HYPERBOLIC FOURIER SERIES AND THE KLEIN-GORDON EQUATION

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ABSTRACT. In an effort to extend classical Fourier theory, Hedenmalm and Montes-Rodríguez (2011) found that the function system

$$e_m(x) = e^{i\pi mx}, \quad e_n^\dagger(x) = e_n(-1/x) = e^{-i\pi n/x}$$

is weak-star complete in $L^\infty(\mathbb{R})$ when m, n range over the integers with $n \neq 0$. It turns out that the system can be used to provide unique representation of functions and more generally distributions on the real line $\mathbb{R}$. For instance, we may represent uniquely the unit point mass at a point $x \in \mathbb{R}$:

$$\delta_x(t) = A_0(x) + \sum_{n \neq 0} \left(A_n(x) e^{i\pi nt} + B_n(x) e^{-i\pi n/t} \right),$$

with at most polynomial growth of the coefficients, so that the sum converges in the sense of distribution theory. In a natural sense, the system $\{A_n, B_n\}_n$ is biorthogonal to the initial system $\{e_n, e_n^\dagger\}_n$ on the real line. More generally, for a distribution f on the compactified real line, we may decompose it in a *hyperbolic Fourier series*

$$f(t) = a_0(f) + \sum_{n \neq 0} \left(a_n(f) e^{i\pi nt} + b_n(f) e^{-i\pi n/t} \right),$$

understood to converge in the sense of distribution theory. Such hyperbolic Fourier series arise from two different considerations. One is the Fourier interpolation problem of recovering a radial function ϕ on $\mathbb{R}^d$ from partial information on ϕ and its Fourier transform $\widehat{\phi}$, studied by Radchenko and Viazovska (2019). Another consideration is the interpolation theory of the Klein-Gordon equation $\partial_x \partial_y u + u = 0$. For instance, the biorthogonal system $\{A_n, B_n\}_n$ leads to a collection of solutions that vanish along the lattice-cross of points $(\pi k, 0)$ and $(0, \pi l)$ save for one of these points. These interpolating solutions allow for restoration of a given solution u from its values on the lattice-cross.

1. OVERVIEW

We develop the theory of hyperbolic Fourier series on the real line in Section 3. We first do this in the sense of distribution theory, where it becomes natural to consider harmonic extensions to the upper half-plane. We consider more general harmonic functions in the upper half-plane, and expand them in harmonic hyperbolic Fourier series, which can be viewed as expansion of the boundary "trace" ultradistributions. We explore the relationship between growth and uniqueness of the coefficients in the expansion. The results are typically stated in Section 3 and deferred to Section 7 for

the uniqueness aspects, and to Section 8 for the existence. The key to the proof of the existence is an application of the Schwarz reflection principle for harmonic functions along a circular boundary. This gives rise to an algorithm which successively expands the solution to the entire upper half-plane, with control of the growth of the solution see Section 8). This permits us to obtain sharper coefficient bounds than what was available with the earlier methods based on integral kernels and Eichler cohomology. The expansion of a point mass on the line is particularly curious, as it gives rise to a biorthogonal system of functions. Here, the main results are stated in Section 3 and the proofs are deferred to Section 9. The hyperbolic Fourier series are intimately connected with solutions to the Klein-Gordon equation, and that aspect is studied in Section 4, with proofs supplied in Sections 6 and 9. We also introduce two kinds of skewed hyperbolic Fourier series: power skewed and exponentially skewed. The power skewed hyperbolic Fourier series are connected with recent advances in Fourier interpolation of radial functions, whereas exponential skewing comes from considering specific perturbations the lattice-cross for the interpolation problem for solutions to the Klein-Gordon equation. As for the details, see Section 10. We throw light on the connection with Fourier interpolation for radial functions in Section 11.

A preliminary version of the present work was announced in 2021, see [4].

2. SOME NOTATION

We write $\mathbb{Z}$ for the integers, $\mathbb{R}$ for the real line, $\mathbb{C}$ for the complex plane, $\mathbb{H}$ for the upper half-plane, and we write, e.g., $\mathbb{R}_{>0}$ for the positive reals. We sometimes separate infinities, $-\infty$ and $+\infty$ should be thought of as different, while ∞ unites them in the complex plane. We write $+i\infty$ in place of $i(+\infty)$.

