Twisted perspectives on Quantum Mechanics

Douglas Lundholm, Uppsala Universitet, 24 November 2022

Motivation for this talk

The Royal Swedish Academy of Sciences has decided to award the **Nobel Prize in Physics 2022** jointly to

Alain Aspect

Institut d'Optique Graduate School – Université Paris-Saclay and École Polytechnique, Palaiseau, France

John F. Clauser

J.F. Clauser & Assoc., Walnut Creek, CA, USA

Anton Zeilinger

University of Vienna, Austria

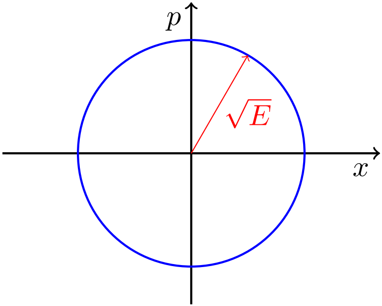
"for experiments with entangled photons, establishing the violation of Bell inequalities and pioneering quantum information science"

Outline

1. "Three pillars" of quantum mechanics in mathematical terms
2. Impossible figures as solutions to impossible problems
3. Contextual reality and quantum games
4. Implications for society?

What is physics?

Physics is about **observables** and **relations**.

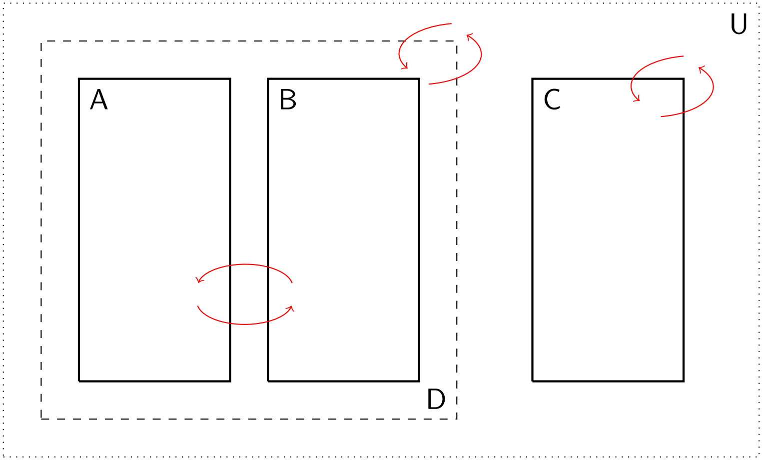


Observables: things that can be measured and have well-defined values, i.e. **properties of reality**, ex. $x \in \mathbb{R}$ position of a particle, $p \in \mathbb{R}$ momentum, $E \in \mathbb{R}$ energy, $t \in \mathbb{R}$ time

Relations: **correlations** between observables, ex. $E = p^2 + x^2$

What is physics?

Physics is about **subsystems/observers** and **information**.



An observable could concern the information that A has on B etc.

What is quantum mechanics?

Quantization $\leftrightarrow$ representation of a **Lie algebra** of observables:

Observables $a, b, \ldots$ usually modeled jointly as self-adjoint linear operators $\hat{a}, \hat{b}, \ldots$ on some Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$, with:

$\operatorname{spec} \hat{a}$ $\leftrightarrow$ values that $\hat{a}$ can take upon measurement

resolution of $\hat{a} = \sum_{a \in \operatorname{spec} \hat{a}} a \operatorname{proj}_{\mathcal{H}_a}$ $\leftrightarrow$ possible information about $\hat{a}$ obtainable from the system

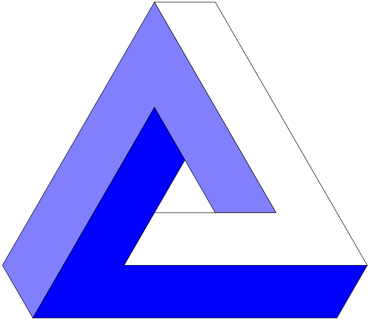
state $0 \neq \Psi \in \mathcal{H} = \bigoplus_{a \in \operatorname{spec} \hat{a}} \mathcal{H}_a$ $\leftrightarrow$ actual info/knowledge i.e. current subjective 'reality'

$$\text{expectation } \frac{\langle \Psi, \hat{a}\Psi \rangle}{\|\Psi\|^2} = \sum_{a \in \operatorname{spec} \hat{a}} a \frac{\|\operatorname{proj}_{\mathcal{H}_a} \Psi\|^2}{\|\Psi\|^2},$$

$i\hat{c} = \hat{a}\hat{b} - \hat{b}\hat{a}$ $\leftrightarrow$ obstacle to coherent information on $\hat{a}$ and $\hat{b}$

What is quantum mechanics?

