Characterizing and computing the $\mathcal{H}_2$ norm of time-delay systems by solving the delay Lyapunov equation

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Abstract*—It is widely known that the solutions of Lyapunov equations can be used to compute the $\mathcal{H}_2$ norm of linear time-invariant (LTI) dynamical systems. In this paper, we show how this theory extends to dynamical systems with delays. The first result is that the $\mathcal{H}_2$ norm can be computed from the solution of a generalization of the Lyapunov equation, which is known as the *delay Lyapunov equation*. From the relation with the delay Lyapunov equation we can prove an explicit formula for the $\mathcal{H}_2$ norm if the system has commensurate delays, here meaning that the delays are all integer multiples of a basic delay. The formula is explicit and contains only elementary linear algebra operations applied to matrices of finite dimension. The delay Lyapunov equations are matrix boundary value problems. We show how to apply a spectral discretization scheme to these equations for the general, not necessarily commensurate, case. The convergence of spectral methods typically depends on the smoothness of the solution. To this end we describe the smoothness of the solution to the delay Lyapunov equations, for the commensurate as well as for the non-commensurate case. The smoothness properties allow us to completely predict the convergence order of the spectral method.

***Index Terms*—Time-delay systems, $\mathcal{H}_2$ norm, Lyapunov equations, robustness, spectral methods**

I. Introduction

The $\mathcal{H}_2$ norm plays a very important role in the field of systems and control. It is important in, for instance, performance analysis and synthesis (see e.g., [1]), linear quadratic and $\mathcal{H}_2$ optimal control (see e.g., [2]), robust optimization (see e.g., [3]) and can be used in combination with model reduction [4], [5]. When designing controllers, an approach based on optimizing criteria expressed in terms of $\mathcal{H}_2$ norms of appropriately defined transfer functions can be particularly useful if the system is affected by additive disturbances that can be accerurately modeled by (filtered) white noise [1, Chapter 4]. In this work we generalize a basic formula for the $\mathcal{H}_2$ norm to continuous-time dynamical systems with delays. Many physical phenomena can indeed be naturally modeled with delays and there are numerous applications; see [6]–[10] for applications and some recent advances in the field of time-delay systems.

Our most general result is for *exponentially stable neutral time-delay systems* with inputs, outputs and multiple delays, described by,

$$\sum_{k=0}^{m} B_k\dot{x}(t-\tau_k) = \sum_{k=0}^{m} A_k x(t-\tau_k) + Bu(t) \quad \text{(1a)}$$

$$y(t) = Cx(t), \quad \text{(1b)}$$

where, $B_1,\ldots,B_m,A_0,\ldots,A_m \in \mathbb{R}^{n\times n}$, $B_0 = I$, $\tau_0 = 0$, $m > 0$, $C \in \mathbb{R}^{n_c\times n}$, $B \in \mathbb{R}^{n\times n_b}$, $u(t) \in \mathbb{R}^{n_b}$ and $y(t) \in \mathbb{R}^{n_c}$. Without loss of generality we will assume that the delays are ordered with increasing order of magnitude, i.e., $0 = \tau_0 < \tau_1 < \cdots < \tau_m$.

The fundamental solution associated with the time-delay system (1) will turn out to be useful in this work. The fundemental solution, denoted $K : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$, is defined as the solution of the matrix delay-differential equation started with identity at time zero and zero elsewhere, i.e.,

$$\sum_{k=0}^{m} B_k\dot{K}(t-\tau_k) = \sum_{k=0}^{m} A_k K(t-\tau_k) \quad (2)$$

$$K(0) = I, \;\; K(\theta) = 0 \;\text{ when }\; \theta < 0. \quad (3)$$

The fundamental solution can be discontinuous (at a countable number of points) if the system is neutral. In the points of discontinuity we will use the convention that the function value at the point of discontinuity is defined as the right limit. The jumps of the discontinuities are defined such that,

$$\sum_{k=0}^{m} B_k K(t-\tau_k) \quad (4)$$

is continuous for $t > 0$. This condition is called the sewing condition. See [11, Section 6.6] for a precise definition of the fundamental solution.

This work is on the $\mathcal{H}_2$ norm of the input-output relation defined by (1). The $\mathcal{H}_2$ norm of an exponentially stable time-delay system can be defined analogous to the delay-free case, i.e.,

$$\|G\|_2^2 := \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Tr}(G(i\omega)^* G(i\omega))\, d\omega,$$

where G is the transfer function of the system. The corresponding time-domain representation is the energy functional

$$\|G\|_2^2 = \int_{t=0}^{\infty} \mathrm{Tr}(h(t)^T h(t))\, dt, \quad (5)$$

where h is the impulse response.

A technical property of neutral systems is that asymptotic stability is not always robust with respect to infinitesimal

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changes in the delays. If the stability of a system is robust against small perturbations then the system is called strongly stable. See [12] for more details, including necessary and sufficient conditions. Note that the $\mathcal{H}_2$ norm is not necessarily finite for unstable systems, hence the $\mathcal{H}_2$ norm may be a discontinuous function of the delay when the system is not strongly stable. Since we wish to compute and characterize a robust quantity, we will assume strong stability.

A fundamental and elegant property of the $\mathcal{H}_2$ norm for systems without delay, is that it can be computed from the solutions of the Lyapunov equation (see e.g. [1, Lemma 4.6]). We will see (in Section II) that this is also the case for systems with delay. Instead of solving the Lyapunov equation we need to solve the *delay Lyapunov equations*, which fortunately have already been studied in the literature (see e.g. [10], [13]–[19]) but mostly in the context of stability assessment with Lyapunov's second method.

The main contributions of the paper are

- a formula for the $\mathcal{H}_2$ norm in terms of the solution to the delay Lyapunov equation (in Section II),
- an explicit formula containing only fundamental linear algebra operations on matrices with finite dimension for the $\mathcal{H}_2$ norm of time-delay systems with commensurate delays, i.e., $\tau_k = k\tau$ (in Section III-A),
- a numerical scheme based on spectral discretization on a Chebyshev grid to solve the delay Lyapunov equation and compute the $\mathcal{H}_2$ norm (in Section III-B),
- a smoothness analysis of the solution to the delay Lyapunov equation for commensurate as well as non-commensurate systems (in Section IV), and
- a convergence analysis of the discretization scheme in terms of the smoothness of the solution of the delay Lyapunov equation (also in Section IV).

The paper is concluded with Section V containing examples illustrating the approaches, smoothness properties and convergence.

II. Characterization of the $\mathcal{H}_2$ norm

Our first result is a formula for the $\mathcal{H}_2$ norm of (1) expressed in terms of the *delay Lyapunov matrix*. There are several definitions of the delay Lyapunov matrix. We will use the energy functional definition[1]

$$U(t) := \int_{s=0}^{\infty} K^T(s) C^T C K(s+t)\, ds. \tag{6}$$

It now turns out that the formula for the $\mathcal{H}_2$ norm of systems without delays extends to systems with delay, where $U(0)$ plays the same role as the solution to the standard Lyapunov equations for systems without delay.