3. HYPERBOLIC FOURIER SERIES

3.1. **Hyperbolic Fourier series: general considerations.** If f is a function on the real line $\mathbb{R}$ which is 2-periodic, i.e. $f(t+2) = f(t)$ holds for all $t \in \mathbb{R}$, we may attempt to represent it in the form of a Fourier series

$$f(t) \sim \sum_{n \in \mathbb{Z}} a_n \, e^{i\pi nt},$$

where we write "$\sim$" in place of "$=$" to indicate that the representation need not be valid pointwise, but is nevertheless correct in a generalized sense. One such reasonable sense is that of distribution theory, which would ask that

$$(3.1.1) \qquad\qquad f(\varphi) = \langle f, \varphi \rangle_{\mathbb{R}} = \sum_{n \in \mathbb{Z}} a_n \langle \varphi, e_n \rangle_{\mathbb{R}},$$

where we write $e_n(t) := e^{i\pi nt}$, provided that φ is a *test function*, which typically could be assumed to be C^∞-smooth and compactly supported. The notation $f(\varphi)$ may be confusing given that we use the function notation $f(t)$ at the same time, so we will prefer to speak of $\langle f, \varphi \rangle_{\mathbb{R}}$ instead. Here, we should specify that we think of the

indicated pairing as an extension of the bilinear form

$$\langle f,g\rangle_{\mathbb{R}} = \int_{\mathbb{R}} f(t)g(t)\mathrm{d}t,$$

which is well-defined, e.g., for f,g measurable with $fg \in L^1(\mathbb{R})$. For smooth test functions φ, the decay $\langle \varphi, e_n \rangle_{\mathbb{R}} = \mathrm{O}(|n|^{-k})$ as $|n| \to +\infty$ holds for any fixed positive integer k, so that convergence of the right-hand side of (3.1.1) holds provided that $|a_n| = \mathrm{O}(|n|^k)$ holds for some fixed k as $|n| \to +\infty$. Please note that by considering a smaller collection of test functions φ, we can get faster decay of $\langle e_n, \varphi \rangle_{\mathbb{R}}$ as $|n| \to +\infty$, which in turn will allow us to make sense of the convergence in (3.1.1) also when the coefficients a_n grow more rapidly as $|n| \to +\infty$. We would however like to represent more general functions or distributions on the line $\mathbb{R}$ which need *not be periodic*. Traditionally, this is done by considering Fourier integrals as the limit of Fourier series with the period tending to infinity. Here, we follow the hunch in [15], which suggests that we should add an additional system of a similar form. As for notation, we let $\mathbb{Z}_{\neq 0} := \mathbb{Z} \setminus \{0\}$ denote the nonzero integers.

Definition 3.1.1. A series of the form

$$a_0 + \sum_{n \in \mathbb{Z}_{\neq 0}} \left(a_n \, \mathrm{e}^{\mathrm{i}\pi n t} + b_n \, \mathrm{e}^{-\mathrm{i}\pi n/t} \right)$$

is called a *hyperbolic Fourier series*. If we agree that $b_0 = 0$, we may also write it in the form

$$\sum_{n \in \mathbb{Z}} \left(a_n \, \mathrm{e}^{\mathrm{i}\pi n t} + b_n \, \mathrm{e}^{-\mathrm{i}\pi n/t} \right).$$

Remark 3.1.2. This defines formal series, no convergence requirement is made whatsoever. It is clear how to add two such series, but it is generally unclear how to multiply them. The difficulty is that the product

$$\mathrm{e}^{\mathrm{i}\pi n t} \mathrm{e}^{-\mathrm{i}\pi m/t} = \mathrm{e}^{\mathrm{i}\pi n t - \mathrm{i}\pi m/t}$$

is not of the required form unless $n = 0$ or $m = 0$.

We want to use hyperbolic Fourier series to represent functions or more generally distributions (or even ultradistributions) on the real line. If we write

$$f_1(t) = a_0 + \sum_{n \in \mathbb{Z}_{\neq 0}} a_n \, \mathrm{e}^{\mathrm{i}\pi n t}, \quad f_2(t) = \sum_{n \in \mathbb{Z}_{\neq 0}} b_n \, \mathrm{e}^{\mathrm{i}\pi n t},$$

which represent 2-periodic distributions on the line if the coefficient growth is at most polynomial, then the hyperbolic Fourier series takes the form

$$f_1(t) + f_2(-1/t) = a_0 + \sum_{n \in \mathbb{Z}_{\neq 0}} \left(a_n \, \mathrm{e}^{\mathrm{i}\pi n t} + b_n \, \mathrm{e}^{-\mathrm{i}\pi n/t} \right).$$