1. **Uncertainty principle** (1D)
2. **Exclusion principle** (2D)
3. **Contextuality** (3D)



Uncertainty principle: incommensurability

(In)**commensurate** observables $\leftrightarrow$ (non)**commuting** operators, ex.

$$A = \begin{bmatrix} +1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

same spectrum $\{+1, -1\}$, $A^2 = B^2 = \mathbb{1}$, but

$$AB - BA = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \neq 0, \qquad AB = -BA.$$

This means that obtaining knowledge of one destroys knowledge of the other:

$$\mathcal{H} = \mathbb{C}^2 = \mathbb{C} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \oplus \mathbb{C} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbb{C} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \oplus \mathbb{C} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Uncertainty principle: Heisenberg's version (1D)

Continuous version: $x \in \mathbb{R}$ and $p \in \mathbb{R}$, **"conjugate"** non-comm.:

$$\hat{x}\hat{p} - \hat{p}\hat{x} = i\mathbb{1}$$

Solution/representation:

$$\mathcal{H} = L^2(\mathbb{R}; \mathfrak{h}) = \int_{\mathbb{R}}^{\oplus} \mathfrak{h}, \quad \text{i.e. } \Psi : \mathbb{R} \to \mathfrak{h},$$

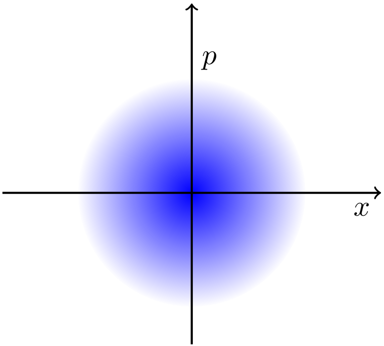
$$\hat{x}\Psi(x) = x\Psi(x), \qquad \hat{p}\Psi(x) = -i\Psi'(x),$$

$$x(-i\Psi'(x)) - (-id/dx)(x\Psi(x)) = i\Psi(x),$$

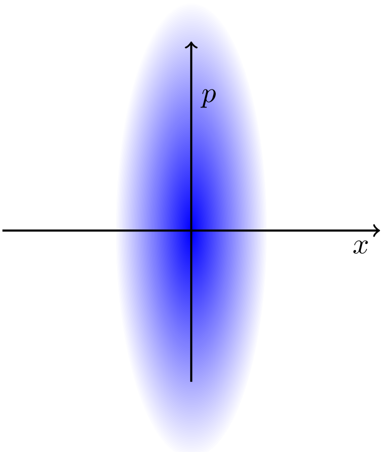
$$\frac{\langle \Psi, \hat{x}\Psi \rangle}{\|\Psi\|^2} = \int_{-\infty}^{\infty} x \frac{|\Psi(x)|_{\mathfrak{h}}^2}{\|\Psi\|^2}\, dx.$$

Cannot simultaneously localize $\hat{E} = \hat{p}^2 + \hat{x}^2$

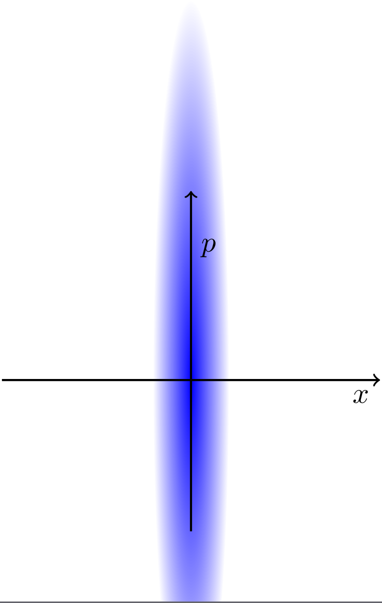
Uncertainty principle: Heisenberg's version (1D)



Uncertainty principle: Heisenberg's version (1D)



Uncertainty principle: Heisenberg's version (1D)



Exclusion principle (2D)

Two commensurate observables $(x_1, x_2) \in \mathbb{R}^2$ with their conjugates (p_1, p_2) and a correlating energy observable, ex.

$$\hat{E} = \hat{p}_1^2 + \hat{p}_2^2 = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}$$

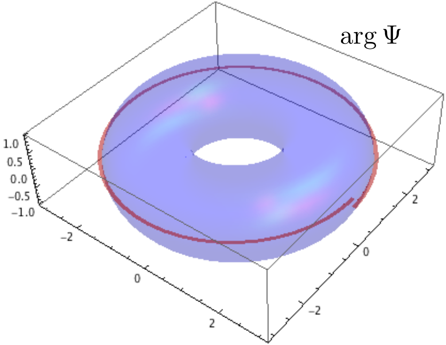
In polar coordinates $(r, \varphi) \in \mathbb{R}_+ \times [0, 2\pi)$ (fibration by circles):

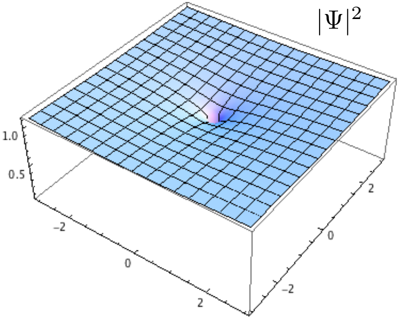
$$\hat{E} = -\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$$

Different ways of representing the observable $\hat{p}_\varphi = -i\partial/\partial\varphi$ on $\mathbb{S}^1$ by boundary condition, or **twist**: $\Psi(r, 2\pi) = e^{i\theta}\Psi(r, 0)$:

$$\hat{E} = \bigoplus_{n \in \mathbb{Z}} \left(-\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}(n + \theta/(2\pi))^2 \right) \otimes \mathbb{1}_{\mathfrak{h}_n}$$

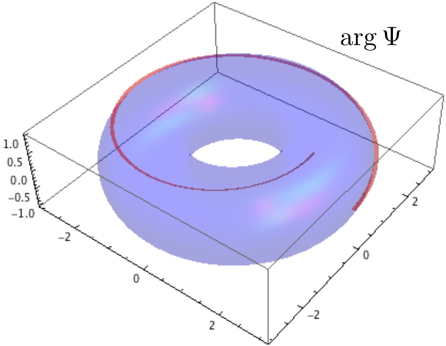
where $L^2(\mathbb{S}^1) = \oplus_{n \in \mathbb{Z}} \mathfrak{h}_n$ (twisted Fourier series $e^{i(n+\theta/(2\pi))\varphi}$).