Theorem 1: Suppose that the time-delay system (1) is exponentially stable. Then, the $\mathcal{H}_2$ norm is

$$\begin{aligned} \|G\|_2^2 &= \mathrm{Tr}(B^T U(0) B) & (7)\\ &= \mathrm{Tr}(C V(0) C^T), & (8) \end{aligned}$$

if $U(t)$, $V(t)$ are unique solutions to the delay Lyapunov equation

$$\sum_{k=0}^{m} U'(t-\tau_k) B_k = U(t) A_0 + \sum_{k=1}^{m} U(t-\tau_k) A_k,\ t \geq 0 \tag{9a}$$

$$U(-t) = U^T(t) \tag{9b}$$

$$-C^T C = \sum_{i=0}^{m}\sum_{j=0}^{m} \left(B_i^T U(\tau_i - \tau_j) A_j + A_j^T U^T(\tau_i - \tau_j) B_i\right) \tag{9c}$$

and the dual delay Lyapunov equation,

$$\sum_{k=0}^{m} V'(t-\tau_k) B_k^T = V(t) A_0^T + \sum_{k=1}^{m} V(t-\tau_k) A_k^T,\ t \geq 0 \tag{10a}$$

$$V(-t) = V^T(t) \tag{10b}$$

$$-BB^T = \sum_{i=0}^{m}\sum_{j=0}^{m} \left(B_i V(\tau_i - \tau_j) A_j^T + A_j V^T(\tau_i - \tau_j) B_i^T\right). \tag{10c}$$

Proof: The underlying idea of the proof is that the solution of the delay Lyapunov equations (9) as well as the $\mathcal{H}_2$ norm can be expressed with the fundamental solution.

The time-domain representation $\mathcal{H}_2$ norm (5) allows a simple formulation with the fundamental solution

$$\|G\|_2^2 = \int_{t=0}^{\infty} \mathrm{Tr}\left(y(t)^T y(t)\right)\, dt = \int_{t=0}^{\infty} \mathrm{Tr}\left(B^T K(t)^T C^T C K(t) B\right)\, dt.$$

We will now see that the integral can be computed with the solutions of the Lyapunov equations.

The definition (6) uniquely defines $U(t)$ for exponentially stable systems. By inserting (6) into (9) one can verify that (9) define the same solution. See also [14, Theorem 4], [19].

We can now express the $\mathcal{H}_2$ norm with $U(0)$,

$$\begin{aligned} \|G\|_2^2 &= \int_{s=0}^{\infty} \mathrm{Tr}\left(B^T K^T(s) C^T C K(s) B\right)\, ds \\ &= \mathrm{Tr}\left(B^T \left(\int_{s=0}^{\infty} K^T(s) C^T C K(s)\, ds\right) B\right) & (11)\\ &= \mathrm{Tr}\left(B^T U(0) B\right). & (12) \end{aligned}$$

We have shown (7). It remains to show that the $\mathcal{H}_2$ norm can be computed from the dual equations which are the same equations but with transposed matrices. This follows from the fact that the operation of transposing all the matrices in the DDE, has the result that the fundamental solution is also transposed. This can be seen more precisely as follows. First note that, since the trace of the product of two matrices is independent of order of the multiplication,

$$\|G\|_2^2 = \int_{t=0}^{\infty} \mathrm{Tr}\left(y(t) y(t)^T\right)\, dt = \mathrm{Tr}\left(C \int_{t=0}^{\infty} K(t) B B^T K(t)^T\, dt C^T\right).$$

[1] In [14] and related works, where the delay Lyapunov matrix is used to construct complete Lyapunov-Krasovskii functionals, U is typically defined as $U(t) := \int_{s=0}^{\infty} K^T(s) W K(s+t)\, ds$, with arbitrary symmetric W. In the context of $\mathcal{H}_2$ norm it is natural to fix W as in (6).

Also note that the order of the multiplication in the definition of the fundamental solution (2) can be switched [18, Corollary 2]. That is, K also fulfills,

$$\sum_{k=0}^{m} \dot{K}(t-\tau_k)B_k = \sum_{k=0}^{m} K(t-\tau_k)A_k. \tag{13}$$

If we again use [14, Theorem 4] or [19] but with the transposed system we find that

$$V(t) = \int_{s=0}^{\infty} K^T(s)BB^TK(s+t)\,ds.$$

We have completed the proof by showing (8). ∎

Remark 2 (Uniqueness): Note that we assumed exponential stability and that the delay Lyapunov equation has a unique solution. For an exponentially stable retarded system with multiple delays it is known [13, Theorem 6.31], [14, Theorem 4] that the delay Lyapunov equations define a unique solution. This is also proved for neutral systems with a single delay [19, Theorem 2]. The general neutral case with multiple delays appears to be open, but the general ideas of the existing proofs are expected to extend to neutral exponentially stable systems with multiple delays [20].

Remark 3 (Relation with delay-free case): The formula for the $\mathcal{H}_2$ norm in Theorem 1 is a generalization of a formula for the $\mathcal{H}_2$ norm of a dynamical system without delay. The formula for the $\mathcal{H}_2$ norm of a stable dynamical system without delay is (see e.g. [21] or [1, Lemma 4.6])

$$\|G\|_2^2 = \mathrm{Tr}(B^TUB) = \mathrm{Tr}(CVC^T)$$

where

$$-C^TC = UA + A^TU \text{ and } -BB^T = AV + VA^T. \tag{14}$$

Theorem 1 is a true generalization of this result in two ways. Suppose $\tau_1 = \cdots = \tau_m = 0$ and $\sum_{i=0}^m B_i = I$ then the algebraic conditions (9c) and (10c) reduce to the Lyapunov equations (14) with $A = \sum_{i=0}^m A_i$. But also, if $A_1 = \cdots = A_m = B_1 = \cdots = B_m = 0$ and $B_0 = I$ then (9c) and (10c) again reduce to (14) with $A = A_0$. We also note that if $B_0 \neq I$ the delay Lyapunov equations (9c) and (10c) reduce to the Lyapunov equations of differential algebraic systems [22], [23].

III. Solving the delay Lyapunov equations

In order to apply Theorem 1 we only need to know the function value of U at the point $t = 0$, The function U, i.e., the solution to the delay Lyapunov equation, is however only implicitly given as the solution to a matrix boundary value problem. Some additional theory will now be presented showing how $U(0)$ can be computed in practice. We find an explicit formula for the case that the delays are commensurate (in Section III-A) and show how one can numerically handle the general case by discretization (in Section III-B).

A. Vectorization for commensurate systems

A well known explicit approach to analyze and sometimes solve the standard Lyapunov equation (14) is based on vectorizing the equation, i.e., reformulating the linear matrix equation into a standard linear equation of squared dimension. We will now see that the analogon of this *vectorization approach* extends to time-delay systems if we assume that the system has commensurate delays. In this context, a time-delay system is said to have commensurate delays if all delays are integer multiples of some delay τ_1, i.e., $\tau_i = i\tau_1$, $i = 1, \dots, m$.

It turns out that the vectorization for the commensurate case is a true generalization of the standard delay-free case. But unlike the delay-free case, the resulting vectorized equation is of dimension $2mn^2 \times 2mn^2$ and contains an expression with the matrix exponential.

Since the dual delay Lyapunov equations (10) have the same structure as the original delay Lyapunov equations (9), the derivations are analogous for U and V. We therefore carry out the derivation only for U.

The derivation of the result that follows is inspired by derivations of other results for commensurate systems in the literature. Some of the important ideas are presented in different generality settings and with different levels of details in [13, Chapter 6], [14], [15], [16] and [17], where the theory is mostly used to study stability by constructing a Lyapunov functional. Due to the fact that we only need to evaluate the Lyapunov matrix in one point to compute the $\mathcal{H}_2$ norm, the solution can be expressed explicitly. The somewhat technical proof is available in appendix.

