Invariance properties in the root sensitivity of time-delay systems with double imaginary roots

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Abstract: If $i\omega \in i\mathbb{R}$ is an eigenvalue of a time-delay system for the delay τ_0 then $i\omega$ is also an eigenvalue for the delays $\tau_k := \tau_0 + k\frac{2\pi}{\omega}$, for any $k \in \mathbb{Z}$. We investigate the sensitivity and other properties of the root $i\omega$ for the case that $i\omega$ is a double eigenvalue for some τ_k. It turns out that under natural conditions, the presence of a double imaginary root $i\omega$ for some delay τ_0 implies that $i\omega$ is a simple root for the other delays τ_k, $k \neq 0$. Moreover, we show how to characterize the root locus around $i\omega$. The entire local root locus picture can be determined from the square root splitting of the double root. We separate the general picture into two cases depending on the sign of a single scalar constant; the imaginary part of the first coefficient in the square root expansion of the double eigenvalue.

Keywords: Time-delay systems, sensitivity, perturbation analysis, imaginary axis, root locus, double roots, critical delays

1. INTRODUCTION

The setting of this paper is a time-delay system with a single constant delay and constant coefficients,

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau),\ A_0, A_1 \in \mathbb{C}^{n\times n}. \quad (1)$$

This time-delay system has the characteristic equation,

$$f(s,\tau) := \det\left(-sI + A_0 + A_1 e^{-\tau s}\right). \quad (2)$$

When analyzing the stability of (1), the delays τ for which (2) has a purely imaginary root $s = i\omega \in i\mathbb{R}$ often play a crucial role. In this work, these delays will be referred to as critical delays. They are important, since the critical delays and the sensitivity of the imaginary roots, i.e., the derivative with respect to the delay, can be used to produce a complete stability picture by keeping track of the number of roots entering and leaving the right half-plane. This type of reasoning is used in several works in the literature, e.g., Cooke and Grossman (1982), Rekasius (1980), Sipahi and Olgac (2005), and many more.

It is widely known, and often exploited, that the presence of an imaginary root at $i\omega$ is periodic in the delay parameter with periodicity $\frac{2\pi}{\omega}$.

As a first result we will see that in general the same type of periodicity property does not hold for the presence of a double imaginary root. That is, if $i\omega$ is a double root for some delay τ_0 then for $\tau_k = \tau_0 + \frac{2\pi k}{\omega}$, $k \neq 0$, $i\omega$ is not a double root. This somewhat unexpected result motivates our study of properties of double imaginary roots and sequences of critical delays $\{\tau_k\}$ for which the time-delay system has a double imaginary root for one of the delays.

Stability conditions based on reasoning with imaginary roots are often some form of elimination of either the exponential $e^{-\tau s}$ or the scalar s in the characteristic equation. The resulting condition is typically expressed in terms of roots of a polynomial (as in e.g., Thowsen (1981), Rekasius (1980), Hertz et al. (1984), Sipahi and Olgac (2005)) or in terms of the eigenvalues of a generalized eigenvalue problem (as in e.g., Louisell (2001), Chen et al. (1995), Fu et al. (2006), Jarlebring and Hochstenbach (2009)). See also (Michiels and Niculescu, 2007, Section 4.3.2), (Niculescu, 2001, Section 4.4) and (Jarlebring, 2008, Section 3.2) for more references to delay-dependent stability results expressed in terms of imaginary eigenvalues. These standard results on imaginary eigenvalues do not reveal properties of a repeated imaginary eigenvalues.

Some results of high-order analysis are given in Fu et al. (2007) including results for multiple imaginary roots (Fu et al., 2007, Theorem 4). The focus in this paper is *invariance properties* of the imaginary root for the *other critical delays* when there is a repeated imaginary root. Invariance properties are not treated in Fu et al. (2007).

Finally, a higher order analysis is adapted for the *direct method* in Sipahi and Olgac (2003), where several involved cases are discussed. For instance, two sequences of critical delays may have the same crossing frequency but for a disjoint sets of critical delays such that the roots are still simple, or for some delay there may be several (distinct) imaginary roots, or the real part of the first derivative may be zero (a similar example will be discussed in Example 6.3).

Throughout this work we will implicitly assume that $i\omega \neq 0$ since otherwise, zero is a root for any delay, and the periodicity of the critical delays is not defined.

The notation is standard. We denote partial derivatives with subscript, e.g., f_s is the partial derivative of f with respect to s. Total derivatives of functions of one variable are denoted by $(\cdot)'$, e.g., the derivative of $s(\tau)$ is $s'(\tau)$. The imaginary unit it denoted i, the real and imaginary parts of a complex number $z \in \mathbb{C}$ is denoted by Re z and Im z correspondingly.

