Mathematical physics of the 2D anyon gas

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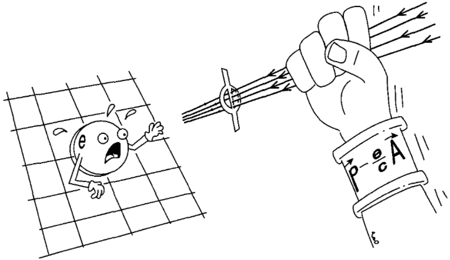
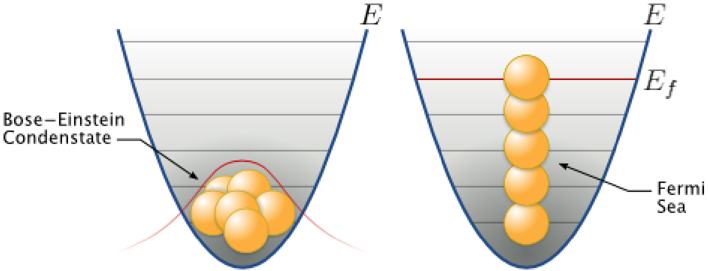


Outline

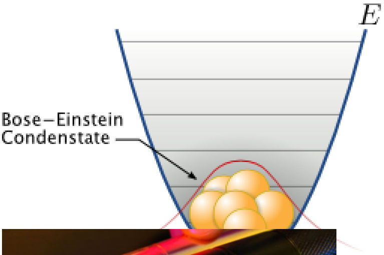
1. 2D quantum statistics and anyons
2. Exchange vs. exclusion ("statistical repulsion") in the ideal anyon gas
3. Almost-bosonic/fermionic extended anyon gas and DFT
4. Emergence of anyons: FQHE, polarons, angulons, ...

Quantum statistics in 3D

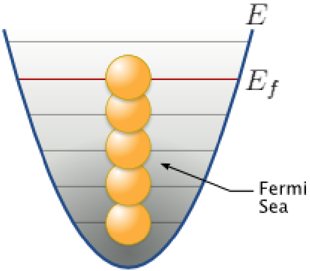


Cold **bosons** Cold **fermions**

Quantum statistics in 3D

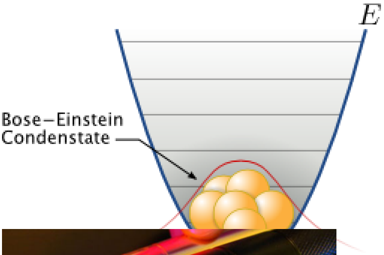


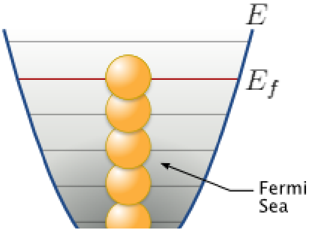
force carriers (coherent/degenerate)



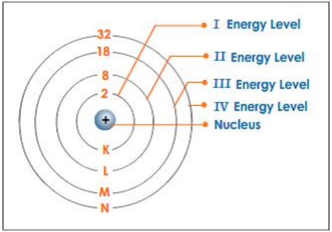
Cold **fermions**

Quantum statistics in 3D





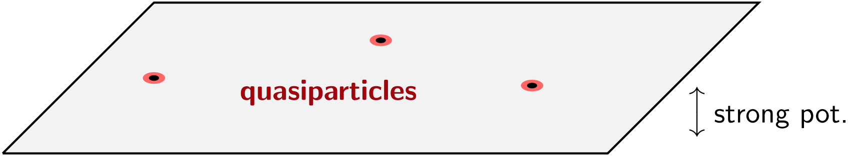
force carriers (coherent/degenerate)



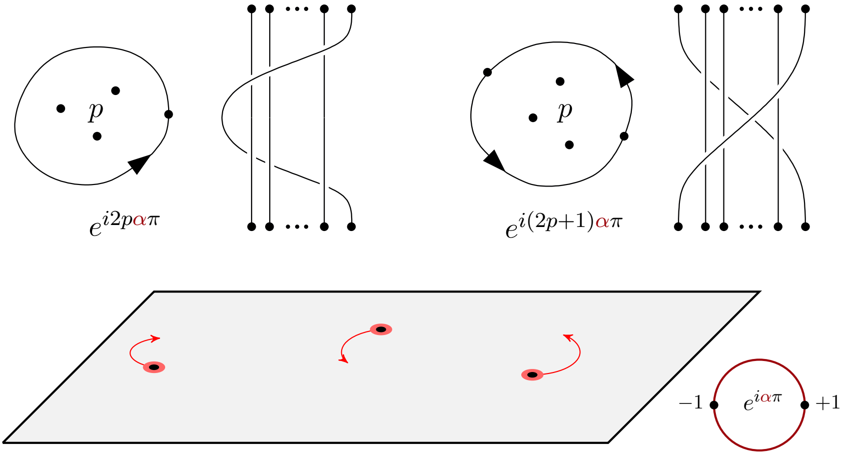
matter (stable/non-degenerate)

Quantum statistics in 2D

Different in 2D! (and in 1D)



Quantum statistics in 2D (exchange a/symmetry)



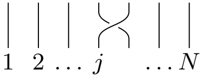
$\Psi(\mathbf{x}_2, \mathbf{x}_1) = e^{\pm i\theta}\Psi(\mathbf{x}_1, \mathbf{x}_2)$ $\qquad$ $\theta = \alpha\pi$ **any** phase $\Rightarrow$ "**anyons**"

The braid group

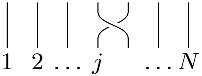
B_N is the **braid group** on N strands:

$$B_N = \left\langle \sigma_1, \ldots, \sigma_{N-1} : \sigma_j \sigma_{j+1} \sigma_j = \sigma_{j+1} \sigma_j \sigma_{j+1},\ \sigma_j \sigma_k = \sigma_k \sigma_j \right\rangle_{|j-k|>1}$$

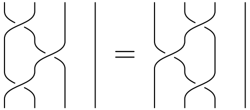
σ_j :



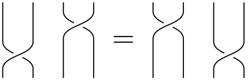
σ_j^{-1} :



Examples in B_4:



$\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$



$\sigma_1 \sigma_3 = \sigma_3 \sigma_1$

If we add the relations $\sigma_j^2 = 1$ we obtain the **permutation group** S_N

Quantum statistics in $\mathbb{R}^d$ (exchange $\overset{?}{\Rightarrow}$ exclusion)

Ψ wave function for N **distinct** particles in $\mathbb{R}^d$ (diagonals $\triangle_N$):

$$|\Psi(\mathrm{x})|^2, \qquad \mathrm{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N) \in \mathbb{R}^{dN} \setminus \triangle_N$$

identical: $\mathbb{R}^d \supset \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\} \in \mathcal{C}^N := (\mathbb{R}^{dN} \setminus \triangle_N)/S_N$

$$\Psi(\sigma.\mathrm{x}) = \rho(\sigma)\Psi(\mathrm{x}), \qquad \sigma \in \pi_1(\mathcal{C}^N) = 1, B_N \text{ or } S_N$$

