$\boxed{\text{Ö3}}$

3.1 5, 13, 23, 25, 35, 42

3.2 5, 19

3.3 5, 15

5) Pb-209 $t_{1/2} = 3.3$ timmar

1 g isotop, initialt.

Hur lång tid tar det för 90% att sönderfalla?

$X(t) =$ antal g. $X_0 = X(0) = 1$.

$$X'(t) = -kX(t)$$

$X(t) = X_0 e^{-kt}$ lösn. $X(3.3) = \frac{1}{2} X_0$ så

$e^{-3.3k} = \frac{1}{2}$, dvs $k = \frac{1}{3.3} \ln 2$.

$0.1g = X(t_1) = X_0 e^{-kt_1}$, $t_1 = \frac{\ln 10}{k} = 3.3 \cdot \frac{\ln 10}{\ln 2} \approx 11$ tim b)

②

$$\frac{dT}{dt} = -k(T - T_m)$$

$T(0) = 70°F$

$T_m = 10°F.$

$T(\tfrac{1}{2}\text{ min}) = 50°F.$

$T(1) = ?$

$$T(t) = T_m + Ce^{-kt}$$

$$T(\tfrac{1}{2}) = 50 = \underbrace{T_m}_{10} + Ce^{-k/2}$$

$$Ce^{-k/2} = 40$$

$$T(0) = 70 = \overset{=10}{\widetilde{T_m}} + C \Rightarrow C = 60$$

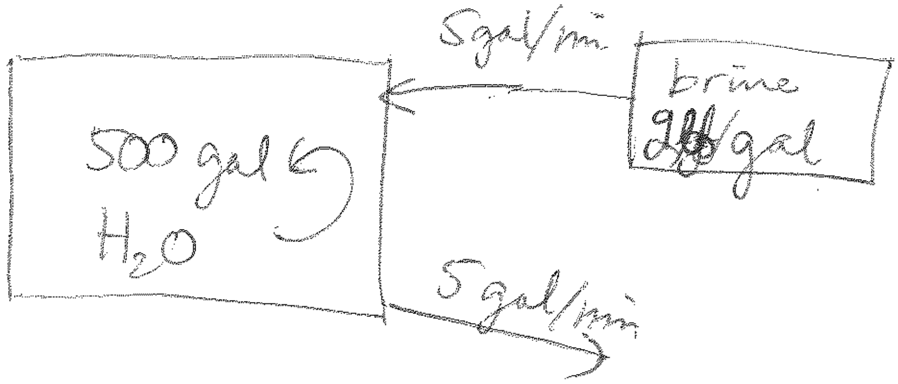
$$60\,e^{-k/2} = 40$$

after 1 min;

$$e^{k/2} = \frac{3}{2} \Rightarrow e^{k} = \frac{9}{4} \Rightarrow T(1) = T_m + Ce^{-k}$$

$$= 10 + 60 \cdot \frac{4}{9} \simeq 36.7°F$$

3)



$A(t)$ = lb's of salt in tank at time t.

$A(0) = 0.$

konc. i tanken

$$A'(t) = 10 - \frac{A}{500} * 5 =$$

$$= 10 - \frac{A}{100}.$$

$$\frac{dA}{10 - \frac{A}{100}} = dt$$

$$\frac{100\,dA}{1000 - A} = dt \qquad 100 \ln(1000 - A) = -t + C_1$$

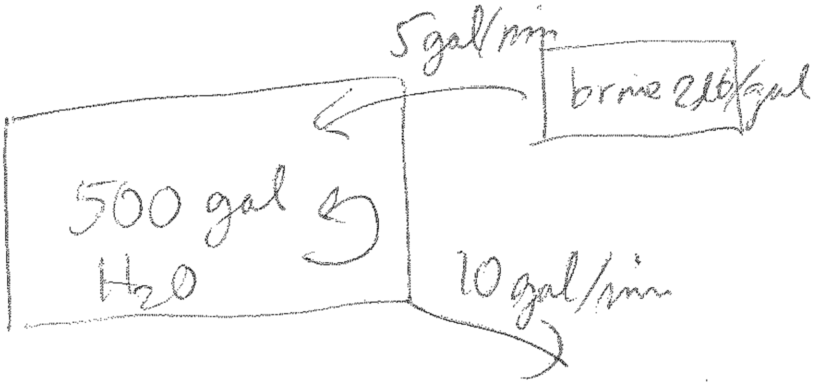
$$1000 - A = C_2 e^{-t/100} \qquad (C_2 > 0)$$

$$A = 1000 - C_2 e^{-t/100}$$

$$C_2 = 1000 : \qquad A = 1000(1 - e^{-t/100})$$

5)