2. FIRST ORDER ANALYSIS

Suppose that for $\tau = \tau_0$ the characteristic equation (2) has a double root at $i\omega$. The following proposition states $i\omega$ is a simple root for all other delays in the sequence $\mathcal{T} = \{\tau_0 + \frac{2\pi k}{\omega}\}_k$ under the conditions that $f_\tau(i\omega, \tau_0) \neq 0$. Since the roots are simple, we can compute the sensitivity, which we give in a closed form. It turns out that the sensitivity is purely imaginary telling us that the root path close to the $i\omega$ is vertical, and at one point, $s(\tau) = i\omega$. It is hence either just touching the imaginary axis or (in case of a saddle point), approaching the imaginary axis vertically, but still cross into the other complex half-plane.

Proposition 1. Let $\mathcal{T} = \{\tau_k\}_{k\in\mathbb{Z}} := \{\tau_0 + k\frac{2\pi}{|\omega|}\}_{k\in\mathbb{Z}}$ be a set of delays for which $i\omega \in i\mathbb{R}$ is an eigenvalue. Let $s(\tau)$ be a continuous eigenvalue path defined in a neighborhood of τ_k for some $k \in \mathbb{Z}\backslash\{0\}$, i.e., $s(\tau_k) = i\omega$. Suppose that for some other delay $\tau_0 \in \mathcal{T}$, $i\omega$ is a double (not triple) eigenvalue and $f_\tau(i\omega, \tau_0) \neq 0$. Then, $i\omega$ is a simple eigenvalue for the delay τ_k and

$$s'(\tau_k) = -i\frac{\omega|\omega|}{2\pi k}. \tag{3}$$

Proof. Without loss of generality we let τ_0 be the delay for which $i\omega$ is a double (not triple) non-semisimple eigenvalue and $k \neq 0$. We denote $s_0 := s(\tau_0) = i\omega$. Since s_0 is a double root for $\tau = \tau_0$, $f_s(s_0; \tau_0) = 0$. We first show that s_0 is a simple eigenvalue for delay τ_k by proving that $f_s(s_0; \tau_k) \neq 0$.

We note that $f(s, \tau) = p(s, e^{-s\tau})$ where $p(s, z)$ is a multivariate polynomial (and does not depend on τ). We denote the partial derivatives by p_s and p_z. The reason for this substitution is that $p(s_0, e^{-s_0\tau_k})$ is independent of k. The same holds for the partial derivatives,

$$\begin{aligned} p_s(s_0, e^{-s_0\tau_k}) &= p_s(s_0, e^{-s_0\tau_0}) \\ p_z(s_0, e^{-s_0\tau_k}) &= p_z(s_0, e^{-s_0\tau_0}). \end{aligned}$$

We can express the derivatives of f with p, p_s, p_z, by partial differentiation,

$$\begin{aligned} f_s(s, \tau) &= p_s(s, e^{-s\tau}) + p_z(s, e^{-s\tau})e^{-s\tau}(-\tau), \\ f_\tau(s, \tau) &= p_z(s, e^{-s\tau})e^{-s\tau}(-s). \end{aligned} \tag{4}$$

We express the first derivative of $s(\tau)$ by rearranging the terms of the chain rule,

$$0 = f_s(s(\tau), \tau)s'(\tau) + f_\tau(s(\tau), \tau)$$

and

$$s'(\tau) = -\frac{p_z(s, e^{-s\tau})e^{-s\tau}(-s)}{p_s(s, e^{-s\tau}) + p_z(s, e^{-s\tau})e^{-s\tau}(-\tau)}.$$

Since we will only evaluate f and p and the derivatives for $s = s_0$ and $z = e^{s_0\tau_k}$ we simplify the notation by $p_s(s_0, e^{-s_0\tau_k}) = p_s$, and p_z correspondingly.

Now let $s = s_0$ and $\tau = \tau_k$ and recall that $f_s(s_0, \tau_0) = 0$, i.e.,

$$p_s = \tau_0 z p_z.$$

Hence

$$\begin{aligned} s'(\tau_k) &= -\frac{p_z z(-s)}{p_s + p_z z(-\tau)} = \\ &- \frac{p_z z(-s)}{\tau_0 z p_z + p_z z(-\tau)} = \frac{s}{\tau_0 - \tau_k} = -i\frac{\omega|\omega|}{2\pi k}. \end{aligned}$$

This completes the proof. □

Remark 2. (Non-semisimple). The time-delay system (1) can be represented as a linear time-invariant (LTI) infinite-dimensional system. In some contexts, it is common to use a classification of the multiplicity of the eigenvalues of the LTI system to understand the local behavior of eigenvalue paths. The interpretation in terms of classification is also possible in this context using the definitions of multiplicities of nonlinear eigenvalue problems.

