POTENTIAL THEORY

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ABSTRACT. We summarize some of the foundations of classic potential theory by studying the Newtonian and Green potentials. After this has been done we look at some recent developments of partial balayage and mother bodies in the last two chapters.

Contents

1. Introduction

This paper was written at the Royal Institute of Technology (Department of Mathematics) under the guidance of Björn Gustafsson. At first, we decided that I should work with mother bodies mainly (a concept due to Björn to a large extent), but before this was possible I had to learn the fundamentals of the subject. It became more and more clear that finding suitable material for this was not easy. There are a lot of good books treating potential theory, but they are usually too old and out of date, too abstract or with different aims than those we were looking for.

So instead we decided that it would be a good idea to first summarize the necessary material, and then look at some recently developed theories.

I would like to take the opportunity to thank Björn for his patient support. He always took time to answer my questions, and as I feel has helped me to grow as a mathematician with his experienced answers and insightful comments on my work. Furthermore, during the last two years I have attended a lot of courses at the department, and have had the pleasure to be inspired and influenced by many competent teachers. So my thanks go out to all of you who have helped me to strengthen my passion for mathematics in one way or another.

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2. Preliminary Material

In this chapter we summarize some material frequently used in the sequel. The purpose is not to introduce the concepts in a complete and rigorous manner, but rather to explain the notation. The most important prerequisites concern measure and integration. As references for this we recommend [B6],[B11],[B13],[B15] and [B17]. Lang's book ([B13]) also contains the basic calculus facts and some distribution theory. We will also frequently need some advanced calculus and distribution theory. For the former any book on the subject will do, or as stated above one can consult [B13]. For the latter we suggest [B2],[B8] or [B16].

2.1. Basic Set-Theoretic Notation.

We freely use symbols like $\in, \forall, \cup, \cap, \subset, \exists, \Rightarrow, \Leftrightarrow$ and so on without explanation. Furthermore we denote the empty set by $\emptyset$. If E is a set we define its characteristic function by

$$\chi_E := \begin{cases} 1 & \text{if} \quad x \in E, \\ 0 & \text{if} \quad x \in E^c. \end{cases}$$

Often we need to study families of sets. This is done by introducing an index set I as usual, and then we look at $\{U_\alpha : \alpha \in I\}$ where U_α is a set for each α in I. If I happens to be the natural numbers we allow ourselves to be rather sloppy with the notation and often just write something like U_n for $n = 1, 2, \ldots$.

Furthermore the extended real number system (with $\pm\infty$) is denoted by $\hat{\mathbb{R}}$. (The real and complex number systems are denoted by $\mathbb{R}$ and $\mathbb{C}$ respectively). Also $\mathbb{R}^N$ will always refer to $N \geq 2$.

2.2. Measures.

Definition 2.1. *Let X be a set. A family Σ of subsets of X is called a σ-algebra on X if:*

1) $\emptyset \in \Sigma$.

2) $A, B \in \Sigma \Rightarrow A \setminus B \in \Sigma$.
3) $A_n \in \Sigma\ \forall\ n=1,2,\ldots \Rightarrow \cup_{n=1}^{+\infty} A_n \in \Sigma$.
4) $X \in \Sigma$.

Definition 2.2. *Let Σ be a σ-algebra on a set X and let $\mu:\Sigma \to [0, +\infty]$ satisfy:*
1) $A_n \in \Sigma\ \forall\ n=1,2,\ldots$ *and* $i \neq j \Rightarrow A_i \cap A_j = \emptyset \Rightarrow \mu(\cup_{n=1}^{\infty} A_n) = \Sigma_{n=1}^{\infty} \mu(A_n)$.
2) $\mu(\emptyset)=0$.
Then we call μ a positive measure (on Σ).
If $\mu:\Sigma \to (-\infty, +\infty]$, or $\mu:\Sigma \to [-\infty, +\infty)$, and μ satisfies 1) and 2) above, then we simply call μ a measure (this is often called a signed measure).

Definition 2.3. *Let Σ be a σ-algebra on X, and μ a measure on Σ. If $E \in \Sigma$ then we define a new measure $\mu_{\lfloor E}$ by: $\mu_{\lfloor E} = \mu(E \cap A)$ for all A in Σ. We call this the restriction of μ to E. (Note that $\{E \cap A : A \in \Sigma\}$ is a σ-algebra on E).*

Theorem 2.4. *If Σ is a σ-algebra on X and μ is a measure on Σ then there exists two sets $A, B \in \Sigma$ such that $\mu_{\lfloor A}(E) \geq 0$ and $\mu_{\lfloor B}(E) \leq 0\ \forall E \in \Sigma$. Furthermore we have $A \cup B = X$, $A \cap B = \emptyset$ and $\mu = \mu_{\lfloor A} + \mu_{\lfloor B}$.*
(Note that although A and B need not be unique , the measures $\mu_{\lfloor A}$ and $\mu_{\lfloor B}$ are!) We can now define $|\mu| = \mu_{\lfloor A} - \mu_{\lfloor B}$.

Definition 2.5. *If X is a topological space we let $\mathcal{B}(X)$ denote the smallest σ-algebra containing all open sets. The sets in $\mathcal{B}(X)$ are called Borel sets. If X is a locally compact Hausdorff space (like $\mathbb{R}^N$ for instance) and if Σ is a σ-algebra containing all Borel sets, then we call a positive measure μ on Σ a positive Radon measure if*
1) $\mu(A) = \inf\{\mu(V) :\ \ V\ \ \text{open},\ \ V \supset A\}\ \forall A \in \Sigma$.
2) $\mu(A) = \sup\{\mu(K) :\ \ K\ \ \text{compact},\ \ K \subset V\}\ \forall A \in \Sigma$.
3) $\mu(K) < +\infty$ *for every compact set K.*
A measure is called a Radon measure if it is the difference of two positive Radon measures.

2.3. **Integration Theory.**

Definition 2.6. *Let Σ be a σ-algebra on X and let μ be a positive measure on Σ. Furthermore let $f : X \to \hat{\mathbb{R}}$. We say that f is (μ-) measurable if the inverse image of every open set in $\hat{\mathbb{R}}$ is in Σ.*
We also say that a property holds (μ-) almost everywhere (a.e.) if the set where it doesn't hold has measure zero. A function

$$f = \sum_{i=1}^{k} \alpha_i \chi_{E_i}$$

where $\alpha_i \in \mathbb{R}$ and $E_i \in \Sigma$ is called a simple function (k is a nonnegative integer). If $\Sigma_{i=1}^{k} |\alpha_i||\mu(E_i)| < \infty$ then the integral of f is defined by

$$\int f d\mu = \sum_{i=1}^{k} \alpha_i \mu(E_i).$$

If f is a general measurable function, then we say that f is integrable if there exists a sequence $\{f_n\}$ of integrable simple functions such that
1) $\forall \varepsilon > 0\ \exists\ N \Rightarrow n,m \geq N \Rightarrow \int |f_n - f_m| d\mu < \varepsilon$
2) $\lim_{n \to 0} f_n(x) = f(x)$ *a.e.*

(Note that we usually denote N-dimensional Lebesgue-measure by $\mathcal{L}^N$ but we write $\int f dx$ instead of $\int f d\mathcal{L}^N$.)

Theorem 2.7. *(Lebesgue's dominated convergence theorem.) If $\{f_n\}$ is a sequence of integrable functions that converges a.e. to a measurable function f, and if we have $|f_n(x)| \le g(x)$ a.e. $\forall$ n where g(x) is integrable, then f is integrable and*

$$\lim_{n\to+\infty} \int |f_n - f| d\mu \to 0.$$

Theorem 2.8. *(Lebesgue's monotone convergence theorem.) Let $\{f_n\}$ be a monotone increasing sequence of integrable functions, and let $f(x) = \lim_{n\to+\infty} f_n(x)$, then*

$$\lim_{n\to+\infty} \int f_n d\mu = \int f d\mu.$$

(If a measurable function g is positive but not integrable, then we define $\int g d\mu = +\infty$).

Theorem 2.9. *(Fatou's lemma.) Let $\{f_n\}$ be a sequence of nonnegative integrable functions, and let $f(x) = \liminf_{n\to+\infty} f_n(x)$, then*

$$\int f d\mu \le \liminf_{n\to+\infty} \int f_n d\mu.$$

(This gives that $\liminf_{n\to+\infty} \int f_n d\mu < +\infty \Rightarrow f$ is integrable.)

(Remark: In the previous theorems we of course work in a fixed space X with a fixed σ-algebra Σ and measure μ.)

Definition 2.10. *Let X and Y be sets, Σ_X a σ-algebra on X and Σ_Y a σ-algebra on Y. Let μ and ν be measures on Σ_X and Σ_Y respectively. We put $Z = X \times Y$ and define Σ_Z to be the smallest σ-algebra containing all sets of the form $A \times B$ where $A \in \Sigma_X$ and $B \in \Sigma_Y$. Now we can define a measure λ called the product measure of μ and ν by $\lambda(A \times B) = \mu(A)\nu(B)$ $\forall A \in \Sigma_X, B \in \Sigma_Y$. (This actually defines λ uniquely although Σ_Z isn't necessarily equal to $\Sigma_X \times \Sigma_Y$).*

Theorem 2.11. *(Fubini's theorem) If we use the notation from above and $h:X \times Y \to \hat{\mathbb{R}}$ is λ-measurable and if either $\int(\int |h| d\mu) d\nu < +\infty$ or $\int(\int |h| d\nu) d\mu < +\infty$ then h is integrable on $X \times Y$ and*

$$\int h d\lambda = \int (\int h d\mu) d\nu = \int (\int h d\nu) d\mu.$$

2.4. **Distribution Theory.**

Definition 2.12. *If α_i are nonnegative integers $\forall$ i=1,...,N , then we call $\alpha = (\alpha_1, ..., \alpha_N)$ a multi-index. We define $D^\alpha := (\frac{\partial}{\partial x_1})^{\alpha_1} ... (\frac{\partial}{\partial x_N})^{\alpha_N}$, $|\alpha| := \alpha_1 + ... + \alpha_N$. Let Ω be an open set in $\mathbb{R}^N$. We say that $\phi \in C^k(\Omega)$ if $\phi : \Omega \to \mathbb{C}$ is continuous and has continuous partial derivatives up to order k in Ω (k is a nonnegative integer). $C^\infty(\Omega) := \cap_{k=0}^\infty C^k(\Omega)$. Now we are in a position to define the space $\mathcal{D}(\Omega)$ consisting of smooth compactly supported test functions. $\mathcal{D}(\Omega) := C_0^\infty(\Omega) := \{\phi : \phi \in C^\infty(\Omega)$, supp(ϕ) is a compact subset of $\Omega\}$. We define a convergence in $\mathcal{D}(\Omega)$ by : if $\phi_n \in \mathcal{D}(\Omega)$ $\forall$ n=1,2,3,.. then we say that $\phi_n \to 0$ in $\mathcal{D}(\Omega)$ as $n \to \infty$ if*

1) all ϕ_n have their supports in a fixed compact set in Ω.

2) for every multi-index α, $D^\alpha \phi_n \to 0$ *uniformly as* $n \to \infty$.
The dual-space $\mathcal{D}'(\Omega)$ *of* $\mathcal{D}(\Omega)$, *consisting of all continuous linear functionals on* $\mathcal{D}(\Omega)$, *is called the space of distributions. In other words:* $\Lambda \in \mathcal{D}'(\Omega) \Leftrightarrow$

1) $\Lambda : \mathcal{D}(\Omega) \to \mathbb{C}$

2) $\Lambda(\alpha\phi_1 + \beta\phi_2) = \alpha\Lambda\phi_1 + \beta\Lambda\phi_2 \; \forall\alpha,\beta \in \mathbb{C} \; \forall\phi_1,\phi_2 \in \mathcal{D}(\Omega)$

3) $\Lambda(\phi_n) \to 0$ *in* $\mathbb{C}$ *if* $\phi_n \to 0$ *in* $\mathcal{D}(\Omega)$.

Definition 2.13. *If* f *is a locally integrable function on* Ω *(with respect to* $\mathcal{L}^N$*), then we identify* f *with the distribution given by*

$$f(\phi) = \int_\Omega \phi(x) f(x) dx.$$

If μ *is a Radon measure , then we identify* μ *with the distribution given by*

$$\mu(\phi) = \int_\Omega \phi d\mu.$$

(Note first that before we can allow ourselves to call this a definition we should prove that this actually gives distributions! Next if we turn to the notation we see that it is unarguably very abusive. But this is almost a necessity to avoid hiding simple ideas in an abundance of letters.)

Definition 2.14. *If* α *is a multi-index and* Λ *is a distribution, then we define its distributional* D^α*-derivative* $D^\alpha\Lambda$ *by*

$$(D^\alpha\Lambda)(\phi) := (-1)^{|\alpha|}\Lambda(D^\alpha\phi)$$

$\phi \in \mathcal{D}(\Omega)$

(This gives a distribution and the reader is invited to verify that when Λ *is a smooth function we get what we ought to.)*

Definition 2.15. *If* $\Lambda \in \mathcal{D}'(\Omega)$, *then we say that* $\Lambda \geq 0$ *if* $\Lambda(\phi) \geq 0$ *whenever* $\phi \in \mathcal{D}(\Omega)$ *and* $\phi \geq 0$.

Lemma 2.16. *A positive distribution is a positive Radon measure in* Ω *(i.e. it has such a representative).*

Lemma 2.17. *(Weyl.) Let* u *be a distribution such that* $\Delta u(\phi) = 0 \; \forall \phi \in \mathcal{D}(\Omega)$ *(where* $\Delta = \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_N^2}$ *). Then* u *has a representative as a* C^2*-function that is harmonic in the ordinary sense (that is* $\Delta u = 0$*).*

2.5. **Calculus.** If $x \in \mathbb{R}^N$ we denote its coordinates such that $x = (x_1, ..., x_N)$ (this has actually already been used). If Ω is a set in $\mathbb{R}^N$ we denote its boundary by $\partial\Omega$. Furthermore we denote surface measure in $\mathbb{R}^N$ by S (that is $(N-1)$-dimensional Hausdorff measure). To make the notation more smooth we usually write $|\Omega|$ instead of $\mathcal{L}^N(\Omega)$ and $|\partial\Omega|$ instead of $S(\partial\Omega)$. $B_r(a) := \{x \in \mathbb{R}^N : |x-a| < r\}$ $(r > 0$, $a \in \mathbb{R}^N)$. Finally we put $\bar{\Omega} :=$ the closure of Ω, $\mathrm{int}(\Omega) =$ interior of Ω .

Definition 2.18. *Let* Ω *be open in* $\mathbb{R}^N$. *Then we say that* $\partial\Omega$ *is* C^m *if* $\overline{\Omega}$ *is a* C^m*-manifold with boundary* $\partial\Omega$. *Furthermore we say that* $u \in C^m(\overline{\Omega})$ *if* $u \in C^m(\Omega)$ *and* u *and all its partial derivatives up to order* m *can be continuously extended to* $\overline{\Omega}$.

Definition 2.19. *If Ω is open in $\mathbb{R}^N$ and $\partial\Omega$ is C^1, then we denote the outward pointing normal of $\partial\Omega$ at $x \in \partial\Omega$ by $N(x)$. Furthermore we define the directional derivative in N's direction by $\frac{\partial u}{\partial N} = N \cdot Du$ if $u \in C^1(\Omega)$ (here $Du = \nabla u = (\frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_N})$, denotes the gradient of u).*

Theorem 2.20. *(Gauss-Green) Suppose that Ω is open and bounded in $\mathbb{R}^N$ and $\partial\Omega$ is C^1. If $u \in C^1(\bar{\Omega})$, then*

$$\int_\Omega \frac{\partial}{\partial x_i} u dx = \int_{\partial\Omega} u \cdot N_i dS$$

(i=1,..,N).

