Lunds Universitet
Matematik MNF
Matematikcentrum

Tentamensskrivning

Linear Functional Analysis
December 14, 1999
10.00–15.00

No calculators, books, etc, allowed; only pencil and paper. Explain solutions as best you can. Partial credit is normally offered for correct partial solutions.

1. Let $\mathcal{D}$ be the set of all complex-valued sequences $\{a_n\}_{n=1}^{\infty}$ with

$$\sum_{n=1}^{\infty} n\,|a_n|^2 < +\infty.$$

Show that we can introduce a norm on $\mathcal{D}$ which makes it a Banach space. Show that in fact, there is an inner product on $\mathcal{D}$ which is compatible with this norm, making $\mathcal{D}$ a Hilbert space.

2. Let $C[0,1]$ be the usual space of complex-valued continuous functions on the interval $[0,1]$, equipped with the uniform (=supremum) norm on $[0,1]$. Let Λ_n be the sequence of bounded linear functionals

$$\Lambda_n(f) = n\int_0^{1/n} f(t)\,dt - n\int_{1-1/n}^{1} f(t)\,dt + \int_{1/4}^{1/2-1/n} f(t)\,dt.$$

a) Show that Λ_n has a weak-star limit as $n \to +\infty$, and find that limit Λ_∞.

b) Does the sequence Λ_n tend to Λ_∞ in the norm?

3. Let ℓ^2 be the usual space of complex-valued square summable sequences $\{x_n\}_{n=0}^{\infty}$, with the standard norm. Let T be the operator

$$T(x_0, x_1, x_2, \ldots) = (x_1, x_2, x_3, \ldots).$$

a) Show that T is a contraction; indeed, $\|T\| = 1$. Is T an isometry?

b) Find the adjoint operator T^*. Is T^* an isometry?

c) Describe the operators T^*T and TT^*.

d) That a nonzero vector $x \in \ell^2$ is an eigenvector for T means that $Tx = \lambda x$ holds for some complex scalar λ; λ is then called the eigenvalue. Find the eigenvalues and eigenvectors of T.

e) Does T^* have any eigenvalues and eigenvectors?

f) The spectrum of T is the set of all complex scalars λ for which the operator $\lambda I - T$ is not invertible, where I stands for the identity operator. What is the spectrum of T?

4. Let H^∞ be the space of bounded analytic functions on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ in the complex plane $\mathbb{C}$. It is known that H^∞ is a Banach space, if it is supplied with the uniform (= supremum) norm on $\mathbb{D}$. A sequence $\{z_1, z_2, z_3, \ldots\}$ of distinct points in $\mathbb{D}$ is said to be an interpolation sequence if to any bounded sequence of complex numbers $\{w_1, w_2, w_3, \ldots\}$ we can find a function $f \in H^\infty$ such that $f(z_j) = w_j$ for $j = 1, 2, 3, \ldots$. Prove that if $\{z_j\}_{j=1}^\infty$ is an interpolation sequence, then there exists a constant M such that the interpolation problem can be solved with a function $f \in H^\infty$ with

$$\sup_{z \in \mathbb{D}} |f(z)| \le M \sup_j |w_j|.$$

5. Let p have $0 < p < +\infty$, and consider the space ℓ_2^p, which as a vector space is $\mathbb{R}^2$, but is supplied with the "norm-type expression"

$$\|(x_1, x_2)\|_p = (|x_1|^p + |x_2|^p)^{1/p}.$$

a) Draw the closed unit ball of ℓ_2^p, and decide for which p the above norm is a Banach space norm on $\mathbb{R}^2$.

b) Let $\Lambda : \mathbb{R}^2 \to \mathbb{R}$ be the linear functional $\Lambda(x_1, x_2) = 5x_1 - 3x_2$. Calculate the norm of Λ as a functional on ℓ_2^p. Recall that as usual, the norm of a functional is the supremum of the (modulus) of the functional on the unit ball.

c) Let M be the subspace $\{(x_1, x_2) : x_1 + 2x_2 = 0\}$. Calculate the norm of $\Lambda|_M$, the restriction of Λ to M.

d) For those p that make ℓ_2^p a Banach space, Hahn-Banach's theorem applies, and we obtain an extension $\widetilde{\Lambda}$ of $\Lambda|_M$ with the same norm. Is it unique? What about those p for which ℓ_2^p is not a Banach space: does there exist a Hahn-Banach-type extension nevertheless?

6. Let X be a Banach space, and let M be a linear subspace of X. Suppose M is not dense in X. Prove that there exists a bounded linear functional $\Lambda \in X^*$, $\Lambda \neq 0$, such that $\Lambda|_M = 0$.

7. Let $C[0,1]$ be the usual space of complex-valued continuous functions on the interval $[0,1]$, equipped with the uniform (=supremum) norm on $[0,1]$. Let $T : C[0,1] \to C[0,1]$ be the (integration) operator

$$Tf(x) = \int_0^x f(t)\,dt, \qquad x \in [0,1].$$

Show that T is a compact operator, which means that the closure of the image of the unit ball under T is compact.