A double eigenvalue can either be *non-semisimple* or *semisimple*. For the time-delay system (1) these cases can be distinguished by the rank of the null-space of the matrix $-sI + A_0 + A_1e^{-\tau s}$. If the rank is one, the eigenvalue is non-semisimple, and if the rank is two it is semisimple. See, e.g. Hryniv and Lancaster (1999) for definitions in the general setting.

In Proposition 1 we assumed that $f_\tau(s, \tau_0) \neq 0$. We will now see that this implies that the eigenvalue is non-semisimple. If the eigenvalue is semisimple, then the rank of the null-space is two. If the rank of the null-space is two, then the dual nonlinear eigenvalue problem, where s is considered a (fixed) parameter and $\tau = \tau_0$ the eigenvalue parameter, has a double eigenvalue in τ. Hence $f_\tau(s, \tau_0) = 0$. By contradiction, it follows that $f_\tau(s, \tau_0) \neq 0$ implies that the eigenvalue is non-semisimple.

3. SECOND ORDER ANALYSIS

We ultimately wish to establish if roots enter or leave the right half-plane. The real part of the sensitivity (3) is zero, and does not give information about the directions the roots are crossing. To establish the crossing direction we need the real part of the second derivative. In the following proposition we see that the real part of the second derivative is generically non-zero which implies that the roots only touch and do not cross the imaginary axis.

Proposition 3. Under the same conditions as in Proposition 1,

$$s''(\tau_k) = 2i\frac{\omega^3}{(2\pi k)^2} + i\frac{\omega^5|\omega|}{(2\pi k)^3}\frac{f_{ss}(i\omega, \tau_0)}{f_\tau(i\omega, \tau_0)} \tag{5}$$

Proof. From the chain-rule we find that second derivative of s is

$$s''(\tau_k) = \frac{2f_{s\tau}(s_0, \tau_k)(s')^2 + f_{\tau\tau}(s_0, \tau_k)s' + f_{ss}(s_0, \tau_k)(s')^3}{f_\tau(s_0, \tau_k)}.$$

In order to show (5) we separate the expression into

$$s'' = T_1 + T_2 + T_3$$

where,

$$T_1 = \frac{2f_{s\tau}(s')^2 + f_{\tau\tau}s'}{f_\tau(s_0, \tau_0)},$$

$$T_2 = \frac{(f_{ss}(s_0, \tau_k) - f_{ss}(s_0, \tau_0))(s')^3}{f_\tau(s_0, \tau_0)}$$

and

$$T_3 = \frac{f_{ss}(s_0, \tau_0)(s')^3}{f_\tau(s_0, \tau_0)}.$$

If we again let $f(s,\tau) = p(s, e^{-s\tau})$ and $z = e^{-s\tau}$ we can express higher partial derivatives of f in terms of higher derivatives of p by differentiating (4) implicitly. We find that

$$\begin{aligned} f_{\tau\tau}(s_0;\tau_k) &= p_{zz}s^2z^2 + p_z s^2 z \\ f_{\tau s}(s_0;\tau_k) &= (p_{zs} + p_{zz}z(-\tau_k))z(-s) + p_z z\tau_k s - p_z z \\ f_{ss}(s_0;\tau_k) &= p_{ss} + p_{sz}z(-\tau_k) + \\ &\quad (p_{zs} + p_{zz}z(-\tau_k))z(-\tau_k) + p_z z\tau_k^2. \end{aligned} \tag{6}$$

If we insert these equations into T_1, T_2 and use that $s' = s/(\tau_0 - \tau_k)$ we can simplify T_1 and T_2 into

$$T_1 = \frac{2f_{s\tau}(s')^2 + f_{\tau\tau}s'}{f_\tau} = \frac{2(-p_{sz}s - p_z + (p_{zz}zs + p_z s)\tau_k)s + (p_{zz}s^2z + p_z s^2)(\tau_0 - \tau_k)}{-(\tau_0-\tau_k)^2 p_z} = \frac{2(-p_{sz}s - p_z)s + (p_{zz}s^2z + p_z s^2)(\tau_0 + \tau_k)}{-(\tau_0-\tau_k)^2 p_z}$$

and

$$T_2 = \frac{(f_{ss}(s,\tau_k) - f_{ss}(s,\tau_0))(s')^3}{f_\tau} = \frac{-(-2p_{sz} + (p_{zz}z + p_z)(\tau_0+\tau_k))s^2}{-(\tau_0-\tau_k)^2 p_z}.$$

Hence,

$$T_1 + T_2 = \frac{2s}{(\tau_0-\tau_k)^2} = \frac{2i\omega^3}{(2\pi k)^2}.$$

We complete the proof by noting that

$$T_3 = \frac{i\omega^5|\omega|}{(2\pi k)^3}\frac{f_{ss}(s_0,\tau_0)}{f_\tau(s_0,\tau_0)}. \ \square$$

Finally we combine Proposition 1 and Proposition 3 into a formula for the first three coefficients of the Taylor expansion.

