

Exchange and exclusion for non-abelian anyons

Douglas Lundholm
KTH Stockholm

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Tübingen

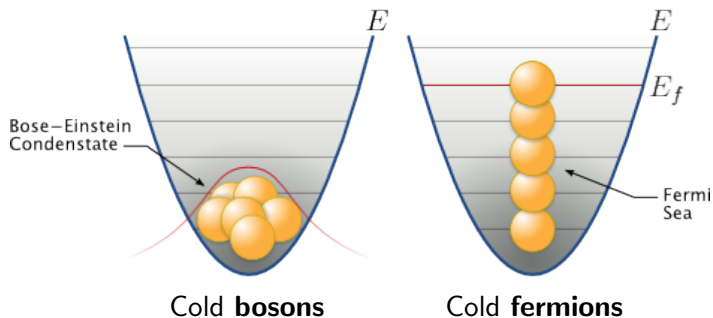
Main references:

- [Q] Viktor Qvarfordt, MSc thesis, KTH/SU, 2017
- [L] D.L., lecture notes 2017-18, [arXiv:1805.03063](https://arxiv.org/abs/1805.03063)
- [LQ] D.L., Viktor Qvarfordt, in preparation.

Outline

- ① Quantum statistics in 2D
- ② Anyon models in general
- ③ Exchange and exclusion (“statistical repulsion”)
- ④ Examples: Fibonacci and Ising
- ⑤ (Algebraic anyon models)

Quantum statistics in 3D



Observable: $|\Psi(\mathbf{x}_1, \dots, \mathbf{x}_j, \dots, \mathbf{x}_k, \dots, \mathbf{x}_N)|^2$, $\mathbf{x}_j \in \mathbb{R}^3$

Exchange symmetry $\rho: S_N \rightarrow U(1)$

+1 : $\rho = 1$ \Rightarrow **bosons** (Bose-Einstein statistics)

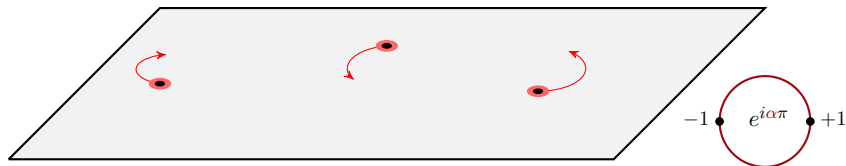
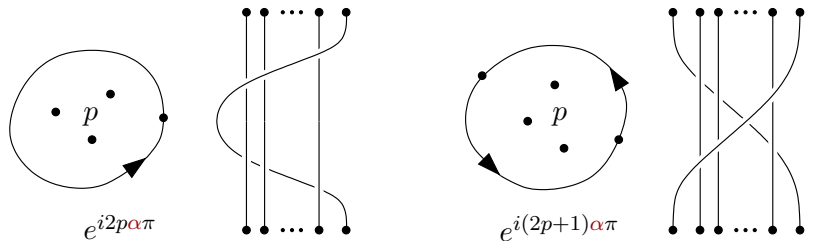
-1 : $\rho = \text{sign}$ \Rightarrow **fermions** (Fermi-Dirac statistics)

Quantum statistics in 2D

Different in 2D!



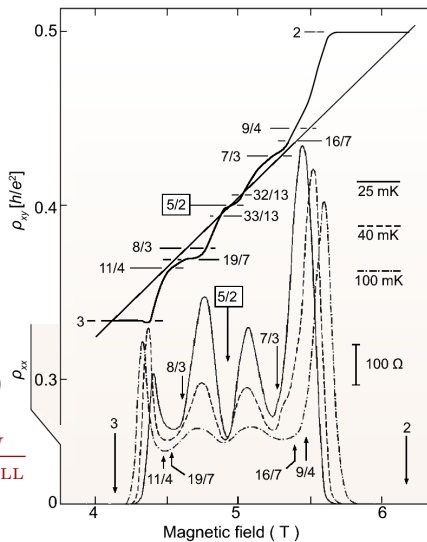
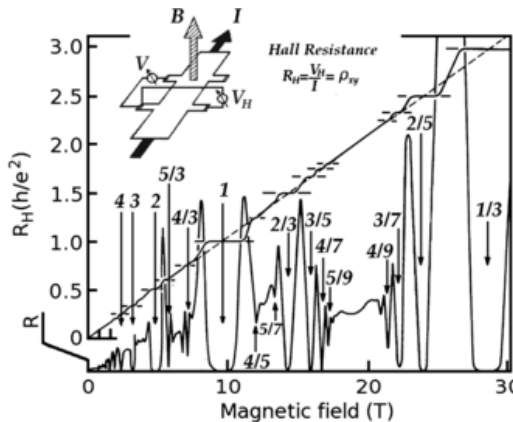
Quantum statistics in 2D



Exchange symmetry $\rho: B_N \rightarrow U(1)$

any phase \Rightarrow “**anyons**”

Fractional quantum Hall effect at $\nu = 5/2$



conductance $_{\perp} \sim$ filling factor $= \frac{N}{\text{dim}_{LL}}$

$\frac{1}{\rho_{xy}} \left[\frac{e^2}{h} \right] \sim \nu = \frac{p}{q}$, usually q odd

[Willett, Eisenstein, Störmer, Tsui, Gossard, English '87]