Theorem 4: Suppose the time-delay system (1) with commensurate delays $\tau_i = i\tau_1$, is exponentially stable and suppose the solution $U(t)$ to the delay Lyapunov equation (9) is unique. Let M_1, M_2, M, $N \in \mathbb{R}^{2nm\times 2nm}$ be the matrices (32) and (34) given in Appendix A. Then, M_1 is regular and

$$\left(M + Ne^{M_1^{-1}M_2\tau_1}\right)\begin{pmatrix}\mathrm{vec}(U_{m-1})\\ \vdots \\ \mathrm{vec}(U_0) \\ \vdots \\ \mathrm{vec}(U_{-m})\end{pmatrix} = \begin{pmatrix}-\mathrm{vec}(C^TC)\\ 0 \\ \vdots \\ 0\end{pmatrix}, \tag{15}$$

has a unique solution $U_i \in \mathbb{R}^{n\times n}$, $i = -m, \dots, m-1$. Moreover, the $\mathcal{H}_2$ norm of (1) is given by

$$\|G\|_2^2 = \mathrm{Tr}\, B^TU_0B. \tag{16}$$

Remark 5 (Finite-dimensional characterization): A time-delay system is often stated as an infinite-dimensional system as e.g. in [24]. It is hence somewhat remarkable that the formula in Theorem 4 is finite-dimensional in the sense that it is a construction where we can compute the $\mathcal{H}_2$ norm of a time-delay system with commensurate delays by only using elementary linear algebra operations applied to finite-dimensional matrices. Even though the time-delay system can be represented as an infinite-dimensional system, the $\mathcal{H}_2$ norm of a time-delay system is of finite-dimensional character. A similar phenomenon appears in the study of stability of time-delay systems, in particular methods to compute the delays such that there is a purely imaginary eigenvalue. In many of these methods the imaginary eigenvalue and corresponding delay can be computed from the eigenvalues of a large matrix. This approach is taken in [25]–[28]. See also the literature reviews in [8, Section 4.3.2], [7, Section 4.4] and [29, Section 3.2].

Remark 6 (Computational complexity): The approach to vectorize the Lyapunov equation is not particularly computationally attractive for delay-free systems with large matrices since the dimension of the linear system to be solved is squared. This is also the case for the approach of this subsection. Consider a general purpose approach to compute the matrix exponential, as e.g., in [30], and suppose the computational complexity of the matrix exponential is roughly cubic. Then the computational complexity to compute the left hand side of (15) is $(2mn^2)^3 = 8m^3n^6$.

Finally, in order to make our result easily accessible for a common case we give a simplified formulation for a system with a single input, a single output and a single delay which follows from manipulations with the Schur complement.

Corollary 7 (Single input, single output, single delay): Suppose the system has a single input and a single output, only one delay and is retarded, i.e., $m = 1$ and $B_1 = 0$. Moreover, let

$$\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} := \exp\left(\tau \begin{pmatrix} A_0^T \otimes I & A_1^T \otimes I \\ -I \otimes A_1^T & -I \otimes A_0^T \end{pmatrix}\right),$$

and assume B_{22} is invertible. Then, the $\mathcal{H}_2$ norm is given by

$$\begin{aligned} \|G\|_2^2 = -\mathrm{vec}(BB^T)^T \big[& A_0^T \otimes I + I \otimes A_0^T \\ & + (A_1^T \otimes I + (I \otimes A_1^T)B_{12})B_{22}^{-1}(I - B_{21}) + \\ & (I \otimes A_1^T)B_{11}\big]^{-1} \mathrm{vec}(C^TC). \end{aligned} \tag{17}$$

B. Numerical solution for the general case

A somewhat remarkable property of the result of the previous subsection is that the $\mathcal{H}_2$ norm of time-delay systems with commensurate delays can be expressed in an explicit exact way by only using elementary linear algebra operations applied to finite-dimensional matrices. Unfortunately, it seems not possible to extend the approach to systems with incommensurate delays. We will now present an approach based on discretizing the Lyapunov equations, suitable if the delays are not commensurate or if the number of delays is large.

The idea to discretize the delay Lyapunov equations is not new and was considered by Ochoa, et al. [16], [17], [31], [32]. However, unlike Ochoa, et al. (and [32]), we use a spectral discretization method with Chebyshev points and Chebyshev polynomials as basis functions. This construction is expected to have more attractive convergence properties than many other discretization approaches. The reason to use a Chebyshev basis functions and a Chebyshev grid are the following.

- It is widely known that the basis used in a representation of a polynomial has a big impact on the numerical stability. A polynomial may be very sensitive to rounding errors in the coefficients, if it is represented in an unsuitable basis, e.g. the nominal basis. See, e.g. [33, Chapter 5]. A nice property of Chebyshev polynomials is that if they are used as basis functions, then the polynomial is typically not sensitive to rounding errors in the coefficients.
- The choice of the discretization grid influences the convergence speed. The rough motivation for the faster convergence with a Chebyshev distribution is that it is denser at the boundaries, preventing large oscillations around the boundaries often present for an equidistant grid. The Chebyshev distribution is in a sense optimal in a more formal setting, as it has an associated potential with minimal energy [34, Chapter 5].

See also [34], [35] for details on spectral discretization methods. We will postpone a discussion of the convergence properties in this setting to Section IV.

Let $\tilde{U}$ be a truncated scaled Chebyshev series approximating U, i.e.,

$$U(t) \approx \tilde{U}(t) = \sum_{j=0}^{N-1} C_j T_j\left(\frac{2}{\tau_m}t - 1\right), \qquad t \in [0, \tau_m], \tag{18}$$

where $T_j : [-1, 1] \to \mathbb{R}$, $j = 0, \dots, N-1$, are the N first Chebyshev polynomials and τ_m denotes the largest delay. In the construction that follows we will find a linear system of equations from which we can compute the unknown coefficient matrices $C_0, \dots, C_{N-1} \in \mathbb{R}^{n\times n}$. The construction consists of requiring that the ansatz (18) fulfills the Lyapunov equations (9) in a number of discretization points.

Note that the ansatz (18) implicitly gives an approximation for U on $t \in [-\tau_m, 0]$ as we can use the symmetry condition (9b) to find that

$$U(t) = U^\top(-t) \approx \tilde{U}^\top(-t) = \sum_{j=0}^{N-1} C_j^\top T_j\left(\frac{2}{\tau_m}(-t) - 1\right), \tag{19}$$

when $t \in [-\tau_m, 0]$. We denote the Chebyshev polynomials of the second kind by S_j. They are defined by the relation $\frac{dT_j}{dt} = jS_{j-1}$ and allow us to construct the derivative of $\tilde{U}(t)$,

$$U'(t) \approx \tilde{U}'(t) = \frac{2}{\tau_m}\sum_{j=0}^{N-1} jC_jS_{j-1}\left(\frac{2}{\tau_m}t - 1\right) \tag{20}$$

when $t \in [0, \tau_m]$ and

$$U'(t) \approx \tilde{U}'(t) = -\frac{2}{\tau_m}\sum_{j=0}^{N-1} jC_j^\top S_{j-1}\left(-\frac{2}{\tau_m}t - 1\right), \tag{21}$$

when $t \in [-\tau_m, 0]$. Consider the discretization points,

$$\theta_i = \frac{\tau_m}{2}(\chi_i + 1) \quad \text{for } i = 0, \dots, N-1, \tag{22}$$

where $\chi_i = \cos(\pi - i\pi/(N-1))$, for $i = 0, \dots, N-1$, are the N Chebyshev points of the second kind on the unit interval $[-1, 1]$. We find $N-1$ matrix equations by requiring that the approximations (18) and (19), and their derivatives (20) and (21), fulfill the Lyapunov equation (9a) on the interval $[-\tau_m, \tau_m]$ in $N-1$ discretization points, namely θ_i, $i = 1, \dots, N-1$. After some manipulations and using of the symmetry condition (9b), we find that,

$$\frac{2}{\tau_m}\sum_{j=1}^{N-1} j\left(\sum_{k=0}^{R_i-1} C_j S_{j-1}\left(\frac{2}{\tau_m}(\theta_i-\tau_k)-1\right)B_k - \sum_{k=R_i}^{m} C_j^\top S_{j-1}\left(\frac{2}{\tau_m}(\tau_k-\theta_i)-1\right)B_k\right) = \sum_{j=0}^{N-1}\left(\sum_{k=0}^{R_i-1} C_j T_j\left(\frac{2}{\tau_m}(\theta_i-\tau_k)-1\right)A_k + \sum_{k=R_i}^{m} C_j^\top T_j\left(\frac{2}{\tau_m}(\tau_k-\theta_i)-1\right)A_k\right), \quad (23)$$

where for each $i=1,\dots,N-1$, R_i is the smallest delay index k for which $\theta_i < \tau_k$. In order for the coefficient matrices $C_0,\dots,C_{N-1}$ to be uniquely defined, we need N matrix equations. The final equation is formed by requiring that the algebraic condition (9c) is satisfied. After inserting (18) into (9c) and several manipulations we find that,

$$-C^TC = \sum_{j=0}^{N-1}\sum_{k=0}^{m}\Bigg(T_j(-1)\left(B_k^\top C_j A_k + A_k^\top C_j^\top B_k\right) + \sum_{l=0}^{k-1} T_j\left(\frac{2}{\tau_m}(\tau_k-\tau_l)-1\right)\left(B_k^\top C_j A_l + A_k^\top C_j B_l + B_l^\top C_j^\top A_k + A_l^\top C_j^\top B_k\right)\Bigg). \quad (24)$$

