Fermi, 1927 — precursor to modern DFT)

$$\langle \Psi_0, (\hat{T}_{\alpha=1} + \hat{V})\Psi_0 \rangle \approx \int_{\mathbb{R}^2} \Big(2\pi \varrho_{\Psi_0}(\mathbf{x})^2 + V(\mathbf{x})\varrho_{\Psi_0}(\mathbf{x}) \Big) d\mathbf{x}$$

Compare with the ideal Fermi gas in 2D

Know: $\Psi_0 = \bigwedge_{k=0}^{N-1} \varphi_k$, φ_k lowest states of $\hat{H}_1 = -\Delta_{\mathbb{R}^2} + V(\mathbf{x})$

The free Fermi gas in a box $Q \subset \mathbb{R}^2$:

$$E_0(N) = \sum_{k=0}^{N-1} \lambda_k \sim 2\pi\,(\underbrace{N/|Q|}_{\bar{\varrho}})^2|Q|,$$

$\Rightarrow$ Thomas–Fermi approximation: (Thomas, Fermi, 1927 — precursor to modern DFT)

$$\langle \Psi_0, (\hat{T}_{\alpha=1} + \hat{V})\Psi_0 \rangle \approx \int_{\mathbb{R}^2} \Big(2\pi \varrho_{\Psi_0}(\mathbf{x})^2 + V(\mathbf{x})\varrho_{\Psi_0}(\mathbf{x}) \Big) d\mathbf{x}$$

The Lieb–Thirring inequality: (Lieb, Thirring, 1975)

$$\langle \Psi, (\hat{T}_{\alpha=1} + \hat{V})\Psi \rangle \geq \int_{\mathbb{R}^2} \Big(C_{\mathsf{LT}}\, \varrho_{\Psi}(\mathbf{x})^2 + V(\mathbf{x})\varrho_{\Psi}(\mathbf{x}) \Big) d\mathbf{x}$$

Compare with the ideal Fermi gas in 2D

Know: $\Psi_0 = \bigwedge_{k=0}^{N-1} \varphi_k$, φ_k lowest states of $\hat{H}_1 = -\Delta_{\mathbb{R}^2} + V(\mathbf{x})$

The free Fermi gas in a box $Q \subset \mathbb{R}^2$:

$$E_0(N) = \sum_{k=0}^{N-1} \lambda_k \sim 2\pi\,\nu(\nu^{-1}\underbrace{N/|Q|}_{\bar{\varrho}})^2|Q|,$$

$\Rightarrow$ Thomas–Fermi approximation: (Thomas, Fermi, 1927 — precursor to modern DFT)

$$\langle \Psi_0, (\hat{T}_{\alpha=1} + \hat{V})\Psi_0 \rangle \approx \int_{\mathbb{R}^2} \Big(2\pi\nu^{-1}\varrho_{\Psi_0}(\mathbf{x})^2 + V(\mathbf{x})\varrho_{\Psi_0}(\mathbf{x}) \Big) d\mathbf{x}$$

The Lieb–Thirring inequality: (Lieb, Thirring, 1975) ν part.s in each state

$$\langle \Psi, (\hat{T}_{\alpha=1} + \hat{V})\Psi \rangle \geq \int_{\mathbb{R}^2} \Big(C_{\mathsf{LT}}\,\nu^{-1}\varrho_{\Psi}(\mathbf{x})^2 + V(\mathbf{x})\varrho_{\Psi}(\mathbf{x}) \Big) d\mathbf{x}$$

Average-field approximation

(see e.g. Wilczek 1990 review)

For anyons one may consider an **average-field** approximation

$$E_0(N) \approx \inf_{\substack{\varrho \geq 0 \\ \int \varrho = N}} \int_{\mathbb{R}^2} \Big(2\pi|\alpha|\varrho(\mathbf{x})^2 + V(\mathbf{x})\varrho(\mathbf{x}) \Big) d\mathbf{x}$$

where $B = \operatorname{curl} \mathbf{A}_j \approx 2\pi\alpha\varrho$ with LLL energy/particle $\sim |B|$.