Anyon (/plekton/nonabelion) models

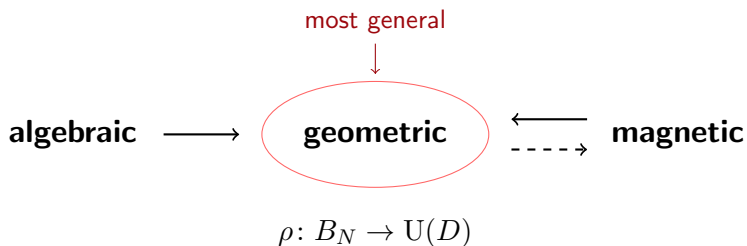
algebraic

geometric

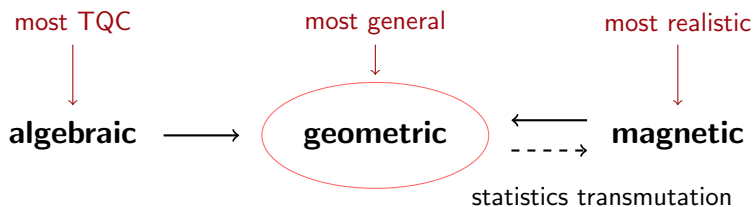


magnetic

Anyon (/plekton/nonabelion) models



Anyon (/plekton/nonabelion) models



$$\rho: B_N \rightarrow U(D)$$

...

Fröhlich '90

Kitaev '97-'06-

+Freedman, Wang '02-

Bonderson '07

+Gurarie, Nayak '11

...

Qvarfordt '17

Leinaas, Myrheim '77

(Goldin, Menikoff, Sharp '81)

...

Dowker '85

Mueller, Doebner, '93

Mund, Schrader '95

Dell'Antonio, Figari, Teta '97

Wilczek '82, Wu '84

+Arovas, Schrieffer '84

Moore, Read '91, +Rezayi '99

Verlinde '91, Lee, Oh '94 (NACS)

Mancarella, Trombettoni, Mussardo '99

DL, Solovej '13,'14

+Rougerie, Larson, Seiringer, '13

Correggi, Duboscq '15-

Geometric anyon models

The **configuration space** of N identical particles in \mathbb{R}^d : [Gibbs]

$$\mathcal{C}^N := \left((\mathbb{R}^d)^N \setminus \Delta \right) / S_N \cong \{N\text{-point subsets of } \mathbb{R}^d\}$$

Distinct points by removal of the **diagonals**:

$$\Delta := \{(\mathbf{x}_1, \dots, \mathbf{x}_N) \in (\mathbb{R}^d)^N : \exists j \neq k \text{ s.t. } \mathbf{x}_j = \mathbf{x}_k\}$$

Exchanges of particles are continuous **loops** in \mathcal{C}^N :

$$\{\text{loops in } \mathcal{C}^N \text{ up to homotopy}\} = \pi_1(\mathcal{C}^N) = \begin{cases} 1, & d = 1, \\ B_N, & d = 2, \\ S_N, & d \geq 3. \end{cases}$$

From now on $d = 2$.

Geometric anyon models: braid group

B_N is the **braid group** on N strands:

$$B_N = \left\langle \sigma_1, \dots, \sigma_{N-1} : \sigma_j \sigma_{j+1} \sigma_j = \sigma_{j+1} \sigma_j \sigma_{j+1}, \sigma_j \sigma_k = \sigma_k \sigma_j \right\rangle_{|j-k|>1}$$

$$\sigma_j : \begin{array}{ccccccc} | & | & | & \text{X} & | & | & | \\ 1 & 2 & \dots & j & \dots & \dots & N \end{array}$$

$$\sigma_j^{-1} : \begin{array}{ccccccc} | & | & | & \text{X} & | & | & | \\ 1 & 2 & \dots & j & \dots & \dots & N \end{array}$$

Ex in B_4 :

$$\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$$

$$\sigma_1 \sigma_3 = \sigma_3 \sigma_1$$

If we add the relations $\sigma_j^2 = 1$ we obtain the **permutation group** S_N

Geometric anyon models: definition

Free anyons: demand that *locally*, i.e. on any topologically *trivial* open subset $\Omega \subseteq \mathcal{C}^N$, the particles behave like usual free non-relativistic *distinguishable* particles (Schrödinger rep.)

$$\hat{T}_\Omega = \sum_{j=1}^N (-i\nabla_{\mathbf{x}_j})^2 \quad \text{on} \quad \Psi \in C_c^\infty(\Omega; \mathcal{F}) \subseteq L^2(\Omega; \mathcal{F}),$$

with some fiber (local/internal) Hilbert space $\mathcal{F} \cong \mathbb{C}^D$.

Fiber bundles: *globally* on \mathcal{C}^N we should consider a hermitian vector bundle $E \xrightarrow{\pi} \mathcal{C}^N$ with fiber \mathcal{F} , endowed with a (locally) *flat* connection \mathcal{A} . **Wave functions** Ψ are L^2 -sections of this bundle.

Theorem: There is a 1-to-1 correspondence between such flat bundles and representations $\rho: \pi_1(\mathcal{C}^N) = B_N \rightarrow \text{U}(\mathcal{F})$.