The equations (23) and (24) constitute N matrix equations in the N unknowns $C_0,\dots,C_{N-1}$. Making use of the perfect shuffle matrix, we can vectorize these equations into a system equation of size $Nn^2 \times Nn^2$ which can be solved with standard numerical software for large linear systems.

After solving this system of equations an approximation of $U(0)$ is easily available

$$U(0) \;\approx\; \tilde{U}(0) = \sum_{j=0}^{N-1} T_j(-1)C_j.$$

The approximation of the $\mathcal{H}_2$ norm is now simply $\mathrm{Tr}(B^T\tilde{U}(0)B)$.

Remark 8 (Computational complexity): The computationally dominating part of this approach is solving the system of linear matrix equations (23)–(24). With a standard method for linear systems, the computational complexity is $\mathcal{O}(N^3n^6)$. In the discussion of the computational complexity of the exact approach (Remark 6) we also found that the computational complexity with respect to n is n^6. However, unlike the exact approach, the method of this section is independent of the number of delays m.

IV. Smoothness properties of Lyapunov matrices and convergence of the discretization scheme

The convergence of spectral methods such as the one presented in Section III-B depends on the smoothness properties of the solution to be computed. More precisely, if the solution is analytic, then a spectral method is expected to have exponential convergence. Spectral methods can however also be used to approximate functions where not all derivatives exist. If the solution is not smooth, then the convergence of the spectral method is slower. In fact, the convergence to a solution which is not smooth is only algebraic, where the order of convergence is determined by the lowest non-existent derivative; see, e.g., [34]. We will now see that the delay Lyapunov matrices are often not smooth and that the characterization of the non-smoothness completely describes the convergence properties of the spectral method proposed in Section III-B.

Note that we will use the smoothness properties to characterize the asymptotic behaviour of the error as a function of the number of discretization points N. This gives a qualitative description of the error, unlike the other approaches [16, Section 4] where an analysis and a computational method for an upper bound is derived.

In order to characterize the smoothness properties of the Lyapunov matrix U we first address the smoothness of the fundamental solution K.

A. Smoothness of the fundamental solution

We will use the property that

$$\dot{K}(t) = \sum_{k=0}^{m} A_k K(t-\tau_k) - \sum_{k=1}^{m} B_k \dot{K}(t-\tau_k), \quad t \geq 0, \quad \text{a.e.}^2, \quad (25)$$

and the sewing condition (4) to prove continuity properties of K. Suppose K has a discontinuity in the lth derivative at some time-point, then from (25) we conclude from the first term in the right-hand side that it also has a discontinuity in the $(l+1)$st derivative at some different time-points. Correspondingly, the second term in (25) introduces other points of discontinuity in the lth derivative. Note that the fundamental solution always has a discontinuity at $t=0$. If the system is retarded, these properties correspond to the well known smoothing property [36]. The argument can be formalized as follows, where we have denoted vectors by $\vec{x}$ and the standard scalar product by $\vec{x}\cdot\vec{y}$.

Lemma 9: Let $\mathcal{I} \subseteq \{1,\dots,m\}$ such that $B_i = \mathbf{0}$ if and only if $i \in \mathcal{I}$. If the function K is not infinitely differentiable at $t=\hat{t}>0$, then $\hat{t} = \vec{n}\cdot\vec{\tau}$ for some $\vec{n} \in \overline{\mathbb{N}}^m$, where $\overline{\mathbb{N}} := \mathbb{N}\cup\{0\}$ and $\vec{\tau} = (\tau_1,\dots,\tau_m)$. Conversely, if $\hat{t} = \vec{n}\cdot\vec{\tau}$ for some $\vec{n}\in\overline{\mathbb{N}}^m$ and $\hat{t}>0$, then the function K generically has a discontinuity in its pth derivative at $t=\hat{t}$, where

$$p = \begin{cases} \min\left\{ \sum_{j=1,\ j\in\mathcal{I}}^{m} |n_j| :\ \vec{n}\in\overline{\mathbb{N}}^m,\ \vec{n}\cdot\vec{\tau} = \hat{t} \right\}, & \mathcal{I}\neq\emptyset, \\ 0, & \mathcal{I} = \emptyset. \end{cases}$$

Furthermore, it satisfies

$$\left| \lim_{t\to\hat{t}-} K^p(t) - \lim_{t\to\hat{t}+} K^p(t) \right| < \infty.$$

In the following subsections we will use Lemma 9 and the definition of U, i.e., (6), to derive the smoothness properties of the Lyapunov matrix U. We distinguish between the commensurate and the non-commensurate delay case as it turns

[2] We use a.e. as an abbreviation for almost everywhere.

out that the smoothness properties for these two cases differ considerably. The effect on the convergence properties of the numerical scheme presented in Section III-B is also discussed.

B. Smoothness of Lyapunov matrices and convergence: commensurate delays

The solution of a linear time-invariant ordinary differential equation is analytic. The segmented and vectorized construction in Theorem 4 is a linear time-invariant ordinary differential equation. It follows that the Lyapunov matrix is piecewise smooth for time-delay systems with commensurate delays.

The size of the jump in the discontinuity for scalar systems has been characterized in [15]. We use a different technique to characterize non-smooth points and the number of continuous derivatives (for the general case), which is the primary interest in our work.

Theorem 10: Assume that the delays $\vec{\tau}$ are commensurate. Let $\mathcal{I} \subseteq \{1,\ldots,m\}$ such that $B_i = \mathbf{0}$ if and only if $i \in \mathcal{I}$. If the function U is not infinitely differentiable at $t = \hat{t} > 0$, then $\hat{t} = \vec{z}\cdot\vec{\tau}$ for some $\vec{z} \in \mathbb{Z}^m$. Conversely, if $\hat{t} = \vec{z}\cdot\vec{\tau}$ for some $\vec{z} \in \mathbb{Z}^m$ and $\hat{t} > 0$, then the function U generically has a discontinuity in its $(p+1)$st derivative at $t = \hat{t}$, where

$$p = \begin{cases} \min\left\{ \sum_{j=1,\ j\in\mathcal{I}}^{m} |z_j| :\ \vec{z} \in \mathbb{Z}^m,\ \vec{z}\cdot\vec{\tau} = \hat{t} \right\}, & \mathcal{I} \neq \emptyset, \\ 0, & \mathcal{I} = \emptyset. \end{cases}$$

Furthermore, it satisfies

$$\left| \lim_{t\to\hat{t}-} U^{(p+1)}(t) - \lim_{t\to\hat{t}+} U^{(p+1)}(t) \right| < \infty.$$

Proof: From (6) we get

$$U(t) = \int_{-\infty}^{\infty} \tilde{K}(s)^T C^T C \tilde{K}(s+t) ds, \tag{26}$$

where $\tilde{K} :\ \mathbb{R} \to \mathbb{R}$ is defined by

$$\tilde{K}(t) = \begin{cases} K(t), & t \geq 0, \\ 0, & t < 0. \end{cases}$$

The assertions follow from Lemma 9, the definition (6) and an application to (26) of the technical Lemma 12, stated in the appendix. Note that the condition (36) is satisfied because the delays are assumed to be commensurate. ∎

From Theorem 10, the set of positive time-instants where the function U is not infinitely differentiable is identified as

$$S := \{\vec{\tau}\cdot\vec{z} :\ \vec{z} \in \mathbb{Z}^m,\ \vec{\tau}\cdot\vec{z} > 0\}. \tag{27}$$

In the commensurate delay case, this set can be equivalently expressed as

$$S = \{kh :\ k \in \mathbb{N}\}, \tag{28}$$

where h is such that $\vec{\tau} = h\vec{n}$ and $\gcd(\vec{n}) = 1$. This follows from a recursive application of Bezout's identity. See also the proof of Proposition 11 in the appendix. The smoothness results described in Table I can be directly derived from Theorem 10. The retarded delay case corresponds to $\mathcal{I} = \{1,\ldots,m\}$.