4



Tanken töms med 5 gal/min så tanken är tom efter 100 minuter.

$$A'(t) = 10 - \frac{A}{500-5t} \cdot 10$$

$$A' + \frac{10}{\underbrace{500-5t}_{5(100-t)}} A = 10.$$

$$\boxed{A' + \frac{2}{100-t} A = 10.}$$

$$\int \frac{2}{100-t} dt = -2 \ln(100-t) + C_0.$$

$$\left((100-t)^{-2} A\right)' = \frac{10}{(100-t)^2}$$

$$(100-t)^{-2} A = \frac{10}{100-t} + C_1.$$

$$A = 10(100-t) + C_1(100-t)^2$$

$$A(0) = 0.$$

$$A(0) = 10\cdot(100-0) + C_1\cdot(100-0)^2 =$$

$$= 1000 + C_1\, 10000 = 0 \Rightarrow C_1 = -\frac{1}{10}$$

$$\boxed{A = 10(100-t) - \frac{1}{10}(100-t)^2}$$

$0 \leq t \leq 100.$

---

35) $\begin{cases} m\dfrac{dv}{dt} = mg - kv \\ v(0) = v_0. \end{cases}$ $(k>0).$

$\frac{dv}{dt} = 0$ (crit.

$mg = kv_{crit}$

$v_{crit} = \frac{mg}{k}.$

$$m v' + kv = mg$$

$$m\left(e^{\frac{k}{m}t} v\right)' = mg\, e^{\frac{k}{m}t}$$

$$m\, e^{\frac{k}{m}t} v = \frac{m^2 g}{k} e^{kt/m} \mp C_1.$$

$$\boxed{v = \frac{mg}{k} - \frac{C_1}{m} e^{-\frac{k}{m}t}}$$

$v_0 = v(0) = \frac{mg}{k} - \frac{C_1}{m}$

$C_1 \geq 0$ $\frac{C_1}{m} = \frac{mg}{k} - v_0$

$$S(t) = \int_0^t \left[\frac{mg}{k} - \left(\frac{mg}{k} - v_0\right) e^{-\frac{k}{m}\tau}\right] d\tau =$$

$$= \frac{mg}{k} t - \left(\frac{m}{k} - v_0\right)\left[-\frac{m}{k} e^{-\frac{k}{m}\tau}\right]_0^t =$$

$$= \frac{mg}{k} t - \frac{m}{k}\left(\frac{m}{k} - v_0\right)\left(1 - e^{-\frac{kt}{m}}\right)$$

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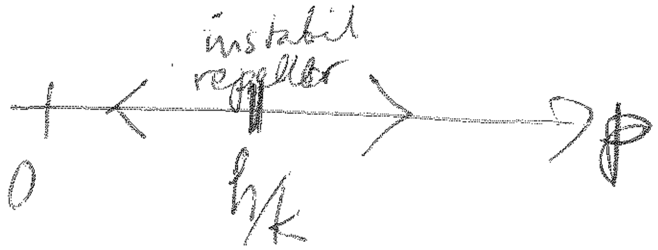
42) fiskestånd $\begin{cases} \frac{dP}{dt} = kP - h \\ P(0) = P_0 \end{cases}$

Hur blir populationen?



$$P' - kP = -h$$

$$\left(e^{-kt} P\right)' = -h e^{-kt}$$

$$e^{-kt} P = \frac{h}{k} e^{-kt} + C_1$$

$$P(0) = \frac{h}{k} + C_1$$

$$P = \frac{h}{k} + C_1 e^{kt} = \frac{h}{k} + \left(P(0) - \frac{h}{k}\right) e^{kt}$$

5) fisk-skörd

$$\begin{cases}\frac{dP}{dt}=P(a-bP)-h\\ P(0)=P_0\end{cases}$$

$a, b, h, P_0 > 0$ konstanter

$a=5,\ b=1,\ h=4$

$$\begin{cases}P'=P(5-P)-4\\ P(0)=P_0\end{cases}$$

Kritisk P:

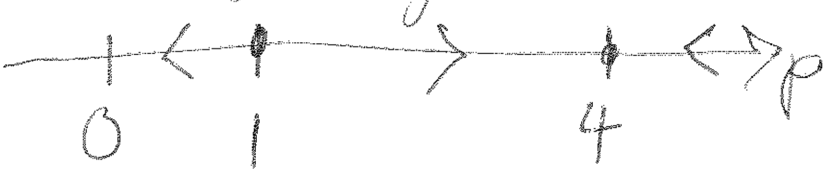
$P(5-P)-4=0.$

$P^2-5P+4=0.$

$$P=\frac{5}{2}\pm\sqrt{\frac{25}{4}-4}=$$

$$=\frac{5}{2}\pm\frac{3}{2}=\begin{cases}4\\1\end{cases}$$

fasdiagram



$P_0 > 4 \Rightarrow$ sjunkande population mot 4

$1 < P_0 < 4 \Rightarrow$ väx mot 4

$0 < P_0 < 1 \Rightarrow$ population dör ut

8

b) $P' = P(5-P) - 4 = -P^2 + 5P - 4$

$= -(P^2 - 5P + 4) =$

$= -(P-1)(P-4)$

$$\frac{dP}{(P-1)(P-4)} = -dt$$

$$\frac{1}{P-1} - \frac{1}{P-4} = \frac{P-4-P+1}{(P-1)(P-4)}$$

$$\frac{1}{3}\left(\frac{1}{P-4} - \frac{1}{P-1}\right) dP = -dt$$

$$\frac{1}{3} \ln \frac{|P-4|}{|P-1|} = -t + C_1.$$

$$\frac{P-4}{P-1} = C_3 e^{-3t}$$

$\frac{P-4}{P-1} = Y, \quad (P-4) = Y(P-1)$

$P(1-Y) = 4 - Y$

$P = \frac{4-Y}{1-Y}$

$$P = \frac{4 - C_3 e^{-3t}}{1 - C_3 e^{-3t}}$$

19) $$\frac{dW}{dx} = W\sqrt{4-2W}$$

$W(x)$ = höjden på tsunami

a) constant solutions :

$$0 = W\sqrt{4-2W}$$

$$\begin{cases} W = 0 \text{ eller} \\ W = 2 \end{cases}$$

$v = \sqrt{u} = \sqrt{2-W}$

b) Lös DE: $$\frac{dW}{W\sqrt{4-2W}} = dx$$

$$\int \frac{dW}{W\sqrt{(4-2W)}} = \left[\begin{matrix} u = 2-W \\ W = 2-u \end{matrix}\right] = -\int \frac{du}{(2-u)\sqrt{2}\sqrt{u}} =$$

$$= \left[u = v^2\right] = -\int \frac{2v\,dv}{(2-v^2)\sqrt{2}\,v} = -\sqrt{2}\int \frac{dv}{2-v^2} =$$

$$= -\frac{1}{2}\int\left(\frac{1}{\sqrt{2}-v} + \frac{1}{\sqrt{2}+v}\right)dv = -\frac{1}{2}\ln\left|\frac{v+\sqrt{2}}{v-\sqrt{2}}\right| + C_1$$

$$\ln\left|\frac{v+\sqrt{2}}{v-\sqrt{2}}\right| = -2x + C_2 \Rightarrow \frac{v+\sqrt{2}}{v-\sqrt{2}} = C_3 e^{-2x}$$

$$\frac{\sqrt{2-W}+\sqrt{2}}{\sqrt{2-W}-\sqrt{2}} = C_3 e^{-2x}$$

$$\cancel{\frac{\sqrt{2-W}+\sqrt{2}}{\sqrt{2}}} \qquad \frac{X+\sqrt{2}}{X-\sqrt{2}} = Y$$

$$X+\sqrt{2} = XY - \sqrt{2}Y$$

$$X(1-Y) = -\sqrt{2}(1+Y)$$

$$X = -\sqrt{2}\,\frac{1+Y}{1-Y}$$

$$\sqrt{2-W} = -\sqrt{2}\,\frac{1+C_3 e^{-2x}}{1-C_3 e^{-2x}} = \sqrt{2}\,\frac{1+C_3 e^{-2x}}{C_3 e^{-2x}-1}$$

$$\boxed{2-W = 2\frac{(1+C_3 e^{-2x})^2}{(1-C_3 e^{-2x})^2}} = 2\left(\frac{e^{-2x}+a_4}{e^{-2x}-a_4}\right)^2$$

W(0)=2 här gäller inte entydighetssatsen!