Definition: A **geometric N -anyon model** is such a rep. $\rho \Rightarrow \hat{T}_\rho$

Magnetic anyon models

Equivalently, sections Ψ are ρ -equivariant functions $\Psi_\rho: \tilde{\mathcal{C}}^N \rightarrow \mathcal{F}$,

$$\Psi_\rho(\gamma \cdot x) = \rho([\gamma])\Psi_\rho(x), \quad \gamma \text{ loop in } \mathcal{C}^N.$$

Local **gauge transformation** $\Psi \rightarrow u\Psi$ where $u: \Omega \rightarrow \text{U}(\mathcal{F})$:

$$\hat{T}_\Omega = -(\nabla + \mathcal{A})^2, \quad \mathcal{A}(x) := u(x)^{-1}\nabla u(x)$$

If $u_\rho: \tilde{\mathcal{C}}^N \rightarrow \text{U}(\mathcal{F})$ is a *global* section of the associated principal bundle,

$$u_\rho(\gamma \cdot x) = \rho([\gamma])u_\rho(x), \quad \gamma \text{ loop in } \mathcal{C}^N,$$

then we have a transformation to *trivial* bundle $\Psi_1 \in L_{\text{sym}}^2(\mathbb{R}^{2N}; \mathcal{F})$:

$$\Psi_\rho = u_\rho\Psi_1 \quad \Leftrightarrow \quad \Psi_1 = u_\rho^{-1}\Psi_\rho$$

Definition: A **transmutable N -anyon model** is an N -anyon model $\rho: B_N \rightarrow \text{U}(\mathcal{F})$ such that its corresponding flat principal bundle $P \rightarrow \mathcal{C}^N$ is topologically trivial.

Magnetic anyon models

So, *transmutable* models ρ may equivalently be described using **bosons** (or fermions) with gauge potentials $\mathcal{A}: \mathbb{R}^{2N} \rightarrow \mathfrak{u}(\mathcal{F})$.

Obstacle: Only some of rep. theory of B_N and topology known.

Theorem: Any *abelian* model is transmutable. [Dowker '85, Mund, Schrader '95]

Theorem: Any rep $\rho: B_N \rightarrow U(D)$, $N > 6$, is abelian if $D < N - 2$. [Formanek '96]

- In the non-abelian case typically $D \sim c^N$ for some $c > 0$.
- NACS: $\rho \sim \rho_1^{\otimes N}$ is transmutable.
- Transmutability of a bundle $E \rightarrow \mathcal{C}^N$ improves with $E \oplus E \dots$

Given ρ , consider **exchange operator** ($2p + 1$ braidings):

$$U_p := \rho(\sigma_1 \sigma_2 \dots \sigma_p \sigma_{p+1} \sigma_p \dots \sigma_2 \sigma_1), \quad p = 0, 1, \dots, N - 2$$

Statistical repulsion \Leftarrow Poincaré inequality

Hardy inequality for fermions: [Hoffmann-Ostenhof², Laptev, Tidblom '08]

$$\hat{T}_{\rho=\text{sign}} \geq \frac{d^2}{N} \sum_{1 \leq j < k \leq N} \frac{1}{|\mathbf{x}_j - \mathbf{x}_k|^2}$$

Local versions can be used to prove extensivity, LT, etc. [DL, Solovej '13]

Poincaré for fermions: $u(-\omega) = -u(\omega)$, $\omega \in \mathbb{S}^{d-1}$ relative angles

$$\int_{\mathbb{S}^{d-1}} |\nabla_{\omega} u|^2 d\omega \geq (d-1) \int_{\mathbb{S}^{d-1}} |u|^2 d\omega$$

Poincaré for 2D fermions: $u(\varphi + \pi) = -u(\varphi)$

$$\int_0^{2\pi} |u'|^2 d\varphi \geq \int_0^{2\pi} |u|^2 d\varphi$$

Statistical repulsion \Leftarrow Poincaré inequality [2 anyons]

Poincaré for 2D fermions: $u(\varphi + \pi) = -u(\varphi)$

$$\int_0^\pi |u'|^2 d\varphi \geq \int_0^\pi |u|^2 d\varphi$$

Poincaré for **abelian anyons**: $u(\varphi + \pi) = e^{i\pi\alpha}u(\varphi)$, $\alpha \in (-1, 1]$

$$\int_0^\pi |u'|^2 d\varphi \geq \alpha^2 \int_0^\pi |u|^2 d\varphi$$

Poincaré for **non-abelian anyons**: $u(\varphi + \pi) = U_0 u(\varphi)$, $U_0 \in U(\mathcal{F})$

$$\int_0^\pi |u'|^2 d\varphi \geq \beta_0^2 \int_0^\pi |u|^2 d\varphi$$

$\beta_0 := \min\{\beta \in [0, 1] : e^{i\beta\pi} \text{ or } e^{-i\beta\pi} \text{ is an eigenvalue of } U_0\}$

\Rightarrow **statistical repulsion** for a pair of anyons if $\beta_0 > 0$

Statistical repulsion \Leftarrow Poincaré inequality [2 + p anyons]

Poincaré for 2D fermions: $u(\varphi + \pi) = -u(\varphi)$

$$\int_0^\pi |u'|^2 d\varphi \geq \int_0^\pi |u|^2 d\varphi$$

Poincaré for **abelian anyons**: $u(\varphi + \pi) = e^{i\pi(2p+1)\alpha}u(\varphi)$,

$$\int_0^\pi |u'|^2 d\varphi \geq \min_{q \in \mathbb{Z}} |(2p+1)\alpha - 2q|^2 \int_0^\pi |u|^2 d\varphi$$

Poincaré for **non-abelian anyons**: $u(\varphi + \pi) = U_p u(\varphi)$, $U_p \in U(\mathcal{F})$

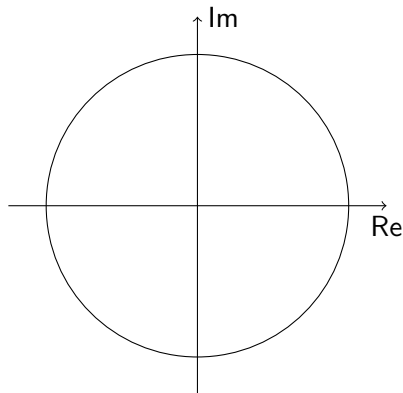
$$\int_0^\pi |u'|^2 d\varphi \geq \beta_p^2 \int_0^\pi |u|^2 d\varphi$$