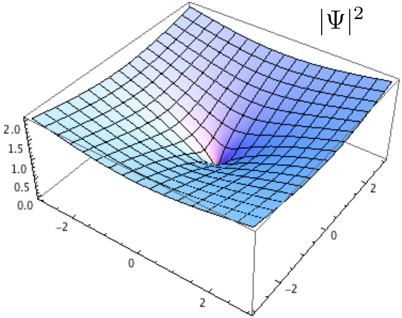




Twist $\Rightarrow$ vortex:

$$\Psi(r,\varphi) \sim r^{\alpha} e^{i\alpha\varphi}, \qquad \alpha = \min_{n\in\mathbb{Z}} |n + \theta/(2\pi)| = 0.04$$

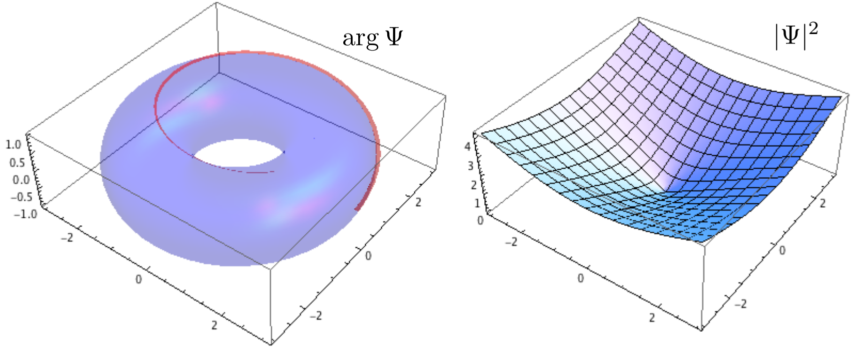




Twist $\Rightarrow$ vortex:

$$\Psi(r,\varphi) \sim r^{\alpha} e^{i\alpha\varphi}, \qquad \alpha = \min_{n\in\mathbb{Z}} |n + \theta/(2\pi)| = 0.25$$

Exclusion principle (2D)



Twist $\Rightarrow$ vortex:

$$\Psi(r,\varphi) \sim r^{\alpha} e^{i\alpha\varphi}, \qquad \alpha = \min_{n\in\mathbb{Z}} |n + \theta/(2\pi)| = 0.5$$

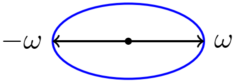
Exclusion principle (2D vs. 3D)

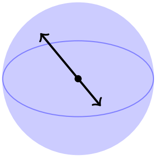
The above can also model two particles in relative coordinates, with identification $\omega \sim -\omega$ on the relative angular sphere $\mathbb{S}^{d-1}$.

2D: $\mathbb{S}^1 \to \mathbb{S}^1/_\sim \;\Rightarrow$ a circle of representations $(\theta) \to$ **"anyons"**

3D: $\mathbb{S}^1 \to \mathbb{S}^2/_\sim \;\Rightarrow$ two reps $\to$ **"bosons"** or **"fermions"**

geometric repulsion & **quantum statistics**





Geometric perspective: Leinaas & Myrheim, 1977
Algebraic perspective: Goldin, Menikoff & Sharp, 1981
Magnetic perspective: Wilczek, 1982

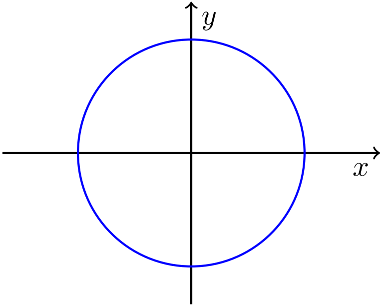
Three or more *locally* commensurate observables that are *globally* incommensurate.

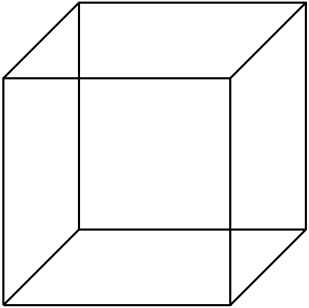
$\Rightarrow$ information can be *locally coherent* but *globally incoherent*.