Theorem 2.21. *(Green's formulas) Let $u, v \in C^2(\bar{\Omega})$ and let Ω be as above,then*
1) $\int_\Omega \Delta u dx = \int_{\partial\Omega} \frac{\partial u}{\partial N} dS$
2) $\int_\Omega Du \cdot Dv dx = -\int_\Omega u \Delta v dx + \int_{\partial\Omega} u \frac{\partial v}{\partial N} dS$
3) $\int_\Omega (u\Delta v - v\Delta u) dx = \int_{\partial\Omega} (u \frac{\partial v}{\partial N} - v \frac{\partial u}{\partial N}) dS$

Definition 2.22. *If Ω is open in $\mathbb{R}^N$ and $\varepsilon > 0$, then we put $\Omega_\varepsilon = \{x \in \Omega : d(x, \partial\Omega) > \varepsilon\}$ Furthermore we put*

$$\eta(x) := \begin{cases} C \exp(\frac{1}{|x|^2 - 1}) & \text{if} \quad |x| < 1, \\ 0 \quad \text{if} \quad |x| \geq 1 \end{cases}$$

and $\eta_\varepsilon(x) := \frac{1}{\varepsilon^N} \eta(\frac{x}{\varepsilon})$. The constant C is chosen so that $\int \eta dx = 1$. We note that η_ε is smooth and has its support in $B_\varepsilon(0)$. If f is locally integrable in Ω we put

$$f^\varepsilon(x) = (\eta_\varepsilon * f)(x) = \int_\Omega \eta_\varepsilon(x - y) f(y) dy$$

$\forall x \in \Omega_\varepsilon$.

(f^ε is called the mollification of f in Ω_ε, η_ε is called a mollifier).

Theorem 2.23. *If we use the notation from above then we have:*
1) $f^\varepsilon \in C^\infty(\Omega_\varepsilon)$
2) $f^\varepsilon \to f$ *a.e. as* $\varepsilon \to 0$
3) $f \in C(\Omega) \Rightarrow f^\varepsilon \to f$ *uniformly on compact subsets of* Ω
4) $1 \leq p < \infty$ *and* $f \in L^p_{loc}(\Omega) \Rightarrow f^\varepsilon \to f$ *in* $L^p_{loc}(\Omega)$

2.6. Semicontinuous functions.

Definition 2.24. *Let Ω be an open set in $\mathbb{R}^N$ and $u : \Omega \to \hat{\mathbb{R}}$. We let T_y denote the set of open neighborhoods of $y \in \mathbb{R}^N$. If $x \in \mathbb{R}^N$, define*
1) $\liminf_{y \to x} u(y) = \sup_{U \in T_x} (\inf_{y \in U} u(y))$
2) $\limsup_{y \to x} u(y) = \inf_{U \in T_x} (\sup_{y \in U} u(y))$.

Definition 2.25. *If Ω is open in $\mathbb{R}^N$ and $u : \Omega \to \hat{\mathbb{R}}$, then u is said to be*
1) lower semicontinuous (l.s.c.) at x if $u(\Omega) \subset (-\infty, \infty]$ and
$u(x) = \liminf_{y \to x} u(y)$.
2) upper semicontinuous (u.s.c.) at x if $u(\Omega) \subset [-\infty, \infty)$ and
$u(x) = \limsup_{y \to x} u(y)$.

Theorem 2.26. *If Ω is open in $\mathbb{R}^N$ and $u : \Omega \to (-\infty, \infty]$, then the following five properties are equivalent:*
1) u is lower semicontinuous (at every point of Ω).
2) $\{x \in \Omega : u(x) > \alpha\}$ is open for each $\alpha \in \mathbb{R}$.

3) $\{x \in \Omega : u(x) \leq \alpha\}$ *is closed for each* $\alpha \in \mathbb{R}$.

4) $u(x) \leq \liminf_{x_n \to x} u(x_n)$

5) For every compact subset $K \subset \Omega$ *there is a sequence* $u_n \in C(K)$ *such that* $u_n \nearrow u$ *on* K *(by this we mean that* u_n *increasingly converges to* u*).*

3. Classic Potential Theory

We now start our investigation of potential theory. For the classical parts in section 3-10 we suggest [B3],[B4],[B7],[B9],[B12] and [B14] as reference sources. Potential theory has a close connection to the theory of analytic functions. Kellogg's book, ([B9]), deals with this in an elementary way. As a standard reference for the theory of analytic functions we recommend however Conway's book ([B1]). Finally, as will be seen soon, classical potential theory is related with the Laplace operator, and this operator is treated in most books in partial differential equations. The books [B5] and [B19] are good sources on this.

3.1. Introduction. Potential theory has its origin in elementary physics in problems concerning for instance gravitational and electrical forces. Let's study the case of two point masses in space . From Newton's law we know that the force between the two particles is given by $F = C\frac{m_1 m_2}{r^2}$ where m_1,m_2 are the masses in appropriate units and r is the distance between the particles. The constant C, on the other hand, depends solely on the units chosen, so for now let's just put $C = 1$. If we change say m_2 for a continuously distributed body Ω we get the force on m_1 by

$$-m_1 \int_\Omega \frac{\rho(y)(x-y)}{|x-y|^3} dy$$

where $\rho(y)$ is the density at y of the given body. Now $\nabla_x(\frac{1}{|x-y|}) = -\frac{(x-y)}{|x-y|^3}$ so that $F = -m_1 \int \rho(y) \nabla_x (\frac{1}{|x-y|}) dy = m_1 \nabla_x \phi(x)$ where $\phi(x) = \int \frac{\rho(y)}{|x-y|} dy$. The function $\phi(x)$ is called the potential of the mass-distribution ρ. On the other hand if we know the function ϕ and it is smooth enough we can get ρ back from $-\Delta\phi = 4\pi\rho$ (Δ = Laplacian = $\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ in $\mathbb{R}^3$).

If we pause a minute to think about the density ρ we might see that there is something artificial about it. Of course, $\rho(y)$ is defined as the limit of the ratio between the mass of a piece of Ω containing y and the volume of that piece as the volume goes to zero. So it is actually more natural to start with a measure μ that is to be thought of as measuring the weight of a given part of Ω. (In our case we have $\mu(A) = \int_A \rho(y)dy$).

Now if we put $\Phi(x) = \frac{1}{|x|}$ we get $\phi = \Phi * \rho = \Phi * \mu$. (Remember the note about the abuse of notation described in the preliminaries).

Φ is called the Newtonian kernel in $\mathbb{R}^3$ (well at least if we overlook a missing constant).

3.2. Generalization to $\mathbb{R}^N$. We now turn our attention to $\mathbb{R}^N$ ($N \geq 2$) and define.

$$\Phi(x) = -\frac{1}{2\pi} \log|x| \qquad (N = 2)$$

$$\Phi(x) = \frac{C_N}{|x|^{N-2}} \qquad (N \geq 3), \qquad C_N := \frac{1}{|\partial B_1(0)|(N-2)}$$

Then $-\Delta\Phi = \delta_0$ (δ_0 = Dirac-measure at the origin). If μ is a Radon measure we define its potential

$$U^\mu(x) = \int \Phi(x-y)d\mu(y) = (\Phi * \mu)(x)$$

if this is well-defined.

Basically, potential theory is the subject of studying the properties of U^μ, related topics and its generalizations. In this section we develop some of the most basic facts about potentials.

Another thing worth mentioning is that Φ is called the Newtonian kernel, and one way to generalize the theory is to study other kernels.

Lemma 3.1. *If μ is a Radon measure with compact support then U^μ and ∇U^μ are defined a.e. and are in L^1_{loc}. (If μ is positive then U^μ is defined everywhere taking values in $(-\infty,\infty]$).*

Proof. (for U^μ, the proof for ∇U^μ is similar). Assume first that μ is positive and note that $\Phi \in L^1_{loc}(\mathbb{R}^N)$.

$$\int_{B_r(0)} |U^\mu(x)|dx = \int_{B_r(0)} (|\int_{\mathbb{R}^N} \Phi(x-y)d\mu(y)|)dx \leq$$

$$\int_{B_r(0)}\int_{\mathbb{R}^N} |\Phi(x-y)|d\mu(y)dx = \int_{\mathbb{R}^N}\int_{B_r(0)} |\Phi(x-y)|dxd\mu(y) < \infty.$$

If μ is signed then the lemma now easily follows by dividing it into positive and negative part. □

Theorem 3.2. *Suppose that μ is a Radon measure with compact support. Then $-\Delta U^\mu = \mu$ (in the distribution sense).*

Proof. First note that if $\phi(x) = f(|x|)$, and f is smooth, then $\Delta\phi(x) = f''(r) + \frac{N-1}{r}f'(r)$ where $r = |x|$. Now apply Green's formula :

$$-\int U^\mu \Delta\phi dx = -\int\int \Phi(x-y)\Delta\phi(x)dxd\mu(y) =$$

$$-\int\int \Phi(z)\Delta\phi(y+z)dzd\mu(y) = -\int\int_{|z|<\epsilon} \Phi(z)\Delta\phi(y+z)dzd\mu(y) -$$

$$\int\int_{|z|\geq\epsilon} \Phi(z)\Delta\phi(y+z)dzd\mu(y) =$$

$$-\int\int_{|z|<\epsilon} \Phi(z)\Delta\phi(y+z)dzd\mu(y) - \int d\mu(y)[\int_{|z|>\epsilon} \phi(y+z)\Delta\Phi(z)dz$$

$$+\int_{|z|=\epsilon} \Phi(z)\frac{\partial\phi}{\partial N}(y+z)dS(z) - \int_{|z|=\epsilon} \phi(y+z)\frac{\partial\Phi(z)}{\partial N}dS(z)]$$

We consider each term separately.

1). That $-\int\int_{|z|<\epsilon} \Phi(z)\Delta\phi(y+z)dzd\mu(y) \to 0$ as $\epsilon \to 0$ since $\Phi \in L^1_l oc$.

2). $\int_{|z|>\epsilon} \phi(y+z)\Delta\Phi(z)dz = 0$ since $\Delta\Phi(z) = 0$ here.

3). $|\int_{|z|=\epsilon} \Phi(z)\frac{\partial\phi}{\partial N}(y+z)dS(z)| \leq \sup_{|z|=\epsilon}|\nabla\phi|\Phi(\epsilon)\epsilon^{N-1}|\partial B_1(0)| \to 0$ as $\epsilon \to 0$.

4).$\int_{|z|=\epsilon} \phi(y+z)\frac{\partial\Phi(z)}{\partial N}dS(z) = -\frac{d\Phi(r)}{dr}\lfloor_{r=\epsilon}\int_{|z|=\epsilon} \phi(y+z)dS(z) \to \phi(y)$ as $\epsilon \to 0$ (Note that $-\frac{d\Phi(r)}{dr}\lfloor_{r=\epsilon} = \frac{1}{|\partial B_\epsilon(0)|}$).

Summing up we get the desired conclusion : $-\int U^\mu \Delta\phi dx = \int \phi d\mu$ □

(We get the statement $-\Delta\Phi = \delta_0$ as a special case of theorem 3.2.)

Corollary 3.3. *If μ is a Radon measure with compact support then U^μ is harmonic outside supp(μ).*

Proof. Follows easily from Weyl's lemma. □

Theorem 3.4. *Let μ be a Radon measure with compact support. Then*
1) $|U^\mu(x)| = O(|x|^{2-N}) \to 0$ as $|x| \to \infty$ if $N \geq 3$.
2) $U^\mu(x) = -\frac{1}{2\pi}\log|x| \int d\mu + O(|x|^{-1})$ as $|x| \to \infty$ if N=2.

Proof. ($N = 2$. The proof for $N \geq 3$ is similar) $U^\mu(x) = -\frac{1}{2\pi}\int \log|x-y|d\mu(y) =$
$-\frac{1}{2\pi}\int \log(|x||e_x - \frac{y}{|x|}|)d\mu(y) =$
$-\frac{1}{2\pi}\int \log|x|d\mu(y) - \frac{1}{2\pi}\int \log|e_x - \frac{y}{|x|}|d\mu(y) =$
$-\frac{1}{2\pi}\log|x|\int d\mu + O(\frac{1}{|x|})$. □

If u is a function such that $-\Delta u = \mu$, where μ is a Radon measure with compact support, then we don't necessarily have u equal to U^μ since we always can add any harmonic function to u.

Theorem 3.5. *Suppose that u is a function such that $-\Delta u = \mu$ for some Radon measure μ with compact support. Furthermore assume that*
1) $u(x) \to 0$ as $|x| \to \infty$ ($N \geq 3$) or
2) $u(x) = -\frac{1}{2\pi}\log|x|\int d\mu + O(\frac{1}{|x|})$ as $|x| \to \infty$ ($N = 2$).
Then $u = U^\mu$

Proof. Put $v = u - U^\mu$. Then $\Delta v = 0$ in $\mathbb{R}^N$ and $v(x) \to 0$ as $|x| \to \infty$. (Note that v is a function by Weyl's lemma). If we now apply Liouville's theorem (that will be proven shortly) we see that v is equal to zero everywhere. □

These last results are very useful, since they give a complete description of what it means for a function to be a potential. That is, if u is the potential of a measure μ, then $-\Delta u = \mu$, and if $-\Delta u = \mu$, then $u = U^\mu$ if and only if it behaves the right way at infinity.

4. Harmonic, sub- and superharmonic functions. Part 1.

In the previous section we saw that potentials are harmonic in free space, and this is certainly motivation enough to study harmonic functions. We will also introduce sub- and superharmonic functions, and the interest in these classes is due to the fact that if μ is a positive Radon measure, then U^μ is superharmonic everywhere (or identically infinite). This is only true however in a more general sense that is introduced in chapter 7. (If μ is a "negative" Radon measure, then U^μ is subharmonic. In fact, due to that many authors have reversed sign on Φ, they have the opposite in their texts).

Definition 4.1. *A function $u \in C^2(\Omega)$ in a domain (open connected set) $\Omega \subset \mathbb{R}^N$ is called*
1) harmonic if $\Delta u = 0$ in Ω.
2) subharmonic if $-\Delta u \leq 0$ in Ω.
3) superharmonic if $-\Delta u \geq 0$ in Ω.
(Note that if N=1 we have : u harmonic $\Leftrightarrow$ u linear, u subharmonic $\Leftrightarrow$ u convex, u superharmonic $\Leftrightarrow$ u concave).

Theorem 4.2. *(Mean-Value Properties) Suppose that u is a superharmonic function in $\Omega \subset \mathbb{R}^N$ and let $\overline{B_R(a)} \subset \Omega$. Then*

1) $u(a) \geq \frac{1}{|B_R(a)|} \int_{B_R(a)} udx$.

2) $u(a) \geq \frac{1}{|\partial B_R(a)|} \int_{\partial B_R(a)} udS$.

Proof. We can without loss of generality assume that $a = 0$.
Put

$$F(r) = \frac{1}{|\partial B_r(0)|} \int_{\partial B_r(0)} udS$$

$= =$ { change to polar coordinates} $=$

$$\frac{1}{|\partial B_r(0)|} \int_{\partial B_1(0)} u(r,\theta) r^{N-1} dS(\theta) = \frac{1}{|\partial B_1(0)|} \int_{\partial B_1(0)} u(r,\theta) dS(\theta)$$

$$\Rightarrow F'(r) = \frac{1}{|\partial B_1(0)|} \int_{\partial B_1(0)} \frac{\partial u}{\partial r}(r,\theta) dS(\theta) =$$

$=$ { change back to usual coordinates} $=$

$$= \frac{1}{|\partial B_r(0)|} \int_{\partial B_r(0)} \frac{\partial u}{\partial r} dS.$$

We also have by Green's formula that

$$\int_{\partial B_r(0)} \frac{\partial u}{\partial r} dS = \int_{B_r(0)} \Delta u dx \leq 0$$

for every r since $\Delta u \leq 0$. So we get (with $F(0) := \lim_{r \to 0} F(r)$) $F(R) - F(0) = F(R) - u(0) = \int_0^R F'(r)dr \leq 0 \Rightarrow F(R) \leq u(0)$.

Hence the proof of 2) is finished. To prove 1) we integrate 2) :

$$\int_{B_R(0)} udx = \int_0^R \int_{\partial B_r(0)} udSdr \leq$$
$$\int_0^R |\partial B_r(0)| u(0) dr = |B_R(0)| u(0).$$

□

Corollary 4.3. *If u is harmonic then the mean-value formulas in the previous theorem hold with equality.*

(Note that u superharmonic ⇔ −u subharmonic).

