LUNDS UNIVERSITET

Matematikcentrum
Matematik MNF

Tentamensskrivning
Analytiska funktioner
Måndagen 26 november 2001, 8.00–13.00

Inga hjälpmedel. Använd institutionens papper och skriv på bara den ena sidan. Fyll i omslaget fullständigt och skriv initialer på varje ark. Skriv tydligt. Ge klara och kortfattade motiveringar; rita gärna figur i förekommande fall.

1. Find all (complex) roots of the equation $\cos z = i$.

2. Let D be an open set in the plane, and suppose that f is holomorphic there; you can assume that this means it is continuously differentiable and satisfies the Cauchy-Riemann equations on D. Show that the squared function $(f)^2$ also is holomorphic in D by verifying the Cauchy-Riemann equations for it.

3. Do a Laurent series expansion of the function

$$f(z) = \sin(2z) + \cos(1/z)$$

around the origin. Also, calculate the integral

$$\int_\Gamma z^{-100} f(z)\, dz,$$

where Γ is the curve $|z| = 2$ in the counter-clockwise direction.

4. Evaluate the integral

$$\int_0^{+\infty} \frac{\sin x}{x}\, dx,$$

interpreted in the generalized sense of Riemann.

5. Determine whether there exist functions f_1, f_2, f_3, with the following properties:
(a) f_1 is holomorphic in the unit disk $|z| < 1$ with $f_1(1/n^2) = 1/n$ for all $n = 2, 3, 4, 5, \ldots$,
(b) f_2 is holomorphic in the unit disk $|z| < 1$ and has $\mathrm{Re}\,[f_2(x)] = \sqrt{1-x}$ for *real* $x \in]-1, 1[$, and
(c) f_3 is holomorphic in the whole plane with real part $e^y \cos x$, where $z = x + iy$ is the complex variable.

6. Suppose we are given that F is an analytic function in the unit disk $\{z \in \mathbb{C} : |z| < 1\}$, and that

$$|F(z)| \le e^{-1/|z|}$$

holds there. What can you say about F?