In the context of solving boundary value problems it is widely known that methods based on spectral collocation have exponential convergence if the solution is analytic (see e.g. [34, Chapter 4]). Moreover, if the solution is p times differentiable, with a piecewise continuous $(p+1)$st derivative with bounded variation, then the convergence is algebraic, with convergence order equal to $(p+1)$. This is consistent with the convergence results observed for the numerical scheme (Section III-B), which are summarized in Table II. The exponential convergence $O(N^{-N})$ for the single delay case is due to fact that the numerical scheme relies on computing the function U on the interval $[0, \tau_m]$ only, on which it is analytic.

C. Smoothness and convergence: non-commensurate delays

Non-commensurate delays can be approximated to arbitrary accuracy by commensurate delays. The Lyapunov matrices have, despite this fact, very different smoothness properties for the non-commensurate and commensurate case. This is indicated by the following result. The proof can be found in the appendix.

Proposition 11: If the delays $(\tau_1,\ldots,\tau_m)$ are not commensurate, then the set S in (27) is dense in $\mathbb{R}^+$.

From similar arguments as spelled out in the proof of Lemma 12 it follows that the *interaction* between discontinuities of K at $t = t_1$ and $t = t_2 > t_1$ (a jump in its q_1th derivative and its q_2th derivative, respectively) contributes to a discontinuity of U at $t = t_2 - t_1$ (a jump in its (q_1+q_2+1)st derivative). From Lemma 9 we have

$$t_i \in \{\vec{n}\cdot\vec{\tau} :\ \ \vec{n} \in \overline{\mathbb{N}}^m\},\ i = 1,2,$$

implying $t_2 - t_1 = \vec{z}\cdot\vec{\tau}$ with $\vec{z} \in \mathbb{Z}^m$. A combination of this result with Proposition 11 yields that U is no longer a piecewise smooth function. In what follows we distinguish between two cases.

In the retarded case, the function $t \geq 0 \mapsto K(t)$ is continuous and its derivative is piecewise continuous, with jumps occurring at $t = \tau_i,\ i = 1,\ldots,m$. Hence, we can express

$$U''(t) = \int_0^{\infty} K(s) C^T C d\dot{K}(t+s)$$

for $t > 0$, provided that the integral is interpreted in a distribution sense, where the integrals are Riemann-Stieltjes integrals. We conclude that the second derivative of U is piecewise continuous. The number of discontinuities of U'' is finite and they occur at $t = \tau_i,\ i = 1,\ldots,m$.

In the neutral case, where K is only piecewise continuous for $t \geq 0$, the derivative of U can still be computed point-wise as

$$U'(t) = \int_0^{\infty} K(s) C^T C dK(t+s),$$

provided that the integral is again interpreted in a distribution sense. However, the function U is nowhere *continuously* differentiable whenever the subset of delays

$$\{\tau_i :\ 1 \leq i \leq m,\ B_i \neq 0\} \tag{29}$$

	$U(t),\ t \in [0,\infty)$	$U'(t),\ t \in (0,\infty)$	$U''(t),\ t \in (0,\infty)$
retarded	continuous piecewise smooth	continuous	piecewise continuous, uniformly bounded
neutral	continuous piecewise smooth	piecewise continuous, uniformly bounded	

TABLE I
SMOOTHNESS PROPERTIES OF LYAPUNOV MATRICES, COMMENSURATE DELAY CASE

	$U(0) - \tilde{U}(0)$
retarded, $m = 1$	N^{-N}
retarded, $m > 1$	N^{-2}
neutral, $m = 1$	N^{-N}
neutral, $m > 1$	N^{-1}

TABLE II
OBSERVED CONVERGENCE RATES

consists of non-commensurate numbers. This implies that U, as defined by (6), is no longer a classical solution of the boundary value problem (9), but must be interpreted as a weak solution. The numerical solution found with the approach of Section III-B converges to this weak solution.

The smoothness properties described above are summarized in Table III. They can again be related with the observed convergence rates of the numerical scheme of Section III-B, stated in Table II. In the retarded case the connection is explained by the same argument as for commensurate delays. In the neutral case this argument fails because U' is no longer a piecewise continuous function. However, it typically *behaves* as a piecewise continuous function, as explained in the remainder of this section.

Consider a discontinuity of K at $t_1 := \vec{n}_1 \cdot \vec{\tau}$, characterized by a jump s_1, and another discontinuity at time $t_2 := \vec{n}_2 \cdot \vec{\tau}$, characterized by a jump s_2. Assume further that $t_2 > t_1$. The interaction of these two discontinuities contributes to a jump of U' at time $t_2 - t_1$, whose size is given by $\Delta := s_1 s_2$. Because the system is assumed exponentially stable there exist constants $C > 0$ and $\lambda > 0$ such that

$$K(t) \leq Ce^{-\lambda t},\ \forall t \geq 0.$$

Hence, we have

$$|s_i| \leq Ce^{-\lambda t_i},\ i = 1, 2,$$

and we arrive at

$$\Delta \leq C^2 e^{-\lambda((\vec{n}_1 + \vec{n}_2)\cdot\vec{\tau})}.$$

This leads us to the following interpretation. The jump of U' at some time instant $\hat{t} = \vec{z} \cdot \vec{\tau}$ is very small when $\sum_{i=1}^{m} |z_i|\tau_i$ is large. This explains for instance why in a plot of the function U, jumps in the derivative are only visible at "strong resonances", where $\sum_{i=1}^{m} |z_i|\tau_i$ is small, like

$$\tau_1,\ \tau_2,\ \tau_1 - \tau_2,\ \tau_1 + \tau_2$$

if the delays are comparable in size. Accordingly, the numerical scheme (Section III-B) behaves in the same way as if the function were piecewise smooth, with a convergence rate as displayed in Table II.

V. EXAMPLES

Example 1: Consider the scalar system with input and output

$$\begin{aligned} \dot{x}(t) &= -ax(t-\tau) + bu(t) \\ y(t) &= cx(t). \end{aligned}$$

This problem is sometimes known as the hot-shower problem [13, Example 6.1]. The matrix exponential can be computed exactly by using (17) and the closed form for the $\mathcal{H}_2$ norm is

$$\|G\|_2^2 = \frac{c^2 b^2}{2a} \frac{\cos(a\tau)}{1 - \sin(a\tau)},$$

if the system is stable. Note that the effect of the delay on the $\mathcal{H}_2$ norm is easily identified since the $\mathcal{H}_2$ norm is the product of the $\mathcal{H}_2$ norm of the corresponding delay free system and the term $\cos(a\tau)/(1 - \sin(a\tau))$.