$\beta_p := \min\{\beta \in [0, 1] : e^{i\beta\pi} \text{ or } e^{-i\beta\pi} \text{ is an eigenvalue of } U_p\}$

\Rightarrow **statistical repulsion** (Hardy, extensivity, ...) for anyons if $\beta_p > 0$

Abelian anyons, ex. $\alpha = 3/5$

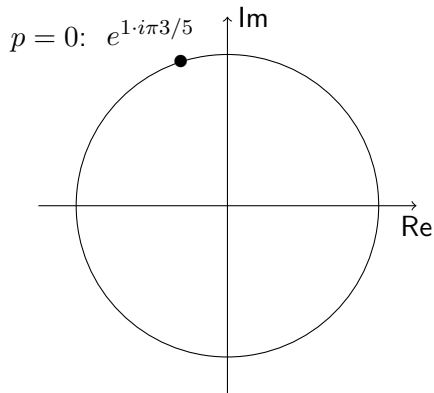
$$U_p = e^{(2p+1)i\pi\alpha}$$



$$\Rightarrow \text{Poincaré inequality with } \beta_p = \begin{cases} 3/5, & p = 0, 4, 5, \dots \\ 1/5, & p = 1, 3, \dots \\ 1, & p = 2, 7, \dots \end{cases}$$

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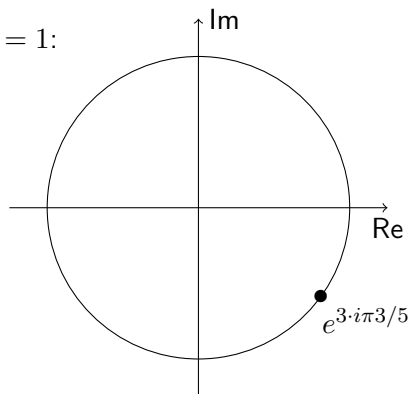


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$$U_p = e^{(2p+1)i\pi\alpha}$$

$p = 1:$

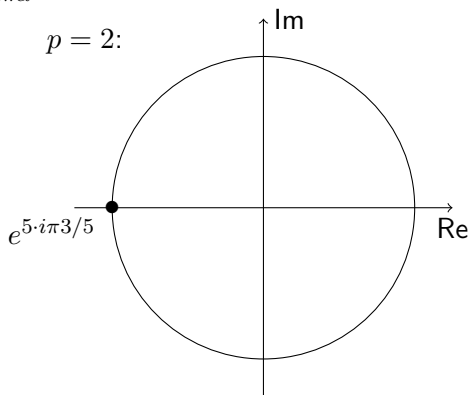


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Abelian anyons, ex. $\alpha = 3/5$

$$U_p = e^{(2p+1)i\pi\alpha}$$

$$p = 2:$$

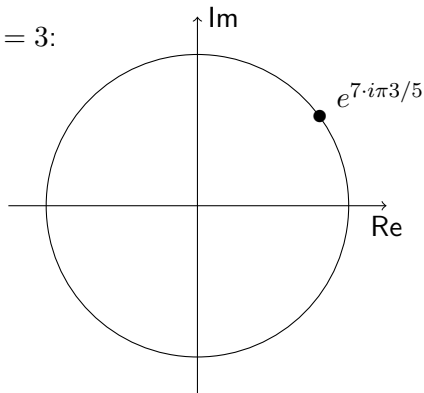


$$\Rightarrow \text{Poincaré inequality with } \beta_p = \begin{cases} 3/5, & p = 0, 4, 5, \dots \\ 1/5, & p = 1, 3, \dots \\ 1, & p = 2, 7, \dots \end{cases}$$

Abelian anyons, ex. $\alpha = 3/5$

$$U_p = e^{(2p+1)i\pi\alpha}$$

$p = 3:$

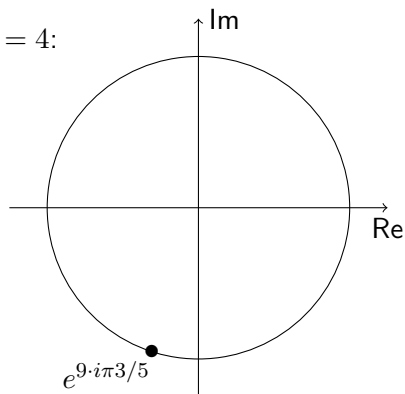


\Rightarrow **Poincaré inequality** with $\beta_p = \begin{cases} 3/5, & p = 0, 4, 5, \dots \\ 1/5, & p = 1, 3, \dots \\ 1, & p = 2, 7, \dots \end{cases}$

Abelian anyons, ex. $\alpha = 3/5$

$$U_p = e^{(2p+1)i\pi\alpha}$$

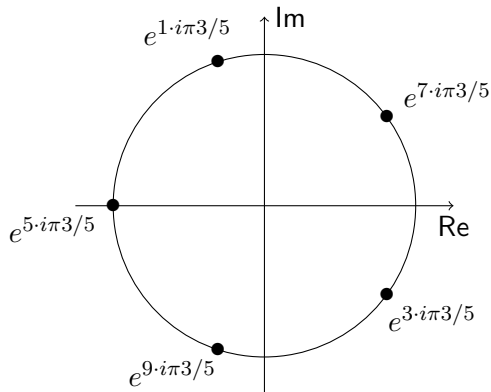
$p = 4:$



\Rightarrow **Poincaré inequality** with $\beta_p = \begin{cases} 3/5, & p = 0, 4, 5, \dots \\ 1/5, & p = 1, 3, \dots \\ 1, & p = 2, 7, \dots \end{cases}$