$\rho(\sigma)$ **exchange phase** (or operator):

bosons $\rho(\sigma) = +1$, symm., ex. independent identically distributed
$\Psi_0 = \otimes^N u_0 \ \in L^2_{\mathrm{sym}}$

fermions $\rho(\sigma) = \mathrm{sign}(\sigma)$, determinantal correlations & Pauli principle
$\Psi_0 = u_0 \wedge u_1 \wedge \ldots \wedge u_{N-1} \ \in L^2_{\mathrm{asym}}$

anyons ρ unitary rep. of B_N ... **intermediate/fractional statistics?**

Leinaas, Myrheim 1977; Goldin, Menikoff, Sharp 1981; Wilczek 1982

Abelian vs. non-abelian representations

An N-**anyon wave function** is *locally* a map $\Psi\colon \mathcal{C}^N \to \mathcal{F}$,
$\mathcal{F}$ Hilbert space of 'internal degrees of freedom' on which B_N acts:

$$\rho\colon B_N \to \mathrm{U}(\mathcal{F})$$

Irreducible abelian anyons: $\mathcal{F} = \mathbb{C}$,

$$\rho(\sigma_j) = e^{i\alpha\pi}$$

Reducible abelian anyons: $\mathcal{F} = \mathbb{C}^D$, $D > 1$,

$$\rho(\sigma_j) \sim \mathrm{diag}(e^{i\beta_1\pi}, \ldots, e^{i\beta_D\pi}) \qquad \forall j$$

Non-abelian anyons: $\mathcal{F} = \mathbb{C}^D$, $D > N - 3$ (if $N > 6$),

$$\rho(\sigma_j)\rho(\sigma_k) \neq \rho(\sigma_k)\rho(\sigma_j) \qquad \text{for some } j \neq k.$$

Anyon (/plekton/nonabelion) models

most computational $\downarrow$ **algebraic** $\longrightarrow$

most general $\downarrow$ **geometric**

$\longleftarrow$ / $\dashrightarrow$ statistics transmutation

most practical $\downarrow$ **magnetic**

$$\boxed{\rho\colon B_N \to \mathrm{U}(\mathcal{F}_N)}$$

Goldin, Menikoff, Sharp '81,'85

Moore, Seiberg, Witten '89
Fröhlich et al. '88-'90
Kitaev '97-'06-
+Freedman, Wang '02-
Bonderson '07
+Gurarie, Nayak '11
...
DL, Qvarfordt '17-'20-

Leinaas, Myrheim '77

Dowker '85
Mueller, Doebner, '93
Mund, Schrader '95
Dell'Antonio, Figari, Teta '97
Goldin, Majid, '04
Maciazek, Sawicki '19
...
Review: DL, Qvarfordt '20

Wilczek '82; Wu '84
+Arovas, Schrieffer '84, +Zee '85

Moore, Read '91, +Rezayi '99
Verlinde '91, Lee, Oh '94 (NACS)
Mancarella, Trombettoni, Mussardo '13
...
DL, Solovej '13,'14
+Rougerie, Larson, Seiringer, Correggi, Duboscq '15-
+Yakaboylu et al '19-
+Lambert 22'

Statistics transmutation in 2D: bosons↔fermions

Convenient: $\mathrm{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_N) \;\leftrightarrow\; \mathrm{z} = (z_1, z_2, \ldots, z_N) \in \mathbb{C}^N \setminus \triangle_N$

$$\Psi = U\tilde{\Psi}, \qquad U(\mathrm{z}) := \prod_{j<k} \frac{z_j - z_k}{|z_j - z_k|} = \exp\left(i \sum_{j<k} \arg(z_j - z_k) \right)$$

transmutes $L^2_{\mathrm{sym}} \leftrightarrow L^2_{\mathrm{asym}}$ at the cost of a gauge potential:

$$-i\nabla\Psi = U(-i\nabla + \mathbf{A})\tilde{\Psi}, \qquad \mathbf{A}_j(\mathrm{x}) = U^{-1}\nabla_{\mathbf{x}_j} U = \sum_{k \neq j} \frac{(\mathbf{x}_j - \mathbf{x}_k)^\perp}{|\mathbf{x}_j - \mathbf{x}_k|^2}$$

$$\hat{T}_0 = \sum_{j=1}^N \hat{\mathbf{p}}_j^2 \qquad \leftrightarrow \qquad \hat{T}_1 = \sum_{j=1}^N \left(-i\nabla_{\mathbf{x}_j} + \mathbf{A}_j\right)^2$$

where $\mathrm{curl}_{\mathbf{x}_j} \mathbf{A}_j = 2\pi \sum_{k \neq j} \delta(\mathbf{x}_j - \mathbf{x}_k)$ Aharonov-Bohm fluxes.

Statistics transmutation in 2D: abelian anyons $e^{i\alpha\pi}$

Convenient: $\mathrm{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_N) \;\leftrightarrow\; \mathrm{z} = (z_1, z_2, \ldots, z_N) \in \mathbb{C}^N \setminus \triangle_N$

$$\Psi = U^{\alpha} \tilde{\Psi}, \qquad U(\mathrm{z}) := \prod_{j<k} \frac{z_j - z_k}{|z_j - z_k|} = \exp\left(i \sum_{j<k} \arg(z_j - z_k) \right)$$

transmutes $L^2_{\mathrm{sym}} \leftrightarrow L^2_{\alpha}$ at the cost of a gauge potential:

$$-i\nabla\Psi = U^{\alpha}(-i\nabla + \alpha\mathbf{A})\tilde{\Psi}, \qquad \mathbf{A}_j(\mathrm{x}) = U^{-1}\nabla_{\mathbf{x}_j} U = \sum_{k \neq j} \frac{(\mathbf{x}_j - \mathbf{x}_k)^{\perp}}{|\mathbf{x}_j - \mathbf{x}_k|^2}$$

$$\hat{T}_0 = \sum_{j=1}^{N} \hat{\mathbf{p}}_j^2 \qquad \leftrightarrow \qquad \hat{T}_{\alpha} = \sum_{j=1}^{N} \left(-i\nabla_{\mathbf{x}_j} + \alpha\mathbf{A}_j\right)^2$$

where $\mathrm{curl}_{\mathbf{x}_j} \alpha\mathbf{A}_j = 2\pi\alpha \sum_{k\neq j} \delta(\mathbf{x}_j - \mathbf{x}_k)$ Aharonov-Bohm fluxes.