Theorem 4. Under the same conditions as in Proposition 1, the Taylor expansion of s around $i\omega$ for $\tau = \tau_k$ is

$$s(\tau) = i\omega - i\frac{\omega|\omega|}{2\pi k}(\tau - \tau_k) + \frac{1}{2}\left(2i\frac{\omega^3}{(2\pi k)^2} + i\frac{\omega^5|\omega|}{(2\pi k)^3}\frac{f_{ss}(i\omega,\tau_0)}{f_\tau(i\omega,\tau_0)}\right)(\tau-\tau_k)^2 + \mathcal{O}((\tau-\tau_k)^3).$$

Proof. This is the result of inserting the first and second derivative given in Proposition 1 and Proposition 3 into the Taylor expansion of s arount $s(\tau_k) = i\omega$. $\square$

The real part as a function of the imaginary part is a also a quadratic function and we can give a formula for the first coefficient.

Corollary 5. Under the same conditions as in Proposition 1, let $a = \text{Re } s$ and $b = \text{Im } s$. The real part as a function of the imaginary part is

$$a(b) = -\frac{\omega|\omega|}{4\pi k}\left(\text{Im } \frac{f_{ss}(i\omega,\tau_0)}{f_\tau(i\omega,\tau_0)}\right)(b-\omega)^2 + \mathcal{O}((b-\omega)^3).$$

Proof. From Theorem 4 we know that

$$a = \text{Re } s(\tau) = -\frac{1}{2}\frac{\omega^5|\omega|}{(2\pi k)^3}\left(\text{Im } \frac{f_{ss}(s_0,\tau_0)}{f_\tau(s_0,\tau_0)}\right)(\tau-\tau_k)^2 + \mathcal{O}(\tau-\tau_k)^3 \tag{7}$$

$$b = \text{Im } s(\tau) = \omega - \frac{\omega|\omega|}{2\pi k}(\tau - \tau_k) + \mathcal{O}(\tau-\tau_k)^2. \tag{8}$$

We eliminate $\tau - \tau_k$ in (7) and (8) by solving the equations for a and find that

$$a = -\frac{1}{2}\frac{\omega|\omega|}{(2\pi k)}\left(\text{Im } \frac{f_{ss}(s_0,\tau_0)}{f_\tau(s_0,\tau_0)}\right)(b-\omega)^2 + \mathcal{O}((b-\omega)^3),$$

which completes the proof. $\square$

Remark 6. If $\text{Im } (f_{ss}(s_0,\tau_0)/f_\tau(s_0,\tau_0)) = 0$ we have a degenerate case. In this case, a second order analysis can not reveal if roots enter or leave the right half-plane. An analysis using higher order derivatives would be necessary to determine this.

4. THE DOUBLE EIGENVALUE

In the previous sections we saw that if $k \neq 0$ then $i\omega$ is a simple eigenvalue and we found formulas for the first terms in the Taylor expansion. When $k = 0$, we have a square root splitting for the double eigenvalue, in the sense that the derivative of the root path at τ_0, i.e., $s'(\tau_0)$, is undefined, but the function $s(\tau)$ can be expanded in a Puiseux series around τ_0 where the first term is a square root. The following result gives a formula for the first coefficient in this expansion, and is a specialization of (Fu et al., 2007, Theorem 4). To make the paper self-contained we present a brief proof.

Theorem 7. Under the same conditions as in Proposition 1, let $s(\tau)$ be a path for which $s(\tau_0) = i\omega$ is the double eigenvalue. Then,

$$s(\tau) = i\omega \pm \left(-2\frac{f_\tau(i\omega,\tau_0)}{f_{ss}(i\omega,\tau_0)}(\tau-\tau_0)\right)^{1/2} + o(\sqrt{\tau-\tau_0}).$$

Proof. The proof consists of two parts. We first show that the first term in the expansion of $s(\tau)$ is $\sqrt{\tau-\tau_0}$, and then find the first coefficient of this expansion.

The condition $f_\tau(s_0,\tau) \neq 0$ allows us to apply (Hryniv and Lancaster, 1999, Theorem 4.2) which implies that the expansion possesses the property *completely regular splitting* (CRS). If we have CRS for a double (not triple) non-semisimple root, the expansion is

$$s(\tau) = s_0 + c_1(\tau-\tau_0)^{1/2} + o(\sqrt{|\tau-\tau_0|}),$$

where $(\cdot)^{1/2}$ denotes both complex branches of the square root. It remains to determine c_1.