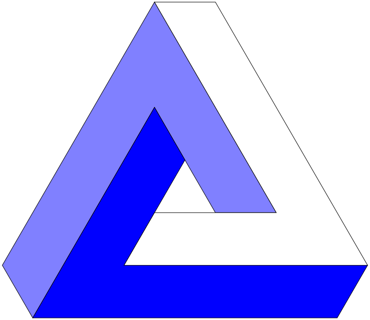
Coherence may then be resolved using **"contextuality"**:

choice of **measurement context** $\leftrightarrow$ choice of **coherent perspective**

Compare the **circle**: resolving the relation $x^2 + y^2 = 1$ by functions requires choice:





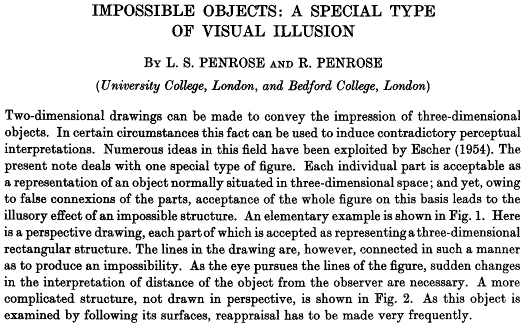


Perspective: a logical coherence or consistency.

The **cube** presents a *choice* of global perspective.

The **tribar** presents "impossibility in its purest form":
Lionel & Roger Penrose, 1956, Oscar Reutersvärd, 1934

Perspectives

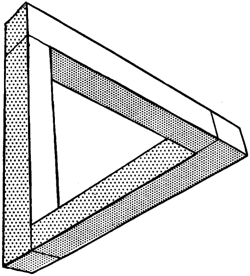


IMPOSSIBLE OBJECTS: A SPECIAL TYPE OF VISUAL ILLUSION

BY L. S. PENROSE AND R. PENROSE

(*University College, London, and Bedford College, London*)

Two-dimensional drawings can be made to convey the impression of three-dimensional objects. In certain circumstances this fact can be used to induce contradictory perceptual interpretations. Numerous ideas in this field have been exploited by Escher (1954). The present note deals with one special type of figure. Each individual part is acceptable as a representation of an object normally situated in three-dimensional space; and yet, owing to false connexions of the parts, acceptance of the whole figure on this basis leads to the illusory effect of an impossible structure. An elementary example is shown in Fig. 1. Here is a perspective drawing, each part of which is accepted as representing a three-dimensional rectangular structure. The lines in the drawing are, however, connected in such a manner as to produce an impossibility. As the eye pursues the lines of the figure, sudden changes in the interpretation of distance of the object from the observer are necessary. A more complicated structure, not drawn in perspective, is shown in Fig. 2. As this object is examined by following its surfaces, reappraisal has to be made very frequently.



Perspectives

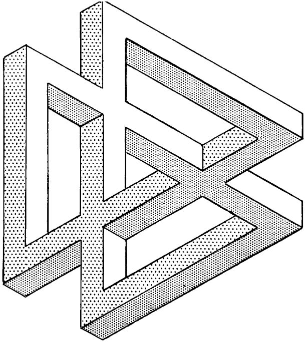


Fig. 2. Diagram of structure with multiple impossibilities.

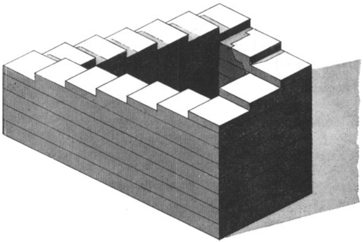
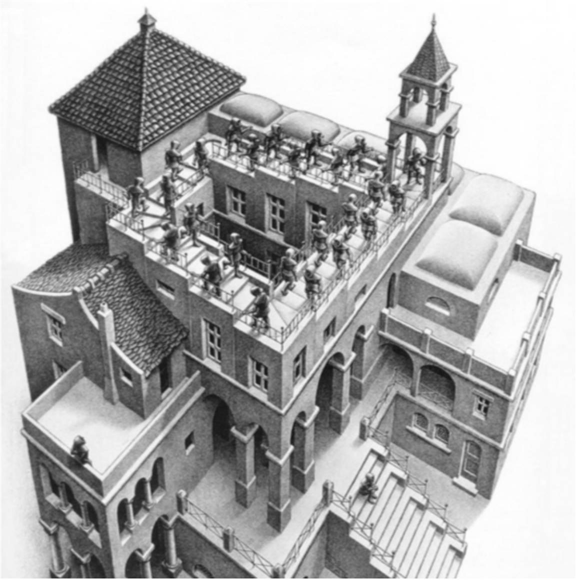
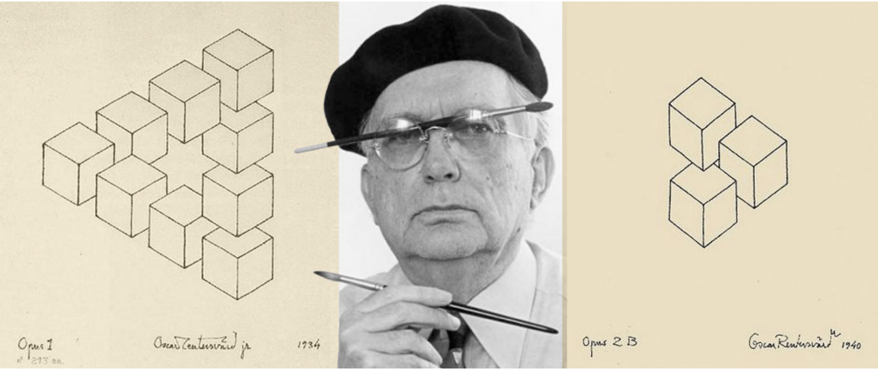


Fig. 3. Continuous flight of steps: shadowed drawing.