Recent experiments

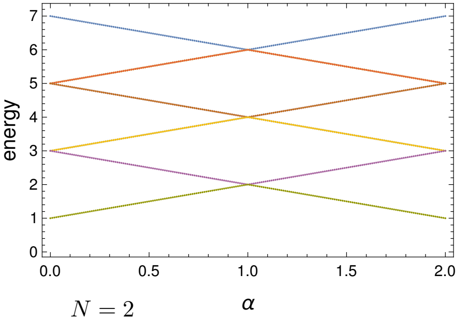
- Nakamura et al. 2020: fractional **exchange** statistics: interferometry in $\nu = 1/3$ FQHE, indicating $\alpha = 2/3$?
- Bartolomei et al. 2020: fractional **exclusion** statistics: bunching in collisions in $\nu = 1/3$ FQHE, indicating $\alpha = 1/3$?
- Google Quantum AI & co. 2022: non-abelian (**Ising**) reps on qubit lattice
- Fan et al. 2022: non-abelian (**Fibonacci**) reps on edge states of a lattice

Issues/obstacles:

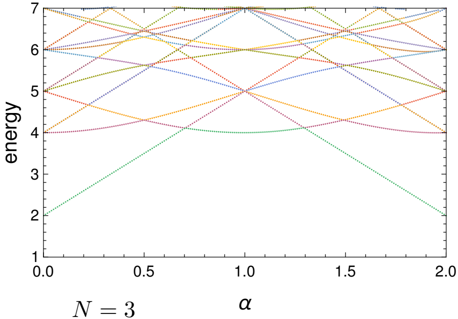
1. fragile phases $(e^{i\alpha\pi})^n$ vs. robust density distributions $\varrho_\Psi(\mathbf{x})$
2. Berry phases, adiabaticity and other phase ambiguities

   Forte '91; Kjønsberg, Myrheim '99; Jain '07; DL, Rougerie '16
3. representations $\rho\colon B_N \to \mathrm{U}(\mathcal{F}_N)$ vs. actual anyons $\hat{T}_\rho$

How do anyons actually behave?



$N = 2$



$N = 3$

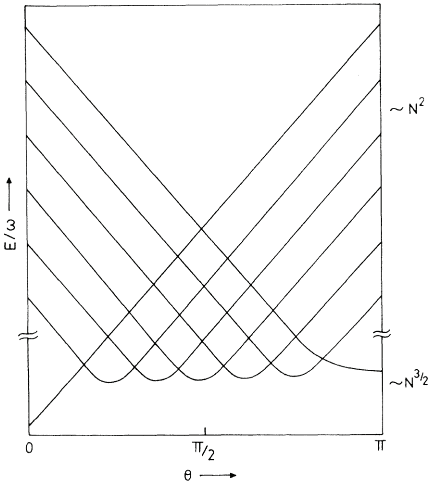
Leinaas, Myrheim '77; Wilczek et al. '82,'85
Murthy, Law, Brack, Bhaduri, '91; Sporre, Verbaarschot, Zahed, '91,'92

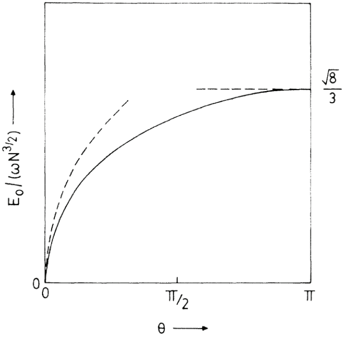
Canright, Johnson '94: "Fractional statistics: α to β"
Yakaboylu et al. 2019

Exactly solvable for $N = 2$, numerics for $N = 3, 4, \ldots$

Exchange vs. exclusion

Potentially interesting dependence on α for $N \to \infty$:





Chitra, Sen, 1992: Schematic $N \to \infty$ spectrum ($\theta = \alpha\pi$, harmonic trap $\omega^2|\mathbf{x}|^2$)

Local exclusion principle

A rigorous and **local approach** to exchange and exclusion.

Statistical repulsion manifests in three ways (at least):

1. effective *scalar* pairwise repulsion $\Rightarrow \Psi \to 0$ at $\triangle_N$

$$\hat{T}_\alpha \;\geq\; \frac{2{\alpha_N}^2}{N} \sum_{j \neq k} \frac{1}{|\mathbf{x}_j - \mathbf{x}_k|^2}$$

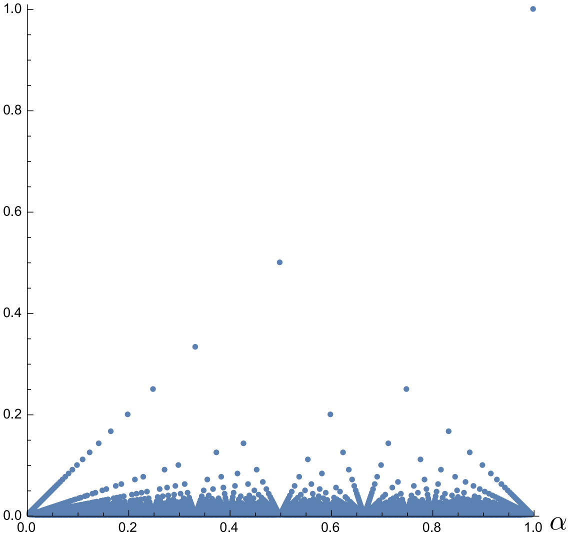
$$\alpha_N := \operatorname{dist}\left(\{(2p+1)\alpha\}_{p=0}^{N-2}, 2\mathbb{Z} \right)$$

2. local exclusion principle: $E_N \gtrsim N - 1$

3. degeneracy pressure, ex. Thomas-Fermi or Lieb-Thirring (uncertainty $\leftrightarrow$ exclusion)

DL, Solovej '13; DL '17

The odd-numerator Thomae/'popcorn' function



$$\alpha_{N\to\infty} = \inf_{p,q\in\mathbb{Z}} |(2p+1)\alpha - 2q|$$

Degeneracy pressure for the ideal anyon gas

The **Thomas-Fermi** approximation for **fermions** in 2D:

$$E_N = \inf_{\Psi \in L^2_{\mathrm{asym}}} \langle \hat{T} + \hat{V} \rangle_\Psi \approx \inf_{\varrho \geq 0 : \int_{\mathbb{R}^2} \varrho = N} \int_{\mathbb{R}^2} \left[2\pi \varrho(\mathbf{x})^2 + V(\mathbf{x}) \varrho(\mathbf{x}) \right] d\mathbf{x},$$

Theorem (**Lieb-Thirring** inequality): For any $\alpha \in \mathbb{R}$ and $\Psi \mapsto \varrho_\Psi$

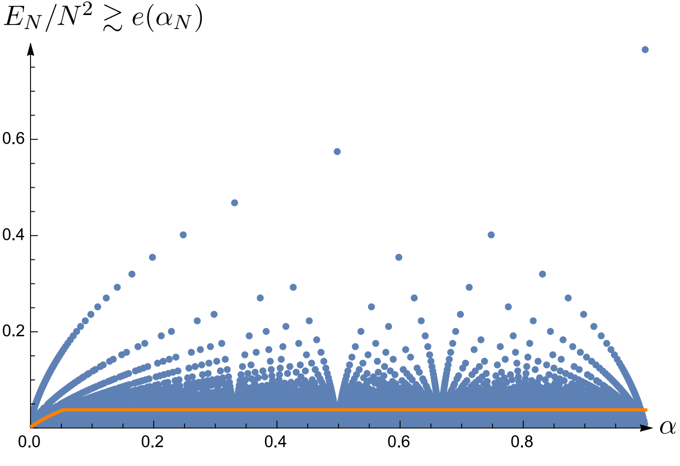
$$\langle \hat{T}_\alpha \rangle_\Psi \gtrsim c(\alpha) \int_{\mathbb{R}^2} \varrho_\Psi(\mathbf{x})^2 \, d\mathbf{x}, \qquad c(\alpha) \sim \mathrm{dist}(\alpha, 2\mathbb{Z}) = \alpha_2.$$

For non-abelian anyons: $\rho(\sigma_j) \sim (e^{i\beta_k \pi})$, dep. on $\mathrm{dist}(\{\beta_k\}, 2\mathbb{Z})$.

