Homework in Potential Theory.

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1 Generalities

The list of problems given below was used for a doctoral course in potential theory in 2002.

2 Problems

1. Assume that u and v are harmonic functions. Show that uv is harmonic if and only if $\nabla u \cdot \nabla v = 0$.

2. Assume u is a harmonic functions. For which values of $p \in \mathbb{R}$ is $|u|^p$ subharmonic? The same question for $|\nabla u|^p$.

3. Let $\Phi(x) = \text{const.}\frac{1}{|x|^{n-2}}$ be the Newtonian kernel in $n \geq 3$ dimensions. (You may restrict to the case $n = 3$ if you like.) Then

$$-{}^*d\Phi = \text{const.}\tau$$

where

$$\tau = \frac{{}^*(x_1 dx_1 + ... + x_n dx_n)}{|x|^n}$$

is the solid angle $(n-1)$-form. (The star is the Hodge star: ${}^*(x_1 dx_1) = x_1 dx_2 ... dx_n$ etc.) Outside the origin this form is locally of the form $d\theta$ for some $(n-2)$-form θ.

   a) Show that θ can be chosen to have harmonic coefficients (not so easy perhaps). To what extent is it uniquely determined?
   b) Find such a θ and figure out what geometric meaning it has.

   Recall that in two dimensions Φ and θ are conjugate harmonic functions, namely $\log |z|$ and $\arg z$.

   If you restrict to $n = 3$, then θ will be a one-form which you may identify with a vector field and you can do everything with ordinary vector analysis.

4. Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain. Considering Ω as a body of density one,

a) show that its gravitational field at an interior point, say $0 \in \Omega$, is given by the formula

$$\nabla U^{\Omega}(0) = \text{const.} \int_0^{2\pi} z(\theta) d\theta.$$

Here $\theta \mapsto z(\theta) = r(\theta)e^{i\theta} \in \partial\Omega$ is the parametrization of $\partial\Omega$ by the polar angle θ (which is actually the same as θ in **3)** above).

b) Generalize the above formula to higher dimensions.

5. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $V(\Omega)$ denote the volume of Ω. Show that

$$\sup_{\mathbb{R}^n} |\nabla U^{\Omega}| \leq \text{const.} V(\Omega)^{1/n}$$

In case $n = 2$ this is the Ahlfors-Beurling inequality (Ransford, Lemma 5.3.6), the best constant then being $\frac{1}{2\sqrt{\pi}}$.

3 Suggested exercises from Ransford

1.2.1
2.2.2
2.4.3
2.5.1
2.5.2
2.6.4
2.7.2
3.1.1
3.2.1
3.3.1
4.3.1
4.5.3
4.5.6
5.2.3
5.5.2
5.5.4