Perspectives

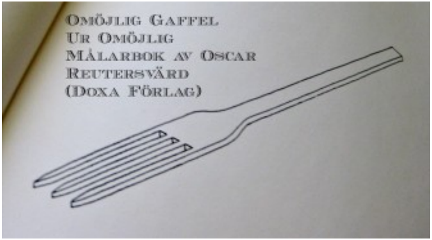


Maurits Cornelis Escher, Ascending and descending, 1960

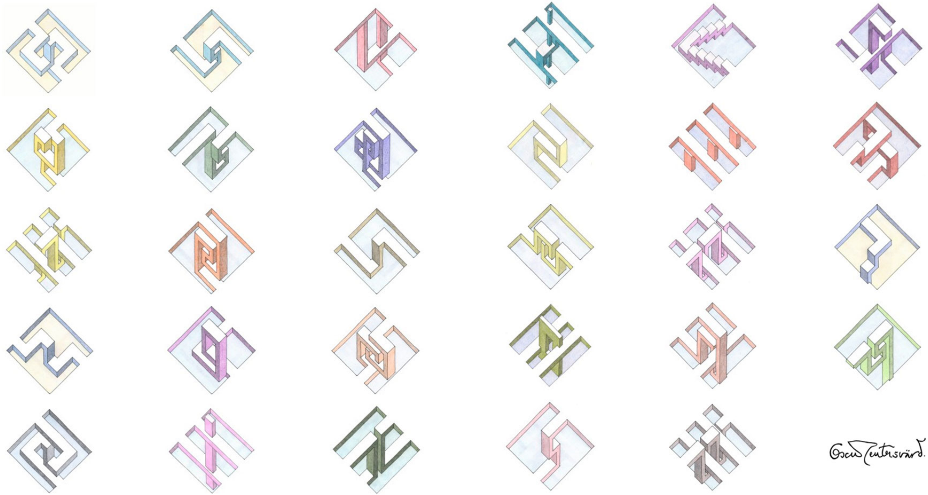


"the father of the impossible figures"

Perspectives



Perspectives

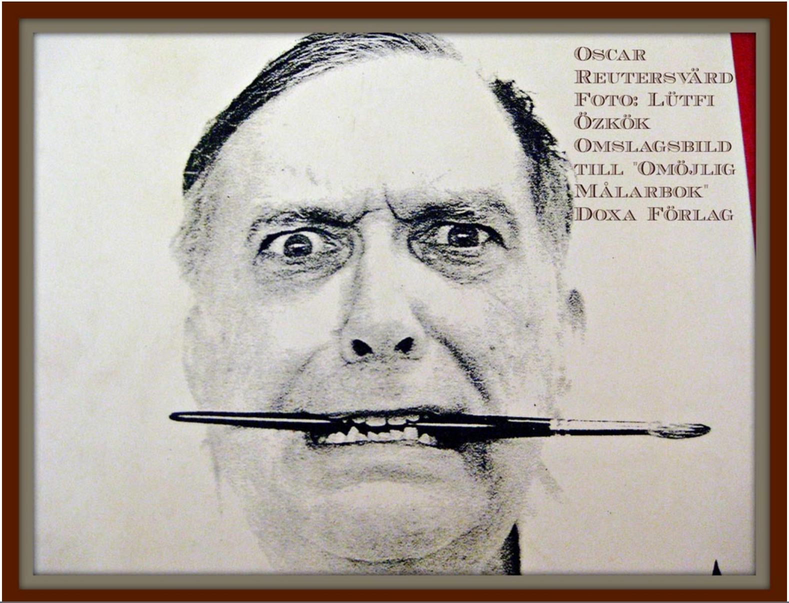


“Window on the Floor” series, 2001-2013

Satire on false perspective

"Whoever makes a Design without the Knowledge of Perspective will be liable to such Absurdities as are shewn in this Frontispiece."
William Hogarth, 1754

Twisted perspectives



Contextuality — dealing with twisted perspectives

measurable observables: $M = \{a, \alpha,\ b, \beta\}$, outcomes: $O = \{0, 1\}$, measurement **contexts**:

$$C \in \big\{\{a, b\}, \{a, \beta\}, \{\alpha, b\}, \{\alpha, \beta\}\big\} \subseteq \mathcal{P}(M)$$

(commensurate measurements, i.e. can be performed together)

empirical model $\mathbb{P}$: contexts $\to$ prob. dist.s on the outcomes

$$C \mapsto \big(\mathbb{P}_C \colon O^C \to [0, 1]\big)$$

marginalization: for any subcontext $D \subseteq C$ and outcomes $t \in O^D$

$$\mathbb{P}_C|_D(t) := \sum_{s \in O^C \,:\, s|_D = t} \mathbb{P}_C(s)$$

local coherence: demand compatibility of all marginals (cf. sheaf):

$$\forall \text{ contexts } C, C' \qquad \mathbb{P}_C|_{C\cap C'} = \mathbb{P}_{C'}|_{C \cap C'}$$