Theorem 4.4. *(Liouville.) If u is harmonic in $\mathbb{R}^N$ and $|u| \leq C$ for some constant C then u is constant.*

Proof. Fix $a \in \mathbb{R}^N$ and take $r > 0$. Then
$|u(a) - u(0)| = \frac{1}{|B_r(0)|} |\int_{B_r(a)} udx - \int_{B_r(0)} udx| \leq$
$\frac{1}{|B_r(0)|} \int_{B_r(a) \triangle B_r(0)} |u| dx \to 0$ as $r \to \infty$. □

Theorem 4.5. *(Maximum principle) If u is subharmonic in a domain $\Omega \subset \mathbb{R}^N$ and u has a maximum somewhere in Ω, then u is constant.*

Proof. If $a \in \Omega$ is a point of maximum then we have for, $x \in B_r(a) \subset \Omega$
$u(x) \leq u(a) \leq \frac{1}{|B_r(a)|} \int_{B_r(a)} udx \Rightarrow u(x) = u(a) \quad \forall x \in B_r(a)$. Thus $\{x : u(x) = u(a)\}$ is open in Ω , but also closed since u is continuous. We conclude that
$\{x : u(x) = u(a)\} = \Omega$. □

(Note. 1. The Laplacian Δ is invariant under rotations,i.e. if A is an orthogonal matrix, then we have with $v(x) = u(Ax)$ that $\Delta u = 0 \Rightarrow \Delta v = 0$.

2. If u and v are harmonic, then uv is harmonic if and only if $\nabla u \cdot \nabla v = 0$.)

Example: (Harnack's Inequality) Let Ω be a domain in $\mathbb{R}^N$, and let V be an open connected set such that $\overline{V}$ is a compact subset of Ω. Then there is a positive constant C (depending only on V) such that $\sup_V u \leq C \inf_V u$ for all nonnegative harmonic functions in Ω. (To prove this, put $r = \frac{1}{4}d(V, \partial\Omega)$, and fix $x, y \in V$such that $|x - y| < r$. Now use the mean-value property in $B_{2r}(x)$ to deduce that $u(x) \geq \frac{1}{2^N}u(y)$. Then cover $\overline{V}$ by finitely many $r-$balls $\{B_i\}_{i=1}^k$ with $B_i \cap B_{i-1} \neq \emptyset$. Now observe that this means that $u(x) \geq \frac{1}{2^{Nk}}u(y)\ \forall x, y \in V$).

5. Dirichlet's Problem

In this section we are going to study Dirichlet's problem, especially in a ball. So given $f \in C(\partial B_r(a))$ we want to find $u \in C(\overline{B_r(a)}) \cap C^2(B_r(a))$ such that,

1) $\Delta u = 0$ in $B_r(a)$.

2) $u = f$ on $\partial B_r(a)$.

This is Dirichlet's problem in a ball.

Theorem 5.1. *To Dirichlet's problem in $B_r(a)$ there exists a unique solution:*

$$u(x) = \int_{\partial B_r(a)} P(x,y) f(y) dS(y) \quad P(x,y) = \frac{r^{N-2}}{|\partial B_r(a)|} \frac{r^2 - |x-a|^2}{|y-x|^N}$$

(P is the Poisson kernel for $B_r(a)$).

The proof of this will not be given directly since a more general result will be proved as lemma 8.1. We remark however that the proof of this lemma is elementary and could have been given here directly.

Now we will define the concept of Green's function for a domain, and then we will use this to prove some theorems about harmonic functions.

Definition 5.2. *If Ω is a domain in $\mathbb{R}^N$ we say that Ω has a Green's function $G : \Omega \times \Omega \to \hat{\mathbb{R}}$, $x, y \in \Omega$, if there exists such a G that fulfills:*

1) $G(x,y) = \Phi(x-y) + h(x,y)$ where $h(x,y)$ is harmonic in x for each fixed y.

2) $\lim_{x_n \to x} G(x_n, y) = 0$ when $x \in \partial\Omega$, $y \in \Omega$ ($x_n \in \Omega$).

(So we can define $G(x,y) = 0$ if $x \in \partial\Omega$ and $y \in \Omega$).

Theorem 5.3. *Suppose that Green's function exists for Ω, $\partial\Omega$ is C^1 and $u \in C(\overline{\Omega})$ is harmonic in Ω. Then*

$$u(y) = -\int_{\partial\Omega} u(x) \frac{\partial G(x,y)}{\partial N_x} dS(x)$$

Proof. Given $y \in \Omega$ and $\epsilon > 0$ such that $B_\epsilon(y) \subset \Omega$ apply Green's formula in $A = \Omega \setminus B_\epsilon(y) \Rightarrow$

$$0 = \int_A u(x) \Delta_x G(x,y) dx - \int_A G(x,y) \Delta_x u(x) dx = \int_{\partial\Omega} u(x) \frac{\partial G(x,y)}{\partial N_x} dS(x) -$$

$$\int_{\partial\Omega} G(x,y) \frac{\partial u(x)}{\partial N_x} dS(x) + \int_{\partial B_\epsilon(y)} u(x) \frac{\partial G(x,y)}{\partial N_x} dS(x) - \int_{\partial B_\epsilon(y)} G(x,y) \frac{\partial u(x)}{\partial N_x} dS(y)$$

We have

$$\int_{\partial\Omega} G(x,y) \frac{\partial u(x)}{\partial N_x} dS(x) = 0$$

$$\int_{\partial B_\epsilon(y)} u(x)\frac{\partial G(x,y)}{\partial N_x}dS(x)$$
$$= -\int_{\partial B_\epsilon(y)} u(x)\frac{\partial \Phi}{\partial r}(x-y)dS(x) + \int_{\partial B_\epsilon(y)} u(x)h(x,y)dS(x)$$
$$\to u(y) \quad \text{as} \quad \epsilon \to 0,$$

since $\{\frac{\partial \Phi}{\partial r}(x-y) = -\frac{C_N(N-2)}{|x-y|^{N-1}}\}$.

$$\int_{\partial B_\epsilon(y)} G(x,y)\frac{\partial u(x)}{\partial N_x}dS(x) \to 0 \quad as \quad \epsilon \to 0$$

Summing up we get $0 = \int_{\partial\Omega} u(x)\frac{\partial G(x,y)}{\partial N_x}dS(x) + u(y)$. □

What we have here is that if Green's function exists for Ω, and $\partial\Omega$ is regular enough, then a harmonic function u in Ω is completely determined by its values on $\partial\Omega$, provided u is continuous up to $\partial\Omega$. This is especially true if u is harmonic in a neighborhood of $\overline{\Omega}$.

Theorem 5.4. *For the ball $B_r(a)$ Green's function exists and is given by:*
1) If $N = 2$:
a) $G(x,y) = -\frac{1}{2\pi}\log\frac{r|x-y|}{|y-a||x-y^*|}$ for $y \in B_r(a)\setminus\{a\}$, $x \in B_r(a)\setminus\{a,y\}$
b) $G(x,y) = -\frac{1}{2\pi}\log\frac{|x-y|}{r}$ for $y = a$, $x \in B_r(a)\setminus\{y\}$
c) $G(x,y) = +\infty$ for $x = y$
2) If $N \geq 3$:
a) $G(x,y) = C_N(|x-y|^{2-N} - \frac{|y-a|^{2-N}}{r^{2-N}}|x-y^*|^{2-N})$ for $y \in B_r(a)\setminus\{a\}$, $x \in B_r(a)\setminus\{a,y\}$
b) $G(x,y) = C_N(|x-y|^{2-N} - r^{2-N})$ for $y = a$, $x \in B_r(a)\setminus\{y\}$
c) $G(x,y) = +\infty$ for $x = y$
where $y^* = a + \frac{y-a}{|y-a|^2}$.

Proof. ($N \geq 3$, the proof for $N = 2$ is similar).
First note that $-C_N\frac{|y-a|^{2-N}}{r^{2-N}}|x-y^*|^{2-N}$ ($y \neq a$, y fixed) and $-C_N r^{2-N}$ are harmonic. Also note that $G(x,y) = G(y,x)$ if $x = y$ or $y = a$.
We may without loss assume that $a = 0$ and $r = 1$ (just study $t := \frac{x-a}{r}$ and $s := \frac{y-a}{r}$ instead). Then $G(x,y) = C_N(|x-y|^{2-N} - |y|^{2-N}|x-y^*|^{2-N})$ (if $y \neq 0$, $\{x\}\cap\{a,y\} = \emptyset$). We only need to prove that $|y|^{2-N}|x-y^*|^{2-N} = |x|^{2-N}|y-x^*|^{2-N}$ to see that $G(x,y) = G(y,x)$. We have $|y|^{2-N}|x-y^*|^{2-N} = ||y|x - \frac{y}{|y|}|^{2-N} = ||y||x|e_x - \frac{|y|e_y}{|y|}|^{2-N} = ||x|y - \frac{x}{|x|}|^{2-N} = |x|^{2-N}|y-x^*|^{2-N}$. Since $C_N(|x-y|^{2-N} - |y-a|^{2-N}|\frac{x-a|}{r} - r(y-a)^*|^{2-N})$ has a continuous extension to $\partial B_r(a)$ when y is fixed inside $B_r(a)$ the theorem follows. (Note that $C_N(|x-y|^{2-N} - \frac{|y-a|^{2-N}}{r^{2-N}}|x-y^*|^{2-N}) = 0$ when $y \in \partial B_r(a)$). □

We proved that $G(x,y) = G(y,x)$, if G is the Green's function for a ball. This is actually true in general. To see this (if we assume that $\partial\Omega$ is smooth enough), we use Green's formula:

$$0 = \int_{\partial\Omega}[G(x,y)\frac{\partial G}{\partial N_x}(x,z) - G(x,z)\frac{\partial G}{\partial N_x}(x,y)]dS(x) =$$
$$\int_\Omega [G(x,y)\Delta_x G(x,z) - G(x,z)\Delta_x G(x,y)]dx = -G(z,y) + G(y,z)$$

(We are a bit careless in the sense that we cannot apply Green's formulas "near" $x = y$ and $z = y$. So to be rigorous we should look at $A = \Omega \setminus (B_\epsilon(y) \cup B_\epsilon(z))$ and take $\epsilon \to 0$ as in earlier proofs. But this is left to the reader).

We can also calculate $\frac{\partial G(x,y)}{\partial N_y}$ if $y \in \partial B_r(a)$ (where G is the Green's function for $B_r(a)$).

$$\frac{\partial G(x,y)}{\partial N_y} = -\frac{r^{N-2}}{|\partial B_r(a)|}\frac{r^2 - |x-a|^2}{|y-x|^N}.$$

(Incidentally this proves theorem 5.1, since $P(x,y) = -\frac{\partial G(x,y)}{\partial N_y}$). If we accept that this function is real analytic as a function of x on $\partial B_r(a)$ we realize that it is possible to expand any harmonic function into a power series. That is, if u is harmonic it is automatically real analytic (see for instance [B4]). But instead of going through the details of this, let's just prove that if u is harmonic it is automatically smooth.

Theorem 5.5. *If u is continuous in an open set $\Omega \subset \mathbb{R}^N$, and u satisfies the mean-value property*

$$u(a) = \frac{1}{|\partial B_r(a)|}\int_{\partial B_r(a)} u(x)dS$$

for each ball $B_r(a) \subset \Omega$, then $u \in C^\infty(\Omega)$.

Proof. We use the results on mollifiers from the preliminaries.

$$u^\varepsilon(a) = \int_\Omega \eta_\varepsilon(a-y)u(y)dy = \frac{1}{\varepsilon^N}\int_{B_\varepsilon(0)} \eta(\frac{|a-y|}{\varepsilon})u(y)dy =$$

$$\frac{1}{\varepsilon^N}\int_0^\varepsilon \eta(\frac{r}{\varepsilon})(\int_{\partial B_\varepsilon(a)} udS)dr =$$

$$\frac{1}{\varepsilon^N}u(a)\int_0^\varepsilon \eta(\frac{r}{\varepsilon})|\partial B_r(a)|dr = u(a)$$

So in Ω_ε we have $u(a) = u^\varepsilon(a)$, and we know that u^ε is smooth in Ω_ε. Hence u is smooth. □

Theorem 5.6. *(Harmonic Continuation). If u is harmonic in a domain Ω, and if u vanishes in a domain $T \subset \Omega$ then $u \equiv 0$ in Ω.*

Proof. Put $S = int\{x : u(x) = 0\}$. Then S is open and nonempty by assumption. But in S, u and all partial derivatives of u vanish. Hence, since u is real analytic, S is also closed in Ω. We get $S = \Omega$. □

(Recall from the theory of analytic functions of one complex variable that we can let T be any set in Ω with a limit point in the theorem above (with harmonic function replaced by analytic function). But this is not enough in theorem 5.5 (even when $N = 2$)).

Green's function for a half-space:
Let $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N : x = (x_1, ..., x_N), x_N > 0\}$ and put $\grave{x} = (x_1, ..., x_{N-1}, -x_N)$. If we define $G(x,y) = \Phi(x-y) - \Phi(\grave{x}-y)$, then G is the Green's function for $\mathbb{R}^N_+$.

6. Harmonic, sub- and superharmonic functions. Part 2.

Now we define the more general notion of sub- and superharmonic functions that is suitable for potential theory. The need for a generalization is easy to understand, it is only in rare cases we can find differentiable functions satisfying everything we want. But we will see that the definition has a striking similarity to the original one, if we understand it in the distribution sense.

Definition 6.1. *Let Ω be a domain in $\mathbb{R}^N$, $u : \Omega \to (-\infty, \infty]$ is said to be superharmonic in Ω if*

1) u is lower semicontinuous.

2) For each compact set K in Ω and for each $h \in C(K)$ such that h is harmonic in $int(K)$ we have $u \geq h$ on $\partial K \Rightarrow u \geq h$ in K.

3) $u \not\equiv +\infty$.

u is said to be subharmonic if $-u$ is superharmonic.

Theorem 6.2. *If $u : \Omega \to (-\infty, \infty]$ is lower semicontinuous and $\not\equiv +\infty$ then* 2) *in the definition above is equivalent to each of the following:*

2′) *$\forall \overline{B_r(a)} \subset \Omega$ we have $u(a) \geq \frac{1}{|\partial B_r(a)|} \int_{\partial B_r(a)} u dS$.*

2″) *$\forall \overline{B_r(a)} \subset \Omega$ we have $u(a) \geq \frac{1}{|B_r(a)|} \int_{B_r(a)} u dx$).*

Proof. 2) $\Rightarrow$ 2′): u is lower semicontinuous $\Rightarrow \exists f_n \in C(B_r(a))$ such that $f_n \nearrow u$ ($B_r(a) \subset \Omega$). Let $P[f_n]$ be the Poisson integral of f_n. Then 2) gives that $u \geq P[f_n]$ in $B_r(a)$ since this is true on $\partial B_r(a)$. So $u(a) \geq P[f_n] = \frac{1}{|\partial B_r(a)|} \int_{\partial B_r(a)} f_n dS \to \frac{1}{|\partial B_r(a)|} \int_{\partial B_r(a)} u dS$.

2′) $\Rightarrow$ 2″): follows from integration as earlier.

2″) $\Rightarrow$ 2): Let K be a compact subset of Ω and suppose that $u \geq h$ on ∂K. Now put $v = u - h$. Since v is lower semicontinuous in K, v takes on its minimum somewhere in K, say at a. If $a \in \partial K \Rightarrow v \geq 0$ and we are done. If $a \in int(K)$ then we have for $0 < r \leq d(a, \partial K)$ that $v(a) \geq \frac{1}{|B_r(a)|} \int_{B_r(a)} v dx \geq v(a)$, hence $v = v(a)$ in $B_r(a)$. If we choose r large enough we have $\partial K \cap \partial B_r(a) \neq \emptyset$, and then we see for $x \in \partial K \cap \partial B_r(a)$ that $v(a) = v(x) \geq 0$. □

Theorem 6.3. *If μ is a positive Radon measure with compact support then $U^\mu(x)$ is harmonic outside $supp(\mu)$ and superharmonic everywhere.*

Proof. That $U^\mu : \Omega \to (-\infty, \infty]$ is obvious.

