Kedjeregeln i flera variabler

Differentialer

Homogena funktioner

Linjärapproximation och diff-barhet

Metod för kedjeregeln via differentialer.

Gå tillbaka till 1 variabel.

$$\frac{d}{dt} f(x(t)) = f'(x(t)) \cdot \underbrace{x'(t)}_{\text{inre derivata}}$$

$x = x(t)$ både funktion och alternativ variabel.

$$\boxed{\frac{df}{dt} = \underbrace{\frac{df}{dx}}_{f'(x)} \cdot \underbrace{\frac{dx}{dt}}_{x'}}$$

Vår fråga: hur blir det i flera variabler?

$f(x,y)$, $\quad \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$

(u,v) relaterade till (x,y)

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$$

$(u,v) \longleftrightarrow (x,y)$

tänk polära koordinater

$\frac{\partial f}{\partial u} = ?$ $\quad \frac{\partial f}{\partial v} = ?$

$\boxed{df}$ förändringen av storheten f.

den är oberoende av koordinat-beskrivningen ((u,v) alt. (x,y)).

Vi minns linjärapprox. vi diskuterade förra gången:

$$f(a+h, b+k) \approx f(a,b) + (h,k) \cdot \nabla f(a,b)$$

$h \approx 0$
$k \approx 0$

$$\nabla f(a,b) = \left(\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b) \right)$$

i (x,y)-koordinater.

$$\underbrace{f(a+h, b+k) - f(a,b)}_{\Delta f} \approx (\underbrace{h}_{\Delta x}, \underbrace{k}_{\Delta y}) \cdot \nabla f(a,b)$$

$(a,b) \rightsquigarrow (x,y)$ $\quad$ $h \rightsquigarrow \Delta x$, $k \rightsquigarrow \Delta y$

$$\Delta f \approx (\Delta x, \Delta y) \cdot \nabla f(x,y) =$$
$$= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \cdot (\Delta x, \Delta y) =$$
$$= \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y$$

Byt $\begin{cases} \Delta x \rightsquigarrow dx \\ \Delta y \rightsquigarrow dy \\ \Delta f \rightsquigarrow df \end{cases}$ Vi tänker oss att dessa storheter är infinitesimala

$$\boxed{df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy}$$ differential

Hur får vi till en kedjeregel?

Förenkla lite först.

Tänk att $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ vi följer en parameter-kurva

$$dx = \frac{dx}{\cancel{dt}} \cancel{dt}$$
$$dy = \frac{dy}{\cancel{dt}} \cancel{dt}$$

vanliga kedjeregeln

Stoppa in detta i differentialformeln:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy =$$
$$= \frac{\partial f}{\partial x}\frac{dx}{dt}dt + \frac{\partial f}{\partial y}\frac{dy}{dt}dt =$$
$$= \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}\right)dt$$

Dela med dt på bägge sidor:

$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}}$$

Kedjeregel för parameterkurvor.

Hur blir det vid ett äkta koordinatbyte (t.ex. polära koordinater)?

(u,v) introduceras då.

Låt först v vara konstant.

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$$

$\left.\begin{array}{l} v \text{ konstant} \\ u \text{ varierar} \end{array}\right\}$ parameterkurva!

Använd u istället för t!

$$\frac{df}{du} = \frac{\partial f}{\partial x}\frac{dx}{du} + \frac{\partial f}{\partial y}\frac{dy}{du}$$ (OBS: v konstant)

$\rightsquigarrow$ Kom i håg att partiella derivator var samma som att hålla variabler konstanta: ger att

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

Byt roller för u och v:

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

Ex: $z = \sin(x^2 y)$. $\begin{cases} x = st^2 \\ y = s^2 + \frac{1}{t} \end{cases}$

Beräkna $\frac{\partial z}{\partial s}$ och $\frac{\partial z}{\partial t}$ genom att:

(a) använda Kedjeregeln i 1 var

(b) använde Kedjeregeln i 2 var.

Lösn. (b) först.