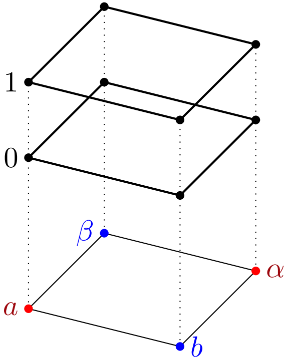
non-contextuality: existence of global assignment of outcomes to all measurable obs. ("hidden variables"/**global coherence**)

$$\exists f \colon O^M \to [0, 1] \text{ s.t } f|_C = \mathbb{P}_C \ \ \forall \text{ contexts } C$$

Contextuality — dealing with twisted perspectives

A B	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a,b)	$1/2$	0	0	$1/2$
(a,β)	$1/2$	0	0	$1/2$
(α,b)	$1/2$	0	0	$1/2$
(α,β)	$1/2$	0	0	$1/2$

(non-contextual model)



Contextuality — dealing with twisted perspectives

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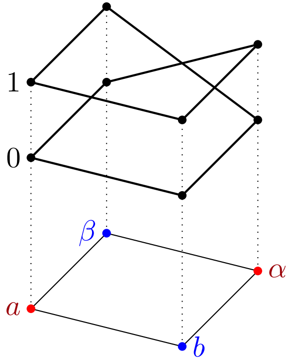
(a,β)	$1/2$	0	0	$1/2$
(α,b)	$1/2$	0	0	$1/2$
(α,β)	0	$1/2$	$1/2$	0

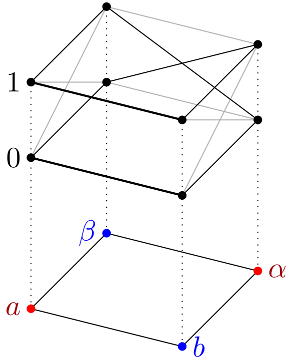
(Popescu-Rohrlich box)



Contextuality — dealing with twisted perspectives

A B	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a,b)	$1/2$	0	0	$1/2$
(a,β)	$3/8$	$1/8$	$1/8$	$3/8$
(α,b)	$3/8$	$1/8$	$1/8$	$3/8$
(α,β)	$1/8$	$3/8$	$3/8$	$1/8$

(Clauser-Horne-Shimony-Holt model)



Summary on QM

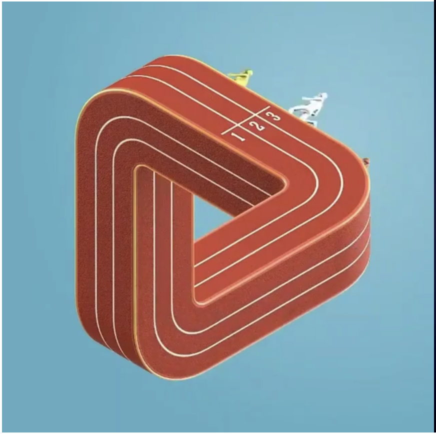
- **Uncertainty** is about the noncommensurability of observables (incompatiblity of perspectives).

- **Entanglement** is about symmetry or correlation in knowledge (compatibility of perspectives).

- Constraints & uncertainty $\Rightarrow$ **twisting** $\Rightarrow$ **vorticity** $\Rightarrow$ correlation/entanglement (geometry–topology–analysis).

- **Contextuality** is about the nonexistence of global sections (necessitates choice of local perspective).

Contextual reality and quantum games



Game theory — using impossible to solve the impossible

Magic square: fill 3×3 grid with $+$ or $-$ such that

- each row has even $-$'s
- each column has odd $-$'s

$+$	$-$	$-$
$-$	$-$	$+$
$+$	$-$	$?$

Impossible!?

Game theory — using impossible to solve the impossible

Constrained linear (binary) system: $x_1, x_2, \ldots, x_9 \in \{0,1\} = \mathbb{Z}_2$

$$\begin{array}{ccccccc} x_1 & + & x_2 & + & x_3 & = & 0 \\ + & & + & & + & & \\ x_4 & + & x_5 & + & x_6 & = & 0 \\ + & & + & & + & & \\ x_7 & + & x_8 & + & x_9 & = & 0 \\ \| & & \| & & \| & & \\ 1 & & 1 & & 1 & & \end{array}$$

Overconstrained:

$$\begin{cases} x_1 + \ldots + x_9 & = 0 \\ x_1 + \ldots + x_9 & = 1 \end{cases}$$

$\Rightarrow$ No solution!

Game theory — using impossible to solve the impossible

Solution as operators on $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$ (spinors):

$A \otimes \mathbb{1}$	$\mathbb{1} \otimes A$	$A \otimes A$
$\mathbb{1} \otimes B$	$B \otimes \mathbb{1}$	$B \otimes B$
$-A \otimes B$	$-B \otimes A$	$-AB \otimes AB$

last row: $-ABAB \otimes BAAB = \mathbb{1} \otimes \mathbb{1}$
last column: $-ABAB \otimes ABAB = -\mathbb{1} \otimes \mathbb{1}$

Compare how one solved $x^2 + 1 = 0$ by lifting $\mathbb{R} \hookrightarrow \mathbb{C}$:

"No one fully understands spinors. Their algebra is formally understood but their general significance is mysterious. In some sense they describe the **"square root" of geometry** *and, just as understanding the square root of -1 took centuries, the same might be true of spinors."*

E-mail from Sir Michael Atiyah, 15 July 2007, quoted in Farmelo, 2009, "The Strangest Man: The hidden life of Paul Dirac, quantum genius".