Hence, to first order, the **degeneracy pressure** for the ideal anyon gas is governed by the 2-anyon simple exchange phase $e^{i\alpha\pi}$.

DL, Solovej '13,'14; Larson, DL 18'; DL, Seiringer '18; DL, Qvarfordt '20

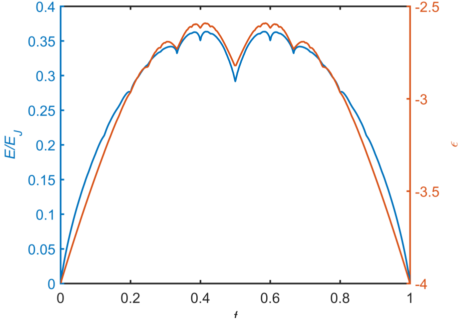
The homogeneous ideal anyon gas



Numerical lower bounds for $e(\alpha) = 4\pi\alpha + O(\alpha^{4/3})$, at $\alpha = \alpha_{N\to\infty}$, versus $c(\alpha_2) = \frac{1}{4}\min\{e(\alpha_2), 0.147\}$

A "wild guess" for the exact ground-state energy

In Figure 6 the lowest energy states of the frustrated XY model are compared to the lowest energy branch of the Hofstadter butterfly which is proportional to the T_c for superconducting networks as predicted by the linearized GL network equations [18–21]. The general shape of the curve is very similar and both show fractal behavior with apparent singularities at rational values of f. However, these singularities seem of different type. In the frustrated XY model these singularities appear logarithmic, while in the Hofstadter butterfly the singularity is approached linearly from both sides. The similarity of these curves suggests there might be a connection between the ground state energy of JJAs and the T_c for superconducting networks.



Lankhorst et al. '18

Towards precise density functionals for anyons

Ground state approximation: $E_N/N \xrightarrow{N\to\infty}$ minimum of $\mathcal{E}[u]$:

The **Gross–Pitaevskii functional** for interacting **bosons**: $(g \in \mathbb{R})$

$$\mathcal{E}^{\mathrm{GP}}[u] := \int_{\mathbb{R}^2} \Big(\big|(-i\nabla + \mathbf{A}_{\mathrm{ext}})u\big|^2 + V|u|^2 + g|u|^4 \Big)$$

The **Thomas–Fermi functional** for **fermions**:

$$\mathcal{E}^{\mathrm{TF}}[u] := \int_{\mathbb{R}^2} \Big(2\pi N|u|^4 + V|u|^2 \Big)$$

An **"average-field" approximation** for **anyons**?

$$\mathcal{E}^{\mathrm{af}}[u] \approx \int_{\mathbb{R}^2} \Big(2\pi\alpha N|u|^4 + V|u|^2 \Big)$$

Mean-field ansatz: $\Psi(\mathrm{x}) = u(\mathbf{x}_1)u(\mathbf{x}_2)\ldots u(\mathbf{x}_N)$
$\Rightarrow$ Single particle $u \in L^2(\mathbb{R}^2)$ in magnetic field $2\pi\alpha N|u(\mathbf{x})|^2$

Almost-bosonic anyons (regularized)

Can be made rigorous in the limit $\alpha = \beta/N \to 0$, fixed β, if anyons are also smeared out to a finite size $R > 0 \longrightarrow$ **extended anyons**.

The idea is then to take $R \sim N^{-\eta} \to 0$ and $\beta = \alpha N$ (flux) large.

$\Rightarrow$ Correct **average-field functional** for any fixed $\beta \in \mathbb{R}$:

$$\mathcal{E}^{\mathrm{af}}_{\beta}[u] = \int_{\mathbb{R}^2} \left(\left| (-i\nabla + \beta \mathbf{A}[|u|^2])u \right|^2 + V|u|^2 \right), \quad \operatorname{curl} \mathbf{A}[\varrho] = 2\pi\varrho$$

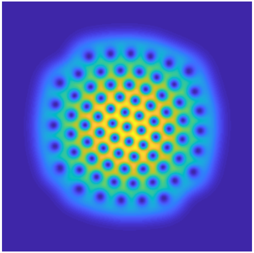
$\Rightarrow$ Effective **Thomas-Fermi-type functional** as $\beta \to \infty$:

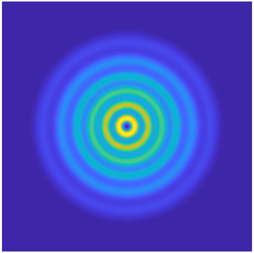
$$\mathcal{E}^{\mathrm{TF}}_{\beta}[\varrho] := \int_{\mathbb{R}^2} \left(C\,\beta\varrho^2 + V\varrho \right)$$

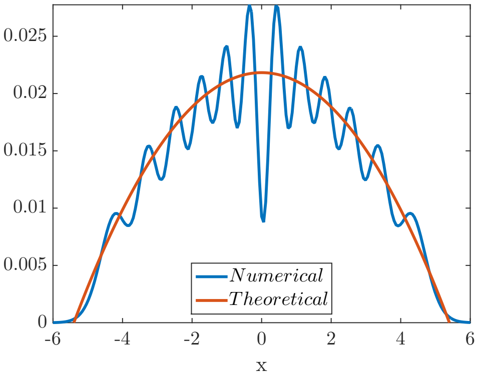
Numerics: g.s. u^{af}_{β} has a vortex lattice distribution with scale set by the TF profile (minimizer) $\varrho^{\mathrm{TF}}_{\beta}$, and $C \approx 4\pi^{3/2}/3 > 2\pi$ (**!**)

DL, Rougerie '15; +Correggi '17; +Duboscq '19

Almost-bosonic anyons: numerics







$\beta = 90$, $V(\mathbf{x}) = |\mathbf{x}|^2$

Averaged $|u_\beta^{\mathrm{af}}|^2$ and comparison with $\varrho_\beta^{\mathrm{TF}}$

Correggi, Duboscq, DL, Rougerie '19

Almost-fermionic anyons

Similar approach close to fermions: $\alpha = 1 - \beta/N \to 1$, $\hbar \sim N^{-1/2}$.
Also "virtually" **extended anyons**, $0 < R \sim N^{-\eta} \to 0$.
$\Rightarrow$ Actual Thomas–Fermi functional for fermions remains relevant:

$$\mathcal{E}^{\mathrm{TF}}[\varrho] := \int_{\mathbb{R}^2} \left(2\pi \varrho^2 + V \varrho \right)$$