The two-variable Taylor expansion for $f(s,\tau)$ is

$$f(s,\tau) = f(s_0,\tau_0) + \Delta s f_s + \Delta\tau f_\tau + \frac{1}{2}\left(f_{ss}\Delta s^2 + 2f_{s\tau}\Delta s\Delta\tau + f_{\tau\tau}\Delta\tau^2\right) + \mathcal{O}(\Delta s,\Delta\tau)^3. \tag{9}$$

Let $\Delta s = s - s_0 = c_1(\tau-\tau_0)^{1/2} + o(\sqrt{\Delta\tau})$ and $\Delta\tau = \tau - \tau_0$, then

$$0 = \Delta\tau f_\tau + \frac{1}{2}\left(f_{ss}c_1^2\Delta\tau + 2f_{s\tau}c_1\Delta\tau^{3/2} + f_{\tau\tau}\Delta\tau^2\right) + \mathcal{O}(\Delta s,\Delta\tau)^3 = \Delta\tau f_\tau + \frac{1}{2}f_{ss}c_1^2\Delta\tau + o(\Delta\tau). \tag{10}$$

Since $f(s_0 + \Delta s, \tau_0 + \Delta\tau) = 0$ must hold for all $\Delta\tau > 0$,

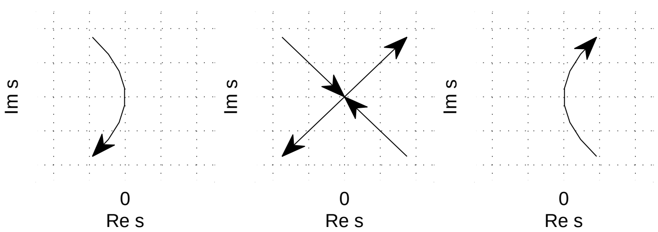
$$c_1^2 = -\frac{2f_\tau}{f_{ss}}.$$

The proof is completed. $\square$

5. COMBINATION OF RESULTS

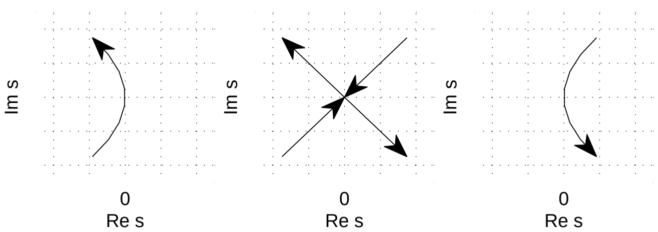
Note that expressions for the coefficients of the expansions in the square root splitting (Theorem 7) and for the Taylor expansion of the simple eigenvalues (Corollary 5 and Theorem 4) both contain the expression $f_{ss}(i\omega, \tau_0)/f_\tau(i\omega, \tau_0)$. Hence, the local root behaviour of all $\tau_k \in \mathcal{T}$ can be determined by the function f (and the derivatives) at $\tau = \tau_0$. This allows us to categorize the local behaviour of the roots into two separate cases. Without loss of generality we assume that $\omega > 0$ for this categorization.

1) If $\mathrm{Im}\, f_{ss}(i\omega, \tau_0)/f_\tau(i\omega, \tau_0) > 0$, then for critical delays $\tau_k \in \mathcal{T}$ greater than τ_0 (positive k) the root path touches the imaginary axis from above and in the left half-plane. For delays $\tau_k \in \mathcal{T}$ less than τ_0 (negative k) the imaginary axis is touched from the left and upward.



(a) $k > 0$ (b) $k = 0$ (c) $k < 0$

2) Analogously, if $\mathrm{Im}\, f_{ss}(i\omega, \tau_0)/f_\tau(i\omega, \tau_0) < 0$, the root path for critical delays $\tau_k \in \mathcal{T}$ less than τ_0 touch the imaginary axis in the left half-plane and for delays greater than τ_0 touch the imaginary axis from above and in the right half-plane.



(d) $k < 0$ (e) $k = 0$ (f) $k > 0$

6. ILLUSTRATIVE EXAMPLES

We illustrate the two cases with two examples.