Average-field approximation

(see e.g. Wilczek 1990 review)

For anyons one may consider an **average-field** approximation

$$E_0(N) \approx \inf_{\substack{\varrho \geq 0 \\ \int \varrho = N}} \int_{\mathbb{R}^2} \Big(2\pi|\alpha|\varrho(\mathbf{x})^2 + V(\mathbf{x})\varrho(\mathbf{x}) \Big) d\mathbf{x}$$

where $B = \operatorname{curl} \mathbf{A}_j \approx 2\pi\alpha\varrho$ with LLL energy/particle $\sim |B|$.

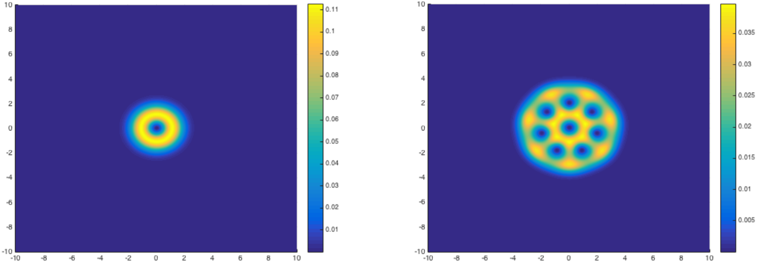
A particular **almost-bosonic** limit $\alpha = \beta/N$ leads to

$$\mathcal{E}^{\mathrm{af}}[u] := \int_{\mathbb{R}^2} \left(\left| \left(-i\nabla + \beta \mathbf{A}[|u|^2] \right) u \right|^2 + V|u|^2 \right), \quad u \in H^1(\mathbb{R}^2)$$

where $\operatorname{curl} \mathbf{A}[|u|^2] = 2\pi|u|^2$ and β the only parameter. DL, Rougerie, 2015

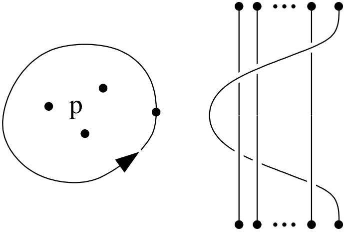
Average-field approximation for almost-bosonic anyons

Continued study of the average-field functional $\mathcal{E}^{\mathrm{af}}[u]$ is work in progress with M. Correggi, R. Duboscq and N. Rougerie.

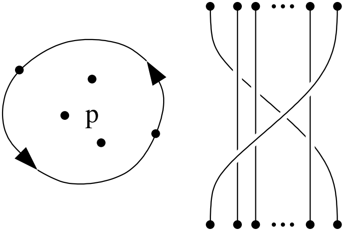


Numerical simulation of $|u^{\mathrm{af}}|^2$ at $\beta = 50$ resp. $\beta = 200$ by Romain Duboscq.

A local exclusion principle for anyons



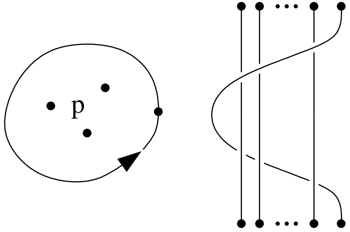
$e^{i2p\alpha\pi}$



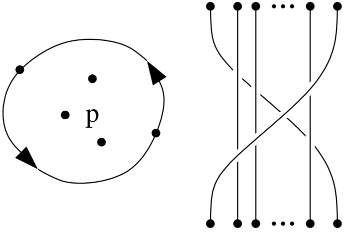
$e^{i(2p+1)\alpha\pi}$

Recall: 2-particle exchange phase $(2p+1)\alpha$ times π.
But anyons can also have pairwise relative angular momenta $\pm 2q$.