Example 2: Consider the time-delay system with

$$A_0 = \begin{pmatrix} -1 & 1 & 2 \\ 1 & -3 & 2 \\ 0 & 0 & -1 \end{pmatrix},\ A_1 = \frac{1}{5}\begin{pmatrix} -3 & 0 & 1 \\ 1 & -2 & 0 \\ 0 & 2 & -2 \end{pmatrix},$$
$$A_2 = \frac{1}{5}\begin{pmatrix} -4 & 1 & 0 \\ 0 & -2 & 1 \\ 2 & 1 & -3 \end{pmatrix},\ B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and $b^T = c = (1, 1, 1)$. We illustrate the convergence table (Table II) with some different cases choices of τ_1, τ_2 and B_1.

(a) Neutral, $m = 1$: $\tau_1 = \tau_2 = 1$ and

$$B_1 = \frac{1}{5}\begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 2 \\ 0 & 1 & 1 \end{pmatrix}. \tag{30}$$

(b) Retarded, $m = 2$: $\tau_1 = \pi/10$, $\tau_2 = 1$, B_1 zero matrix.

(c) Neutral, $m = 2$: $\tau_1 = \pi/10$, $\tau_2 = 1$ and B_1 as in (30).

The convergence of the different cases can be observed in Figure 1. We observe exponential convergence for Case (a). Case (a) is a single delay and can also be solved with the exact approach in Section III-A. This is not the case for Case (b) and (c) since the delays in Case (b) and (c) are not commensurate. We observe quadratic convergence $O(N^{-2})$ for the retarded case non-commensurate case, i.e., Case (b), and linear convergence $O(N^{-1})$ for Case (c).

In order to illustrate the advantage of using a Chebyshev grid, we have also compared it with the an equidistant grid. The error is visualized in Figure 2. Note that the approach based on Chebyshev grids converges faster and reaches a higher accuracy. Moreover, unlike the equidistant case, the error is essentially monotone in the number of grid points.

Example 3: The discretization approach in Section III-B is expected to be faster than the approach for the commensurate

	$U(t),\ t \in [0, \infty)$	$U'(t),\ t \in (0, \infty)$	$U''(t),\ t \in (0, \infty)$
retarded	continuous not piecewise smooth	continuous	piecewise continuous, uniformly bounded
neutral	continuous not piecewise smooth	nowhere continuous (if (29) non-commensurate), uniformly bounded	

TABLE III
SMOOTHNESS PROPERTIES OF LYAPUNOV MATRICES, NON-COMMENSURATE DELAY CASE

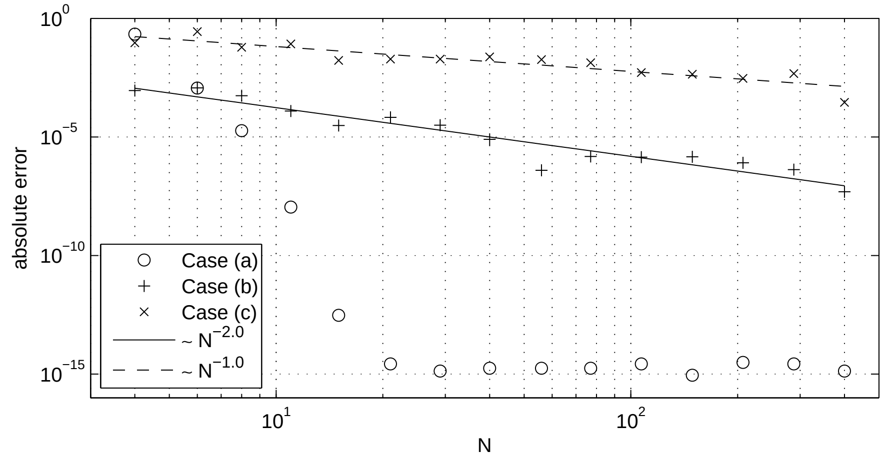


Fig. 1. The convergence of the discretization method for the different cases in Example 2.

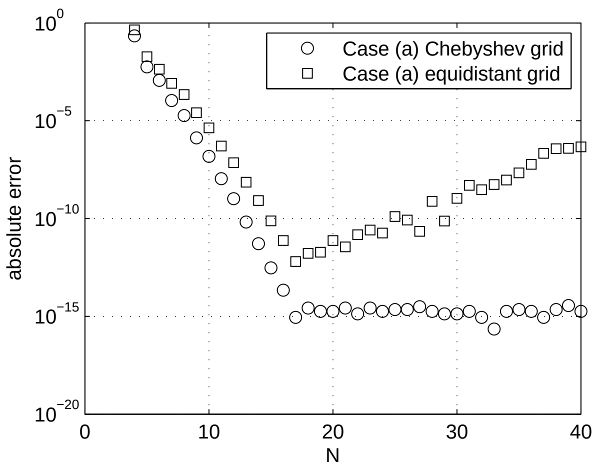


Fig. 2. The convergence for Example 2 Case (a), with different grids.

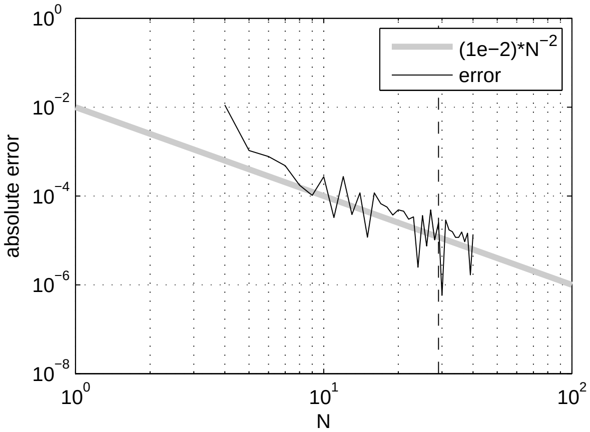
case in Section III-A if τ_m/h is large, where h is the basic delay, i.e., the largest h such that $\vec{\tau} = h\vec{n}$, $\vec{n} \in \mathbb{N}^m$ where $\gcd(n_1, \ldots, n_m) = 1$. We illustrate this with a slightly larger example (from [29, Section 2.4.1]) where we let $n = 9$, $\tau_1 = 0.25 = 5h$, $\tau_2 = 0.3 = 6h$, $b = (1, 0, \ldots, 0)^T$, $c = (1, \ldots, 1)$.

We observe approximately quadratic convergence $O(N^{-2})$ in Figure 3a and the discontinuity in $U''(t)$ is shown in Figure 4. Note that for this example, if the user is satisfied with a solution with accuracy 10^{-4} then it is more efficient to use the discretization approach. On the other hand, it does not make sense to solve the problem with $N > 30$ discretization points since the commensurate approach is then more efficient and more accurate. The cpu-time for solving the problem with the commensurate approach in Section III-A was $2.9s$ and is marked with a dashed line in Figure 3. Note also that if we perturb a delay, τ_m/h will generically increase and so will the computation time with the commensurate approach. Small perturbations in the delay will generally not change the computation time for the discretization approach.

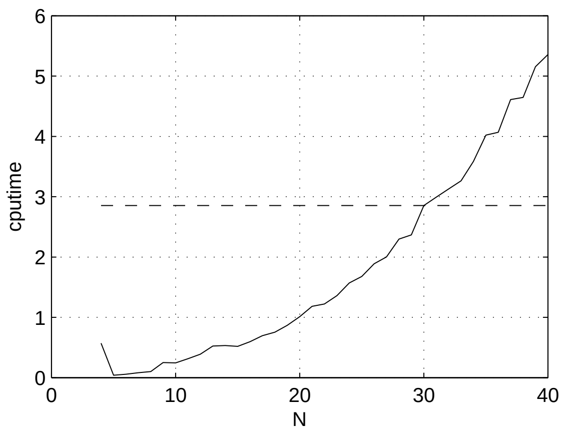
VI. CONCLUDING REMARKS

The Lyapunov equations are often used to analyze the $\mathcal{H}_2$ norm of LTI dynamical systems. We have shown that this relation extends as a relation between time-delay systems and delay Lyapunov equations. It is shown how this can be used in practice by proving explicit formulas for special cases and proposing a discretization scheme for the general case. Even though we focus on the $\mathcal{H}_2$ norm, some results, e.g., the discussion of the smoothness and the discretization scheme, are applicable to the delay Lyapunov equations in general.