Example 6.1. (Case 1: Parabolas left). Let

$$A_0 = \begin{pmatrix} 0 & 1 \\ -9\pi^2 & 2 \end{pmatrix} \text{ and } A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}.$$

In order to classify the root locus we need the partial derivatives of f,

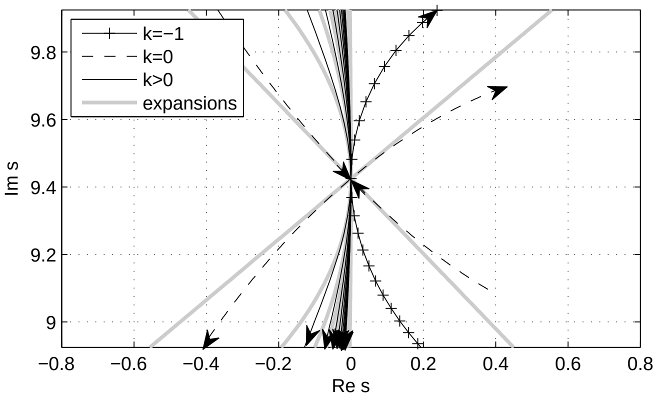


Fig. 1. The root locus for Example 6.1 close to the imaginary eigenvalue $s = 3\pi i$. This corresponds to case 1.

$$\begin{aligned} f(s,\tau) &= s^2 - 2s - 2se^{-\tau s} + 9\pi^2, \\ f_s(s,\tau) &= 2s - 2 - 2e^{-\tau s} + 2s\tau e^{-\tau s}, \\ f_\tau(s,\tau) &= 2s^2 e^{-\tau s}, \\ f_{ss}(s,\tau) &= 2 + 4\tau e^{-\tau s} - 2s\tau^2 e^{-\tau s}. \end{aligned}$$

Hence, $s = i\omega = 3\pi i$ is a double (not triple) root for $\tau = \tau_0$ since $f(3\pi i, 1) = 0$ and $f_s(3\pi i, 1) = 0$ but $f_{ss}(3\pi i, 1) = -2 + 6\pi i \neq 0$. Moreover, $f_\tau(3\pi i, 1) = 18\pi^2$. Note that

$$\mathrm{Im}\, \frac{f_{ss}(3\pi i, 1)}{f_\tau(3\pi i, 1)} = \mathrm{Im}\, \frac{-2 + 6\pi i}{18\pi^2} = \frac{1}{3\pi} > 0.$$

Case 1 in the behavior described in Section 5 can be observed in Figure 1. We see that for $k = -1$, i.e., $\tau = 1/3$ (which is the only negative k for which the delay is positive) the root path is in the right half plane whereas for $k > 0$ all root paths lie in the left half-plane. The truncated expansions from Theorem 4 and Theorem 7 are also visualized in Figure 1. In Figure 1 we have also plotted the truncated Taylor and Puiseux expansion for the roots touching the imaginary axis by using Corollary 5 and Theorem 7.

In Figure 2 we see that the parabola corresponding to the critical delay to the left of the double eigenvalue is from above, i.e., the path lies in the right half-plane. Conversely,

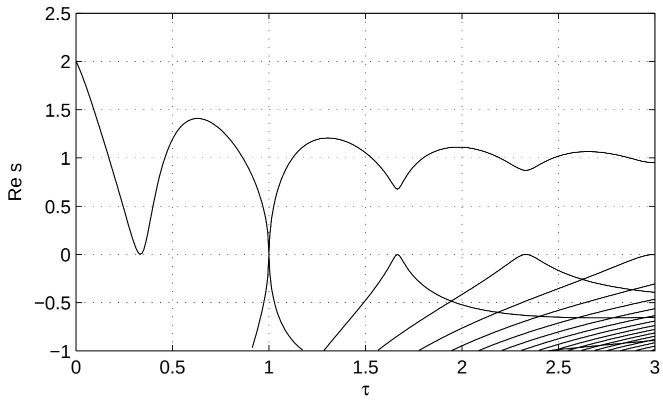


Fig. 2. The real part vs τ for Example 6.1. The touching points to the right of the $\tau_0 = 1$ are from below. This corresponds to case 1.

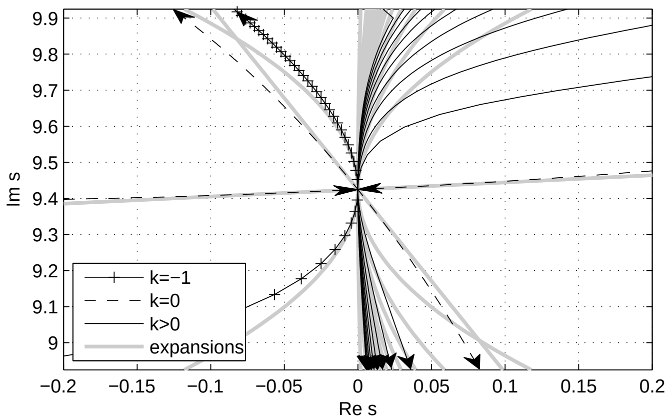


Fig. 3. The root locus for Example 6.2 close to the imaginary eigenvalue $s = 3\pi i$. This corresponds to case 2.