Abelian anyons, ex. $\alpha = 3/5$

$$U_p = e^{(2p+1)i\pi\alpha}$$

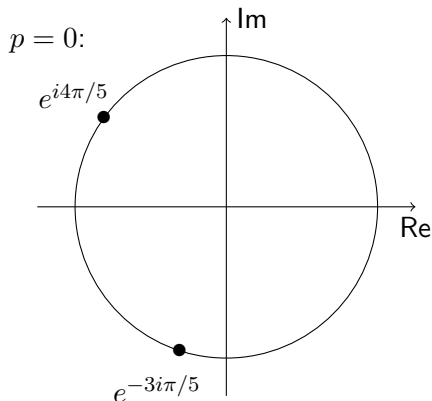


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Fibonacci anyons: Exchange eigenvalues

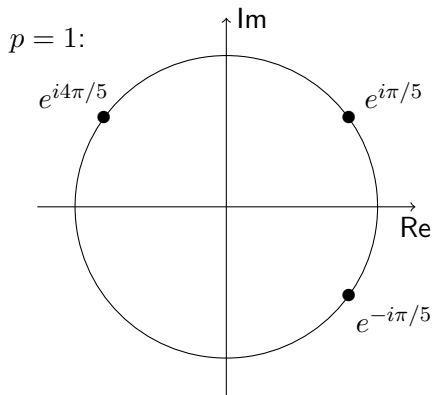
$$U_p \sim U_{\tau,1,\tau}^{\oplus \text{Fib}(p-1)} \oplus U_{\tau,\tau,\tau}^{\oplus \text{Fib}(p)},$$

$$\text{spec}(U_{\tau,1,\tau}) = \{e^{4\pi i/5}, e^{-3\pi i/5}\}, \quad \text{spec}(U_{\tau,\tau,\tau}) = \{e^{4\pi i/5}, e^{\pi i/5}, e^{-\pi i/5}\}$$



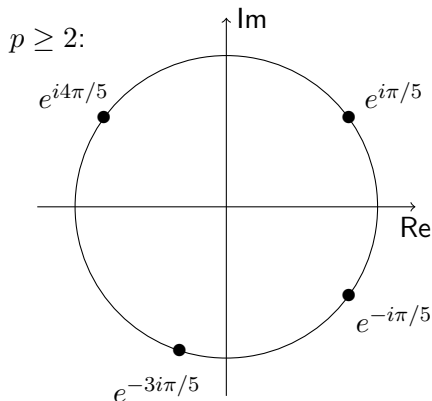
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Fibonacci anyons: Exchange eigenvalues

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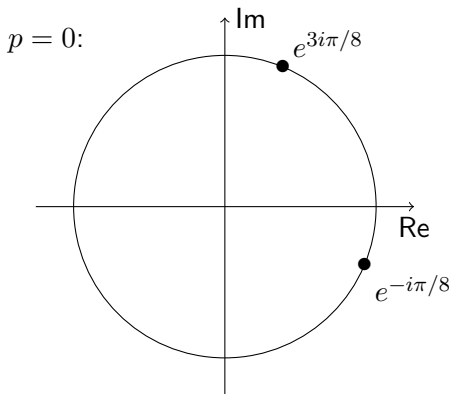


\Rightarrow **Poincaré inequality** with $\beta_0 = 3/5$ and $\beta_{p \geq 1} = 1/5$

Ising anyons: Exchange eigenvalues

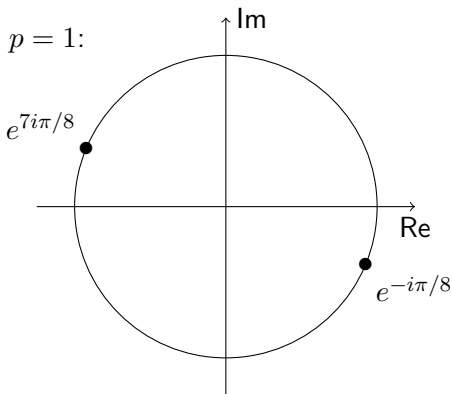
$$U_{p=2n+1} \sim U_{\sigma,\sigma,\sigma}^{\oplus 2^n}, \quad U_{p=2n} \sim U_{\sigma,1,\sigma}^{\oplus 2^{n-1}} \oplus U_{\sigma,\psi,\sigma}^{\oplus 2^{n-1}},$$

$$\text{spec}(U_{\sigma,\sigma,\sigma}) = \{e^{-\pi i/8}, e^{7\pi i/8}\}, \quad \text{spec}(U_{\sigma,1,\sigma}) = \{e^{-\pi i/8}, e^{3\pi i/8}\}, \quad \text{spec}(U_{\sigma,\psi,\sigma}) = \{e^{-5\pi i/8}, e^{7\pi i/8}\}$$



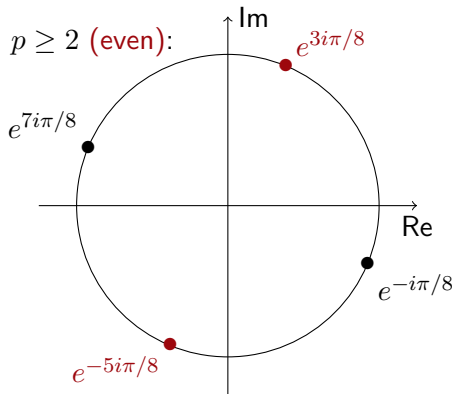
Ising anyons: Exchange eigenvalues

$$U_{p=2n+1} \sim U_{\sigma,\sigma,\sigma}^{\oplus 2^n}, \quad U_{p=2n} \sim U_{\sigma,1,\sigma}^{\oplus 2^{n-1}} \oplus U_{\sigma,\psi,\sigma}^{\oplus 2^{n-1}},$$
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Ising anyons: Exchange eigenvalues