A local exclusion principle for anyons



$e^{i2p\alpha\pi}$



$e^{i(2p+1)\alpha\pi}$

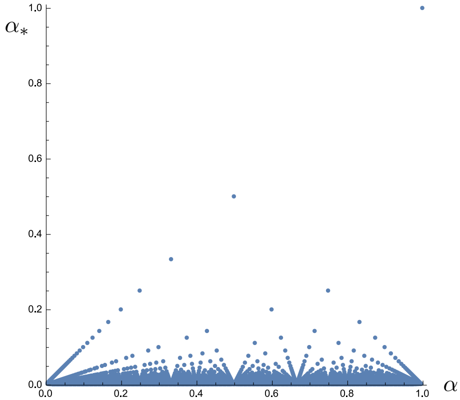
Recall: 2-particle exchange phase $(2p+1)\alpha$ times π.

But anyons can also have pairwise relative angular momenta $\pm 2q$.

$\Rightarrow$ effective **statistical repulsion** DL, Solovej, 2013

$$V_{\mathrm{stat}}(r) = |(2p+1)\alpha - 2q|^2 \frac{1}{r^2} \;\geq\; \frac{\alpha_N^2}{r^2}$$

A local exclusion principle for anyons



$$\alpha_N := \min_{p \in \{0,1,\ldots,N-2\},\, q \in \mathbb{Z}} |(2p+1)\alpha - 2q|$$

$$\underset{N\to\infty}{\longrightarrow} \alpha_* := \begin{cases} \frac{1}{\nu}, & \text{if } \alpha = \frac{\mu}{\nu} \text{ is a reduced fraction with } \mu \text{ odd,} \\ 0 & \text{otherwise.} \end{cases}$$

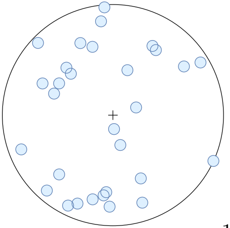
Extended case

We use a magnetic Hardy inequality **with symmetry**

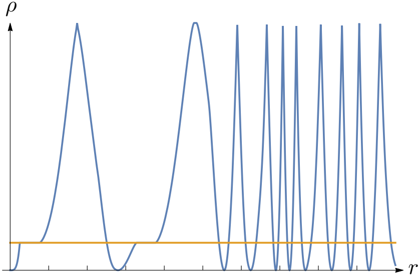
(cf. Laptev, Weidl, 1998; Hoffmann-Ostenhof^2, Laptev, Tidblom, 2008; Balinsky...)

to consider the enclosed flux inside a two-particle exchange loop subtracted with arbitrary pairwise angular momenta. Unwanted oscillation can be controlled by smearing (but analysis is tricky!)

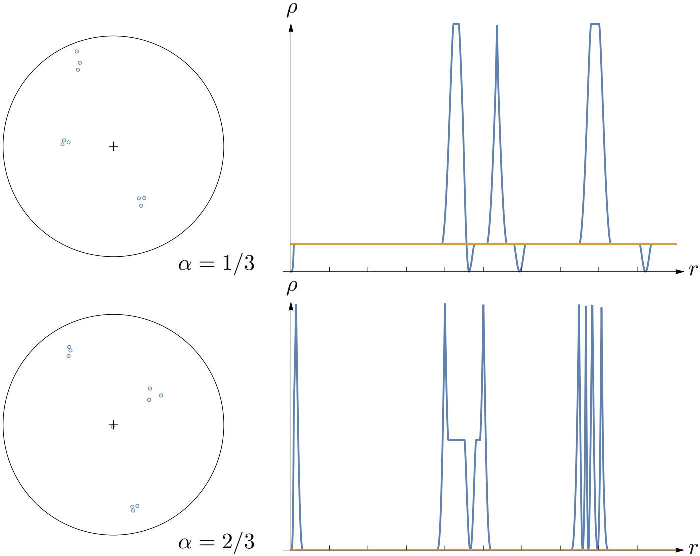
$$V_{\text{stat}}(r) = \rho(r)\frac{1}{r^2}, \qquad \rho(r) = \min_{q\in\mathbb{Z}} \left|\frac{\Phi(r)}{2\pi} - 2q\right|^2$$