Both computational approaches in this paper are directly or indirectly based on vectorization of matrix equations. For the commensurate case, the vectorization and the partitioning of the delay Lyapunov matrix allow us to reformulate the delay Lyapunov equation as a standard linear boundary value problem such that it can be solved with the matrix exponential. The vectorization approach is known to be computationally expensive for large systems. Efficient numerical methods for the standard Lyapunov equations which are not explicitly based on vectorization, such as [37], are based on triangularization of the system matrix. Such an approach appears to be not directly



(a) Absolute error



(b) CPU-time

Fig. 3. Number of discretization points N vs error and CPU-time for Example 3.

generalizable since there is no clear operation corresponding to triangularization for the entire delay Lyapunov equations.

Finally, we wish to mention some generalizations of the time-delay system (1) which have not been treated here. We have assumed that the input and the output do not have delays. However, the principles of this paper carry over naturally to the case where the inputs and outputs are delayed. Note also that we have only treated discrete and not distributed delays.

APPENDIX
PROOF OF THEOREM 4

The general idea of the main part of the proof consists of partitioning the delay Lyapunov matrix U into $2m$ segments of equal lengths, which will be denoted $U_i(t) := U(t + i\tau_1)$. The vectorization of all the function segments will be denoted $z(t) = (\operatorname{vec}(U_{m-1}(t))^T, \ldots, \operatorname{vec}(U_{-m}(t))^T)^T$.

With this notation, the standard differential equation with constant coefficients,

$$M_1 z'(t) = M_2 z(t), \tag{31}$$

is equivalent to the differential equation in the delay Lyapunov equation (9a). After several manipulations, we find that the matrices M_1 and M_2 can be stated explicitly and constructively as follows. We first introduce some notation. Let G be a function from the space of matrix polynomials $C(\lambda) = C_0 + C_1\lambda + \cdots + C_m\lambda^m \in \mathbb{R}^{n\times n}$ to an upper part of a block Sylvester matrix defined by

$$G(C) := \begin{pmatrix} C_0 & \cdots & C_m & & \\ & \ddots & & \ddots & \\ & & C_0 & \cdots & C_m \end{pmatrix} \in \mathbb{R}^{mn\times 2mn}.$$

We will use a notation common in the field of matrix polynomials, where the matrix polynomial with reversed order of the matrix coefficients is denoted by $\operatorname{rev} C$, i.e., $\operatorname{rev}(C)(\lambda) := C_m + C_{m-1}\lambda + \cdots + C_0\lambda^m$. Let

$$\begin{aligned} P(\lambda) &= I + B_1^T\lambda + \cdots + B_m^T\lambda^m, \\ Q(\lambda) &= A_0^T + A_1^T\lambda + \cdots + A_m^T\lambda^m. \end{aligned}$$

The matrices in the differential equation (31) are

$$M_1 = \begin{pmatrix} G(P\otimes I) \\ G(I\otimes \operatorname{rev} P) \end{pmatrix}, \quad M_2 = \begin{pmatrix} G(Q\otimes I) \\ -G(I\otimes \operatorname{rev} Q) \end{pmatrix}. \tag{32}$$

Correspondingly, the boundary conditions (9c) combined with the symmetry condition (9b) can be equivalently written as

$$Nz(\tau_1) + Mz(0) = \begin{pmatrix} -\operatorname{vec}(W) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{33}$$

where the matrices are given by

$$M = \begin{pmatrix} 0 & \cdots & 0 & C_1 & E_{-1} & \cdots & E_{-m} \\ I & & & & & & \\ & \ddots & & & & & \\ & & \ddots & & & & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & & & & & I & \end{pmatrix},$$

$$N = \begin{pmatrix} D_{m-1} & \cdots & D_0 & C_2 & 0 & \cdots & 0 \\ & -I & & & & & \\ & & \ddots & & & & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & & & & & & -I \end{pmatrix}. \tag{34}$$

The matrices C_1, C_2, E_i, $i = -m, \ldots, -1$ and D_i, $i = 0, \ldots, m-1$ are formed by the matrix coefficients of the

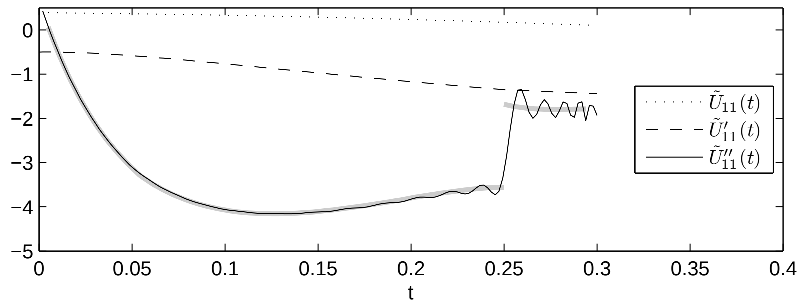


Fig. 4. The approximation $\tilde{U}''$ exhibits Gibbs phenomenon ($N = 50$) since U'' (gray) is discontinuous for Example 3.

vectorization of (9c), i.e.,

$$\begin{aligned}-\mathrm{vec}(W) &= \sum_{i=0}^{m}(A_i^T\otimes B_i^T + B_i^T\otimes A_i^T)\mathrm{vec}(X_0(0)) \\ &\quad+\sum_{i=0}^{m-1}\sum_{j=i+1}^{m} A_i^T\otimes B_j^T\mathrm{vec}(X_{j-i-1}(\tau)) \\ &\qquad + A_j^T\otimes B_i^T\mathrm{vec}(X_{i-j}(0)) \\ &\quad+\sum_{i=0}^{m-1}\sum_{j=i+1}^{m} B_j^T\otimes A_i^T\mathrm{vec}(X_{i-j}(0)) \\ &\qquad + B_i^T\otimes A_j^T\mathrm{vec}(X_{j-i-1}(\tau)) \\ &= (C_1+C_2)\mathrm{vec}(X_0(0)) + \sum_{i=0}^{m-1} D_i\mathrm{vec}(X_i(\tau)) \\ &\qquad + \sum_{i=-m}^{-1} E_i\mathrm{vec}(X_i(0))\end{aligned}$$

and

$$C_1=\sum_{i=0}^{m} A_i^T\otimes B_i^T,\ \ C_2=\sum_{i=0}^{m} B_i^T\otimes A_i^T.$$

The approach to vectorize the segmented problem was presented with explicit matrices for the retarded case in [13, Problem 6.72] and with an implicit representation of the matrices for neutral systems in [16, Section 3.1]. Note that the construction of M_1, M_2, M and N is not unique. Our construction is consistent with [13, Problem 6.72].

The regularity of M_1 can be derived as follows. Since the neutral system is exponentially stable, the difference equation is also exponentially stable [8, Proposition 1.23]. The roots of $\det(P(\lambda)) = 0$ are hence less than one in magnitude. The matrix M_1 is a block Sylvester (or resultant) matrix and the corresponding set of roots of the two polynomials constructing M_1 are the roots of $\det(P(\lambda))$ and $\det(\lambda^m P(\lambda^{-1}))$. They can hence never share roots. From a property of block resultant matrices [13, Proposition 6.73] we find that the Sylvester matrix M_1 is non-singular.

Now note that the standard differential equation (31) can be expressed with the matrix exponential,

$$z(t) = e^{M_1^{-1}M_2 t}z(0). \tag{35}$$

Also note that the uniqueness of $z(t)$ is equivalent to uniqueness of $U(t)$ and hence, since the unknown in (15) is $z(0)$ the equation must have a unique solution. We prove (16) by inserting (35) into (33).