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \quad \text{ger att}$$

$$\begin{cases} \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\ \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \end{cases}$$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(\sin(x^2y)\right) \overset{\text{kedje}}{=} \cos(x^2y)\cdot 2xy$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\sin(x^2y)\right) \overset{\text{kedje}}{=} \cos(x^2y)\cdot x^2$$

$$\begin{cases} x = st^2 \\ y = s^2 + \frac{1}{t} \end{cases} \quad \text{ger} \quad \begin{cases} \frac{\partial x}{\partial s} = t^2, \ \frac{\partial x}{\partial t} = 2st \\ \frac{\partial y}{\partial s} = 2s+0, \ \frac{\partial y}{\partial t} = 0 - \frac{1}{t^2} \end{cases}$$

Stoppa in:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} =$$

$$= 2xy\cos(x^2y)\cdot t^2 + x^2\cos(x^2y)\cdot 2s$$

$$= \cos(x^2y)\left[2xyt^2 + 2sx^2\right]$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} =$$

$$= 2xy\cos(x^2y)\cdot 2st + x^2\cos(x^2y)\cdot\left(-\frac{1}{t^2}\right) =$$

$$= \cos(x^2y)\left[4xyst - \frac{x^2}{t^2}\right]$$

Kommentar: här har vi mixat variablerna (x,y) och (s,t). Mer renlärigt borde vi gå över till (s,t) enbart!

$$\begin{cases} x = st^2 \\ y = s^2 + \frac{1}{t} \end{cases}.$$

Del (a): gör hemma! Kolla att svaret blir samma.

Polära koordinater

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

f som beror på (x,y) alt (r,θ).

$$\begin{cases} \frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} \\ \frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} \end{cases}$$

$$\frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial x}{\partial \theta} = -r\sin\theta$$

$\frac{\partial y}{\partial r} = \sin\theta, \quad \frac{\partial y}{\partial \theta} = r\cos\theta$

Då blir formlerna

$$\begin{cases} \frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta \\ \frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y} r\cos\theta \end{cases}$$

Kolla i specialfallet $f = xy$.

$\frac{\partial f}{\partial x} = y, \quad \frac{\partial f}{\partial y} = x.$

$$\frac{\partial f}{\partial r} = y\cos\theta + x\sin\theta = r\sin\theta\cos\theta + r\cos\theta\sin\theta$$
$$= 2r\sin\theta\cos\theta$$

$$\frac{\partial f}{\partial \theta} = -yr\sin\theta + xr\cos\theta =$$
$$= -r\sin\theta\, r\sin\theta + r\cos\theta\, r\cos\theta =$$
$$= r^2(\cos^2\theta - \sin^2\theta)$$

Alt. direkt metod: $f = xy = r\cos\theta\, r\sin\theta =$
$$= r^2\cos\theta\sin\theta$$

$\frac{\partial f}{\partial r} = 2r\cos\theta\sin\theta, \quad \frac{\partial f}{\partial \theta} = r^2(-\sin\theta\sin\theta + \cos\theta\cos\theta)$

SAMMA!

Ex: $\frac{\partial}{\partial x}\left(f\left(\overbrace{x^2y}^{u}, \overbrace{x+2y}^{v}\right)\right) = ?$

$\frac{\partial}{\partial y}\left(f\left(x^2y, x+2y\right)\right) = ?$

$f(u,v)$ en funktion av två variabler.

$$\begin{cases} \dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial x} \\[2ex] \dfrac{\partial f}{\partial y} = \dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial y} + \dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial y} \end{cases}$$