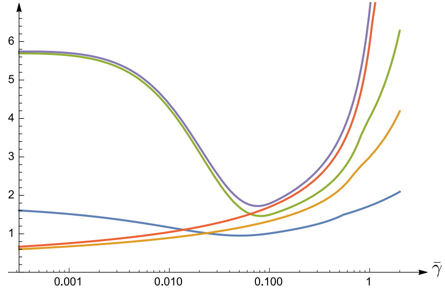
$\alpha = 1/3$

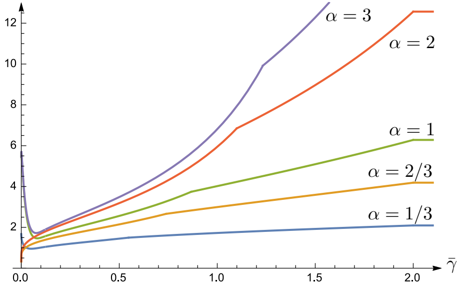


Extended case (clustering)



Universal bounds for the extended anyon gas





Theorem: [Larson, DL, 2016]

There exists $C > 0$ s.t.
$$\liminf_{\substack{N,L\to\infty \\ N/L^2=\bar{\varrho}}} \frac{E_0(N)}{N} \geq C\, e\big(\alpha, \bar{\gamma} := R\sqrt{\bar{\varrho}}\big)\, \bar{\varrho},$$

where
$$e(\alpha,\gamma) \sim \begin{cases} \dfrac{2\pi}{|\ln\gamma|} + \pi (j'_{\alpha_*})^2 \geq 2\pi\alpha_*, & \gamma \to 0, \\ 2\pi|\alpha|, & \gamma \gtrsim 1. \end{cases}$$

Lieb–Thirring inequalities for anyons

Theorem ([DL-Solovej '13] Lieb–Thirring inequality for anyons)

Let Ψ *be an* N*-anyon wavefunction on* $\mathbb{R}^2$ *with any* $\alpha \in \mathbb{R}$*. Then*

$$\langle \Psi, \hat{T}_\alpha \Psi \rangle \;\geq\; C\,\alpha_N^2 \int_{\mathbb{R}^2} \rho_\Psi(\mathbf{x})^2\, d\mathbf{x},$$

for a constant $C > 0$,

So for $\alpha = \mu/\nu$ with **odd** μ and $\nu \geq 1$,

$$\langle \Psi, \hat{H}_N \Psi \rangle \geq \int_{\mathbb{R}^2} \Big(C\nu^{-2} \varrho_\Psi(\mathbf{x})^2 + V(\mathbf{x}) \varrho_\Psi(\mathbf{x}) \Big) d\mathbf{x}$$

Lieb–Thirring inequalities for anyons

DL, Solovej, 2013; LT with general local exclusion developed by DL, Nam, Portmann, Solovej, 2013-'15

Theorem ([Larson-DL '16] Lieb–Thirring inequality for anyons)

Let Ψ be an N-anyon wavefunction on $\mathbb{R}^2$ with any $\alpha \in \mathbb{R}$. Then

$$\langle \Psi, \hat{T}_\alpha \Psi \rangle \;\geq\; C\,(j'_{\alpha_N})^2 \int_{\mathbb{R}^2} \rho_\Psi(\mathbf{x})^2\, d\mathbf{x},$$

for a constant $C > 0$, where $j'_\nu \geq \sqrt{2\nu}$ is first zero of J'_ν Bessel.