More precisely, a semi-classical **Vlasov functional:**

$$\mathcal{E}^{\mathrm{Vla}}[\mu] := (2\pi)^{-2} \int_{\mathbb{R}^4} \left| \mathbf{p} + \beta \mathbf{A}[\varrho] \right|^2 \mu(\mathbf{x}, \mathbf{p}) d\mathbf{x} d\mathbf{p} + \int_{\mathbb{R}^2} V \varrho \, d\mathbf{x}$$

for $0 \leq \mu(\mathbf{x}, \mathbf{p}) \leq 1$ a measure on phase space $\mathbb{R}^4$, $\int_{\mathbb{R}^4} \mu = (2\pi)^2$.
Minimizer:

$$\mu(\mathbf{x}, \mathbf{p}) = \mathbb{1} \left(|\mathbf{p} + \beta \mathbf{A}[\varrho](\mathbf{x})|^2 \leq 4\pi \varrho(\mathbf{x}) \right)$$

with **spatial density** *independent* of β:

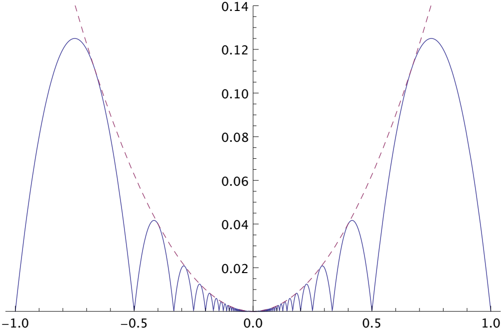
$$\varrho(\mathbf{x}) = (2\pi)^{-2} \int_{\mathbb{R}^2} \mu(\mathbf{x}, \mathbf{p}) d\mathbf{p} = (4\pi)^{-1} (\lambda^{\mathrm{TF}} - V(\mathbf{x}))_+$$

but **momentum density** dep. on β. Chitra, Sen '92; Girardot, Rougerie '21

Intermediate anyons? Try magnetic TF theory

Fermi sea of the Landau Hamiltonian in LDA with a self-generated field $B(\mathbf{x}) = 2\pi\beta\varrho(\mathbf{x})$, $\varrho = \sum_n \varrho_n$, $0 \le \varrho_n \le |B|/(2\pi)$:

$$\mathcal{E}^{\mathrm{mTF}}[\varrho] := \int_{\mathbb{R}^2} \sum_{n=0}^{\infty} \Big(|B|(2n+1)\varrho_n + V\varrho_n \Big) \ge \int_{\mathbb{R}^2} \Big(2\pi(1+M(\beta))\varrho^2 + V\varrho \Big)$$



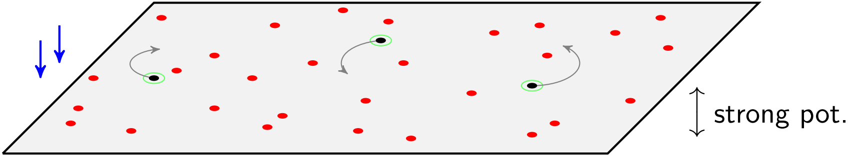
$$M(\beta) := \beta^2(1 - \{\beta^{-1}\})\{\beta^{-1}\} \;\in [0, \beta^2/4)$$

Girardot, Levitt, Rougerie '21; DL '23, cf. Chen et al.'89, Hu et al.'21

Emergence: QHE model for statistics transmutation

Tracers in a bath:

1. large 2D bath of non-interacting **fermions**, $N \gg 1$
2. n tracers/impurities, $2 \leq n \ll N$
3. strong external transverse magnetic field, $b \to \infty$
4. strong short-range repulsive bath-tracer interaction, $g \to \infty$



DL, Rougerie '16; +Lambert '22

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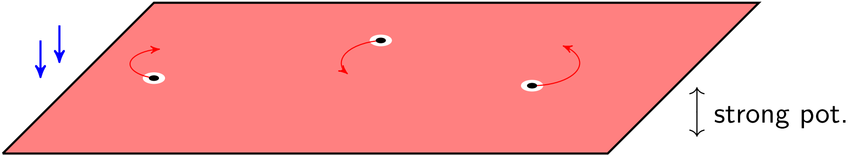
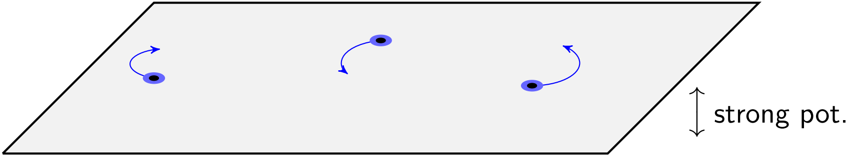


DL, Rougerie '16; +Lambert '22

Emergence: QHE model for statistics transmutation

Tracers in a bath:

1. large 2D bath of non-interacting **fermions**, $N \gg 1$
2. n tracers/impurities, $2 \leq n \ll N$
3. strong external transverse magnetic field, $b \to \infty$
4. strong short-range repulsive bath-tracer interaction, $g \to \infty$



DL, Rougerie '16; +Lambert '22

Model Hamiltonian

$$H_{n\oplus N} := \frac{1}{2}\sum_{k=1}^{N}\big(-i\nabla_{\mathbf{x}_k} - b\mathbf{x}_k^{\perp}\big)^2 \ + \ \frac{1}{2m}\sum_{j=1}^{n}\big(-i\nabla_{\mathbf{y}_j} - qb\mathbf{y}_j^{\perp}\big)^2$$

$$+ \ g\sum_{k=1}^{N}\sum_{j=1}^{n}\delta(\mathbf{x}_k - \mathbf{y}_j) \ + \ W(\mathbf{y}_1, \ldots, \mathbf{y}_n)$$

acting on $\Psi(\mathbf{y}_1, \ldots, \mathbf{y}_n; \mathbf{x}_1, \ldots, \mathbf{x}_N) \in L^2(\mathbb{R}^{2n}) \otimes L^2_{\mathrm{asym}}(\mathbb{R}^{2N})$.

Model Hamiltonian

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acting on $\Psi(\mathbf{y}_1,\ldots,\mathbf{y}_n;\mathbf{x}_1,\ldots,\mathbf{x}_N) \in L^2(\mathbb{R}^{2n}) \otimes L^2_{\mathrm{asym}}(\mathbb{R}^{2N})$.