A TECHNICAL LEMMA

Lemma 12: Let $\left\{t_i^{(1)}\right\}_{i\geq 1}$ and $\left\{t_i^{(2)}\right\}_{i\geq 1}$ be sequences of real numbers, satisfying

$$\inf\left\{\left|t_i^{(1)} - t_j^{(2)} + t\right| :\ i,j\in\mathbb{N},\ t_i^{(1)} \neq t_j^{(2)} + t\right\} > 0,\ \forall t\geq 0. \tag{36}$$

Let $\left\{q_i^{(1)}\right\}_{i\geq 1}$ and $\left\{q_i^{(2)}\right\}_{i\geq 1}$ be sequences of non-negative integers. Assume that the function $f_1\in\mathcal{L}_2(\mathbb{R})$ ($f_2\in\mathcal{L}_2(\mathbb{R})$) is smooth everywhere, excepting at the time-instants $\{t_i^{(1)}\}_{i\geq 1}$ $\left(\{t_i^{(2)}\}_{i\geq 1}\right)$, with a discontinuity in its $q_i^{(1)}$th $\left(q_i^{(2)}\text{th}\right)$ derivative occurring at time $t_i^{(1)}$ $\left(t_i^{(2)}\right)$, such that

$$\left|\lim_{h\to 0+} f_1^{q_i^{(1)}}(t_i^{(1)}+h) - \lim_{h\to 0-} f_1^{q_i^{(1)}}(t_i^{(1)}+h)\right| < \infty$$
$$\left(\left|\lim_{h\to 0+} f_2^{q_i^{(2)}}(t_i^{(2)}+h) - \lim_{h\to 0-} f_2^{q_i^{(2)}}(t_i^{(2)}+h)\right| < \infty\right),$$

for $i\geq 1$. Let the function $F:\mathbb{R}\to\mathbb{R}$ be defined by

$$F(t) = \int_{-\infty}^{\infty} f_1(s)f_2(t+s)ds.$$

If the function F is not infinitely differentiable at $t = \hat{t}$, then $\hat{t}$=$\left(t_i^{(2)} - t_j^{(1)}\right)$ for some $i,j\in\mathbb{N}$. Conversely, if $\hat{t}$=$\left(t_i^{(2)} - t_j^{(1)}\right)$ for some $i,j\in\mathbb{N}$, then the function is $(p-1)$ times differentiable and generically has a discontinuity in its pth derivative at $t=\hat{t}$, where

$$p = \min\left\{q_k^{(1)} + q_l^{(2)} + 1 :\ k,l\in\mathbb{N},\ t_l^{(2)} - t_k^{(1)} = t_i^{(2)} - t_j^{(1)}\right\}.$$

Furthermore,

$$\left|\lim_{h\to 0+} F^p(\hat{t}+h) - \lim_{h\to 0-} F^p(\hat{t}+h)\right| < \infty.$$

Proof: Fix t_0 and define

$$g_1(\xi) := F(t_0+\xi),\ \xi > 0,$$
$$g_2(\xi) := F(t_0+\xi),\ \xi < 0.$$

Let $\{s_i\}_{i\geq 1}$ be the set of time-instants for which either $f_1(\cdot)$ or $f_2((\cdot)+t_0)$ is not infinitely differentiable. For each $i\in\mathbb{N}$, assume that at $t=s_i$, the function $f_1(\cdot)$ has a discontinuity in its $r_{i,1}$th derivative and the function $f_2((\cdot)+t_0)$ a discontinuity in its $r_{i,2}$th derivative, with $r_{i,1}, r_{i,2}\in\mathbb{N}\cup\{\infty\}$ (a value equal to infinity implies that it is infinitely differentiable). Furthermore, let $\tilde{g}_1$ and $\tilde{g}_2$ be the smooth extensions of g_1 and g_2 on an interval $(-\epsilon,\epsilon)$ with $\epsilon>0$ sufficiently small (these exist because of the assumption (36)), and let $g=\tilde{g}_1-\tilde{g}_2$. For small values of ξ we can write

$$g(\xi)=\sum_{i\geq 1}\int_0^{-\xi}\left(f_1^L(s+s_i)-f_1^R(s+s_i)\right)\left(f_2^L(s+s_i+t_0)\right.$$
$$\left.-f_2^R(s+s_i+t_0)\right)\,ds.$$

Here f_1^L, respectively f_1^R, is obtained by a local smooth extension of the left branch, respectively right branch at the points where f_1 is not smooth. The function f_2^L and f_2^R are defined similarly. We have

$$f_1^L(s+s_i)-f_1^R(s+s_i)=a_{i,1}s^{r_{i,1}}+O(s^{r_{i,1}+1}),$$
$$f_2^L(s+s_i+t_0)-f_1^R(s+s_i+t_0)=a_{i,2}r^{r_{i,2}}+O(s^{r_{i,2}+1}),$$

for constants $a_{i,1}\neq 0$ and $a_{i,2}\neq 0$, which implies

$$g(\xi)=\sum_{i\geq 1}\int_0^{-\xi}(a_{i,1}s^{r_{i,1}}+O(s^{r_{i,1}+1}))(a_{i,2}s^{r_{i,2}}$$
$$+O(s^{r_{i,2}+1}))ds$$
$$=\sum_{i\geq 1}\frac{a_{i,1}a_{i,2}}{r_{i,1}+r_{i,2}+1}(-\xi)^{r_{i,1}+r_{i,2}+1}+O\left(\xi^{r_{i,1}+r_{i,2}+2}\right). \tag{37}$$

Thus, thus function g is $(r_{i,1}+r_{i,2})$ times differentiable at zero and generically has a discontinuity in its rth derivative, where

$$r=\min_i(r_{i,1}+r_{i,2}), \tag{38}$$

unless we are in a degenerate case where coefficients corresponding to the dominant term in (37) cancel each other out. The jump in the rth derivative is finite and can be directly obtained from (38).

Note that for any $k\in\mathbb{N}$, the function F is k times differentiable at t_0 if and only if the function g is k times differentiable at $\xi=0$. This property, along with the relation between sequences $\left\{t_i^{(1)}\right\}_{i\geq 1}$, $\left\{t_i^{(2)}\right\}_{i\geq 1}$ and $\{s_i\}_{i\geq 1}$, lead to the assertion to be proven. ■

Proof of Proposition 11

Let $T=\{\vec{z}\cdot\vec{\tau}:\ \vec{z}\in\mathbb{Z}^m\}$ and let

$$s=\inf\{t\in T:\ t>0\}.$$

Then we have $s\in T$. Indeed, if $s=0$, then this is trivial. If $s>0$, then we get

$$\inf\{|a-b|:\ a,b\in T,\ a\neq b\}\geq s,$$

implying that an element from $\mathbb{R}\setminus T$ cannot be approximated arbitrarily well by elements of T.

In what follows, we can distinguish between two potential cases.

Case 1: $\mathbf{s>0}$. First, we have $s\mathbb{Z}\subseteq T$ because $s\in T$. Second, for every $\kappa\in T$ there exists an integer α such that $|\kappa-\alpha s|<s$. We also have $|\kappa-\alpha s|\in T$. From the definition of s it follows that $|\kappa-\alpha s|=0$. Finally, we conclude that $T=s\mathbb{Z}$. This implies on its turn that $\tau_i\in s\mathbb{Z},\ i=1,\ldots,m$, i.e., the delays are commensurate.

Case 2: $\mathbf{s=0}$. Since T contains arbitrarily small strictly positive elements and since T is a group for the addition, the set T is dense in $\mathbb{R}$. Consequently, the set (27) is dense in $\mathbb{R}^+$.

Because the delays are assumed non-commensurate in Proposition 11, the first case can be excluded, and the assertion follows.

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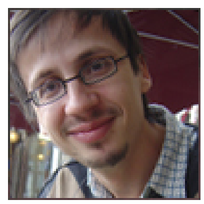

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