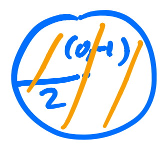
IDAG: TRIPPELINTEGRALER

Börjar med ett exempel från 2 variabler.

Ex $$\iint_D \left(3-2y-x^2-y^2\right)dxdy = ?$$

$$D: \overbrace{x}^{u}{}^2+(\overbrace{y+1}^{v})^2<4$$



Polära koordinater med centrum i $(0,-1)$:

$$\begin{cases} x = r\cos\theta \\ y = -1+r\sin\theta \end{cases}$$

$$\boxed{dxdy = r\,dr\,d\theta}$$

Alternativt:

$$\begin{cases} x = u \\ y = -1+v \end{cases}$$

$(x,y) \longleftrightarrow (u,v)$

$\updownarrow$ polära

(r,θ)

$$\frac{\partial(x,y)}{\partial(u,v)} = 1$$

$$\boxed{dxdy = dudv = r\,dr\,d\theta}$$

$$\iint_D \left(4-x^2-(y+1)^2\right)dxdy = \iint_{\tilde{D}:\, u^2+v^2\le 4} \left(4-u^2-v^2\right)dudv$$

$$= [\text{polära}] = \iint_{\substack{0\le r\le 2\\ 0\le\theta\le 2\pi}} \left(4-r^2\right)r\,dr\,d\theta =$$

$$= [\text{produktformeln}] = \int_0^2 (4-r^2) r\,dr \int_0^{2\pi} d\theta$$

$$= \int_0^2 (4r - r^3)\,dr \, [\theta]_0^{2\pi} = \left[2r^2 - \frac{r^4}{4}\right]_0^2 (2\pi - 0)$$

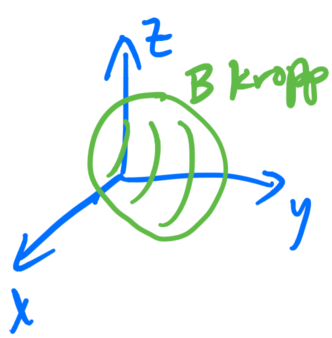
$$= \Big(\underbrace{2\cdot 2^2}_{8} - \underbrace{\frac{2^4}{4}}_{4} - 0\Big) \cdot \underbrace{2\pi}_{2\pi} = 4 \cdot 2\pi = 8\pi.$$

TRIPPELINTEGRALER

tre variabler (x, y, z)

funktion $f(x,y,z)$

Kropp B



$$\iiint_B f(x,y,z) \underbrace{dx\,dy\,dz}_{dV} = ?$$ Hur räknar vi ut?

Tidigare :
$$\frac{\iint_D f(x,y)\,dx\,dy}{\iint_D dx\,dy} = \langle f \rangle_D$$

$\iint_D dx\,dy$: area (D)

$\langle f \rangle_D$: medelvärde av f på D.

Tolkning : $\iint_D f(x,y)\,dx\,dy = \langle f \rangle_D \, \text{area}(D)$.

Fungerar i 3 var också :

$$\iiint_B f(x,y,z)\,dx\,dy\,dz = \langle f \rangle_B \cdot \text{vol}(B).$$

Specialfall : $\rho(x,y,z) \geq 0$ täthet densitet $[kg/m^3]$

totala massan : $\iiint_B \rho(x,y,z)\,dx\,dy\,dz$ $[kg]$

ρ: kg/m^3, $dx\,dy\,dz$: $m \cdot m \cdot m$

I samband med en given täthet:

TYNGDPUNKT $(\langle x \rangle, \langle y \rangle, \langle z \rangle)$

$$\langle x \rangle = \frac{\iiint_B x\rho\,dV}{\iiint_B \rho\,dV}, \quad \langle y \rangle = \frac{\iiint_B y\rho\,dV}{\iiint_B \rho\,dV}$$

$$\langle z\rangle = \frac{\iiint_B z\rho\, dV}{\iiint_B \rho\, dV}$$

$(\langle x\rangle, \langle y\rangle, \langle z\rangle)$ är en punkt i rummet.

