Reduced Variance by Robust Design of Boundary Conditions for an Incompletely Parabolic System of Equations

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Important areas

The study of partial differential equations with uncertainty in the boundary and initial data is an important task in

- Climatology
- Turbulent combustion
- Flow in porous media
- Electromagnetics
- Seismic activity

The problem

An Incompletely parabolic system:

$$\begin{array}{rcll} u_t + Au_x - \epsilon B u_{xx} & = & F(x,t,\xi) & 0 < x < 1, \quad t > 0 \\ H_0 u & = & g_0(t,\xi) & x = 0, \quad t \geq 0 \\ H_1 u & = & g_1(t,\xi) & x = 1, \quad t \geq 0 \\ u(x,0,\xi) & = & f(x,\xi) & 0 \leq x \leq 1, \quad t = 0. \end{array} \tag{1}$$

- A and B are symmetric $M \times M$ matrices
- B is positive semi-definite
- ϵ is a positive constant
- H_0 and H_1 are boundary operators
- g_0, g_1, f and F are data

Stochastic formulation

Taking the expected value of (1) and letting $v = \mathbb{E}[u]$

$$\begin{array}{rcllr} v_t + Av_x - \epsilon B v_{xx} & = & \mathbb{E}[F](x,t) & 0 < x < 1, \quad t > 0 & \\ H_0 v & = & \mathbb{E}[g_0](t) & x = 0, \quad t \geq 0 & \\ H_1 v & = & \mathbb{E}[g_1](t) & x = 1, \quad t \geq 0 & (2) \\ v(x,0,\xi) & = & \mathbb{E}[f](x) & 0 \leq x \leq 1, \quad t = 0. & \end{array}$$

Now taking the difference between (1) and (2)

$$\begin{array}{rcll} e_t + Ae_x - \epsilon B e_{xx} & = & \delta F(x,t,\xi) & 0 < x < 1, \quad t > 0 \\ H_0 e & = & \delta g_0(t,\xi) & x = 0, \quad t \geq 0 \\ H_1 e & = & \delta g_1(t,\xi) & x = 1, \quad t \geq 0 \\ e(x,0,\xi) & = & \delta f(x,\xi) & 0 \leq x \leq 1, \quad t = 0, \end{array} \quad (3)$$

where $e = u - v$.

Variance formulation

Multiplying (3) with e^T and integrating in space gives

$$\|e\|_t^2 + 2\epsilon \int e_x^T B e_x \, dx = [e^T A e - 2\epsilon e^T B e_x]_0^1. \quad (4)$$

Taking the expected value of (4) and notice that $\mathbb{E}[\|e\|^2] = \|Var[u]\|_1$ we obtain

$$\frac{d}{dt}\|Var[u]\|_1 + 2\epsilon \int \mathbb{E}[e_x^T B e_x] \, dx = [\mathbb{E}[e^T A e] - 2\epsilon \mathbb{E}[e^T B e_x]]_0^1. \quad (5)$$

Finally, by imposing boundary conditions in (5) we are able to analyze their effects on the variance of the solution.