1) $x_n \to x \Rightarrow \Phi(x_n - y) \to \Phi(x - y)$ pointwise. Hence, by Fatou's lemma, $U^\mu(x) \leq \liminf_{n \to \infty} U^\mu(x_n)$, so U^μ is lower semicontinuous.

2) follows from Fubini's theorem and:

$$\frac{1}{|\partial B_r(a)|} \int_{\partial B_r(a)} \Phi(x - y) dS(x) =$$

$\{G$ is Green's function for $B_r(a)\}$ =

$$-\int_{\partial B_r(a)} \Phi(x - y) \frac{\partial G(x, a)}{\partial N_x} dS(x) =$$

$$-\int_{\partial B_r(a)} G(x, a) \frac{\partial \Phi}{\partial N_x}(x - y) dS(x)$$

$$+ \int_{B_r(a)} [G(x, a) \Delta_x \Phi(x - y) - \Phi(x - y) \Delta_x G(x, a)] dx \leq \Phi(a - y)$$

3) is obvious.
So now we see that U^μ is superharmonic everywhere, but that it is harmonic outside $supp(\mu)$ is merely a matter of differentiating under the integral sign. □

Theorem 6.4. *(Minimum Principle) Suppose that u is superharmonic in a domain Ω and that there is an $a \in \Omega$ such that $u(a) \leq u$ in Ω. Then $u(a) \equiv u$ in Ω.*

Proof. $u(a) \geq \frac{1}{|B_r(a)|}\int_{B_r(a)} udS \geq u(a) \Rightarrow u = u(a)$ a.e. in $B_r(a)$. But u is lower semicontinuous, hence $u \equiv u(a)$ in $B_r(a)$. So the set where $u = u(a)$ is both open and closed in Ω showing that $u = u(a)$ in all Ω. □

From the following theorem we can see the connection between the original C^2–definition of superharmonic functions, and the definition given in this chapter. Incidentally, it also provides a good example to why distributions are so nice to work with.That is, they give us exactly what we want.

Theorem 6.5. *(Riesz's structure theorem for superharmonic distributions). Let u be a distribution in $\Omega \subset \mathbb{R}^N$. Then the following are equivalent:*

1) $-\Delta u \geq 0$.

2) $\exists u \in L^1_{loc}(\Omega)$ that represents u as a distribution, and $\forall \overline{B_r(a)} \subset \Omega$ there is a harmonic function h in $B_r(a)$ and a positive Radon measure μ ($\mu = -\Delta u_{\lfloor B_r(a)}$) such that $u = h + U^\mu$ in $B_r(a)$.

3) $\exists$ a superharmonic function u that represents u as a distribution.

Proof. 1) $\Rightarrow$ 2): Put $\mu = -\Delta u$ in $B_r(a)$ (this is a measure since a positive distribution is always a measure). Furthermore $\Delta(u - U^\mu) = 0 \Rightarrow$ we can apply Weyl's lemma and see that 2) follows from 1).

2) $\Rightarrow$ 3): We know that U^μ is superharmonic and h is obviously also superharmonic, and we get 3) immediately from 2).

3) $\Rightarrow$ 1): We use property 2″) for superharmonic functions and put $H_r(x) = \frac{1}{|B_r(0)|}\chi_{B_r(0)}(x)$. Then $(H_r * u)(x) = \frac{1}{|B_r(0)|}\int_{B_r(0)} u(x-y)dy = \frac{1}{|B_r(x)|}\int_{B_r(x)} udy$. So we see that $u(x) \geq (H_r * u)(x)$ in Ω when $d(x, \partial\Omega) > r$. Since we know that u^ε (mollification of u) converges to u in the distribution sense it is enough to prove that $-\Delta u^\varepsilon \geq 0 \quad \forall \varepsilon > 0$. So let's assume the opposite : $\exists$ a domain $A \subset \Omega$ and $\varepsilon > 0$ such that $-\Delta u^\varepsilon < 0$ in A. $\Rightarrow u^\varepsilon < H_r * u^\varepsilon$ in some subdomain of A. But we have already seen that $u \geq H_r * u \Rightarrow \eta_\varepsilon * u \geq \eta_\varepsilon * (H_r * u) = H_r * (\eta_\varepsilon * u) \Rightarrow -\Delta u^\varepsilon \geq 0 \quad \forall \varepsilon > 0 \Rightarrow -\Delta u \geq 0$. □

Suppose $u \geq 0$ in all of $\mathbb{R}^N$. If u is harmonic then by Liouville's theorem (and depending on interpretation perhaps also Weyl's lemma) u is necessarily constant. The same is not true in general if u is only supposed to be either subharmonic or superharmonic.

7. Sequences of superharmonic functions

Although the theorems on sequences of superharmonic functions are very neat in themselves (after all, analysis is the art of taking limits), we are mainly interested in them because they give us some powerful results in later chapters, as we shall see.

We also prove a continuity principle and Frostman's maximum principle here, even though these theorems are not chiefly concerned with sequences of superharmonic functions.

Theorem 7.1. *Let $\{u_n\}$ be a sequence of superharmonic functions in a domain $\Omega \subset \mathbb{R}^N$, and suppose that $u_n \nearrow u$ pointwise. Then either u is superharmonic or $u \equiv +\infty$.*

Proof. It is obvious that $u : \Omega \to (-\infty, \infty]$ and that u is lower semicontinuous. Finally we apply Lebesgue's monotone convergence theorem and obtain:

$$u(a) \geq u_n(a) \geq \frac{1}{|\partial B_r(a)|} \int_{\partial B_r(a)} u_n dS \to \frac{1}{|\partial B_r(a)|} \int_{\partial B_r(a)} u dS$$

□

Theorem 7.2. *Let $\{u_\alpha\}$ be an arbitrary family of superharmonic functions that are locally uniformly bounded from below. Then $u = \inf_\alpha u_\alpha$ is superharmonic if u is lower semicontinuous.*

Proof. For every α we have:

$$u_\alpha \geq \frac{1}{|\partial B_r(a)|} \int_{\partial B_r(a)} u_\alpha dS \geq \frac{1}{|\partial B_r(a)|} \int_{\partial B_r(a)} u dS$$

□

Given any function $u : \Omega \to \hat{\mathbb{R}}$, then there is a largest lower semicontinuous minorant to u :

$$\breve{u}(x) = \sup\{\phi(x) : \phi \text{ is l.s.c. and } \quad \phi \leq u\} = \liminf_{x_n \to x} u(x_n)$$

(It is trivial to see that sup over any family of l.s.c. functions is l.s.c.).

Theorem 7.3. *(Fundamental Convergence.) Let $\{u_\alpha : \alpha \in I\}$ be a family of superharmonic functions in Ω that are locally uniformly bounded from below. Define $u = \inf_\alpha u_\alpha$ and $\breve{u}$ as above. Then we have:*
1) $\breve{u} = u$ Lebesgue-a.e.
2) $\breve{u}$ is superharmonic in Ω.

Proof. We have $\forall \alpha \in I$, $\forall x \in \Omega$ (r such that $B_r(a) \subset \Omega$):

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)dy \leq \frac{1}{|B_r(x)|} \int_{B_r(x)} u_\alpha(y)dy \leq u_\alpha(x) \Rightarrow$$

$$\frac{1}{|B_r(x)|} \int_{B_r(X)} u(y)dy \leq u(x)$$

Note that $F(x) = \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)dy$ is continuous. It follows that

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} \breve{u}(y)dy \leq \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)dy \leq \breve{u}(x) \leq u(x)$$

Now let $r \to 0$ and use that $\lim_{r\to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)dy = u(x)$ a.e. The conclusion is that $\breve{u} \leq u \leq \breve{u} \leq u$ a.e. □

Remark: If μ is a positive Radon measure then

1) If $N \geq 3$ we have that U^μ is superharmonic or $\equiv +\infty$. If $\mu(\mathbb{R}^N) < \infty$ then U^μ is superharmonic

2) If $N = 2$, then U^μ need not be well-defined if μ is a positive Radon measure, but if it fulfills $\int_{\mathbb{R}^2 \setminus B_1(0)} log|x| d\mu(x) < \infty$ then U^μ is superharmonic.

Definition 7.4. *A subset $A \subset \mathbb{R}^N$ is called a polar set if for each $a \in A$ there is a $r > 0$ and a superharmonic function u in $B_r(a)$ such that $A \cap B_r(a) \subset \{x : u(x) = +\infty\}$. A property is said to hold quasi everywhere (q.e.) if it holds except on a polar set.*

Note that all countable sets are polar sets, and all polar sets have Lebesgue measure zero. To see the first statement, put $\mu = \sum_{n=1}^{\infty} a_n \delta_{x_n}$ where $\{x_n\}$ is the set in question, and $a_n > 0$, $\sum_{n=1}^{\infty} a_n = 1$ for instance. Then U^μ is superharmonic. As a special case consider $x_n = 1/n$, $a_n = \frac{1}{n2^n}$. This gives an example of a superharmonic function discontinuous at the origin. The second statement is obvious.

The next theorem gives a nice result on the continuity of a potential (of course, we already know that it is continuous at free space, we wish to know when it is continuous everywhere). Roughly speaking, U^μ is continuous if μ is a nice measure, for instance if $\mu \in L^\infty$. Consider a nice domain with unit mass in $\mathbb{R}^N$ say. From what we said above we see that its potential is continuous everywhere. On the other hand, it might very well be possible to extend the harmonic function defined by its potential outside the domain to a larger domain. It might even be the case that there is another domain inside the original one with constant density giving the same potential outside the original domain. This is in fact the case for concentric balls (with smaller radius than the original one). To see some of the rich structure of potential theory, we note that in this case we get a family of continuous functions that are equal outside the original ball. Notice here also that there is no contradiction in the fact that we often can continue the potential harmonically across the support of the measure, but this continuation can not coincide with the potential inside the support of the measure since $-\Delta U^\mu = \mu$. This example becomes even more interesting when we equip a sphere, for instance, with surface measure. Then the potential is harmonic both outside and inside the sphere and it is continuous everywhere, the outer part of the potential can be continued across the boundary of the sphere but the above still holds.

Theorem 7.5. *(Continuity principle) Suppose μ is a positive Radon measure with compact support in $\mathbb{R}^N$. If $U^\mu_{\lfloor supp(\mu)}$ is continuous, then U^μ is continuous in $\mathbb{R}^N$.*

Proof. We already know that U^μ is continuous outside $supp(\mu)$, so what remains is to consider when $a \in \partial supp(\mu)$. If $U^\mu(a) = +\infty$ there is nothing to prove. (Because by the l.s.c. of U^μ we have $\liminf_{y \to a} U^\mu(y) \geq U^\mu(a)$ and in the case $U^\mu(a) = +\infty$ we obviously have $\limsup_{y \to a} U^\mu(y) \leq U^\mu(a)$). So now we assume that $U^\mu(a) < +\infty$.

Given $\epsilon > 0$, then there is a ball $B_r(a)$ such that (with $\mu_r = \mu_{\lfloor B_r(a)}$)

$$U^{\mu_r}(a) = \int \Phi(a - y) d\mu_r(y) < \epsilon$$

(because $\mu(\{a\}) = 0$). Furthermore, given x let Z_x be a chosen nearest point to x on $\partial supp(\mu)$. (That is, we take a one-valued map $x \mapsto Z_x$ $\forall x \in \mathbb{R}^N$). Then we have $|Z_x - y| \leq |Z_x - x| + |x - y| \leq 2|x - y|$ $\forall y \in supp(\mu)$. Now $|U^\mu(x) - U^\mu(a)| \leq |U^{\mu-\mu_r}(x) - U^{\mu-\mu_r}(a)| + |U^{\mu_r}(x) - U^{\mu_r}(a)| \leq |U^{\mu-\mu_r}(x) - U^{\mu-\mu_r}(a)| + U^{\mu_r}(x) + U^{\mu_r}(a)$. If $N = 2$ we have $U^{\mu_r}(x) = \int_{B_r(a)} -\frac{1}{2\pi} \log|x - y| d\mu(y) \leq \int_{B_r(a)} -\frac{1}{2\pi} \log|\frac{Z_x - y}{2}| d\mu(y) = \int -\frac{1}{2\pi} \log|Z_x - y| d\mu(y) + \frac{1}{2\pi} \log(2)\mu(B_r(a))$. If $N \geq 3$ we have $U^{\mu_r}(x) = \int_{B_r(a)} \frac{C_N}{|x-y|^{N-2}} d\mu(y) \leq 2^{N-2} \int_{B_r(a)} \frac{C_N}{|Z_x - y|} d\mu(y)$. By assumption

U^{μ_r} is continuous at a when restricted to $supp(\mu)$, so if we are given $\epsilon > 0$ we can choose r so small that $U^{\mu_r}(Z_x)$, $U^{\mu_r}(a)$ and $\mu(B_r(a))$ is as small as we please.

Now fix r so that $U^{\mu_r}(x) + U^{\mu_r}(a) < \frac{\epsilon}{2}$ $\forall x \in B_r(a)$. With this r fixed we know that $U^{\mu-\mu_r}$ is continuous in $B_r(a)$, since $\mu - \mu_r$ has support outside this ball. Hence, we can choose a ball $B_\delta(a) \subset B_r(a)$ such that $|U^{\mu-\mu_r}(x) - U^{\mu-\mu_r}(a)| < \frac{\epsilon}{2}$ $\forall x \in B_\delta(a)$. The conclusion is: $|U^\mu(x) - U^\mu(a)| \leq |U^{\mu-\mu_r}(x) - U^{\mu-\mu_r}(a)| + U^{\mu_r}(x) + U^{\mu_r}(a) < \epsilon$ $\forall x \in B_\delta(a)$ □

Theorem 7.6. *(Frostman's maximum principle). Suppose μ is a positive Radon measure with compact support in $\mathbb{R}^N$. If $U^\mu \leq M$ on $supp(\mu)$ then $U^\mu \leq M$ everywhere. (M is a constant).*

Proof. Put $\Phi_m(x) = \min\{\Phi(x), m\}$. Then $\Phi_m \nearrow \Phi$ and Φ_m is continuous. We have

$$\int \Phi_m(x-y)d\mu(y) \nearrow \int \Phi(x-y)d\mu(y) = U^\mu(x) \quad \forall x \in \mathbb{R}^N$$

Let $\epsilon > 0$ be given. By Egoroff's theorem there is a closed set $F \subset supp(\mu)$ with $\mu(F^c) < \epsilon$ and

$$\int \Phi_m(x-y)d\mu(y) \nearrow \int \Phi(x-y)d\mu(y)$$

uniformly in x on F. Setting $\mu_1(E) = \mu(E \cap F)$, it follows that $\int \Phi_m(x-y)d\mu_1(y) \nearrow \int \Phi(x-y)d\mu_1(y)$ uniformly in x on F.

Since each $\int \Phi_m(x-y)d\mu_1(y)$ is continuous everywhere it follows that $\int \Phi(x-y)d\mu_1(y)$is continuous on F. But $supp(\mu_1) \subset F$, and if we apply the continuity principle we see that U^{μ_1} is continuous in all of $\mathbb{R}^N$. Furthermore U^{μ_1} is harmonic outside F, if $N \geq 3$ it vanishes at infinity and if $N = 2$ it goes towards $-\infty$ uniformly. If we apply the usual maximum principle in any ball large enough , we see that $U^{\mu_1} \leq M$ in $\mathbb{R}^N$. Now take $x \in supp(\mu)^c$ and put $d = d(x, supp(\mu))$. Then $U^\mu(x) = \int \Phi(x-y)d\mu_1(y) + \int \Phi(x-y)d(\mu-\mu_1)(y) \leq M + \epsilon\Phi(d)$. Since ϵ is arbitrary we get the desired conclusion. □

8. Perron's method and applications.

Mainly, our aim in this section is to prove that a domain in $\mathbb{R}^N$, under very general conditions has a Green's function. This statement ought to suggest that we have to have less strict requirements in the definition of Green's function. But since every open set in $\mathbb{R}^N$ looks locally the same, it might come less as a surprise that we only need to free it from some of the regularity requirements when we reach the boundary, and this is precisely what we shall do.