$C_3 = -1$

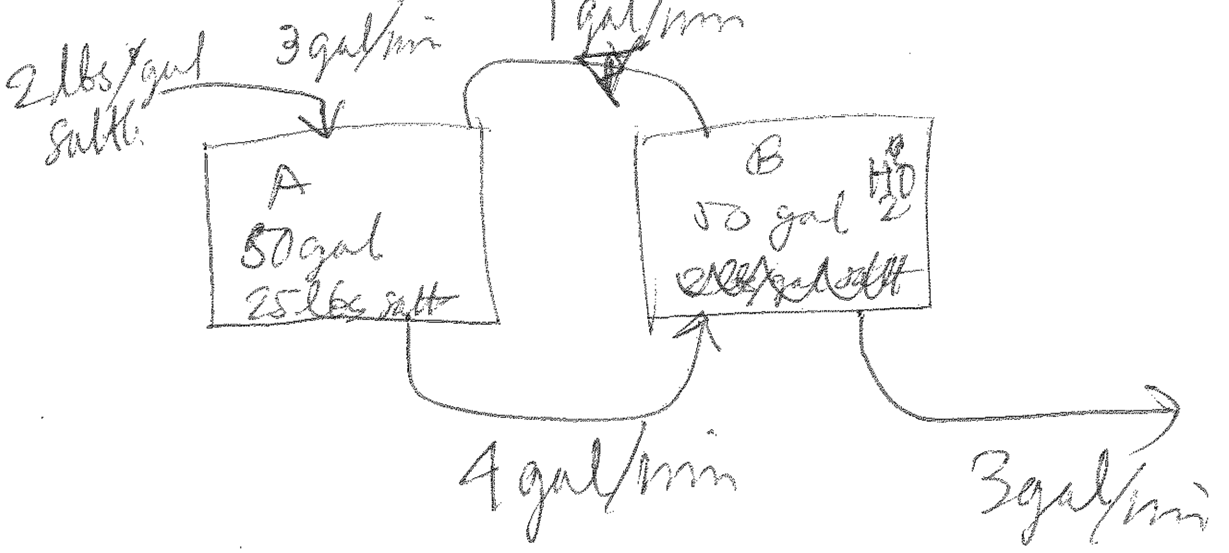
$$W = 2\left[1-\frac{(1-e^{2x})^2}{(1+e^{2x})^2}\right]$$

$$W(t) = \begin{cases} 2 & \text{om } a \le x \le b \\ 2\left[1-\left(\frac{1-e^{2(a-x)}}{1+e^{2(a-x)}}\right)^2\right] & \text{om } x < a \\ 2\left[1-\left(\frac{1-e^{2(b-x)}}{1+e^{2(b-x)}}\right)^2\right] & \text{om } x > b \end{cases}$$

3.3

5)
$$\begin{cases} \dfrac{dx_1}{dt} = -\dfrac{2}{25}x_1 + \dfrac{1}{50}x_2 \\[2ex] \dfrac{dx_2}{dt} = \dfrac{2}{25}x_1 - \dfrac{2}{25}x_2 \end{cases}$$

Vad ändras?



$$\begin{cases} \dfrac{dx_1}{dt} = 3\,\text{gal/min} \cdot 2\,\text{lb/gal} + 1\,\text{gal/min} \cdot \dfrac{x_2}{50}\,\text{lb/gal} \\ \qquad - 4\,\text{gal/min} \cdot \dfrac{x_1}{50} \\[2ex] \dfrac{dx_2}{dt} = 4\dfrac{x_1}{50} - 3\dfrac{x_2}{50} - 1 \cdot \dfrac{x_2}{50} \end{cases}$$

$$\begin{cases} \dfrac{dx_1}{dt} = 6 + \dfrac{x_2}{50} - \dfrac{2}{25}x_1 \\[2ex] \dfrac{dx_2}{dt} = \dfrac{2x_1}{25} - \dfrac{2x_2}{25} \end{cases}$$

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15) $s(t)$ — susceptible
$i(t)$ — infected
$r(t)$ — recovered

$$\begin{cases} \dfrac{ds}{dt} = -k_1 s i \\[2ex] \dfrac{di}{dt} = -k_2 i + k_1 s i = i(k_1 s - k_2) \\[2ex] \dfrac{dr}{dt} = k_2 i \end{cases}$$

SIR model.

Ge rimliga initialvillkor.

Om $s_0 < \dfrac{k_2}{k_1}$ så kommer $i \searrow$ avt.

$s \searrow$ alltid

Om istället $s_0 > k_2/k_1$ så växer i ett tag, tills s avtar till dess att $s < \dfrac{k_2}{k_1}$.

$s > \dfrac{k_2}{k_1}$ i början!