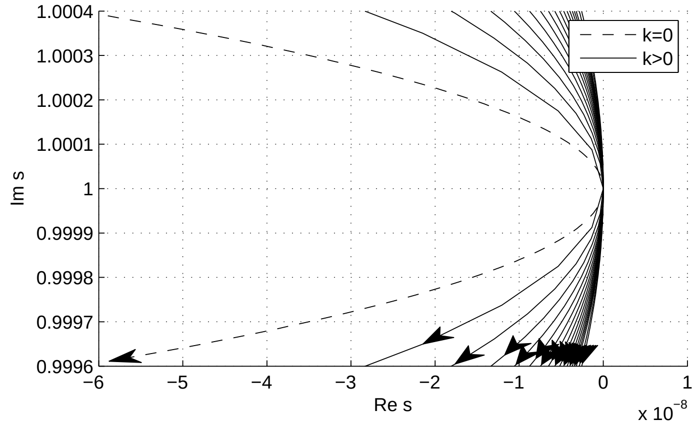


Fig. 4. The root locus close to $s = i$ for the Example 6.3.

all the critical delays (touching points) to the right of the double eigenvalue are touching $\mathrm{Re}\, s = 0$ from below, which means that they lie in the left half-plane.

Example 6.2. (Case 2: Parabolas right). In order to illustrate the case where the root locus has an infinite number of parabolas touching the imaginary axis in the right half-plane, we consider the following example. Let

$$A_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{pmatrix},\ A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -b_3 & -b_2 & -b_1 \end{pmatrix},$$

where

$$a_1 = \frac{2}{5}\frac{(65\pi + 32)}{8 + 5\pi} \approx 3.98,$$
$$a_2 = \frac{9\pi^2(13 + 5\pi)}{8 + 5\pi} \approx 108,$$
$$a_3 = \frac{324}{5}\frac{\pi^2(5\pi + 4)}{8 + 5\pi} \approx 531,$$

$$b_1 = \frac{260\pi + 128 + 225\pi^2}{10(8 + 5\pi)} \approx 13.6,$$
$$b_2 = \frac{45\pi^2}{10(8 + 5\pi)} \approx 18.7 \text{ and}$$
$$b_3 = \frac{81\pi^2(40\pi + 32 + 25\pi^2)}{10(8 + 5\pi)} \approx 1363.$$

This example is constructed (with software for symbolic manipulations) such that $f_s(3\pi i, 1) = 0$ and

$$\mathrm{Im}\, f_{ss}(3\pi i, 1)/f_\tau(3\pi i, 1) \approx -0.0667 < 0.$$

In Figure 3 we observe the expected behavior that for positive k, the imaginary axis is touched from the right. This is what we expect from case 2 in Section 5. The other properties of case 2 are also easily verified in Figure 3.

Example 6.3. (A converse example). We have seen in the theory and examples above that the generic situation is that a double imaginary root implies that the root locus close to the other critical delays are curves touching but not crossing the imaginary axis. With this example we show that the converse is not true. That is, this example [1] from (Thowsen, 1981, Example 2) and (Hertz et al., 1984, Example 5) shows that touching (but not crossing) the imaginary axis does not in general imply that we have a double root for some other critical delay.

Suppose

$$A_0 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix},\ A_1 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}. \tag{11}$$

If $\tau = \pi$ then $s_0 = i$ is an eigenvalue. The partial derivatives of f are

$$f_s(i, \tau_k) = (2 + \pi + 2\pi k)i,\ f_\tau(i, \tau_k) = -1.$$

Note that $s = i$ is a simple root for all critical delays, since $f_s(i, \tau_k) \neq 0$ for all k. The sensitivity (which is also given in Hertz et al. (1984)) is

$$s'(\tau_k) = -\frac{f_\tau(i, \tau_k)}{f_s(i, \tau_k)} = \frac{-i}{(2 + \pi + 2\pi k)}.$$

This shows that the derivative is purely imaginary for all critical delays. It can also be shown that the real part of the second derivative is non-zero, such that the root locus is a parabola touching the imaginary axis (see Figure 4), similar to the main phenomena of this paper (a double root for some critical delay). In this example, there is however no double root. Hence, even if a root path is touching the imaginary axis, it is not necessary that another critical delay has a double imaginary root.

We also note that a similar phenomenon (touching not crossing) can be observed for an example in (Sipahi and Olgac, 2003, Equation 19).