After this goal has been achieved we can define the Green potential, and we will see that this has a lot of advantages since it allows us to state theorems in bounded domains amounting to almost the same thing as in all $\mathbb{R}^N$. It is especially useful when $N = 2$, because then Φ is not bounded from below, but goes towards $-\infty$ uniformly as $|x| \to \infty$.

Recall the definition of the Poisson kernel in theorem 5.1. If $f \in L^1(\partial B_r(a))$, let

$$u = P[f](x) = \frac{r^{N-2}}{|\partial B_r(a)|} \int_{\partial B_r(a)} \frac{r^2 - |x-a|^2}{|y-x|^N} f(y)dS(y) =$$

$$= \int_{\partial B_r(a)} P(x,y) f(y) dS(y)$$

Lemma 8.1. *If $f \in L^1(\partial B_r(a))$ then $u = P[f]$ is harmonic in $B_r(a)$,*

$$\liminf_{x \in \partial B_r(a) \quad x \to y} f(x) \leq \liminf_{x \in B_r(a) \quad x \to y} u(x)$$

$$\leq \limsup_{x \in B_r(a) \quad x \to y} u(x) \leq \limsup_{x \in \partial B_r(a) \quad x \to y} f(x)$$

$(y \in \partial B_r(a))$.

Proof. That u is harmonic follows from the fact that $\frac{r^2 - |x-a|^2}{|y-x|^N}$ is harmonic in $B_r(a)$ when $y \in \partial B_r(a)$.

$$\limsup_{x \in B_r(a) \quad x \to y} \frac{r^{N-2}}{|\partial B_r(a)|} \Big(\int_{\partial B_r(a) \cap B_\epsilon(y)} \frac{r^2 - |x-a|^2}{|t-x|^N} f(t) dS(t) +$$

$$+ \int_{\partial B_r(a) \setminus B_\epsilon(y)} \frac{r^2 - |x-a|^2}{|t-x|^N} f(t) dS(t) \Big) =$$

$$= \limsup_{x \in B_r(a) \quad x \to y} \frac{r^{N-2}}{|\partial B_r(a)|} \int_{\partial B_r(a) \cap B_\epsilon(y)} \frac{r^2 - |x-a|^2}{|t-x|^N} f(t) dS(t) \leq$$

$$\leq \limsup_{x \in B_r(a) \quad x \to y} \frac{r^{N-2}}{|\partial B_r(a)|} \Big(\sup_{t \in \partial B_r(a) \cap B_\epsilon(y)} f(t) \Big) \int_{\partial B_r(a)} \frac{r^2 - |x-a|^2}{|t-x|^N} dS(t) \leq$$

$$\leq \sup_{t \in \partial B_r(a) \cap B_\epsilon(y)} f(t).$$

Because

$$\limsup_{x \in B_r(a) \quad x \to y} \frac{r^{N-2}}{|\partial B_r(a)|} \int_{\partial B_r(a)} \frac{r^2 - |x-a|^2}{|t-x|^N} dS(t) \leq$$

$$\limsup_{x \in B_r(a) \quad x \to y} \Big(\frac{r^{N-2} r^2}{|\partial B_r(a)|} \int_{\partial B_r(a+x)} \frac{1}{|t|^N} dS(t) = 1$$

So we have that

$$\limsup_{x \in B_r(a) \quad x \to y} u(x) \leq \sup_{t \in \partial B_r(a) \cap B_\epsilon(y)} f(t) \quad \forall \epsilon > 0.$$

Since

$$\sup_{t \in \partial B_r(a) \cap B_\epsilon(y)} f(t) \to \limsup_{x \in \partial B_r(a) \quad x \to y} f(x) \qquad as \qquad \epsilon \to 0,$$

we have

$$\limsup_{x \in B_r(a) \quad x \to y} u(x) \leq \limsup_{x \in \partial B_r(a) \quad x \to y} f(x).$$

The remaining inequalities now become obvious. □

Theorem 8.2. *Suppose u is superharmonic in $\Omega \subset \mathbb{R}^N$, let $\overline{B_r(a)} \subset \Omega$ and define*

$$\tau_{B_r(a)}(u) := \begin{cases} P[u] & in \quad B_r(a), \\ u & in \quad \Omega \setminus B_r(a). \end{cases}$$

Then $\tau_{B_r(a)}(u)$ is well-defined and superharmonic in Ω, harmonic in $B_r(a)$ and finally $\tau_{B_r(a)}(u) \leq u$.

Proof. Since u is superharmonic it is bounded from below on $\partial B_r(a)$. Furthermore there is a sequence $f_n \in C(\partial B_r(a))$ such that $f_n \nearrow u$ on $\partial B_r(a)$. This implies that $P[f_n] \nearrow P[u]$. If we use the minimum principle we get $u - P[f_n] \geq 0$ in $B_r(a)$. Since $u \in L^1(\partial B_r(a))$ we obtain $P[u] \leq u$ in $B_r(a)$ by taking $n \to \infty$. So the part that $\tau_{B_r(a)}(u) \leq u$ is clear.

Now $\tau_{B_r(a)}(u)$ is lower semicontinuous by the previous lemma. At last we prove the super mean-value property which will end the proof: Let h be continuous in the compact set $K \subset \Omega$, and harmonic in $int(K)$, $h \leq \tau_{B_r(a)}(u)$ on ∂K. Then $h \leq \tau_{B_r(a)}(u)$ in K. This follows from $h \leq \tau_{B_r(a)}(u) \leq u$ on ∂K, showing that $h \leq u$ in K, especially $h \leq \tau_{B_r(a)}(u)$ in $K \setminus B_r(a)$. Thus $h \leq \tau_{B_r(a)}(u)$ on $\partial(K \cap B_r(a)) \subset \partial K \cup K \setminus B_r(a)$. Now $h \leq \tau_{B_r(a)}(u)$ in $K \cap B_r(a)$ by the usual maximum principle, hence $h \leq \tau_{B_r(a)}(u)$ in K. □

Theorem 8.3. *Suppose that u is superharmonic in Ω and that there is a subharmonic minorant $f \leq u$ in Ω. Then there is a largest subharmonic minorant v, and v is harmonic in Ω.*

Proof. By the fundamental convergence theorem there is a largest subharmonic minorant v. If $\overline{B_r(a)} \subset \Omega$ then $v \leq \tau_{B_r(a)}(v) \leq \tau_{B_r(a)}(u) \leq u$ and $\tau_{B_r(a)}(v)$ is subharmonic. Thus $v = \tau_{B_r(a)}(v)$, so v is harmonic in $B_r(a)$. Since $B_r(a)$ is an arbitrary ball in Ω v is harmonic in all of Ω. □

Definition 8.4. *(Green's function). Let Ω be a domain in $\mathbb{R}^N$. If $G : \Omega \times \Omega \to \hat{\mathbb{R}}$ satisfies*

1) $G(x,y) = \Phi(x-y) + h(x,y)$ where $h(x,y)$ is harmonic in x for each fixed y.

2) $G \geq 0$.

3) if for $y \in \Omega$, $V(x)$ is a non-negative superharmonic function that is the sum of $\Phi(x-y)$ and some superharmonic function, then $V(x) \geq G(x,y)$.

Then we call G the Green's function for Ω

(Note that G is unique if it exists, and furthermore $G(x,y) = G(y,x)$, but the last part is perhaps not trivial to see).

Theorem 8.5. *Every domain in $\mathbb{R}^N$ $(N \geq 3)$ and every bounded domain in $\mathbb{R}^2$ has a Green's function.*

Proof. Let Ω be an arbitrary domain in $\mathbb{R}^N$ $(N \geq 3)$, or a bounded domain in $\mathbb{R}^2$. Then Φ is a superharmonic function in Ω and is bounded from below. Therefore $\Phi(x-y)$ has a largest subharmonic minorant $h(x,y)$ for each fixed y. If we put $G(x,y) = \Phi(x-y) - h(x,y)$, then G is the Green's function for Ω. (Note that if $N \geq 3$ and $\Omega = \mathbb{R}^N$, then $G(x,y) = \Phi(x-y)$ by Liouville's theorem). □

If $\partial\Omega$ is smooth enough we get $G(x,y) = 0$ when $y \in \partial\Omega$, on the other hand we always have that 0 is the largest subharmonic minorant to G. (Note that $-\log|x|$ doesn't have a subharmonic minorant in $\mathbb{R}^2$). It's easy to see that the new definition of Green's function generalizes the old one.

Now we have all the material needed to define the Green potential. We shall also state a few theorems after the definition, and the proofs are left out simply because they are direct consequences of the definition of Green's function and the equivalent theorems for the Newtonian potentials. (In the rest of this section Ω is supposed to be a domain in $\mathbb{R}^N$ with a Green's function $G(x,y)$).

Definition 8.6. *Let Ω be a domain in $\mathbb{R}^N$ with a Green's function $G(x,y)$. Suppose that μ is a positive Radon measure in Ω (or just a Radon measure with compact support in Ω for instance), then we define the Green potential of μ in Ω by*

$$G^{\mu}(x) = \int_{\Omega} G(x,y)d\mu(y)$$

Theorem 8.7. *Suppose that μ is a positive Radon measure. Then $G^{\mu}(x)$ is superharmonic in Ω , and harmonic outside supp(μ). (We have $-\Delta G^{\mu} = \mu$ in distribution sense).*

Theorem 8.8. *(Riesz's structure theorem) If f is a superharmonic function in Ω, then it locally is on the form $G^{\mu} + h$ for some positive Radon measure ($\mu = -\Delta f$) and some harmonic function h.*

Theorem 8.9. *(Continuity principle). If μ is a positive Radon measure with compact support in Ω, and G^{μ} is continuous on supp(μ), then G^{μ} is continuous everywhere in Ω.*

Theorem 8.10. *(Frostman's maximum principle). Suppose μ is a positive Radon measure with compact support in Ω. If $G^{\mu} \leq M$ on supp(μ) then $G^{\mu} \leq M$ everywhere in Ω.*

9. Capacity And Balayage.

We now proceed with the notions from the previous chapter to briefly introduce some important concepts. The two most important ones being capacity and balayage. Only a few theorems will be proved, and I actually consider capacity to be outside the scope of this paper. The only reason to include it is that it is one of the most central concepts in potential theory, but it is therefore also many books giving complete treatments of it for the interested. Helms book [B7] is a good source on this (and indeed all the material in this paper except for the last two chapters). Balayage, on the other hand, will be dealt with in the more general setting (with some mild restrictions) of partial balayage in chapter 11.
Throughout this chapter we will assume that Ω is a bounded domain in $\mathbb{R}^N$, and we will denote its Green's function by G.

Definition 9.1. *If $f : \partial\Omega \to \hat{\mathbb{R}}$ and Ω is a bounded domain in $\mathbb{R}^N$, then we define*
1) $Su(f) := \{u : \quad u \text{ superharmonic} \quad \text{or} \quad u \equiv +\infty \quad \text{in} \quad \Omega\}$
2) $Sl(f) := \{u : \quad u \text{ subharmonic} \quad \text{or} \quad u \equiv -\infty \quad \text{in} \quad \Omega\}$
3) $UC(f) := \{u \in Su(f) : \quad u \text{ bounded from below in } \Omega$ $\text{and} \quad \liminf_{x\to y} u(x) \geq f(y) \quad \forall y \in \partial\Omega\}$
4) $LC(f) = \{u \in Sl(f) : \quad u \text{ bounded from above in } \Omega$ $\text{and} \quad \limsup_{x\to y} u(x) \leq f(y) \quad \forall y \in \partial\Omega\}$
5) $\overline{H}_f = \inf\{u : u \in UC(f)\}$
6) $\underline{H}_f = \sup\{u : u \in LC(f)\}$

f is called resolutive if $\overline{H}_f = \underline{H}_f$. In this case we have $f(y) \leq \liminf_{x\to y} u(x) \leq \limsup_{x\to y} u(x) \leq f(y)$ $\forall y \in \partial\Omega$ when $u = \overline{H}_f = \underline{H}_f$ (Note that $\overline{H}_f$ ($\underline{H}_f$)is either harmonic or $\equiv +\infty$ ($-\infty$). To see this, just compare with theorem 8.3). So u solves Dirichlet's problem in the classical sense. $\partial\Omega$ is called resolutive if every $f \in C(\partial\Omega)$ is resolutive.

Definition 9.2. *Let $y \in \partial\Omega$. A barrier in y is a superharmonic function $\phi \geq 0$ in Ω such that $\lim_{x \to y} \phi(x) = 0$ and $\inf_{\Omega \setminus B_r(y)} \phi > 0 \ \forall r > 0$.*

Theorem 9.3. *A sufficient condition for $\partial\Omega$ to be resolutive is that there is a barrier at every point $y \in \partial\Omega$. (The condition is actually also necessary).*

Proof. Let $y \in \partial\Omega$ and ϕ is a barrier in y. Take $r > 0$ and put $M = \sup_{\partial\Omega \cap B_r(y)} f$. As Ω is bounded , $M + n \inf_{\Omega \setminus B_r(y)} \phi \geq \sup_{\partial\Omega} f$ if n is large enough. Hence $M + n\phi \geq f$ on $\partial\Omega$ and $M + n\phi$ is superharmonic, so $\limsup_{x \to y} \overline{H}_f(x) \leq \limsup_{x \to y} M + n\phi(x) = M$ and $\limsup_{x \to y} \overline{H}_f(x) = f(y)$. The proof of the other inequality is similar. □

Definition 9.4. *Suppose that Ω has a resolutive boundary. Given $a \in \Omega$, consider $\Lambda : C(\partial\Omega) \to \hat{\mathbb{R}}$ defined by $\Lambda f = \underline{H}_f(a)$ (alternatively $\overline{H}_f(a)$). The functional Λ is positive , continuous and linear. So there is a measure $\mu_a \geq 0$ on $\partial\Omega$ such that $\underline{H}_f(a) = \int_{\partial\Omega} f d\mu_a$. This is the harmonic measure of the point a. (If $\partial\Omega$ is smooth we have $d\mu_a = -\frac{\partial G(x,a)}{\partial N_x} dS(x)$).*

Suppose that $u \geq 0$ is superharmonic in Ω and $A \subset \Omega$ .Put

$R_u^A = \inf\{v \geq 0 : v \quad \textit{superharmonic} \quad \textit{in} \quad \Omega \quad \textit{and} \quad v \geq u \quad \textit{on} \quad A\}$.

R_u^A is called the reduced function of u w.r.t. A.

$\breve{R}_u^A := \sup\{\phi : \phi \leq R_u^A, \quad \phi \quad \textit{is} \quad \textit{l.s.c.} \quad \textit{in} \quad \Omega\}$.

We already know the following

1) $\breve{R}_u^A \leq R_u^A \leq u$
2) $\breve{R}_u^A = u$ a.e. in A.
3) $\breve{R}_u^A$ is harmonic in $\Omega \setminus \overline{A}$.
4) $\breve{R}_u^A$ is superharmonic in Ω.

Definition 9.5. *Let K be a compact subset of Ω. Put $\mu = -\Delta \breve{R}_1^K$. We define the capacity of K (w.r.t. Ω) by $Cap(K) := \mu(K)$.*

By 3) and 4) above we see that $\mu \geq 0$ and $supp(\mu) \subset K$.

Theorem 9.6. *With the notation from the definition above the following holds.*

1) $\breve{R}_1^K$ is superharmonic.
2) μ is a positive measure on K.
3) $G^\mu = 1$ a.e. on K.
4) $\breve{R}_1^K = G^\mu$.