OBS: Vid homogent material tag $\rho = 1$.

Notera att $\iiint_B dx\,dy\,dz = \text{vol}(B)$.

Hur räknar vi ut trippelintegraler?

Beskriv B:

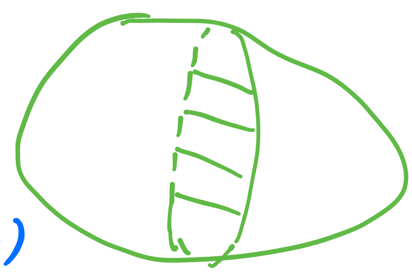
$$(*)\begin{cases} a_1 \le x \le b_1 \\ a_2(x) \le y \le b_2(x) \\ a_3(x,y) \le z \le b_3(x,y) \end{cases}$$

en av 6 st möjligheter byt roller

strimlingsmetod.

Alternativt t.ex.

$$\begin{cases} a_3 \le z \le b_3 \\ a_2(z) \le y \le b_2(z) \\ a_1(y,z) \le x \le b_1(y,z) \end{cases}$$



Iterationsmetoden för $(*)$:

$$\iiint_B f(x,y,z)\,dx\,dy\,dz =$$

$$= \int_{a_1}^{b_1} \left(\int_{y=a_2(x)}^{y=b_2(x)} \left(\int_{z=a_3(x,y)}^{z=b_3(x,y)} f(x,y,z)\,dz \right) dy \right) dx.$$

Speciellt rätblock : $\begin{cases} a_1 \le x \le b_1 \\ a_2 \le y \le b_2 \\ a_3 \le z \le b_3 \end{cases}$

$$\iiint_B f(x,y,z)\,dx\,dy\,dz = \int_{x=a_1}^{x=b_1} \left(\int_{y=a_2}^{y=b_2} \left(\int_{z=a_3}^{z=b_3} f(x,y,z)\,dz \right) dy \right) dx$$

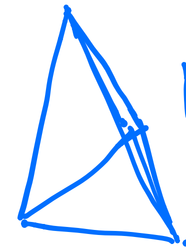
OBS : I det här fallet blir det samma resultat i vilken ordning vi integrerar.

Produktfunktion vid rätblock:

$$f(x,y,z) = F(x)\,G(y)\,H(z):$$

$$\iiint_{\substack{a_1 \le x \le b_1 \\ a_2 \le y \le b_2 \\ a_3 \le z \le b_3}} F(x)\,G(y)\,H(z)\,dx\,dy\,dz = \int_{a_1}^{b_1} F(x)\,dx \cdot \int_{a_2}^{b_2} G(y)\,dy \cdot \int_{a_3}^{b_3} H(z)\,dz.$$

Ex. T tetraeder

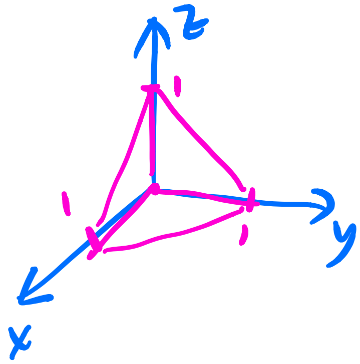


pyramid

hörn $(0,0,0)$ $(0,1,0)$
$(1,0,0)$ $(0,0,1)$

orig. plus längs koordinataxlarna.

$$I = \iiint_T y \, dV = ?$$



beskriv T så vi kan räkna ut integralen!