7. CONCLUSIONS

For simple imaginary roots, the root tendency, i.e., the sign of the derivative of the root path sign(Re $s'(\tau)$), is independent of k for $\tau_k \in \mathcal{T} = \{\tau_0 + \frac{2\pi k}{\omega}\}_k$. That is, the root tendency is invariant in the sense that the root tendency for some delay $\tau_k \in \mathcal{T}$ determines the root tendency for all $\tau \in \mathcal{T}$.

We have considered the case where the time-delay system has a double imaginary root for $\tau_0 \in \mathcal{T}$, and shown that the

[1] We have corrected the typographical error in the matrix formulation in (Hertz et al., 1984, Equation 51) where the term $-x_2(t)$ is missing in the second equation.

multiplicity is not the same for all $\tau \in \mathcal{T}$. The multiplicity is not invariant with respect to k. However, a consequence of the results in this paper is that an invariance property similar to the case of simple roots still holds: The crossing behavior of all $\tau \in \mathcal{T}$ are completely determined from the crossing directions at $\tau = \tau_0$.

More precisely, we have found a formula for the first coefficients in the Taylor expansion of $s(\tau)$ around $\tau = \tau_k \in \mathcal{T}$, $k \neq 0$, in terms of the first coefficient of the Puiseux series around $\tau = \tau_0$. This has made it possible to classify the local behavior of all $\tau \in \mathcal{T}$ *only* depending on the sign of Im $f_{ss}(i\omega, \tau_0)/f_\tau(i\omega, \tau_0)$.

As a final remark we comment on the generality of this work. The setting has been time-delay systems with a single delay (1). The results do however hold for much more general systems. The only property of the characteristic equation (2) needed for the proofs and discussions is that the characteristic equation is a (sufficiently smooth) multivariate function in s and $z = e^{-s\tau}$, i.e., $f(s,\tau) = p(s, e^{-s\tau})$. The characteristic equation of neutral systems and systems with multiple commensurate delays also have this property. Hence, the results are valid for these system, and also systems with multiple delays if perturbation in only one of the delays is considered.

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REFERENCES

Chen, J., Gu, G., and Nett, C.N. (1995). A new method for computing delay margins for stability of linear delay systems. *Syst. Control Lett.*, 26(2), 107–117.

Cooke, K.L. and Grossman, Z. (1982). Discrete delay, distributed delay and stability switches. *J. Math. Anal. Appl.*, 86, 592–627.

Fu, P., Chen, J., and Niculescu, S.I. (2007). High-order analysis of critical stability properties of linear time-delay systems. In *Proceedings of the 2007 American Control Conference, New York City, USA, July 11-13*, 4921–4926.

Fu, P., Niculescu, S.I., and Chen, J. (2006). Stability of linear neutral time-delay systems: Exact conditions via matrix pencil solutions. *IEEE Trans. Autom. Control*, 51(6), 1063–1069.

Hertz, D., Jury, E., and Zeheb, E. (1984). Simplified analytic stability test for systems with commensurate time delays. *IEE Proc., Part D*, 131, 52–56.

Hryniv, R. and Lancaster, P. (1999). On the perturbation of analytic matrix functions. *Integral Equations Oper. Theory*, 34(3), 325–338.

Jarlebring, E. (2008). *The spectrum of delay-differential equations: numerical methods, stability and perturbation.* Ph.D. thesis, TU Braunschweig.

Jarlebring, E. and Hochstenbach, M.E. (2009). Polynomial two-parameter eigenvalue problems and matrix pencil methods for stability of delay-differential equations. *Linear Algebra and its Applications*, 431(3), 369–380.

Louisell, J. (2001). A matrix method for determining the imaginary axis eigenvalues of a delay system. *IEEE Trans. Autom. Control*, 46(12), 2008–2012.

Michiels, W. and Niculescu, S.I. (2007). *Stability and Stabilization of Time-Delay Systems: An Eigenvalue-Based Approach.* Advances in Design and Control 12. SIAM Publications, Philadelphia.

Niculescu, S.I. (2001). *Delay effects on stability. A robust control approach.* Springer-Verlag London.

Rekasius, Z. (1980). A stability test for systems with delays. In *Proc. of joint Autom. Contr. Conf San Francisco*, TP9–A.

Sipahi, R. and Olgac, N. (2003). Degenerate cases in using the direct method. *J. Dyn. Syst.-T. ASME*, 125(2), 194–201.

Sipahi, R. and Olgac, N. (2005). Complete stability robustness of third-order LTI multiple time-delay systems. *Automatica*, 41(8), 1413–1422.

Thowsen, A. (1981). An analytic stability test for class of time-delay systems. *IEEE Trans. Autom. Control*, 26(3), 735–736.