So for $\alpha = \mu/\nu$ with **odd** μ and $\nu \geq 1$,

$$\langle \Psi, \hat{H}_N \Psi \rangle \geq \int_{\mathbb{R}^2} \Big(C\nu^{-1} \varrho_\Psi(\mathbf{x})^2 + V(\mathbf{x}) \varrho_\Psi(\mathbf{x}) \Big) d\mathbf{x}$$

Anyons in a harmonic trap

Harmonic oscillator Hamiltonian:

$$\hat{H}_N = \hat{T}_\alpha + \hat{V} = \sum_{j=1}^{N} \left(\frac{1}{2m} (-i\nabla_j + \alpha \mathbf{A}_j)^2 + \frac{m\omega^2}{2} |\mathbf{x}_j|^2 \right).$$

Anyons in a harmonic trap

Harmonic oscillator Hamiltonian:

$$\hat{H}_N = \hat{T}_\alpha + \hat{V} = \sum_{j=1}^{N} \left(\frac{1}{2m} (-i\nabla_j + \alpha \mathbf{A}_j)^2 + \frac{m\omega^2}{2} |\mathbf{x}_j|^2 \right).$$

Rigorous bounds for the ground-state energy $E_0(N)$:

$$\hat{H}_N \big|_{\text{ang. mom. } L} \geq \omega \left(N + \left| L + \alpha \frac{N(N-1)}{2} \right| \right)$$ (Chitra, Sen, 1992)

Anyons in a harmonic trap

Harmonic oscillator Hamiltonian:

$$\hat{H}_N = \hat{T}_\alpha + \hat{V} = \sum_{j=1}^{N} \left(\frac{1}{2m}(-i\nabla_j + \alpha \mathbf{A}_j)^2 + \frac{m\omega^2}{2}|\mathbf{x}_j|^2 \right).$$

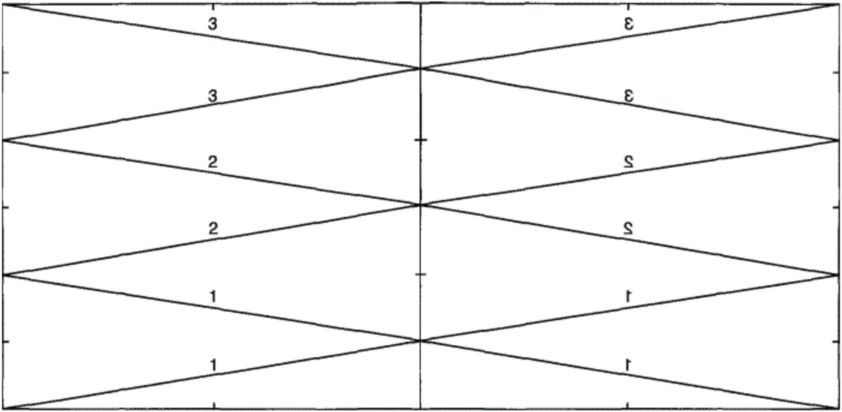
Rigorous bounds for the ground-state energy $E_0(N)$:

$$\hat{H}_N|_{\text{ang. mom. } L} \ \geq \ \omega \left(N + \left| L + \alpha \frac{N(N-1)}{2} \right| \right)$$ (Chitra, Sen, 1992)

$$C_1 \, j'_{\alpha_N} \ \leq \ E_0(N)/(\omega N^{\frac{3}{2}}) \ \leq \ C_2 \quad \forall \alpha, N$$ (DL, Solovej, 2013; Larson, DL, 2016)