$$\begin{cases} 0 \le x \le 1 \\ 0 \le y \le 1 \\ 0 \le z \le 1 \\ \underbrace{x+y+z \le 1}_{z \le 1-x-y} \end{cases}$$

$$\begin{cases} 0 \le x \le 1 \\ 0 \le y \le 1-x \\ 0 \le z \le 1-x-y \end{cases}$$ beskriver tetraedern

$$\iiint_T y \, dV = \int_{x=0}^{x=1} \left(\int_{y=0}^{y=1-x} \left(\int_{z=0}^{z=1-x-y} y \, dz \right) dy \right) dx$$

$$= \int_0^1 \left(\int_{y=0}^{y=1-x} \Big[y z \Big]_{z=0}^{z=1-x-y} dy \right) dx =$$

$$= \int_0^1 \left(\int_{y=0}^{y=1-x} \Big(\underbrace{y(1-x-y) - 0}_{y-xy-y^2} \Big) dy \right) dx =$$

$$= \int_0^1 \left[\frac{y^2}{2} - x\frac{y^2}{2} - \frac{y^3}{3} \right]_{y=0}^{y=1-x} dx =$$

$$= \int_0^1 \left(\frac{(1-x)^2}{2} - x\frac{(1-x)^2}{2} - \frac{(1-x)^3}{3} - 0 \right) dx$$

$$= \begin{bmatrix} \text{variabelbyte } t = 1-x \\ dt = -dx \end{bmatrix} =$$

$$= \int_1^0 \left(\frac{t^2}{2} - (1-t)\frac{t^2}{2} - \frac{t^3}{3} \right)(-dt)$$

$$= \int_0^1 \left(\frac{t^2}{2} - \overbrace{(1-t)\frac{t^2}{2}}^{\frac{t^2}{2} - \frac{t^3}{2}} - \frac{t^3}{3} \right) dt =$$

$$= \int_0^1 \left(\frac{t^3}{2} - \frac{t^3}{3}\right) dt = \int_0^1 \frac{t^3}{6} dt = \left[\frac{t^4}{24}\right]_0^1 = \frac{1}{24}.$$

$\left(\frac{1}{2} - \frac{1}{3} = \frac{1}{6}\right)$

Ex. (bakvänt!)

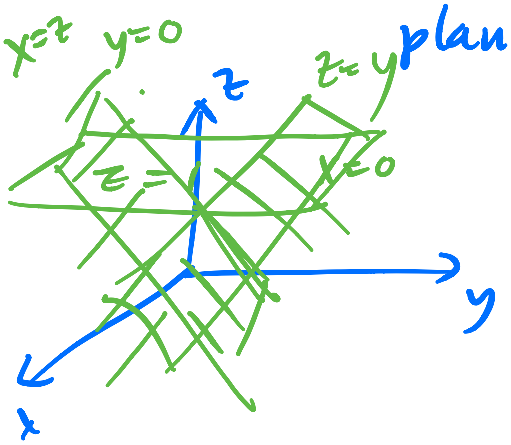
$$I = \int_0^1 \left(\int_y^1 \left(\int_0^z f(x,y,z)\, dx \right) dz \right) dy$$

Beskriv I som $\iiint_B$, vilket B?

$$\begin{cases} 0 \le x \le z \\ y \le z \le 1 \\ 0 \le y \le 1 \end{cases} \Leftrightarrow \begin{cases} 0 \le y \le 1 \\ y \le z \le 1 \\ 0 \le x \le z \end{cases}$$

Kropp som begränsas av planen $\begin{cases} x=0, \quad x=z \\ z=y, \quad z=1 \\ y=0 \end{cases}$

"diamant"



VARIABELBYTE I 3 VARIABLER

$$dV = dx\,dy\,dz$$

$$\begin{cases} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w) \end{cases} \Leftrightarrow \begin{cases} u = u(x,y,z) \\ v = v(x,y,z) \\ w = w(x,y,z) \end{cases}$$

$$dV = \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{pmatrix} x'_u & x'_v & x'_w \\ y'_u & y'_v & y'_w \\ z'_u & z'_v & z'_w \end{pmatrix}$$

$$\boxed{\iiint_B f(x,y,z)\,dx\,dy\,dz = \iiint_{\tilde{B}} f(x(u,v,w), y(u,v,w), z(u,v,w)) \cdot \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw}$$

$\tilde{B}$ är kroppen B uttryckt i (u,v,w)-koordinaterna.