Proof. 1) follows from the fundamental convergence theorem.
2) We only need to see that since $\breve{R}_1^K$ is harmonic on $\Omega \setminus K$, we have that μ has compact support. Because by Riesz's structure theorem we have $\breve{R}_1^K = G^\mu + h$, but $h \equiv 0$ necessarily, and $-\Delta \breve{R}_1^K = -\Delta G^\mu = \mu$.
3) and 4) is obvious from the definition and the proof of part 2). □

Definition 9.7. *Let D be a bounded domain in $\mathbb{R}^N$ with Green's function G_D, and suppose that $\overline{\Omega} \subset D$. Put $A = D \setminus \Omega$. Suppose that μ is a positive Radon measure with compact support in Ω, and put $u := G_D^\mu$. We now define the balayage of μ with respect to Ω by $Bal_\Omega(\mu) := -\Delta \breve{R}_u^A$.*

Theorem 9.8. *With the notation from the previous definition, set* $\nu = Bal_\Omega(\mu)$. *Then*

1) $supp(\nu) \subset \partial\Omega$.
2) $G_D^\nu = G_D^\mu$ *a.e. on* A.

Proof. We already know the following: $\check{R}_u^A = u$ everywhere on A (since u is harmonic here), so $-\Delta\check{R}_u^A = 0$ in A.
$\check{R}_u^A$ is harmonic in Ω, so $-\Delta\check{R}_u^A = 0$ in Ω. Hence $supp(\nu) \subset \partial\Omega$. The theorem now follows immediately from Riesz's structure theorem. □

(Note that we can define the harmonic measure in a as $Bal_\Omega(\delta_a)$).

10. Energy

The purpose of introducing energy , which is an inner product on a space of suitably chosen measures, in a domain with a Green's function is that it leads to a different point of view of many problems. And it also allows us to apply some of the highly developed machinery from functional analysis.

Definition 10.1. *Let* Ω *be a domain with a Green's function* $G(x,y)$ *in* $\mathbb{R}^N$ $(N \geq 2)$. *Then if* μ *and* ν *are positive Radon measures on* Ω *we define*

$$(\mu,\nu)_e := \int\int G(x,y)d\mu(x)d\nu(y)$$

and

$$||\mu||_e^2 := (\mu,\mu)_e$$

Now we put $\varepsilon^+(\Omega) := \{\mu : \mu$ *is a positive Radon measure with* $||\mu||_e < \infty\}$.

Theorem 10.2. *Let* Ω *be a domain with a Green's function. Then*

1) $(\mu,\nu)_e = (\nu,\mu)_e \; \forall \mu,\nu \in \varepsilon^+(\Omega)$.
2) $(t\mu,\nu)_e = t(\mu,\nu)_e \; \forall t \geq 0, \forall \mu,\nu \in \varepsilon^+(\Omega)$.
3) $(\mu,\nu)_e \leq ||\mu||_e||\nu||_e < \infty \; \forall \mu,\nu \in \varepsilon^+(\Omega)$.
4) $||\mu||_e \geq 0$ *with equality if and only if* $\mu = 0$.

Proof. 1) and 2) are obvious (1) by Fubini and the fact that $G(x,y) = G(y,x)$). To prove 3), let K_n be a sequence of compact subsets exhausting Ω. Then we can look at the restriction of the measures to K_n and then take the limit as $n \to \infty$. Therefore we can assume without loss of generality that $supp(\mu)$ and $supp(\nu)$ are contained in a fixed compact subset K of Ω. These assumptions make it possible to define $\lambda = \mu - \nu$ which then becomes a signed measure on Ω. Consider

$$\int\int G(x,y)d\lambda(x)d\lambda(y) = \int\int G(x,y)d(\mu-\nu)(x)d(\mu-\nu)(y) =$$

$$||\mu||_e^2 + ||\nu||_e^2 - 2(\mu,\nu)_e.$$

Suppose that μ and ν happens to be "nice measures". In that case we have:

$$\int\int G(x,y)d\mu(x)d\nu(y) = \int G^\mu(y)d\nu(y) = \int G^\mu(-\Delta G^\nu)dy =$$

$$= \int \nabla G^\mu \cdot \nabla G^\nu dy.$$

We especially get, if λ happens to be "nice", $(\lambda,\lambda)_e \geq 0$. It is possible to prove that we can approximate λ with "nice measures" to yield this result in general. (One way is to define $\lambda^\varepsilon(x) = \int \eta_\varepsilon(x-y)d\lambda(y)$. Then $\lambda^\varepsilon \to \lambda$ in the weak*-topology. It is easily seen that λ^ε is "nice" in the above sense, and we can take a limit to get what we want.)

We get $||\mu||_e^2 + ||\nu||_e^2 \geq 2(\mu,\nu)_e$, so $(\mu,\nu)_e \leq 1$ if $||\mu||_e$ and $||\nu||_e$ is ≤ 1. Hence

$$(\frac{\mu}{||\mu||_e}, \frac{\nu}{||\nu||_e})_e \leq 1$$

and the theorem follows. □

We now define

$$\varepsilon = \{\lambda : \lambda = \mu - \nu \quad where \quad \mu, \nu \in \varepsilon^+ \quad under \quad equivalence\}.$$

If $\lambda_1, \lambda_2 \in \varepsilon$ define $(\lambda_1,\lambda_2)_e = \int\int G(x,y)d\lambda_1(x)d\lambda_2(y)$. Now we see that ε is an inner product space, but it is not a Hilbert space, i.e. it is not complete. (Although ε^+ is complete with the induced metric.)

To motivate section 11, let's assume that μ and ν are "nice measures". We then have (as in the proof above) by Green's formulas :

$$(\mu,\nu)_e = \int G^\mu d\nu = -\int G^\mu \Delta G^\nu dx =$$

$$= \int \nabla G^\mu \cdot \nabla G^\nu dx$$

So $||\mu||_e^2 = \int |\nabla G^\mu|^2 dx$. But this integral always makes sense when μ is a positive Radon measure for instance, simply because then $\nabla G^\mu \in L^1_{loc}(\Omega)$.

Existence of equilibrium distributions

Let $K \subset \Omega \subset \mathbb{R}^N$ where K is compact and Ω is an open domain with a Green's function $G(x,y)$. Consider the problem: Minimize $||\mu||_e^2$ over all positive Radon measures μ with $supp(\mu) \subset K$ and $\mu(K) = 1$. Put $B = \{\mu : \mu$ is a positive Radon measure with support in K , $\mu(K) = 1, ||\mu||_e < \infty\}$.

Theorem 10.3. *With the notation from above we have that either $B = \emptyset$ (which occurs if and only if K has capacity zero), or $B \neq \emptyset$. If the latter holds, then there is a measure $\mu \in B$ such that*

$$||\mu||_e^2 = \inf\{||\mu||_e^2 : \quad \mu \in B\}.$$

Proof. There exists a sequence μ_n such that $||\mu_n||_e^2 \to \inf ||\mu||_e^2$, by well known properties of $\mathbb{R}$. Since the unit ball in $C(K)^*$ is $weak^* - compact$ (by Alaoglu's theorem) there is a positive Radon measure with $supp(\mu) \subset K$, $\mu(K) = 1$ such that $\mu_n \to \mu$ in the $weak^* - topology$. So if we let $G_m(x,y) = \min\{G(x,y), m\}$ then we have that for each fixed m $\int\int G_m(x,y)d\mu_n(x)d\mu_n(y) \to \int\int G_m(x,y)d\mu(x)d\mu(y)$ as $n \to \infty$. If we let $m \to \infty$ we get :

$$\int\int G_m(x,y)d\mu(x)d\mu(y) \leq \liminf_{n\to\infty} ||\mu_n||_e^2 \quad \forall m,$$

hence

$$||\mu||_e^2 \leq \liminf_{n\to\infty} ||\mu_n||_e^2$$

and the assertion follows. □

Now B is a convex set in $\varepsilon^+(\Omega)$, so we have that the μ that solves the problem above fulfills $(\nu - \mu, \mu)_e \geq 0 \ \forall \nu \in B$, i.e., $\int_K G^\mu d\mu \leq \int_K G^\mu d\nu \ \forall \nu \in B$. Put $\gamma = ||\mu||_e^2 = \int G^\mu d\mu$, and for $\alpha \in \mathbb{R}$, $F_\alpha := \{x \in K : G^\mu(x) \leq \alpha\}$ (here μ is the minimizing measure as above). Since G^μ is l.s.c. F_α is closed. If $Cap(F_\alpha) > 0$ (for a fixed α), then we have that there is a $\nu \in B$ with $supp(\nu) \subset F_\alpha$. This gives $\gamma = \int G^\mu d\mu \leq \int G^\mu d\nu \leq \alpha$, so $Cap(F_\alpha) = 0 \ \forall \alpha < \gamma$, especially $\mu(F_\alpha) = 0 \ \forall \alpha < \gamma$.

The conclusion is now that $G^\mu \geq \gamma$ on $K \setminus E$ where $E = \cup_{\alpha<\gamma} F_\alpha = \cup_{n=1}^\infty F_{\gamma - 1/n}$. We see that $\mu(E) = 0$, hence $G^\mu \geq \gamma \ \mu - a.e.$ on K. But from the definition of γ we have $\int (G^\mu - \gamma) d\mu = 0$ so $G^\mu = \gamma \ \mu - a.e.$ on K. So $\mu(F_\gamma) = 1$, that is $supp(\mu) \subset F_\gamma$. If we use Frostman's maximum principle together with the definition of F_γ we have $G^\mu \leq \gamma$ everywhere in Ω.

To summarize: if $Cap(K) > 0$ we have found $\mu \in B$ such that $G^\mu \leq \gamma$ everywhere in Ω, $G^\mu = \gamma$ on $K \setminus E$ with E as above (γ is a positive constant). Moreover $\mu(E) = 0$, and E is a countable union of compact sets with capacity zero. Now we come to the point of it all. Recall that we defined the capacity of K by $Cap(K) = \nu(K)$ where $\nu = -\Delta u$, $u = \breve{\mathbb{R}}_1^K$. Also recall that this means that $G^\nu = 1$ a.e. on K. But the minimization we justified gives us just such a measure times some suitable constant! So we have found another way to determine the capacity of a compact set K.

11. Sobolev Spaces and Variational Inequalities

Let Ω be a bounded domain in $\mathbb{R}^N$. Then we have $C_0^\infty(\Omega) \subset L^2(\Omega) \subset \mathcal{D}'(\Omega)$. Define $\langle f, g \rangle = \int_\Omega fg dx$ if $f, g \in L^2(\Omega)$, and $\langle \Lambda, \phi \rangle = \Lambda(\phi)$ if $\Lambda \in \mathcal{D}'(\Omega)$ and $\phi \in C_0^\infty(\Omega)$. This is consistent when $f \in L^2(\Omega)$ is seen as a distribution as well. Let $(\cdot, \cdot)$ be another inner product on $C_0^\infty(\Omega)$ such that $\langle \phi, \phi \rangle \leq C(\phi, \phi)$ for some constant C and $\forall \phi \in C_0^\infty(\Omega)$, and let V be the completion of $C_0^\infty(\Omega)$ with respect to this inner product. We then have $C_0^\infty(\Omega) \subset V \subset L^2(\Omega)$.

Example 1: $(\phi, \psi) = \int_\Omega \nabla\phi \cdot \nabla\psi dx$. We have $\int_\Omega |\phi|^2 dx \leq C \int_\Omega |\nabla\phi|^2 dx$ by Poincare's inequality. We denote the completion of $C_0^\infty(\Omega)$ with respect to this inner product $H_0^1(\Omega)$.

Since $\langle \cdot, \cdot \rangle$ is defined on $L^2(\Omega)$, we still have that V is an inner product space with this norm. So if we let V' be the dualspace to V under $\langle \cdot, \cdot \rangle$ we obviously have $V' \subset \mathcal{D}'(\Omega)$. Now, since $L^2(\Omega) = L^2(\Omega)'$ we have that $L^2(\Omega) \subset V'$. The conclusion is that $C_0^\infty(\Omega) \subset V \subset L^2(\Omega) \subset V' \subset \mathcal{D}'(\Omega)$.

Example 2: With $V = H_0^1(\Omega)$, then we put $V' = H^{-1}(\Omega)$.

Given $v \in V$, then $u \mapsto (u, v)$ is a continuous linear functional on V, and all functionals on V under $(\cdot, \cdot)$ is on this form. But we then see that there is a $w \in V'$ such that $(u, v) = \langle u, w \rangle$, so this gives a linear mapping $A : V \to V'$ such that $(u, v) = \langle u, Av \rangle \ \forall u, v \in V$.

Lemma 11.1. *A is an isometric isomorphism.*

Proof. If $u \neq 0$, $u \in V$, then $0 < (u, u) \leq \langle u, Au \rangle$, so A is injective. If $w \in V'$, then Riesz-Fischer's theorem gives that there is a $v \in V$ such that $\langle u, w \rangle = (u, v) \ \forall u \in V$. So A is surjective. We also have $||Av||_{V'} = \sup_{u \in V} \frac{\langle u, Av \rangle}{||u||_V} = \sup_{u \in V} \frac{(u, v)}{||u||_V} = ||v||_V$, so A is an isometry. $\square$

Example 3: With $V = H_0^1(\Omega)$ and $V' = H^{-1}(\Omega)$ we have $A = -\Delta$, since by Green's formula we have $(\phi, \psi) = \int_\Omega \nabla\phi \cdot \nabla\psi dx = -\int_\Omega \phi\Delta\psi dx = \langle \phi, -\Delta\psi \rangle$ $\forall \phi, \psi \in C_0^\infty(\Omega)$. (Note that $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$).

Lemma 11.2. *Let $G : H^{-1}(\Omega) \to H_0^1(\Omega)$ be the inverse of $-\Delta$. If $f \in H^{-1}(\Omega)$ is a positive Radon measure or a Radon measure with compact support, then we have $Gf = G^f =$ the Green potential of f. (We assume that Ω has a Green's function).*

Proof. We know that $-\Delta G^f = f$. So if we notice that $G^f \in H_0^1(\Omega)$ we are done (see for instance [B19] for details). □

Theorem 11.3. $f \in H^{-1}(\Omega) \Leftrightarrow \exists f_0, ..., f_N \in L^2(\Omega), \quad f = f_0 + \sum_{j=1}^N \frac{\partial f_j}{\partial x_j}$ *(in distribution sense).*

Proof. If $f = f_0 + \sum_{j=1}^N \frac{\partial f_j}{\partial x_j}$, then $|\langle f, \phi \rangle| = |\int_\Omega f_0 \phi dx - \sum_{j=1}^N \int_\Omega f_j \frac{\partial \phi}{\partial x_j} dx| \leq ||f_0||_{L^2} ||\phi||_{L^2} + \sum_{j=1}^N ||f_j||_{L^2} ||\frac{\partial \phi}{\partial x_j}||_{L^2} \leq C||\phi||_{H_0^1}$, so $f \in H^{-1}(\Omega)$. Now suppose that $f \in H^{-1}(\Omega)$. Then $f = -\Delta u$ for some $u \in H_0^1(\Omega)$. With $-\nabla u = (f_1, ..., f_N)$ we have $f = \sum_{j=1}^N \frac{\partial f_j}{\partial x_j}$. □

Theorem 11.4. *(The Obstacle Problem). Let Ω be a bounded domain in $\mathbb{R}^N$ and let an obstacle $\psi \in H_0^1(\Omega)$ be given. Put $K = \{u \in H_0^1(\Omega) : u \geq \psi\}$. Then the following are equivalent for a function $u \in H_0^1(\Omega)$.*

1) $u \in K$ and minimizes $||u||^2 = \int_\Omega |\nabla u|^2 dx$ among all $u \in K$.

2) $u \in K$ and $(v - u, u) \geq 0$ $\forall v \in K$. (This is an example of a variational inequality).

3) $u \in K$ and satisfies $-\Delta u \geq 0$ and $\langle -\Delta u, u - \psi \rangle = 0$. (By this we mean that $-\Delta u \in H^{-1}$ acts on $(u - \psi) \in H_0^1$). (This is an example of a complementarity problem).

4) u is the smallest superharmonic function in $H_0^1(\Omega)$ such that $u \geq \psi$ in Ω.

5) u is superharmonic and $(v-u, u-\psi) \geq 0$ for every superharmonic $v \in H_0^1(\Omega)$.

6) u is superharmonic and minimizes $\int_\Omega |\nabla(u-\psi)|^2 dx$ among all superharmonic $u \in H_0^1(\Omega)$.