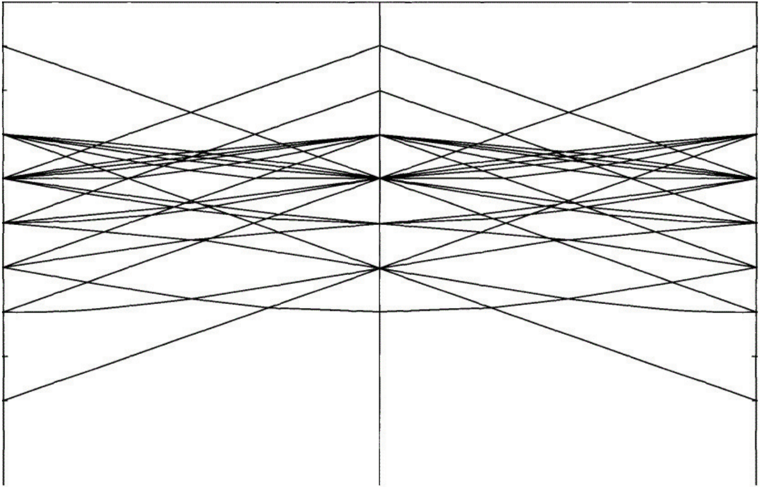
cp. with fermions in 2D: $E_0(N) \sim \frac{\sqrt{8}}{3}\omega N^{\frac{3}{2}}$ as $N \to \infty$

Anyons in a harmonic trap — exact spectrum



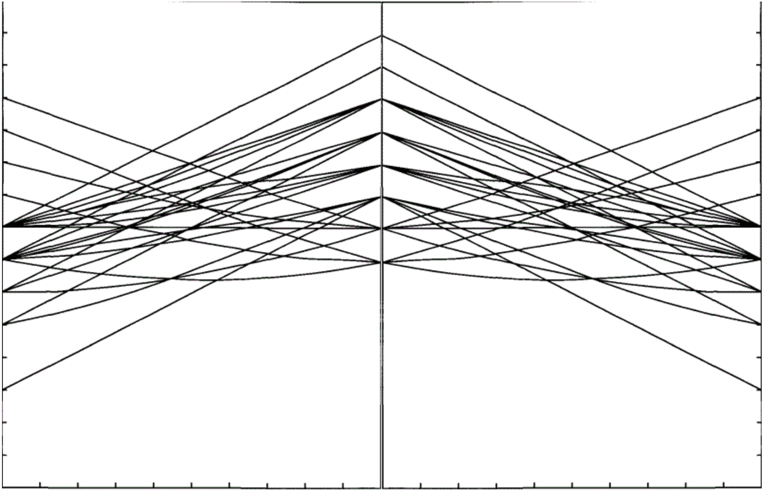
Exact $N = 2$ spectrum: Leinaas, Myrheim, 1977

Anyons in a harmonic trap — exact spectrum



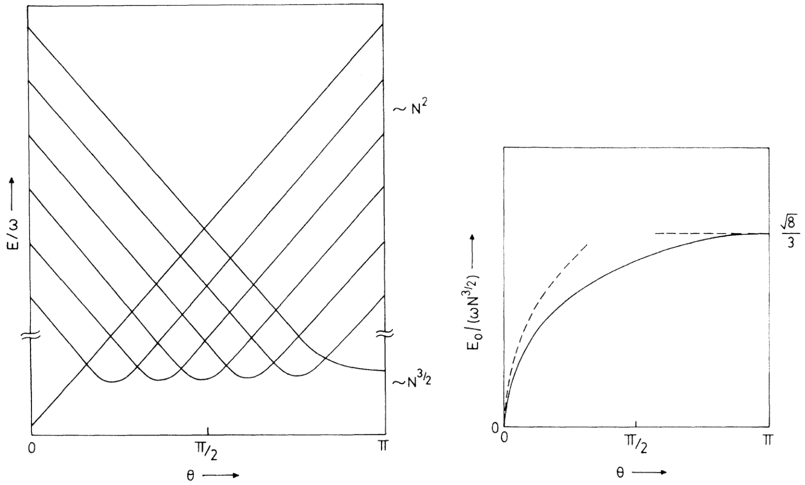
Numerical $N = 3$ spectrum: Murthy, Law, Brack, Bhaduri, 1991; Sporre, Verbaarschot, Zahed, 1991