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\Rightarrow **Poincaré inequality** with $\beta_{p \geq 0} = 1/8$

Further references on the math-phys of anyons

Introduction and reviews of recent (abelian) results:

D. L., *Methods of modern mathematical physics: Uncertainty and exclusion principles in quantum mechanics*, lecture notes for a master-level course given at KTH in 2017 (revised May 2018), arXiv:1805.03063

N. Rougerie, *Some contributions to many-body quantum mathematics*, habilitation thesis, 2016, arXiv:1607.03833

D. L., *Many-anyon trial states*, Phys. Rev. A 96 (2017) 012116, arXiv:1608.05067

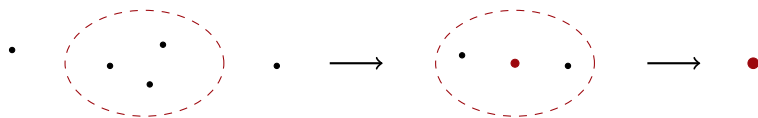
M. Correggi, R. Duboscq, D. L., N. Rougerie, *Vortex patterns in the almost-bosonic anyon gas*, arXiv:1901.10739

Bonus Part

Algebraic anyon models

Braided fusion categories...

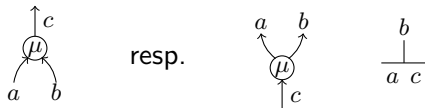
Idea: **zoom out**



Labels / topological charges / particle types:

$$\mathcal{L} = \{a, b, c, \dots\} = \{1, a, \bar{a}, b, \bar{b}, \dots\}$$

Fusion / splitting diagrams:



Span spaces $V_{ab}^c \cong V_c^{ab}$ of dimension $N_{ab}^c =$ number of ways fusion/splitting can occur.

Algebraic anyon models: Fusion

Fusion algebra: $a, b \in \mathcal{L}$,

$$a \times b = \sum_{c \in \mathcal{L}} N_{ab}^c c$$

The model turns out to be abelian if there is a unique result of fusion,

$$a \times b = c$$

Typically, $N_{ab}^c \in \{0, 1\}$, i.e. *multiplicity-free* models.

We write $c \in a \times b$ if $N_{ab}^c \neq 0$. Sums only over allowed indices.

Algebraic anyon models: Fusion

Associativity of fusion: $(a \times b) \times c = a \times (b \times c)$

\Rightarrow **F operator** (isomorphism on 2-split diagrams):

$$F: \frac{\begin{array}{cc} b & c \\ | & | \\ a & e & d \end{array}}{\quad} \mapsto \frac{\begin{array}{cc} b & c \\ \cup & \\ & e \\ | & \\ a & d \end{array}}{\quad} = \sum_f F_{d;fe}^{abc} \frac{\begin{array}{cc} b & c \\ | & | \\ a & f & d \end{array}}{\quad}$$

Algebraic anyon models: Braiding

Commutativity of fusion: $a \times b = b \times a$

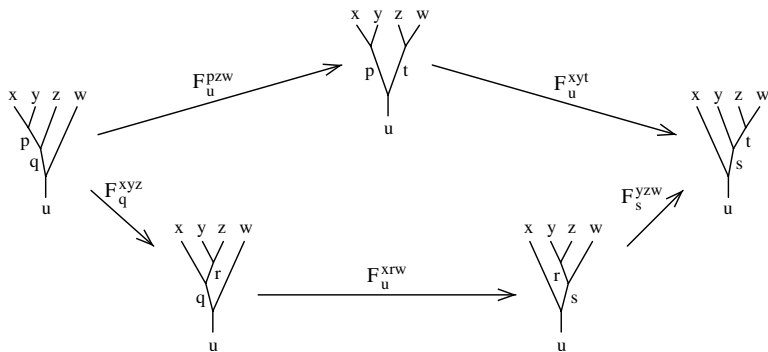
\Rightarrow **R operator** (isomorphism on 1-split diagrams): $R_c^{ab} \in U(V_c^{ab})$

$$R^{ab} : \begin{array}{c} a \quad b \\ \diagdown \quad / \\ \mu \\ | \\ c \end{array} \mapsto \begin{array}{c} a \quad b \\ | \quad | \\ \diagdown \quad / \\ \mu \\ | \\ c \end{array} = \sum_{\nu} [R_c^{ab}]_{\nu\mu} \begin{array}{c} a \quad b \\ \diagdown \quad / \\ \nu \\ | \\ c \end{array} .$$

\Rightarrow **B operator** (isomorphism on 2-split diagrams): $B := F R F^{-1}$

$$\begin{aligned} \frac{\begin{array}{c} b \quad c \\ \diagdown \quad / \\ a \quad e \quad d \end{array}}{a \quad e \quad d} &= \sum_f (F^{-1})_{d;fe}^{acb} \frac{\begin{array}{c} b \quad c \\ | \quad | \\ \diagdown \quad / \\ f \\ | \\ a \quad d \end{array}}{a \quad d} = \sum_f R_f^{bc} (F^{-1})_{d;fe}^{acb} \frac{\begin{array}{c} b \quad c \\ \diagdown \quad / \\ a \quad f \quad d \end{array}}{a \quad f \quad d} \\ &= \sum_g \sum_f F_{d;gf}^{abc} R_f^{bc} (F^{-1})_{d;fe}^{acb} \frac{\begin{array}{c} b \quad c \\ | \quad | \\ a \quad g \quad d \end{array}}{a \quad g \quad d} \end{aligned}$$