$\boxed{b \sim N}$ and $N \gg n \quad \Rightarrow \quad$ **assume** bath entirely in LLL:

$$\mathcal{H}^{n\oplus N}_{\mathrm{sym/asym}} := L^2_{\mathrm{sym/asym}}(\mathbb{R}^{2n}) \otimes \bigwedge^N LLL$$

Ground-state energy $\;E_{n\oplus N} := \inf_{\Psi\in\mathcal{H}^{n\oplus N}_{\mathrm{sym/asym}}} \langle H_{n\oplus N}\rangle_\Psi$

Lowest Landau Level (LLL) & Laughlin states

Hamiltonian for a charged particle in a constant magnetic field:

$$H_1 = \frac{1}{2m}\big(-i\nabla - b\mathbf{x}^\perp\big)^2 = \omega_c(a^\dagger a + 1/2)$$

$$LLL := \ker a = \Big\{\psi \in L^2(\mathbb{R}^2) : \psi(\mathbf{x}) = f(z)e^{-\frac{b}{2}|z|^2}, f \text{ analytic}\Big\}$$

ON basis: $u_k(z) = c_k z^k e^{-\frac{b}{2}|z|^2}, \qquad k = 0, 1, 2, \ldots$

$$\Psi(\mathrm{z}) = (u_0 \wedge u_1 \wedge \ldots \wedge u_{N-1})(z_1, \ldots, z_N)$$

$$\propto \det \begin{bmatrix} 1 & 1 & \ldots & 1 \\ z_1 & z_2 & \ldots & z_N \\ \vdots & & & \\ z_1^{N-1} & z_2^{N-1} & \ldots & z_N^{N-1} \end{bmatrix} e^{-\frac{b}{2}|\mathrm{z}|^2} = \prod_{j>k}(z_j - z_k)\ e^{-\frac{b}{2}|\mathrm{z}|^2}$$

IQHE: "filled LLL" (**Fermi sea**) in a radial trap

Lowest Landau Level (LLL) & Laughlin states

Hamiltonian for a charged particle in a constant magnetic field:

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$\Psi(\mathrm{z}) =$

$$\det \begin{bmatrix} 1 & 1 & \ldots & 1 \\ z_1 & z_2 & \ldots & z_N \\ \vdots & & & \\ z_1^{N-1} & z_2^{N-1} & \ldots & z_N^{N-1} \end{bmatrix}^{\mu} e^{-\frac{b}{2}|\mathrm{z}|^2} \;=\; \prod_{j>k}(z_j - z_k)^{\mu} e^{-\frac{b}{2}|\mathrm{z}|^2}$$

FQHE: "fractionally filled LLL", $\mu = 3, 5, 7, \ldots$ **Laughlin states**

Quasi-hole ansatz

For $g \to \infty$ we take as our ansatz for the ground state $(\ker \delta)$:

$$\Psi_\Phi(\mathrm{y};\mathrm{x}) := \Phi(\mathrm{y}) c_{\mathrm{qh}}(\mathrm{w}) \Psi_{\mathrm{qh}}(\mathrm{w};\mathrm{z}), \quad c_{\mathrm{qh}} > 0,$$

$$\int_{\mathbb{C}^n} |\Phi|^2 = 1, \;\; c_{\mathrm{qh}}(\mathrm{w})^{-2} := \int_{\mathbb{C}^N} |\Psi_{\mathrm{qh}}(\mathrm{w};\mathrm{z})|^2 d\mathrm{z} \;\Rightarrow\; \int_{\mathbb{C}^{n+N}} |\Psi_\Phi|^2 = 1$$

$$\Psi_{\mathrm{qh}}(w_1, \ldots, w_n; z_1, \ldots, z_N) = \prod_{j=1}^{n} \prod_{k=1}^{N} (w_j - z_k) \prod_{1 \le k < l \le N} (z_k - z_l) \;\; e^{-\frac{b}{2}|\mathrm{z}|^2}$$

Aim:

$$\langle H_{n \oplus N} \rangle_{\Psi_\Phi} = bN + \langle H_n^{\mathrm{eff}} \rangle_\Phi + \text{error?}$$

$$H_n^{\mathrm{eff}} = \frac{bn}{m} + \frac{1}{2m} \sum_{j=1}^{n} \left(-i\nabla_{\mathbf{y}_j} - (q-1) b \mathbf{y}_j^\perp - \mathbf{A}_j \right)^2 + W(\mathrm{y})$$

Quasi-hole ansatz (integer case)

For $g \to \infty$ we take as our ansatz for the ground state $(\ker \delta)$:

$$\Psi_\Phi(\mathrm{y};\mathrm{x}) := \Phi(\mathrm{y}) c_{\mathrm{qh}}(\mathrm{w}) \Psi_{\mathrm{qh}}(\mathrm{w};\mathrm{z}), \quad c_{\mathrm{qh}} > 0,$$

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$$\Psi_{\mathrm{qh}}(w_1,\ldots,w_n;z_1,\ldots,z_N) = \prod_{j=1}^{n}\prod_{k=1}^{N}(w_j - z_k) \prod_{1\le k<l\le N}(z_k - z_l) \; e^{-\frac{b}{2}|\mathrm{z}|^2}$$

Aim:

$$\langle H_{n\oplus N}\rangle_{\Psi_\Phi} = bN + \langle \tilde{H}_n^{\mathrm{eff}}\rangle_{\tilde{\Phi}} + \text{error?}$$

$$\tilde{H}_n^{\mathrm{eff}} = \frac{bn}{m} + \frac{1}{2m}\sum_{j=1}^{n}\left(-i\nabla_{\mathbf{y}_j} - (q-1)b\mathbf{y}_j^{\perp}\right)^2 + W(\mathrm{y})$$

Quasi-hole ansatz (fractional case)

For $g \to \infty$ we take as our ansatz for the ground state $(\ker \delta)$:

$$\Psi_\Phi(\mathrm{y};\mathrm{x}) := \Phi(\mathrm{y})c_{\mathrm{qh}}(\mathrm{w})\Psi_{\mathrm{qh}}(\mathrm{w};\mathrm{z}), \quad c_{\mathrm{qh}} > 0,$$

$$\int_{\mathbb{C}^n} |\Phi|^2 = 1, \;\; c_{\mathrm{qh}}(\mathrm{w})^{-2} := \int_{\mathbb{C}^N} |\Psi_{\mathrm{qh}}(\mathrm{w};\mathrm{z})|^2 d\mathrm{z} \;\Rightarrow\; \int_{\mathbb{C}^{n+N}} |\Psi_\Phi|^2 = 1$$

$$\Psi_{\mathrm{qh}}(w_1,\ldots,w_n;z_1,\ldots,z_N) = \prod_{j=1}^{n}\prod_{k=1}^{N}(w_j - z_k)^p \prod_{1\le k<l\le N}(z_k - z_l)^\mu e^{-\frac{b}{2}|\mathrm{z}|^2}$$

Conjecture: (Theorem for $p = \mu = 1$) [Lambert, DL, Rougerie, '22]

$$\langle H_{n\oplus N}\rangle_{\Psi_\Phi} = bN + \langle H_n^{\mathrm{eff}}\rangle_\Phi + \mathsf{error}\left(\frac{n}{N}\right)$$

$$H_n^{\mathrm{eff}} = \frac{bn}{m}\frac{p}{\mu} + \frac{1}{2m}\sum_{j=1}^{n}\left(-i\nabla_{\mathbf{y}_j} - (q - \tfrac{p}{\mu})b\mathbf{y}_j^\perp - \tfrac{p^2}{\mu}\mathbf{A}_j\right)^2 + W(\mathrm{y})$$