Quantum pseudo-telepathy

Cooperative game theory: Alice $\leftrightarrow$ Bob solve the impossible together. Ex: Alice gets to generate a row and Bob a column.

Foundations of Physics, Vol. 35, No. 11, November 2005 (© 2005)
DOI: 10.1007/s10701-005-7353-4

Quantum Pseudo-Telepathy

Gilles Brassard,[1,*] Anne Broadbent,[1,†] and Alain Tapp[1,‡]

Received April 22, 2005

Quantum information processing is at the crossroads of physics, mathematics and computer science. It is concerned with what we can and cannot do with quantum information that goes beyond the abilities of classical information processing devices. Communication complexity is an area of classical computer science that aims at quantifying the amount of communication necessary to solve distributed computational problems. Quantum communication complexity uses quantum mechanics to reduce the amount of communication that would be classically required. Pseudo-telepathy is a surprising application of quantum information processing to communication complexity. Thanks to entanglement, perhaps the most nonclassical manifestation of quantum mechanics, two or more quantum players can accomplish a distributed task with no need for communication whatsoever, which would be an impossible feat for classical players. After a detailed overview of the principle and purpose of pseudo-telepathy, we present a survey of recent and not-so-recent work on the subject. In particular, we describe and analyse all the pseudo-telepathy games currently known to the authors.

KEY WORDS: entanglement; nonlocality; Bell's theorem; quantum information processing; quantum communication complexity; pseudo-telepathy.

ARTICLE

doi:10.1038/nature13460

Contextuality supplies the 'magic' for quantum computation

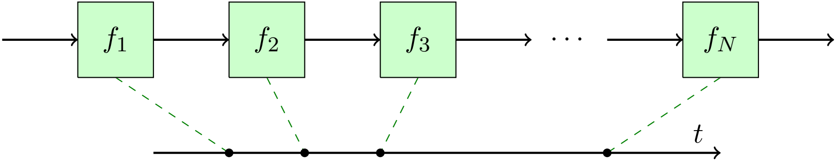
Mark Howard[1,2], Joel Wallman[2], Victor Veitch[2,3] & Joseph Emerson[2]

Quantum computers promise dramatic advantages over their classical counterparts, but the source of the power in quantum computing has remained elusive. Here we prove a remarkable equivalence between the onset of contextuality and the possibility of universal quantum computation via 'magic state' distillation, which is the leading model for experimentally realizing a fault-tolerant quantum computer. This is a conceptually satisfying link, because contextuality, which precludes a simple 'hidden variable' model of quantum mechanics, provides one of the fundamental characterizations of uniquely quantum phenomena. Furthermore, this connection suggests a unifying paradigm for the resources of quantum information: the non-locality of quantum theory is a particular kind of contextuality, and non-locality is already known to be a critical resource for achieving advantages with quantum communication. In addition to clarifying these fundamental issues, this work advances the resource framework for quantum computation, which has a number of practical applications, such as characterizing the efficiency and trade-offs between distinct theoretical and experimental schemes for achieving robust quantum computation, and putting bounds on the overhead cost for the classical simulation of quantum algorithms.

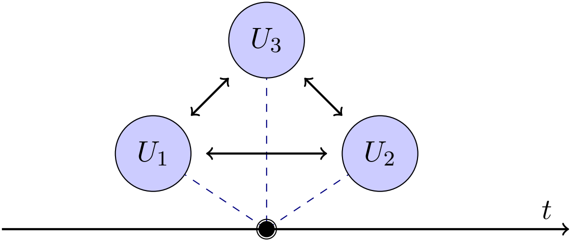
Contextual resource: a measurement-based quantum computer which computes a nonlinear Boolean function $f\colon \mathbb{Z}_2^n \to \mathbb{Z}_2^m$ with a high probability is necessarily contextual:

avg. failure probability $\geq$ deg. of noncontextuality $\times$ dist. from linear.

Resolution of simultaneity — **serialism** vs. **parallelism**:



Fermionic clock (ordered/stacked) vs. bosonic clock (unordered)



Further implications for society: Free will (whim)

The Strong Free Will Theorem

John H. Conway and Simon Kochen

The two theories that revolutionized physics in the twentieth century, relativity and quantum mechanics, are full of predictions that defy common sense. Recently, we used three such paradoxical ideas to prove "The Free Will Theorem" (strengthened here), which is the culmination of a series of theorems about quantum mechanics that began in the 1960s. It asserts, roughly, that if indeed we humans have free will, then elementary particles already have their own small share of this valuable commodity. More precisely, if the experimenter can freely choose the directions in which to orient his apparatus in a certain measurement, then the particle's response (to be pedantic—the universe's response near the particle) is not determined by the entire previous history of the universe.