Proof. 1) $\Leftrightarrow$ 2) : follows from the fact that K is a closed convex set in $H_0^1(\Omega)$ (K is even a cone).
2) $\Rightarrow$ 3) : Take $v = u+(u-\psi)$ in 2), then we get $(u-\psi, u) \geq 0$, that is $\langle -\Delta u, u-\psi \rangle \geq 0$. Now take $v = u + \phi$ where ϕ is a testing function with $\phi \geq 0$. Then $(\phi, u) \geq 0$, that is $-\Delta u \geq 0$. $v = \psi$ gives trivially that$(u - \psi, u) \leq 0$, hence $\langle -\Delta u, u - \psi \rangle = 0$.
3) $\Rightarrow$ 2) : $\forall v \in K$ we have $(v - u, u) = (v - \psi - (u - \psi), u) = (v - \psi, u) - 0 = \langle -\Delta u, v - \psi \rangle \geq 0$
2) $\Rightarrow$ 4) :Suppose that $-\Delta v \geq 0$, $v \geq \psi$ and put $w = \min(u, v)$. It is known that if $u, v \in H_0^1(\Omega)$ then $\min(u, v) \in H_0^1(\Omega)$.This gives that $-\Delta w \geq 0$,$w \geq \psi$ and $\langle -\Delta w, w - \psi \rangle \geq 0$. Furthermore $||u - w||^2 = \int_\Omega \nabla(u - w) \cdot \nabla(u - w) dx = \int_\Omega \nabla(u - w) \cdot \nabla(u - v) dx = -(w - u, u) - \langle u - w, -\Delta v \rangle \leq 0$. Hence $u = w$, so $u \leq v$.
4) $\Rightarrow$ 2) : There is no question of existence of the solution of 2), since it is merely the projection of the origin onto K. This solution solves 4) as has already been seen, and furthermore the solution to 4) is obviously unique. So from this we see that 4) $\Rightarrow$ 2).
3) $\Rightarrow$ 5) : $(v - u, u - \psi) = (v, u - \psi) - (u, u - \psi) = \langle -\Delta v, u - \psi \rangle + \langle -\Delta u, \psi - u \rangle \geq 0$

by assumption since $-\Delta v \geq 0$, $-\Delta u \geq 0$ and $u \geq \psi$.
5) $\Rightarrow$ 3) : u is superharmonic by assumption, and with $v = 0$ we have $(-u, u-\psi) = -\langle -\Delta u, u-\psi\rangle \geq 0$, and with $v = 2u$ we get $(u, u-\psi) = \langle -\Delta u, u-\psi\rangle \geq 0$, hence $\langle -\Delta u, u-\psi\rangle = 0$.
5) $\Leftrightarrow$ 6) : is analogous to 1) $\Leftrightarrow$ 2). This completes the proof. □

It seems well motivated to mention a few weaknesses of the Sobolev space approach introduced in this chapter. We have already seen that H_0^1 is a Hilbert space, and the well developed theory of these spaces makes a lot of theorems less painful to prove. But there are two rather obvious disadvantages. The first is that not every potential is in H_0^1, but only those with finite energy. The second is that not every element in H_0^1 is a potential of a signed measure.
Another shortcoming is that in many applications in potential theory (for instance when dealing with reduced functions and capacity) we want to use something similar to the obstacle problem above. But then our obstacle is in general not itself in H_0^1.

The interested reader who wish to know more about variational inequalities is recommended to turn to [B10].

12. Partial Balayage

Recall that classical balayage was all about "sweeping" a measure with support inside a domain to another measure with support on the boundary, producing the same external potential as the former. The idea of partial balayage is very similar, but one allows to "sweep" the measure with more general conditions to "where" it should end up in some sense. In fact, with partial balayage one wants to sweep the given measure μ to a measure with prescribed density ρ, but with support on an unknown open set $\Omega(\mu)$. It is always possible to perform the balayage, but the naturally desired form $\rho\chi_{\Omega(\mu)}$ is only taken under certain assumptions on μ.

The concept of partial balayage (and also that of a mother body) is essentially due to Zidarov, and his ideas can be found in [B20]. But the rigorous exposition treated here, is to be found in several pages by Gustafsson and Sakai. As reference source we recommend [P3], but also [P2] and [P6] contains information about this.

Theorem 12.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a resolutive boundary. Furthermore assume that $\rho \in L^\infty(\Omega)$, and put $\rho = +\infty$ in Ω^c. $M_c := \{\mu \ : \ \mu$ is a positive Radon measure with compact support in $\mathbb{R}^N\}$.*

$F_\rho^\mu := \{u \in \mathcal{D}'(\mathbb{R}^N) \ : \ u \leq U^\mu \text{ in } \mathbb{R}^N, \ -\Delta u \leq \rho \text{ in } \Omega\}$.

(Note that the following holds: $F_{\rho+\sigma}^{\mu+\sigma} = F_\rho^\mu + U^\sigma$ $\forall \sigma \in M_c$, and $F_{t\rho}^{t\mu} = tF_\rho^\mu$ $\forall t > 0$). With these notations we have that F_ρ^μ contains a largest element V^μ ($=V_\rho^\mu$ when necessary) that fulfills:

1) $V^\mu = U^\mu$ outside a compact set.

2) $$-\Delta V^\mu \geq \lambda := \begin{cases} \min(\rho, \mu) & \text{in } \Omega \\ \mu & \text{in } \Omega^c \end{cases} .$$

By 2), V^μ is superharmonic and hence has a lower semicontinuous representative. In the sequel we always refer to this representative.

3) V^μ is continuously differentiable in Ω, and $V^\mu = U^\mu$ on Ω^c.

4) $-\Delta V^\mu = \rho$ in $\omega(\mu) := \{x \in \Omega \ : \ V^\mu(x) < U^\mu(x)\}$.

Definition 12.2. *We define the partial balayage of μ with respect to ρ by $Bal(\mu;\rho) = -\Delta V^{\mu}$ (in $\mathbb{R}^N$).*

(Compare this definition with the one of classical balayage. Notice that we actually "go from the other direction" here).

Proof. $\mu' = \mu - \rho$, μ' is a Radon measure in Ω. Define

$$V^{\mu'} = \begin{cases} U^{\mu'} & in \quad \Omega^c \\ u & in \quad \Omega, \end{cases}$$

where u is the largest subharmonic function in Ω such that $u \leq U^{\mu'}$ (note that U^{ρ} is continuous, so this makes sense by the fundamental convergence theorem). Put

$$V^{\mu} = \begin{cases} U^{\mu} & in \quad \Omega^c \\ u + U^{\rho} & in \quad \Omega. \end{cases}$$

Since all $v \in F_0^{\mu'}$ are subharmonic by definition,and since it is obvious that $V^{\mu} \in F_{\rho}^{\mu}$, we immediately see that V^{μ} is the supremum of all functions in F_{ρ}^{μ}. So V^{μ} exists and satisfies 1). In-fact $V^{\mu} = U^{\mu}$ in Ω^c by definition. Let $x \in \omega(\mu)$, and note that there is a closed ball $\overline{B_r(x)} \subset \omega(\mu)$ by the semicontinuity of V^{μ} and U^{μ}. In $\overline{B_r(x)}$ we must have $-\Delta V^{\mu} = \rho$ since otherwise we can apply the Poisson formula in $\overline{B_r(x)}$ to $V^{\mu'}$ to get a contradiction to the maximality of $V^{\mu'}$. We conclude that $-\Delta V^{\mu} = \rho$ in $\omega(\mu)$, and now part 4) is clear.

Since $F_{\rho}^{\mu} = F_{\rho-\lambda}^{\mu-\lambda} + U^{\lambda}$ it is enough to prove that $-\Delta V^{\mu} \geq 0$ if $\mu \geq 0$ and $\rho \geq 0$ in order to see that 2) holds. Let W^{μ} be the smallest superharmonic majorant of V^{μ}. Then $V^{\mu} \leq W^{\mu} \leq U^{\mu}$, $-\Delta W^{\mu} \geq 0$. We want to prove that $-\Delta W^{\mu} \leq \rho$ in Ω, because then it follows that $W^{\mu} \in F_{\rho}^{\mu}$, hence that $W^{\mu} = V^{\mu}$ and so $-\Delta V^{\mu} \geq 0$, which is what we want.

Put $v = W^{\mu} - V^{\mu}$ in Ω, $I = \{x \in \omega \ : \ v(x) = 0\}$, $D = \{x \in \omega \ : \ v(x) > 0\}$. Then I is closed and D is open, $I \cup D = \Omega$. Clear is that $D \subset \omega(\mu) \Rightarrow -\Delta V^{\mu} = \rho$ in D. The same argument as for V^{μ} yields $-\Delta W^{\mu} = 0$ in $D \Rightarrow -\Delta v = \rho$ in D.

Let $x \in D$, $\overline{B_r(x)} \subset D$, put $\eta = \rho - \Delta v$ in $\overline{B_r(x)} \Rightarrow supp(\eta) \subset \overline{B_r(x)} \cap I$. Since $-\Delta(U^{\eta} - v) = \rho \in L^{\infty}(B_r(x))$ we have that $U^{\eta} - v$ is continuous in $B_r(x)$. But $U^{\eta}_{\lfloor supp(\eta)} = (U^{\eta} - v)_{\lfloor supp(\eta)} \Rightarrow U^{\eta}_{\lfloor supp(\eta)}$ is continuous, so U^{η} is continuous by the continuity principle, hence v is continuous. Let $K \subset \Omega$ be compact and let $h \in C(K)$, h harmonic in $int(K)$ and $v \leq h$ on ∂K. Since $v \geq 0 \Rightarrow h \geq 0$ on ∂K $\Rightarrow h \geq 0$ in K, hence $v \leq h$ on $K \cap I$. In $int(K) \cap D$ v is subharmonic, and on $\partial((int(K)) \cap D) \subset \partial K \cup (K \cap I)$ the inequality is also obvious. So $v \leq h$ in K. The conclusion is that v is subharmonic in Ω, that is $-\Delta v \leq 0 \Rightarrow -\Delta W^{\mu} + \Delta V^{\mu} \leq 0$ $\Rightarrow -\Delta W^{\mu} \leq \rho$ and 2 is proved.
That V^{μ} is continuously differentiable in Ω follows from the inequality $\min(\rho, \mu) \leq -\Delta V^{\mu} \leq \rho \in L^{\infty}(\Omega)$. □

Theorem 12.3. *Let Ω, Ω_j , $j = 1, 2, ...$ be open and bounded in $\mathbb{R}^N$, let $\rho \in L^{\infty}(\Omega)$, $\rho_j \in L^{\infty}(\Omega_j)$, $\rho, \rho_j \geq 0$, $\mu, \mu_j \in M_c$. Then*
1) $Bal(\mu + \sigma; \rho + \sigma) = Bal(\mu; \rho) + \sigma$ $\forall \sigma \in M_c \cap L^{\infty}(\Omega)$.
2) $Bal(t\mu; t\rho) = tBal(\mu; \rho)$ $\forall t > 0$.
3) $Bal(Bal(\mu_1; \rho_2) + \mu_2; \rho_1) = Bal(\mu_1 + \mu_2; \rho_1)$ if $\Omega_2 \subset \Omega_1$, $\rho_1 \leq \rho_2 + \mu_2$ in Ω_2.
4) $\min(\rho, \mu) \leq Bal(\mu; \rho) \leq \rho$ in Ω, $\mu \leq Bal(\mu; \rho)$ in Ω^c.

5) $\mu_1 \le \mu_2 \Rightarrow Bal(\mu_1;\rho_2) \le Bal(\mu_2;\rho_2)$.
6) $\mu_n \nearrow \mu$ *weakly* $\Rightarrow Bal(\mu_n;\rho) \nearrow Bal(\mu;\rho)$ *weakly.*

Proof. 1) and 2) are obvious and 4) follows from the definition and part 2) of the previous theorem.

3) We want to show that $V_{\rho_1}^{Bal(\mu_1;\rho_2)+\mu_2} = V_{\rho_1}^{\mu_1+\mu_2}$. By definition we have $V_\rho^\mu = U^{Bal(\mu;\rho)} =$ the largest function $\le U^\mu$ satisfying $-\Delta V_\rho^\mu \le \rho$ in Ω, so we get $V_{\rho_1}^{Bal(\mu_1;\rho_1)+\mu_2} \le U^{Bal(\mu_1;\rho_2)+\mu_2} = U^{Bal(\mu_1;\rho_2} + U^{\mu_2} = V_{\rho_2}^{\mu_1} + U^{\mu_2} \le U^{\mu_1} + U^{\mu_2} = U^{\mu_1+\mu_2}$, and since $-\Delta V_{\rho_1}^{Bal(\mu_1;\rho_2)+\mu_2} \le \rho_1$ in Ω_1,it follows that $V_{\rho_1}^{Bal(\mu_1;\rho_2)+\mu_2} \le V_{\rho_1}^{\mu_1+\mu_2}$.

To prove the opposite inequality we have: $V_{\rho_1}^{\mu_1+\mu_2} - U^{\mu_2} \le U^{\mu_1}$ and $-\Delta(V_{\rho_1}^{\mu_1+\mu_2} - U^{\mu_2}) \le \rho_1 - \mu_2 \le \rho_2$ in $\Omega_2 \subset \Omega_1$, which shows that $V_{\rho_1}^{\mu_1+\mu_2} - U^{\mu_2} \le V_{\rho_2}^{\mu_1}$. Hence $V_{\rho_1}^{\mu_1+\mu_2} \le V_{\rho_2}^{\mu_1} + U^{\mu_2} = U^{Bal(\mu_1;\rho_2)} + U^{\mu_2} = U^{Bal(\mu_1;\rho_2)+\mu_2}$. Since $-\Delta V_{\rho_1}^{\mu_1+\mu_2} \le \rho_1$ in Ω_1 this gives that $V_{\rho_1}^{\mu_1+\mu_2} \le V_{\rho_1}^{Bal(\mu_1;\rho_2)+\mu_2}$ and 3 is proved.

5) Put

$$\lambda := \begin{cases} \min(Bal(\mu_1;\rho_2) + \mu_2 - \mu_1, \rho_2) & in \quad \Omega_2, \\ Bal(\mu_1;\rho_2) + \mu_2 - \mu_1 & in \quad \Omega_2^c \end{cases}$$

Using 3) and 4) we get: $Bal(\mu_1;\rho_2) \le \lambda \le Bal(Bal(\mu_1;\rho_2) + \mu_2 - \mu_1;\rho_2) = Bal(\mu_1 + \mu_2 - \mu_1;\rho_2) = Bal(\mu_2;\rho_2)$.

6) By 5), $Bal(\mu_1;\rho) \le Bal(\mu_2;\rho) \le ... \le Bal(\mu;\rho)$.
Now note that $\int d\mu = \int d(Bal(\mu;\rho))$ holds $\forall \mu \in M_c$, so that $0 \le \int \phi d(Bal(\mu;1) - Bal(\mu_n;1)) \le ||\phi||_\infty \int d(\mu - \mu_n) \to 0 \ \forall 0 \le \phi \in \mathcal{D}(\mathbb{R}^N)$. □

Theorem 12.4. *If we use the notation from the previous theorem, and assume that* μ *has finite energy, then* $Bal(\mu;\rho)$ *has finite energy and is characterized as the unique minimizer of* $||\mu - \eta||_e$ *among all measures* $\eta \in M_c$ *with finite energy such that* $\eta \le \rho$ *in* Ω.