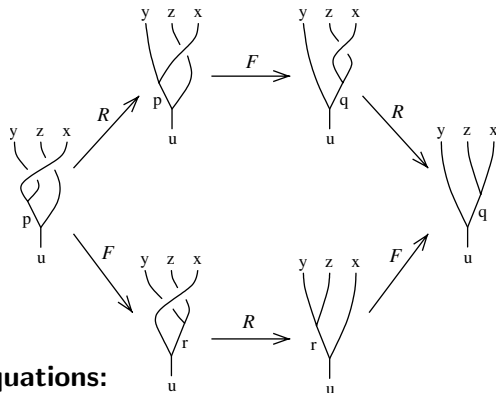
Algebraic anyon models: Consistency conditions



Pentagon equation:

$$F_{e;xu}^{aby} F_{e;yv}^{ucd} = \sum_{w \in \mathcal{L}} F_{v;wu}^{abc} F_{e;xv}^{awd} F_{x;yw}^{bcd}$$

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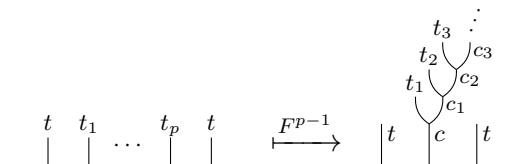


Hexagon equations:

$$R_p^{yx} F_{u;qp}^{yxz} R_q^{zx} = \sum_{r \in \mathcal{L}} F_{u;rp}^{xyz} R_u^{rx} F_{u;qr}^{yzx}$$

$$(R_p^{yx})^{-1} F_{u;qp}^{yxz} (R_q^{zx})^{-1} = \sum_{r \in \mathcal{L}} F_{u;rp}^{xyz} (R_u^{rx})^{-1} F_{u;qr}^{yzx}$$

Exchange in algebraic anyon models



Theorem ([DL, Qvarfordt])

The exchange operator of two t 's around t_1, \dots, t_p is given by

$$U_{t, \{t_1, \dots, t_p\}, t} \sim \bigoplus_{c \in t_1 \times \dots \times t_p} U_{t, c, t}$$

where c is a possible result of the fusion $t_1 \times t_2 \times \dots \times t_p$, counted with multiplicity.

Fibonacci anyons

Model: $\mathcal{L} = \{1, \tau\}$,

$$\tau \times \tau = 1 + \tau$$

N -particle basis states:

$$\frac{\tau}{1\tau}, \quad \frac{\tau\tau}{1\tau 1}, \frac{\tau\tau}{1\tau\tau}, \quad \frac{\tau\tau\tau}{1\tau\tau 1}, \frac{\tau\tau\tau}{1\tau 1\tau}, \frac{\tau\tau\tau}{1\tau\tau\tau}, \quad \dots$$

$$\tau^N = \text{Fib}(N-1)1 + \text{Fib}(N)\tau,$$

where $\text{Fib}(0) = 0$, $\text{Fib}(1) = 1$, $\text{Fib}(n) = \text{Fib}(n-2) + \text{Fib}(n-1)$.

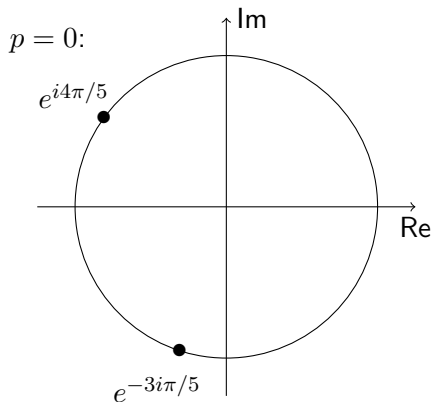
$$F_1^{\tau\tau\tau} = 1 \quad \text{and} \quad F_\tau^{\tau\tau\tau} = \begin{bmatrix} \phi^{-1} & \phi^{-1/2} \\ \phi^{-1/2} & -\phi^{-1} \end{bmatrix}, \quad \phi = \frac{1 + \sqrt{5}}{2},$$

$$R_1^{\tau\tau} = e^{4\pi i/5}, \quad R_\tau^{\tau\tau} = e^{-3\pi i/5}.$$

An N -anyon **Fibonacci model**: $\mathcal{F} = V_*^{1, \tau^N} \cong \mathbb{C}^D$, $D = \text{Fib}(N+1)$

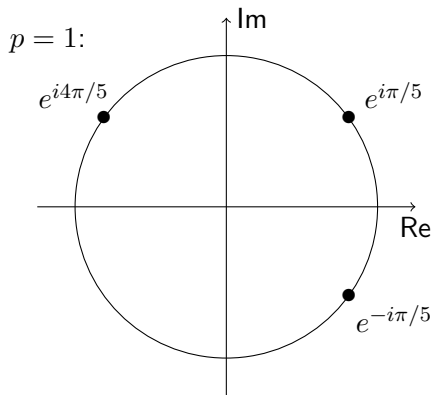
Fibonacci anyons: Exchange eigenvalues

$$U_{\tau,\tau^p,\tau} \sim U_{\tau,1,\tau}^{\oplus \text{Fib}(p-1)} \oplus U_{\tau,\tau,\tau}^{\oplus \text{Fib}(p)},$$
$$\text{spec}(U_{\tau,1,\tau}) = \{e^{4\pi i/5}, e^{-3\pi i/5}\}, \quad \text{spec}(U_{\tau,\tau,\tau}) = \{e^{4\pi i/5}, e^{\pi i/5}, e^{-\pi i/5}\}$$