Another illustrative model for statistics transmutation

Hamiltonian H_0 on Hilbert space $\mathcal{H}_0$ of N **bosons/fermions**.
Add a collective degree of freedom: $[a, a^\dagger] = 1$, $\mathcal{N} = a^\dagger a$, $|n\rangle$

$$\boxed{H_\omega := H_0 + \omega a^\dagger a + \gamma\omega(F a^\dagger + F^{-1} a) + \gamma^2 \omega}$$

Two parameters: $\omega > 0$, $\gamma \in \mathbb{R}$

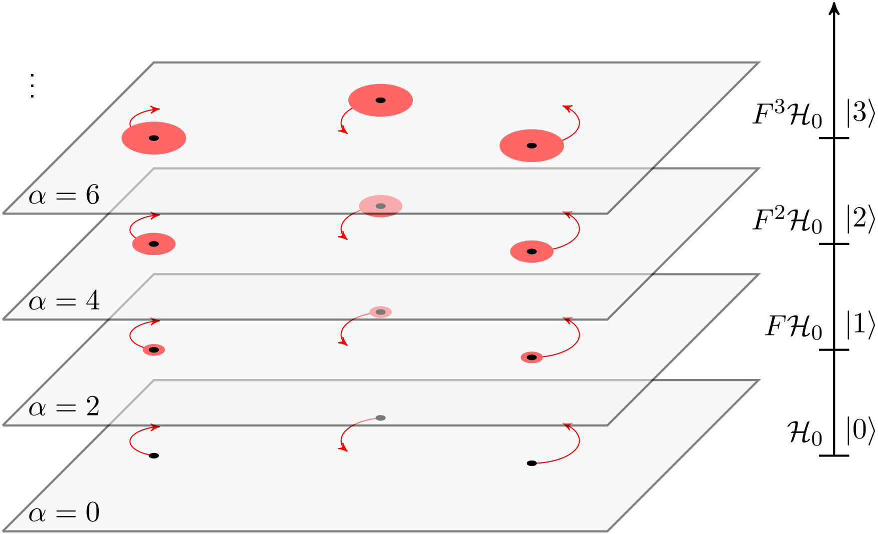
Two model choices:

1. $F = (Z/|Z|)^2 = U^2$ **flux** attachment
2. $F = Z^2$ **vortex** attachment

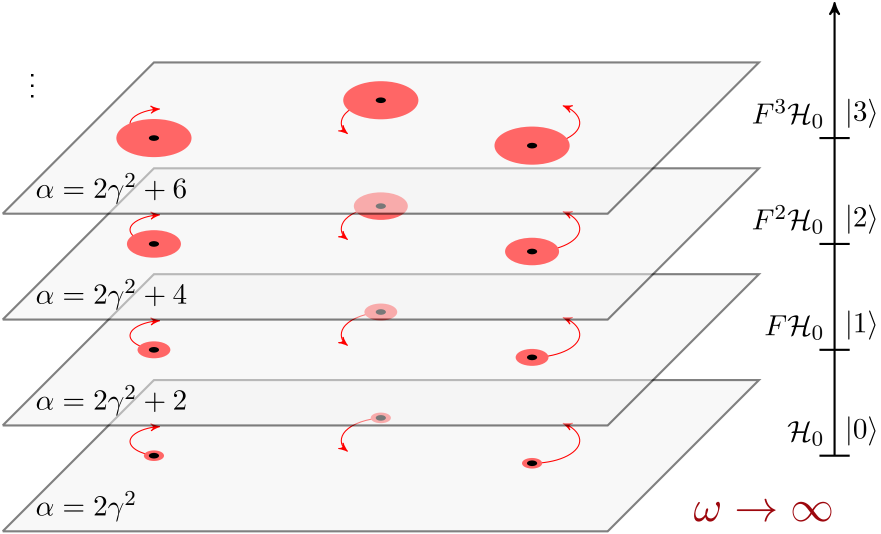
$\Rightarrow$ Hilbert spaces of **composite bosons/fermions**: $\mathcal{H}^n = F^n \mathcal{H}_0 |n\rangle$
$\alpha = 2n$, $n = 0, 1, 2, \ldots$

Now take the 'adiabatic' limit $\omega \to \infty$ with γ fixed.
Claim: in the bottom of the spectrum of H_ω we obtain **anyons** (interacting resp. free) with $\alpha = 2\gamma^2 + 2n$

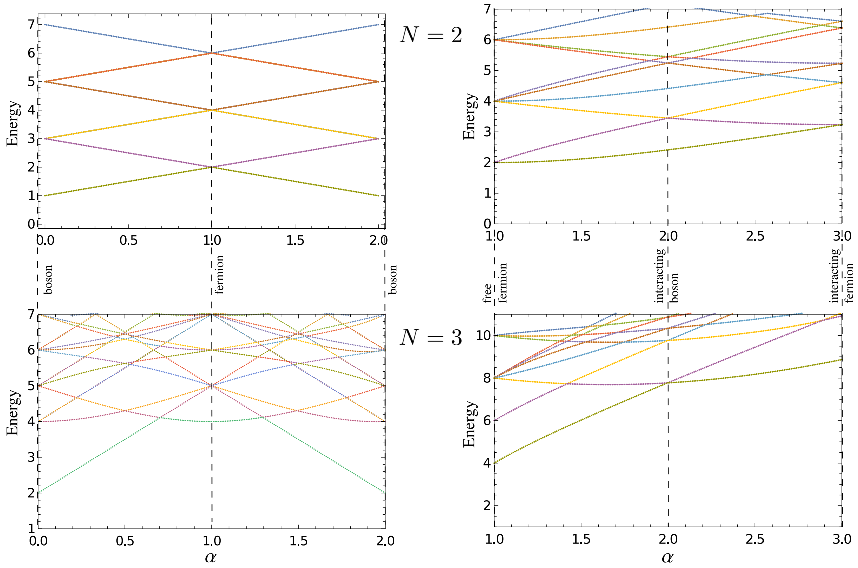
Composite bosons/fermions: ladder of integer bundles



Emergent anyons: ladder of fractional bundles

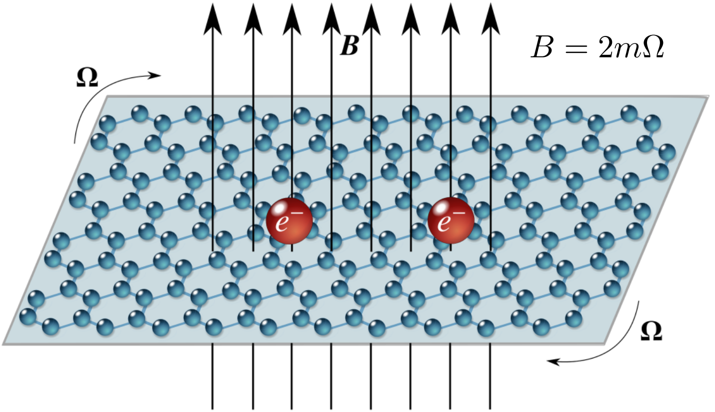


Computation of spectrum: free vs. interacting (harm. osc.)