Ex. Volymen av en ellipsoid

$B: \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1, \quad a,b,c>0.$

$$\iiint_B dx\,dy\,dz = \mathrm{vol}(B)$$

Nya variabler: $\begin{cases} x = au \\ y = bv \\ z = cw \end{cases}$ linjärt variabelbyte.

$\tilde{B}: \quad \frac{(au)^2}{a^2} + \frac{(bv)^2}{b^2} + \frac{(cw)^2}{c^2} \leq 1$ dvs

$u^2 + v^2 + w^2 \leq 1$ sfäriskt klot.

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{pmatrix} x'_u & x'_v & x'_w \\ y'_u & y'_v & y'_w \\ z'_u & z'_v & z'_w \end{pmatrix} =$$

$$= \det \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = abc > 0$$

Detta ger att $\left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| = abc.$

Vi får att

$$\iiint_{B} dx\,dy\,dz = \iiint_{\tilde{B}} \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw =$$

$$= \iiint_{\tilde{B}} abc\; du\,dv\,dw = abc \iiint_{\tilde{B}} du\,dv\,dw$$

$$= abc \cdot vol(\tilde{B}) = abc \cdot \frac{4\pi}{3} \cdot 1^3 =$$

$$= \frac{4\pi}{3} abc \text{ blir volymen.}$$

Cylindriska och sfäriska koordinater i trippelintegraler.

$dV = ?$

Cylindriska : $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z' \end{cases}$ $(x,y,z) \leftrightarrow (r,\theta,z)$

$$dV = dx\,dy\,dz = \left| \frac{\partial(x,y,z)}{\partial(r,\theta,z')} \right| dr\,d\theta\,dz'$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z')} = \det\begin{pmatrix} x'_r & x'_\theta & x'_{z'}\\ y'_r & y'_\theta & y'_{z'} \\ z'_r & z'_\theta & z'_{z'}\end{pmatrix} =$$

$$= \det\begin{pmatrix}\cos\theta & -r\sin\theta & 0\\ \sin\theta & r\cos\theta & 0\\ 0 & 0 & 1\end{pmatrix} = \text{(kalkyl)}$$

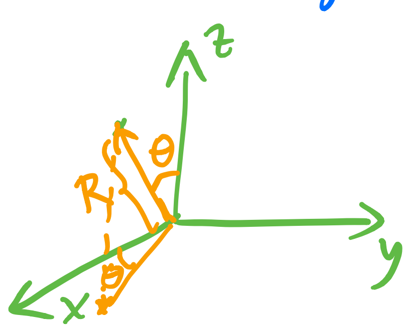
$$= r$$

$dV = r\,dr\,d\theta\,dz$ som polära i (x,y)

Sfäriska koordinater

$$\begin{cases} x = R\sin\phi\cos\theta \\ y = R\sin\phi\sin\theta \\ z = R\cos\phi \end{cases} \qquad (x,y,z) \leftrightarrow (R,\phi,\theta)$$

$$dV = dx\,dy\,dz = \underbrace{R^2\sin\phi}_{\text{Jakobianen}}\, dR\,d\phi\,d\theta$$



Kan vi använda sfäriska koordinater för att räkna ut volymen på en sfär av radie 1?

$$\text{volym} = \iiint_{x^2+y^2+z^2 \le 1} dx\,dy\,dz = [\text{sfäriska}]$$

$$= \iiint_{\substack{0 \le R \le 1 \\ 0 \le \phi \le \pi \\ 0 \le \theta \le 2\pi}} R^2 \sin\phi \, dR\, d\phi\, d\theta = [\text{produktformel}]$$

$$= \int_0^1 R^2\,dR \int_0^{\pi} \sin\phi\, d\phi \int_0^{2\pi} d\theta =$$

$$= \left[\frac{R^3}{3}\right]_0^1 \left[-\cos\phi\right]_0^{\pi} \left[\theta\right]_0^{2\pi} =$$

$$= \left(\frac{1}{3} - 0\right)\Big(\underbrace{-(-1)+1}_{2}\Big)(2\pi - 0) = \frac{4\pi}{3}.$$