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial}{\partial x}(x^2y) = 2xy \\[2ex] \dfrac{\partial v}{\partial x} = \dfrac{\partial}{\partial x}(x+2y) = 1+0 = 1 \\[2ex] \dfrac{\partial u}{\partial y} = \dfrac{\partial}{\partial y}(x^2y) = x^2 \\[2ex] \dfrac{\partial v}{\partial y} = \dfrac{\partial}{\partial y}(x+2y) = 0+2 = 2 \end{cases}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\cdot 2xy + \frac{\partial f}{\partial v}\cdot 1$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\cdot x^2 + \frac{\partial f}{\partial v}\cdot 2$$

$$\begin{cases} \frac{\partial}{\partial x}\left(f(x^2y, x+2y)\right) = 2xy\, f_1'(x^2y, x+2y) \\ \qquad\qquad + f_2'(x^2y, x+2y) \\ \frac{\partial}{\partial y}\left(f(x^2y, x+2y)\right) = x^2 f_1'(x^2y, x+2y) \\ \qquad\qquad + 2 f_2'(x^2y, x+2y) \end{cases}$$

SVAR

Homogena funktioner

Spec. homogena polynom.

x^2+2xy hom. polynom i 2 var., grad 2

$x^3-2xy^2+y^3$ hom pol. i 2 var, grad 3,

$\boxed{\begin{array}{l}\text{DEF. } f(x,y) \text{ är homogen av grad } k \\ \text{om } \quad f(tx,ty) = t^k f(x,y) \\ \text{för alla } t>0.\end{array}}$

Ex. $f(x,y) = \sqrt{x^2+y^2}$ homogen av grad 1.

$$f(tx,ty) = \sqrt{(tx)^2+(ty)^2} =$$
$$= \sqrt{t^2(x^2+y^2)} = \sqrt{t^2}\sqrt{x^2+y^2} =$$
$$= [t>0] = t\sqrt{x^2+y^2} = t f(x,y).$$

EULERS SATS. Om $f(x,y)$ är "snäll" och av grad k så gäller att

$$\boxed{x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = kf.}$$

"snäll" = C^1-glatt

Varför?

$$f(tx,ty) = t^k f(x,y)$$

$\frac{\partial}{\partial t}$ på bägge sidor (x,y konstanta)

Kedjeregeln (2 var)

$$\frac{\partial}{\partial t}(f(tx,ty)) = \frac{\partial f}{\partial x}(tx,ty)\cdot x + \frac{\partial f}{\partial y}(tx,ty)\cdot y$$

$$\frac{\partial}{\partial t}\left(t^k f(x,y)\right) = k t^{k-1} f(x,y)$$

lika! $\quad x \frac{\partial f}{\partial x}(tx,ty) + y\frac{\partial f}{\partial y}(tx,ty) =$

$$= k t^{k-1} f(x,y)$$

Stoppa in $t=1$:

$$x\frac{\partial f}{\partial x}(x,y) + y\frac{\partial f}{\partial y}(x,y) = k f(x,y)$$

KLART!

Vad är differentierbarhet?
deriverbarhet?

$\frac{\partial f}{\partial x}(a,b)$ finns
$\frac{\partial f}{\partial y}(a,b)$ finns
$\Big\}$ deriverbar i meningen att partialderivatorna finns.

Ex. $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & \text{om } (x,y)=(0,0). \end{cases}$

$(a,b)=(0,0)$.

$f(x,0)=0$ samt $f(0,y)=0$.

Detta ger att $f'_x(0,0) = 0$ och $f'_y(0,0) = 0$.

Samtidigt är f inte kont. i origo!

Skiljer sig från 1-variabelskursen!

$$\text{Deriverbar} \overset{1\text{var}}{\Longrightarrow} \text{kont}$$

Hitta ett deriverbarhetsbegrepp som är bättre!

$$f(x,y) - L(x,y) = o\Big(\big|\overbrace{(x,y)-(a,b)}^{|(h,k)|}\big|\Big)$$

för linjärapprox $L(x,y)$ till f i punkten (a,b). <u>Differentierbarhet</u>

$$L(x,y) = f(a,b) + (\overbrace{x-a}^{h})\,f'_x(a,b) + (\overbrace{y-b}^{k})\,f'_y(a,b)$$

Jfr förra gången
$$x = a + h$$
$$y = b + k$$