Yakaboylu et al. '20

Application: Polarons ($\mathbb{R}^2$)



$$H_\omega = \frac{1}{2m}\sum_{j=1}^{N}\mathbf{p}_j^2 + W(\mathrm{x}) + \sum_{\mathbf{k}}\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{\mathbf{k}}\lambda_{\mathbf{k}}(\mathrm{x})\left(e^{-i\beta_{\mathbf{k}}(\mathrm{x})} b_{\mathbf{k}}^\dagger + h.c.\right)$$

Application: Polarons ($\mathbb{R}^2$)

For $N = 2$ with relative coordinates (r, φ) and radial interaction:

$$J_z = L_z + \Lambda_z, \qquad L_z = -i\partial_\varphi, \qquad \Lambda_z = \sum_{k,\mu} \mu b^\dagger_{k\mu} b_{k\mu}$$

$$\Psi = \psi_{\mathrm{A}}(r,\varphi) S(\varphi) U(r)|0\rangle$$

$$S(\varphi) = \exp\left[-i\varphi\Lambda_z\right], \qquad U(r) = \exp\left[-\sum_{k,\mu} \frac{\lambda_{k\mu}(r)}{\omega_{k\mu}}(b^\dagger_{k\mu} - b_{k\mu})\right]$$

Fixed total angular momentum but shift in relative:

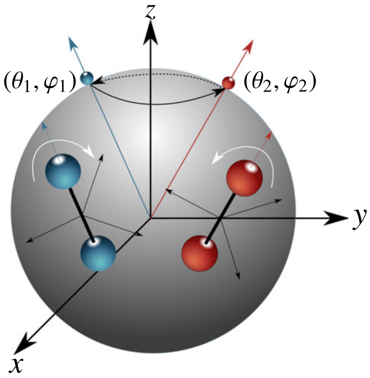
$$\mathbb{Z} \ni j = \langle J_z \rangle_\Psi = \langle L_z \rangle_\Psi + \langle \Lambda_z \rangle_\Psi \;\Rightarrow\; \langle L_z \rangle_\Psi = j - \langle \Lambda_z \rangle_\Psi$$

$$\mathbf{A}(r,\varphi) = -\langle 0|U^{-1}\Lambda_z U|0\rangle \frac{1}{r}\mathbf{e}_\varphi \quad \text{i.e.} \quad \alpha(r) = -\langle \Lambda_z \rangle_{\text{coherent state}\,\gamma(r)}$$

Can be computed for suitable interaction, $\alpha \sim$ const.(Ω)

Application: Angulons ($\mathbb{S}^2$)

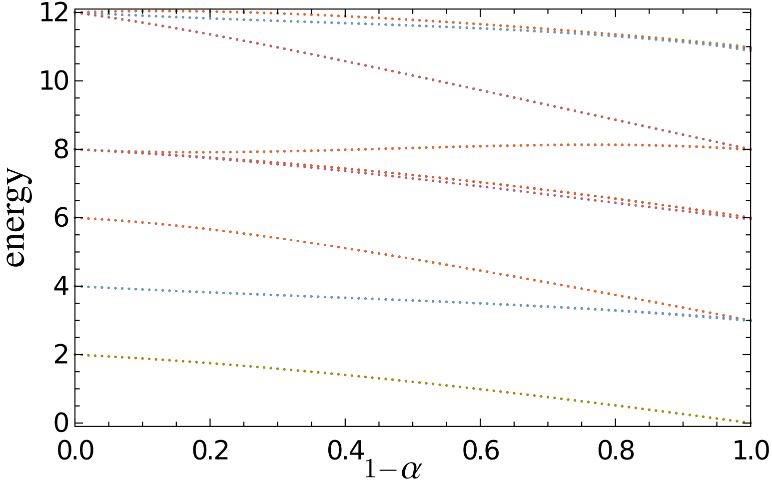
Molecular orientation



$$H_\omega = \sum_{j=1}^{N} \mathbf{L}_j^2 + W(\mathrm{x}) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{\mathbf{k}} \lambda_{\mathbf{k}}(\mathrm{x}) \left(e^{-i\beta_{\mathbf{k}}(\mathrm{x})} b_{\mathbf{k}}^\dagger + h.c. \right)$$

Application: Angulons ($\mathbb{S}^2$)

Spectrum of $N = 2$ anyons on $\mathbb{S}^2$ with monopole field $2B = (N-1)\alpha$



Brooks et al. '21; cf. also Ouvry, Polychronakos '19-'20

Quantum gravity: anyons on the event horizon...

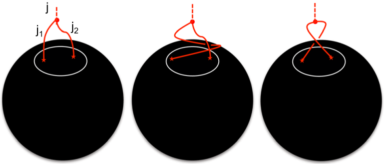


FIG. 3 (color online). Two incident bulk edges piercing the horizon: unbraided vs upon the application of $\hat{M}$ and $\hat{B}$, respectively.

knotting of the spin network at least in the vicinity of the horizon as in Fig. 3, which cannot be unraveled through a (small) bulk diffeomorphism. In the following we want to investigate whether such a different knotting of the spin network in the neighborhood of the horizon has any observable consequences. The area operator would not be of great help here, since $\hat{A}$ is a function of the $su(2)$-Casimir operator and thus commutes with all the generators of this Lie algebra. For a representation $\hat{\rho}$ of a generic element of the braid group one has

$$\langle\hat{\rho}\psi|\hat{A}|\hat{\rho}\psi\rangle = \langle\psi|\hat{\rho}^{-1}\hat{A}\,\hat{\rho}\,|\psi\rangle = \langle\psi|\hat{A}|\psi\rangle. \qquad (61)$$

unless $n = 2$ and $k \to \infty$. For $n = 1$ and $k \to \infty$ (65) reduces to expression (43). For example, when $n = 2$ the commutator yields

$$\begin{aligned}[\hat{F}^i, \hat{M}] = {} & i\frac{4\pi}{k+2}\frac{4\pi}{k} \\ & \times i\epsilon^i_{jk}(\delta^2(x,x_1)\hat{J}^k_{\rho_1} \otimes \hat{J}^j_{\rho_2} + \hat{J}^j_{\rho_1} \otimes \hat{J}^k_{\rho_2}\delta^2(x,x_2)) \\ & + \mathcal{O}(k^{-3}).\end{aligned} \qquad (66)$$

A local stationary observer who resides on the node in Fig. 3 at proper distance ℓ to the horizon will be able to discern braided from unbraided states e.g. by measuring differences in the expectation values of the field strength operator. When considering large black holes the effect of the braiding onto the field strength would be negligible but it would become relevant for smaller (and smaller getting) black holes.

The physical picture behind the statistical phase is very similar to what happens in electromagnetism when dealing with the Aharonov-Bohm effect. To see this we use the ideas presented in [49] and consider a locally flat connection on $S^2 - \{p\}$

$$A_{i\varphi}(x) = \frac{\phi_i}{}\alpha_{\varphi}(x) \qquad (67)$$

Some math-phys references on the 2D anyon gas

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