Anyons in a harmonic trap — exact spectrum



Numerical $N = 4$ spectrum: Sporre, Verbaarschot, Zahed, 1992

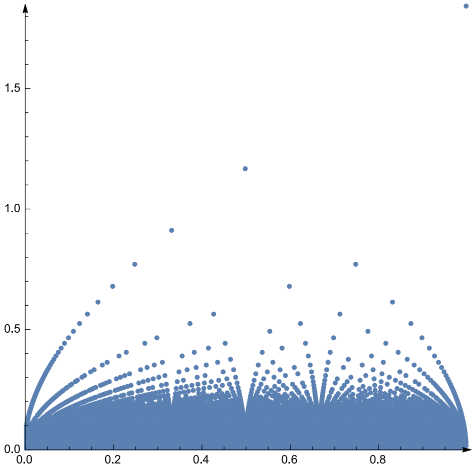
Anyons in a harmonic trap — qualitative spectrum



Schematic $N \to \infty$ spectrum: Chitra, Sen, 1992 $(\theta = \alpha\pi)$

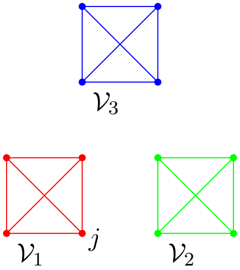
Anyons in a harmonic trap — current lower bound

$$\frac{E_0(N)}{\omega N^{\frac{3}{2}}} \gtrsim$$



Rigorous lower bound: DL, Solovej, 2013/'14, improved in Larson, DL, 2016

Upper bounds: many-anyon trial states



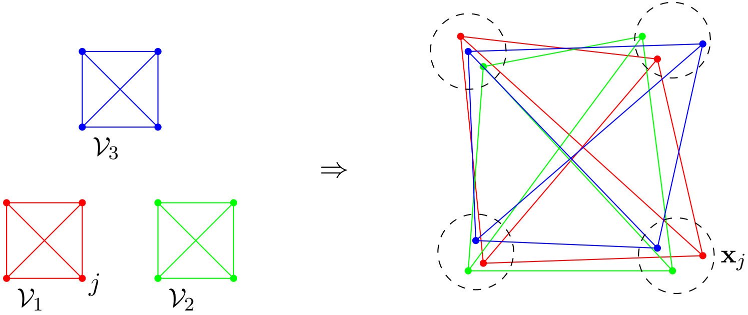
$N = \nu K$ particles arranged into ν complete graphs $(\mathcal{V}_q, \mathcal{E}_q)$

$\alpha = \frac{\mu}{\nu}$ **even**:

$$\psi_\alpha(\mathrm{z}) := \prod_{j<k} |z_{jk}|^{-\alpha}\, \mathcal{S}\left[\prod_{q=1}^{\nu} \prod_{(j,k)\in\mathcal{E}_q} (\bar{z}_{jk})^{\mu}\right] \prod_{k=1}^{N} \varphi_0(z_k)$$

(cf. Moore–Read (Pfaffian), Read–Rezayi)

Upper bounds: many-anyon trial states



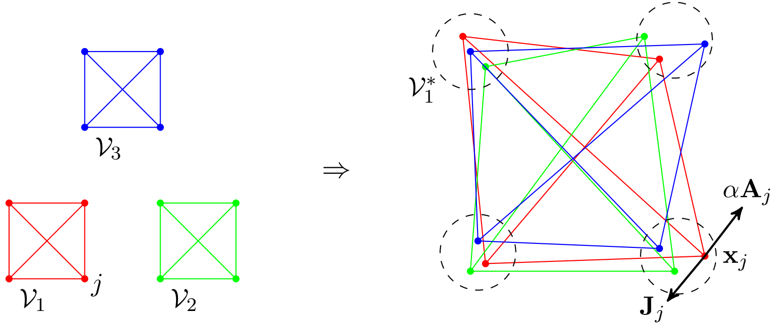
$N = \nu K$ particles arranged into ν complete graphs $(\mathcal{V}_q, \mathcal{E}_q)$