Proof. Put $\eta = Bal(\mu;\rho)$. Then $\eta \ge 0$ and $U^\eta = V^\mu \le U^\mu$ so that $||\eta||_e^2 = \int U^\eta d\eta \le \int U^\mu d\eta \le \int U^\mu d\mu = ||\mu||_e^2 < \infty$. Now we only need to prove that η is the minimizer we're looking for. But this merely amounts to proving that $(\mu - \eta, \eta - \nu)_e \ge 0$ for all $\nu \in M_c$ with finite energy, satisfying $\nu \le \rho$ in Ω. But using the first theorem of this chapter we get: $(\mu - Bal(\mu;\rho), Bal(\mu;\rho) - \nu)_e = \int (U^\mu - V^\mu) d(Bal(\mu;\rho) - \nu) = \int_{\omega(\mu)} (U^\mu - V^\mu) d(Bal(\mu;\rho) - \nu) = \int_{\omega(\mu)} (U^\mu - V^\mu) d(Bal(\mu;\rho) - \rho) + \int_{\omega(\mu)} (U^\mu - V^\mu) d(\rho - \nu)$. Here the first term vanishes and the second term is ≥ 0, so the proof is done. □

Above we have limited our study only to positive μ and bounded domains Ω. We have also mainly considered positive ρ. These restrictions has been made to avoid the proofs and notation from becoming too involved. Let's just state that all the theory above extends with only minor modifications to unbounded domains, to measures of the form $\mu^+ - \mu^-$ where $\mu^+ \in M_c$ and $\mu^- \in M_c \cap L^\infty$, and to the case where ρ isn't necessarily nonnegative.
(When Ω is unbounded we have to assume that ρ is not to small at infinity, so we assume that outside a sufficiently large ball we have $\rho \ge C > 0$ where C is constant).

Example 1: If μ has support inside Ω, and $\rho = 0$, then we get classical balayage. (We said earlier that partial balayage is more general than classical balayage. But this is not entirely true since we have an additional regularity assumption about the boundary of Ω).

Example 2: Let $\Omega = \mathbb{R}^N$ and $\rho = 1$. We shall discuss without proofs some properties of this special case, which is of interest in several applications, for instance in inverse problems concerning bodies of constant density, problems in analytic function theory and Hele-Shaw flows.

As to the structure of $Bal(\mu; 1)$ ($\mu \in M_c$), if we define

$\Omega(\mu)$:=the largest open set where $Bal(\mu; 1) = 1$,

then $Bal(\mu; 1) = \chi_{\Omega(\mu)} +$ some remainder.

If $\mu \in L^\infty$, then we have $Bal(\mu; 1) = \chi_{\Omega(\mu)} + \mu\chi_{(\Omega(\mu))^c}$. If furthermore μ is concentrated enough, for instance singular with respect to Lebesgue measure, then $\mu\chi_{(\Omega(\mu))^c} = 0$. That is, in this case we have $Bal(\mu; 1) = \chi_{\Omega(\mu)}$.

Moreover, some geometric properties of $Bal(\mu; 1)$ can be proved. For instance, if H is any closed half-space that contains $supp(\mu)$, then the part of $Bal(\mu; 1)$ that lies outside H is of the form $\chi_{\Omega(\mu)\setminus H}$. Furthermore, if we for simplicity (and without loss of generality) assume that $H = \{x \ : \ x_N \leq 0\}$ then $\Omega(\mu) \setminus H = \{x \ : \ 0 < x_N < g(x_1, ..., x_{N-1})\}$, where g is real analytic. If K denotes the closed convex hull of $supp(\mu)$, then from the above we can see that outside K, $Bal(\mu; 1)$ is on the form $\chi_{\Omega(\mu)}$, $\partial\Omega(\mu) \setminus K$ is real analytic and $\forall x \in (\partial\Omega(\mu)) \setminus K$, the inner pointing normal to $\partial\Omega(\mu)$ at x crosses K.

In close connection to this example is the following theorem, stated without proof: If $\mu \in M_c$, then there is at most one open set Ω (up to null sets) such that

$$\begin{cases} U^{\chi_\Omega} = U^\mu & \text{a.e.} \quad \textit{outside} \quad \Omega \\ U^{\chi_\Omega} \leq U^\mu & \text{in} \quad \mathbb{R}^N \end{cases} \tag{1}$$

These two statements are equivalent to the one that $Bal(\mu; 1) = \chi_{\Omega(\mu)}$. If μ satisfies one of the following conditions, then there really is an Ω satisfying (1)above, and $\Omega = \Omega(\mu)$ up to null sets.

a) $d\mu = f dx$, where for some open set $D \subset \mathbb{R}^N$ we have

$$\begin{cases} f \geq 1 & \text{in} \quad D, \\ f = 0 & \text{a.e.} \quad \text{outside} \quad D \end{cases}$$

b) μ is singular with respect to $\mathcal{L}^N$.

c) $supp(\mu) \subset B_r(a)$, $\mu(B_r(a)) \geq 2^N |B_r(a)|$.

Example 3: Here we shall consider a physical problem called Hele-Shaw flow. Assume that the plane domain Ω_0 is filled with a viscous incompressible fluid, and assume that we have a point source δ_0 at a point inside Ω_0, which we choose to be the origin. Then we get a growing family of domains Ω_t that is filled with fluid at time t. The problem is now to determine the domain Ω_t as a function of t and Ω_0. The Hele-Shaw equation states that $\partial\Omega_t$ has a velocity in the point $x \in \partial\Omega_t$ given by the harmonic measure in x w.r.t. 0. That is, it is given by $-\frac{\partial G_{\Omega_t}(x,o)}{\partial N_x}$. Now assume that ϕ is subharmonic and Ω_t is a regular solution of the Hele-Shaw problem. Then

$$\frac{d}{dt}\int_{\Omega_t}\phi dx = -\int_{\partial\Omega_t}\phi(x)\frac{\partial G_{\Omega_t}(x,0)}{\partial N_x}dS(x) =$$

$$-\int_{\partial\Omega_t}G_{\Omega_t}(x,0)\frac{\partial\phi(x)}{\partial N_x}dS(x) - \int_{\Omega_t}\phi\Delta G_{\Omega_t}dx + \int_{\Omega_t}G_{\Omega_t}\Delta\phi dx \geq \phi(0)$$

By integrating from 0 to t we get

$$\int_{\Omega_t}\phi dx - \int_{\Omega_0}\phi dx \geq t\phi(0).$$

Put $\mu_t = t\delta_0 + \chi_{\Omega_0}$. Then $\int_{\Omega_0}\phi d\mu_t \leq \int_{\Omega_t}\phi dx$. Choosing $\phi(x) = \pm\Phi(x-y)$ for $y \notin \Omega_t$ and $\phi(x) = -\Phi(x-y)$ for $y \in \Omega_t$ we see from (1) in Example 2 that $Bal(\mu_t;1) = \chi_{\Omega_t}$, hence $\Omega_t = \Omega(\mu_t)$.

Example 4:
Consider $Bal(\mu;e^{-t})$ where t goes from $-\infty$ to 0. In this way we can see partial balayage as a continuous process. (It is reasonable that $Bal(\mu;e^{-t}) \to \mu$ as $t \to -\infty$ in some sense). This idea can be applied to a similar Hele-Shaw problem to the one above. If we now study two parallel plates with a drop of viscous incompressible fluid between, and wants to describe the domain Ω_t that the fluid covers at time t, when the plates are pressed together such that the distance between them are given by some function $f(t)$. Assuming that $f(t) = e^{-t}$, it turns out that we have $\chi_{\Omega_t} = Bal(e^{t-s}\chi_{\Omega_s};1)$ $\forall s < t$ $(s,t \geq 0)$. Especially we have $\chi_{\Omega_t} = Bal(e^t\chi_{\Omega_0};1)$ $\forall t \geq 0$. Note that as above we have that $Bal(e^{t-s}\chi_{\Omega_s};1)$ really is on the form χ_Ω for some open set Ω.

Example5: There is a ball $B_r(0)$ such that we have $U^{\delta_0} = U^{\chi_{B_r(0)}}$.

Example 6: Let $N = 2$, $r > 1$. Put $\mu_r = \pi r^2(\delta_{-1} + \delta_1)$, and $\Omega_r = \{(x,y) \in \mathbb{R}^2 : (x^2+y^2) - 2(x^2-y^2) - 2r^2(x^2+y^2) < 0\}$. Then $Bal(\mu_r;1) = \chi_{\Omega_r}$.

13. Mother Bodies

Reference material for this section is [B18], [P1], [P4] and [P5].

We gave in the previous section a brief description of Hele-Shaw flow. For the Hele-Shaw problem in Example 4 we have (in some sense) that $e^{-t}\chi_{\Omega_t} = Bal(\mu;e^{-t}) \to \mu$ as $t \to -\infty$, and if $supp(\mu)$ is small enough. Running the Hele-Shaw process backwards (i.e. letting $t \to -\infty$) one can retrieve from the initial domain Ω_0 the measure μ, which may be interpreted, under suitable circumstances, as a potential theoretic skeleton (mother body) of Ω_0.

To fix ideas let's study a ball in $\mathbb{R}^N$, with unit density say. It's well known that a smaller ball with the same center and same total mass (constant density) gives the same potential field outside the original ball. The same is of course also true for the Dirac measure at the center times the balls total mass. What we wish to call a mother body for the ball is this point mass. With this vague motivation we give the definition.

Definition 13.1. *We call a bounded domain Ω a body if it satisfies $\Omega = int(\overline{\Omega})$. A mother body for Ω is a positive Radon measure μ with support in Ω such that:*
1) $U^\mu = U^{\chi_\Omega}$ in $(\overline{\Omega})^c$,
2) $U^\mu \geq U^{\chi_\Omega}$ in $\mathbb{R}^N$,
3) $\mathcal{L}^N(supp(\mu)) = 0$,

4) Each component of $\mathbb{R}^N \setminus supp(\mu)$ *meets* $(\overline{\Omega})^c$.
(Note as in (1) *in Example 2 in the previous section that 1) and 2) are equivalent to* $Bal(\mu; 1) = \chi_\Omega$*).*

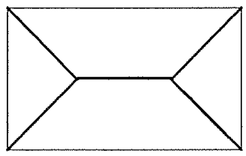
We show that a mother body necessarily possess some extremality properties ($supp(\mu)$ is minimal, U^μ is maximal).

Theorem 13.2. *Let* μ *satisfy 1),3),4) above with respect to a given body* Ω*, and suppose that* η *is another Radon measure.*
a) If η *satisfies 1) and* $supp(\eta) \subset supp(\mu)$*, then* $\eta = \mu$.
b) If η *satisfies 1),3) and* $U^\eta \geq U^\mu$ *in* $\mathbb{R}^N$*, then* $\eta = \mu$.

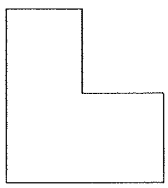
Proof. Put $u = U^\eta - U^\mu$ and let D be a component of $\mathbb{R}^N \setminus supp(\mu)$. In a) u is harmonic in D, so by harmonic continuation we have $u = 0$ in D, hence the statement follows. In b) u is superharmonic and nonnegative, but it must take on the value 0, so $u = 0$ in D in this case as well. □

An important question now comes up naturally: given a body Ω, does it have a mother body and is it unique? We can immediately say that far from all bodies have a mother body, and those which have one might very well have several. One positive result is that a convex polyhedron always has a unique mother body.

Example 1: For a rectangle in $\mathbb{R}^2$, there is a mother body with support on the diagonals like in the figure below.



From this it is easy to see that a domain like the one below has a mother body, but it is not unique.



(In general for a convex polyhedron, the unique mother body has support on a finite set of hyperplanes reaching the boundary only at corners and edges).

Example 2: Let $\Omega \subset \mathbb{R}^2$ be a conformal image of the unit disc under a polynomial of degree (strictly) greater than one. Then Ω does not have a mother body!

Example 3: Mother bodies has a close resemblance to the more classical notion of quadrature domains. The most narrow definition of a quadrature domain would be about the same as for a mother body, but with 3)-4) strenghtened to the much stronger requirement that *supp*(μ) be a finite set in Ω. A wider definition is the following.

Definition 13.3. *Let Ω be an open domain in $\mathbb{R}^N$, and assume that $F = \{u_\alpha : \alpha \in I\}$ is a family of harmonic functions in $L^1(\Omega)$. Then we say that Ω is a quadrature domain with respect to F if there is a distribution μ with compact support inside Ω such that $\int_\Omega u_\alpha dx = \mu(u_\alpha)\ \forall \alpha \in I$.*

As an important special case we can consider $F = F_\Phi = \{u_y(x) = \Phi(x-y) : y \in (\overline{\Omega})^c\}$, and see the connection with mother bodies. But there are some obvious differences. We allow μ to be more general in the sense that it is only required to be a distribution, on the other hand the support of a mother body can very well reach $\partial\Omega$ (as in the case of convex polyhedra).

Example 4: We wish to study a ball in $\mathbb{R}^N$, say $B_1(0)$ for simplicity.

If $N \geq 3$ put $u(x) = \{\frac{k}{|x|^{N-2}},\ \ k = \frac{1}{N(N-2)},\ \ |x| \geq 1 : \frac{|x|^2}{2N} + b,\ \ b = \frac{-1}{2(N-2)},\ \ |x| < 1\}$. Then we have $u \in C^1(\mathbb{R}^N)$, $\Delta u = \chi_{B_1(0)}$ and $u(x) \to 0$ as $|x| \to \infty$. So $u(x) = \int_{B_1(0)} \Phi(x-y)dy$, but outside $B_1(0)$ we have $u(x) = \frac{1}{N(N-2)|x|^{N-2}} = \frac{|B_1(0)|}{|\partial B_1(0)|(N-2)|x|^{N-2}} = |B_1(0)|\Phi(x)$. That is, $B_1(0)$ gives rise to the same external potential as a point mass (of the same total weight) at the origin.

If $N = 2$ we use another approach. Note that if Ω is an open bounded domain and $\partial\Omega$ is C^1 say, then if we use the Gauss-Green theorem and the Cauchy-Riemann equations we obtain $\int_\Omega f dxdy = (2i)^{-1}\int_{\partial\Omega} f(z)\bar{z}dz$, for all functions f that are analytic in some domain $D \supset \overline{\Omega}$. Now on $\partial B_1(0)$ we have $\frac{1}{z} = \bar{z}$, so $\int_{B_1(0)} f dxdy = (2i)^{-1}\int_{\partial B_1(0)} f(z)\frac{1}{z}dz$.

But $f(z)\frac{1}{z}$ is analytic in $D\setminus\{0\}$, so we can take any smaller ball, for instance, and get $\int_{\partial B_1(0)} f(z)\frac{1}{z}dz = \int_{\partial B_r(0)} f(z)\frac{1}{z}dz = \int_{\partial B_r(0)} f(z)\frac{\bar{z}}{r}dz = \frac{1}{r}\int_{B_r(0)} f dxdy$. So any ball $B_r(0)$ with density $\frac{1}{r}$ $(r \leq 1)$ gives the same potential outside $B_1(0)$. Especially as a limiting case we get our point mass at the origin producing the same potential as stated earlier.

So the ball $B_1(0)$ (and any other ball as well) is an example of a quadrature domain in its strongest sense. Now the point mass producing the same potential outside the ball is a mother body for the ball. Note that the minimal support of this mother body easily gives uniqueness in this case!

Example 5: Now we shall consider an ellipse instead of a ball (in $\mathbb{R}^2$) using similar methods as in the previous example. Put $\Omega = \{(x,y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1\}$, where $a > b > 0$, and $S(z) := \frac{a^2+b^2}{a^2-b^2}z + \frac{2ab}{a^2-b^2}(z^2 - a^2 + b^2)^{1/2}$. Notice that $S(z) = \bar{z}$ on $\partial\Omega$ ($S(z)$ is the so called Schwarz function to $\partial\Omega$). So if f is analytic in $D \supset \overline{\Omega}$ we have as in the previous example $\int_\Omega f dxdy = \int_{\partial\Omega} f(z)S(z)dz$. Now notice that $S(z)$ is analytic everywhere except at the two poles in $\pm(a^2 - b^2)^{1/2} := \pm c$. This gives that any two ellipses with the same value of c, constant density and same total mass gives the same potential outside the largest one. Taking a limit we obtain $\int_\Omega f dxdy = \frac{2ab}{c^2}\int_{[-c,c]} f(x)(x^2 - c^2)^{1/2}dx$.

So the ellipse is a quadrature domain , and the last equation gives us a mother body for it as well (with $\mu = \frac{2ab}{c^2}(x^2 - c^2)^{1/2}$ on $[-c, c]$ and 0 elsewhere). However, uniqueness is not as trivial as in the previous example, because we have to consider that the support of another mother body might split $\Omega \setminus supp(\mu)$ into several parts, where some might not connect with Ω^c.

In [P7] a similar (although more geometric) definition of a mother body is used, and some problems of uniqueness is treated (including that of an ellipse).

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