Our argument combines the well-known consequence of relativity theory, that the time order of space-like separated events is not absolute, with the EPR paradox discovered by Einstein, Podolsky, and Rosen in 1935, and the Kochen-Specker Paradox of 1967 (See [2].) We follow Bohm in using a spin version of EPR and Peres in using his set of 33 directions, rather than the original configuration used by Kochen and Specker. More contentiously, the argument also involves the notion of free will, but we postpone further discussion of this to the last section of the article.

Note that our proof does not mention "probabilities" or the "states" that determine them, which is fortunate because these theoretical notions have led to much confusion. For instance, it is often said that the probabilities of events at one location can be instantaneously changed by happenings at another space-like separated location, but whether that is true or even meaningful is irrelevant to our proof, which never refers to the notion of probability.

For readers of the original version [1] of our theorem, we note that we have strengthened it by replacing the axiom FIN together with the assumption of the experimenters' free choice and temporal causality by a single weaker axiom MIN. The earlier axiom FIN of [1], that there is a finite upper bound to the speed with which information can be transmitted, has been objected to by several authors. Bassi and Ghirardi asked in [3]: what precisely is "information", and do the "hits" and "flashes" of GRW theories (discussed in the Appendix) count as information? Why cannot hits be transmitted instantaneously, but not count as signals? These objections miss the point. The only information to which we applied FIN is the choice made by the experimenter and the response of the particle, as signaled by the orientation of the apparatus and the spot on the screen. The speed of transmission of any other information is irrelevant to our argument. The replacement of FIN by MIN has made this fact explicit. The theorem has been further strengthened by allowing the particles' responses to depend on past half-spaces rather than just the past light cones of [1].

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The Axioms

We now present and discuss the three axioms on which the theorem rests.

Bell-Kochen-Specker paradox

Theorem (Kochen-Specker)

There exists an explicit, finite set of vectors in $\mathbb{R}^3$ that cannot be $\{0,1\}$-colored in such a way that both of the following conditions hold simultaneously:

1. *For every orthogonal pair of vectors, at most one is colored 0.*
2. *For every mutually orthogonal triple of vectors, at least one of them (and therefore exactly one) is colored 0.*

Proof by contradiction on an explicit set $E \subseteq \mathbb{S}^2$, i.e. non-existence of such a function (coloring) $f\colon E \to \{0,1\}$.

Apply this to a **choice of frame** for measuring the polarization of entangled photons. This contextual setup may again be applied in pseudo-telepathic strategies.

Bell-Kochen-Specker paradox

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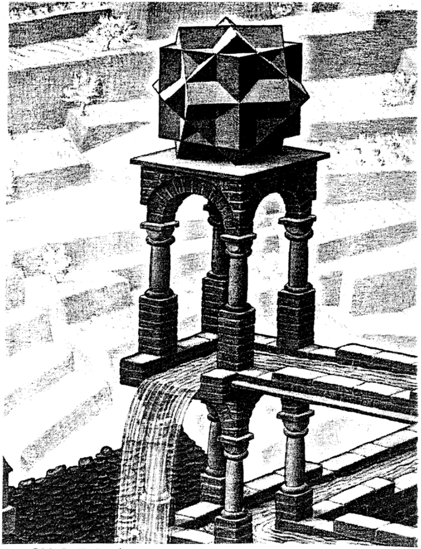

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FIG. 2. The tower on the left of M. C. Escher's engraving "Waterfall." © M. C. Escher/ Cordon Art, Baarn, Holland. The ornament atop the tower consists of three superimposed cubes. One of the cubes has all its edges horizontal or vertical. The other two are given by rotating this one through 90 degrees about each of the two perpendicular horizontal lines that connect the midpoints of opposite vertical edges. The 33 uncolorable directions used in the proof of the Bell-KS theorem in Peres, 1991, lie along the lines connecting the common center of the cubes to their vertices and the centers of their edges and faces.

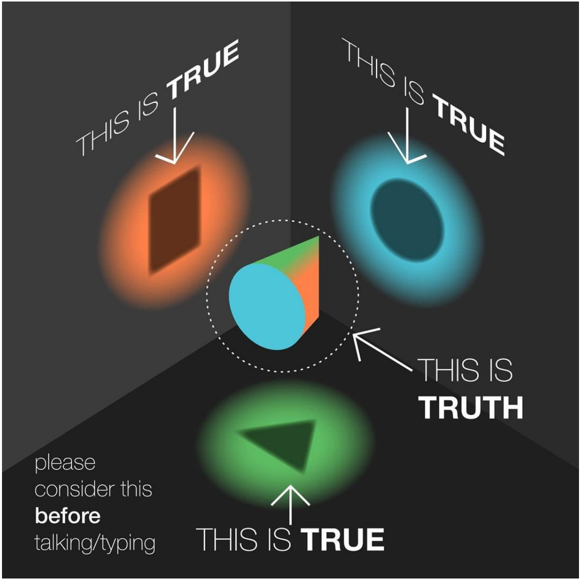
Further implications for society: Realism

Local vs *global* **realism**

Contextuality forces a **relational worldview**, e.g.

- the state of Bob w.r.t. Alice in her perspective α.
- the state of Alice w.r.t. Bob in his perspective β.
- the state of Bob w.r.t. Charlie in its perspective γ.

Perspectives / worldviews



Implications

$0 \to \infty \to$ 

You are not a function!

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