$\alpha = \frac{\mu}{\nu}$ **even**:

$$\psi_\alpha(\mathrm{z}) := \prod_{j<k} |z_{jk}|^{-\alpha} \, \mathcal{S} \left[\prod_{q=1}^{\nu} \prod_{(j,k)\in\mathcal{E}_q} (\bar{z}_{jk})^{\mu} \right] \prod_{k=1}^{N} \varphi_0(z_k)$$

(cf. Moore–Read (Pfaffian), Read–Rezayi)

Upper bounds: many-anyon trial states

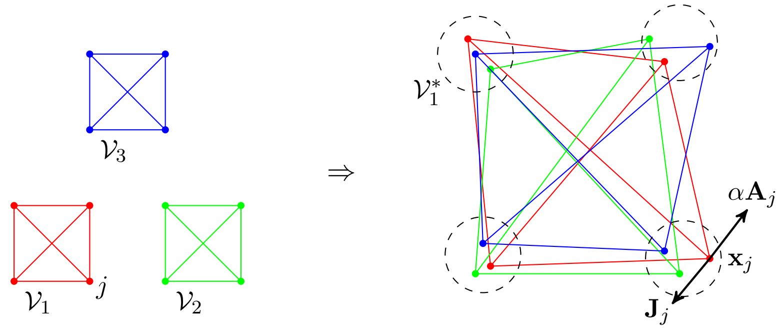


$N = \nu K$ particles arranged into ν complete graphs $(\mathcal{V}_q, \mathcal{E}_q)$
$\alpha = \frac{\mu}{\nu}$ **even**:

$$\psi_\alpha(\mathrm{z}) := \prod_{j<k} |z_{jk}|^{-\alpha} \, \mathcal{S} \left[\prod_{q=1}^{\nu} \prod_{(j,k)\in\mathcal{E}_q} (\bar{z}_{jk})^{\mu} \right] \prod_{k=1}^{N} \varphi_0(z_k)$$

(cf. Moore–Read (Pfaffian), Read–Rezayi)

Upper bounds: many-anyon trial states



$N = \nu K$ particles arranged into ν complete graphs $(\mathcal{V}_q, \mathcal{E}_q)$

$\alpha = \frac{\mu}{\nu}$ **odd**:

$$\psi_\alpha(\mathrm{z}) := \prod_{j<k} |z_{jk}|^{-\alpha} \, \mathcal{S} \left[\prod_{q=1}^{\nu} \prod_{(j,k) \in \mathcal{E}_q} (\bar{z}_{jk})^{\mu} \bigwedge_{k=0}^{K-1} \varphi_k \left(z_{j \in \mathcal{V}_q} \right) \right]$$

(cf. Moore–Read (Pfaffian), Read–Rezayi)

References

D. L., J.P. Solovej, *Hardy and Lieb-Thirring inequalities for anyons*, Commun. Math. Phys. 322 (2013) 883, arXiv:1108.5129

References ↬ Thanks, and happy birthday, Barry!

D. L., J.P. Solovej, *Hardy and Lieb-Thirring inequalities for anyons*, Commun. Math. Phys. 322 (2013) 883, arXiv:1108.5129

D. L., J.P. Solovej, *Local exclusion principle for identical particles obeying intermediate and fractional statistics*, Phys. Rev. A 88 (2013) 062106, arXiv:1205.2520

D. L., J.P. Solovej, *Local exclusion and Lieb-Thirring inequalities for intermediate and fractional statistics*, Ann. Henri Poincaré 15 (2014) 1061, arXiv:1301.3436

D. L., N. Rougerie, *The Average Field Approximation for Almost Bosonic Extended Anyons*, J. Stat. Phys. 161 (2015) 1236, arXiv:1505.05982

D. L., N. Rougerie, *Emergence of fractional statistics for tracer particles in a Laughlin liquid*, Phys. Rev. Lett. 116 (2016) 170401, arXiv:1601.02508

S. Larson, D. L., *Exclusion bounds for extended anyons*, arXiv:1608.04684

D. L., *Many-anyon trial states*, arXiv:1608.05067