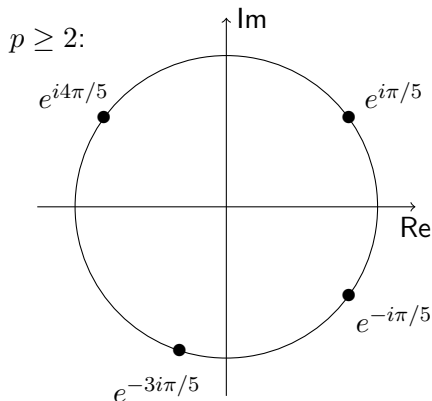
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\Rightarrow **Poincaré inequality** with $\beta_0 = 3/5$ and $\beta_{p \geq 1} = 1/5$

Ising anyons

Model: $\mathcal{L} = \{1, \psi, \sigma\}$,

$$\sigma \times \sigma = 1 + \psi, \quad \sigma \times \psi = \sigma, \quad \psi \times \psi = 1$$

$$\frac{\sigma}{1\sigma}, \quad \frac{\sigma\sigma}{1\sigma 1}, \quad \frac{\sigma\sigma}{1\sigma\psi}, \quad \frac{\sigma\sigma\sigma}{1\sigma 1\sigma}, \quad \frac{\sigma\sigma\sigma}{1\sigma\psi\sigma}, \quad \frac{\sigma\sigma\sigma\sigma}{1\sigma 1\sigma 1}, \quad \frac{\sigma\sigma\sigma\sigma}{1\sigma\psi\sigma 1}, \quad \dots$$

$$\sigma^{2n+1} = 2^n \sigma, \quad \sigma^{2n} = 2^{n-1} (1 + \psi)$$

$$F_{\sigma}^{\sigma\sigma\sigma} = \pm \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad F_{\psi;\sigma\sigma}^{\sigma\psi\sigma} = F_{\sigma;\sigma\sigma}^{\psi\sigma\psi} = -1,$$

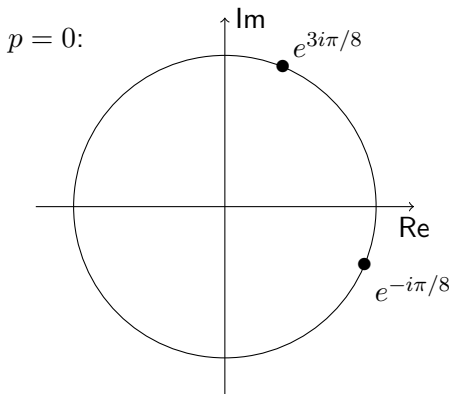
$$R_1^{\sigma\sigma} = e^{-\pi i/8}, \quad R_{\psi}^{\sigma\sigma} = e^{3\pi i/8}, \quad R_{\sigma}^{\sigma\psi} = R_{\sigma}^{\psi\sigma} = -i, \quad R_1^{\psi\psi} = -1.$$

An **N -anyon Ising model**: \hat{T}_{ρ_N} with $\mathcal{F} = V_*^{1,\sigma^N} \cong \mathbb{C}^D$, $D = 2^{\lfloor N/2 \rfloor}$

Ising anyons: Exchange eigenvalues

$$U_{\sigma, \sigma^{2n+1}, \sigma} \sim U_{\sigma, \sigma, \sigma}^{\oplus 2^n}, \quad U_{\sigma, \sigma^{2n}, \sigma} \sim U_{\sigma, 1, \sigma}^{\oplus 2^{n-1}} \oplus U_{\sigma, \psi, \sigma}^{\oplus 2^{n-1}},$$

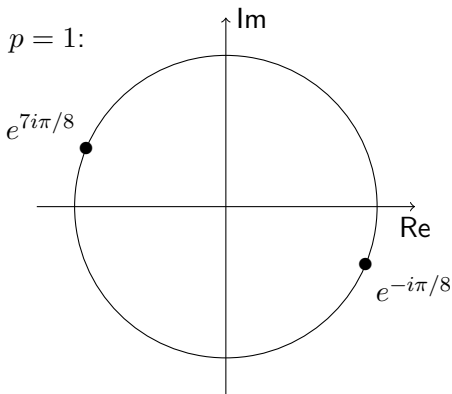
$$\text{spec}(U_{\sigma, \sigma, \sigma}) = \{e^{-\pi i/8}, e^{7\pi i/8}\}, \quad \text{spec}(U_{\sigma, 1, \sigma}) = \{e^{-\pi i/8}, e^{3\pi i/8}\}, \quad \text{spec}(U_{\sigma, \psi, \sigma}) = \{e^{-5\pi i/8}, e^{7\pi i/8}\}$$



Ising anyons: Exchange eigenvalues

$$U_{\sigma,\sigma^{2n+1},\sigma} \sim U_{\sigma,\sigma,\sigma}^{\oplus 2^n}, \quad U_{\sigma,\sigma^{2n},\sigma} \sim U_{\sigma,1,\sigma}^{\oplus 2^{n-1}} \oplus U_{\sigma,\psi,\sigma}^{\oplus 2^{n-1}},$$

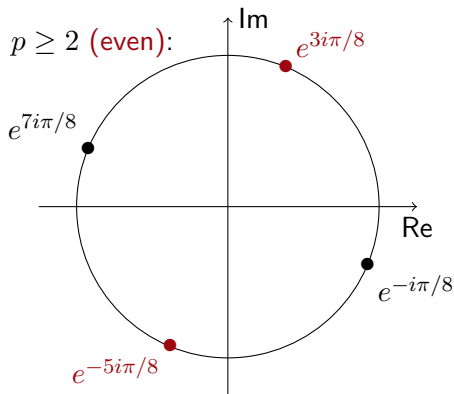
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\Rightarrow **Poincaré inequality** with $\beta_{p \geq 0} = 1/8$