UNDERLYING PHILOSOPHY: If we think in terms of f_1, f_2 as functions, then each of them is pretty much arbitrary on the interval $]-1,1[$, with periodic extensions beyond that interval. In particular, $f_2(-1/t)$ is an arbitrary function on $\mathbb{R} \setminus [-1,1]$, so together with $f_1(t)$, which is arbitrary on $]-1,1[$, there is a chance that we may be able to represent more or less any function or distribution on the line. This is inspired
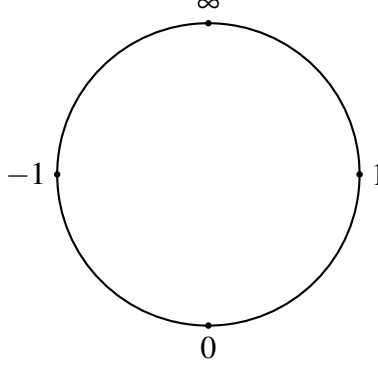
FIGURE 3.1.1. The compactified real line thought of as a circle

by ideas from differential geometry, where we use local patches to describe a given manifold. Here, we would use as coordinate functions $t \mapsto t$ on $]-1,1[$ and $t \mapsto -1/t$ on $\mathbb{R} \setminus [-1,1]$. See Figure 3.1 for a visualization of the compactified real line $\mathbb{R} \cup \{\infty\}$ as a great circle on the Riemann sphere.

3.2. **Convergence of hyperbolic Fourier series to a distribution.** The above definition of hyperbolic Fourier series is formal. We now turn to convergence issues. Let $\mathbf{J}_1, \mathbf{J}_1^*$ denote the transformations

$$(3.2.1) \qquad \mathbf{J}_1 \varphi(t) := t^{-2} \varphi(-1/t), \quad \mathbf{J}_1^* \psi(t) := \psi(-1/t),$$

for real t. By the change-of-variables formula, we know that

$$(3.2.2) \qquad \langle \mathbf{J}_1 \varphi, \psi \rangle_{\mathbb{R}} = \langle \varphi, \mathbf{J}_1^* \psi \rangle_{\mathbb{R}},$$

whenever any of the two sides is well-defined as a Lebesgue integral. If ψ is a distribution on $\mathbb{R}$, we can use (3.2.2) as the definition of $\mathbf{J}_1^* \psi$ as a distribution. This suggests that we should introduce as space of test functions

$$C_1^\infty(\mathbb{R} \cup \{\infty\}) := \{ \varphi \in C^\infty(\mathbb{R}) : \mathbf{J}_1 \varphi \in C^\infty(\mathbb{R}) \},$$

which amounts to considering the space of test functions that give rise to C^∞-smooth 1-forms on the great circle model of $\mathbb{R} \cup \{\infty\}$ (see Figure 3.1). The dual space to $C_1^\infty(\mathbb{R} \cup \{\infty\})$ we call the *distributions on the extended real line* $\mathbb{R} \cup \{\infty\}$. Note that the test function space $C_1^\infty(\mathbb{R} \cup \{\infty\})$ contains the well-known Schwartzian test function space $\mathscr{S}(\mathbb{R})$ as a subspace. We will write

$$(3.2.3) \qquad e_n(t) := e^{i\pi nt}, \quad \mathbf{J}_1^* e_n(t) = e^{-i\pi n/t},$$

to have a condensed notation for the basis elements of hyperbolic Fourier series.

Definition 3.2.1. For a distribution f on the extended real line, we say that the hyperbolic Fourier series expansion

$$f(t) = a_0 + \sum_{n \in \mathbb{Z}_{\neq 0}} \left(a_n e^{i\pi nt} + b_n e^{-i\pi n/t} \right) = a_0 + \sum_{n \in \mathbb{Z}_{\neq 0}} \left(a_n e_n(t) + b_n \mathbf{J}_1^* e_n(t) \right)$$

converges in the sense of distribution theory if for each $\varphi \in C_1^\infty(\mathbb{R} \cup \{\infty\})$,

$$\langle \varphi, f \rangle_\mathbb{R} = a_0 \langle \varphi, 1 \rangle_\mathbb{R} + \sum_{n \in \mathbb{Z}_{\neq 0}} a_n \langle \varphi, e_n \rangle_\mathbb{R} + \sum_{n \in \mathbb{Z}_{\neq 0}} b_n \langle \mathbf{J}_1 \varphi, e_n \rangle_\mathbb{R}$$

holds with absolute convergence.

Remark 3.2.2. The convergence in the above sense of distribution theory holds if and only if the coefficients grow at most polynomially:

(3.2.4) $$\exists k: \ |a_n|, |b_n| = \mathrm{O}(|n|^k) \quad \text{as} \quad |n| \to +\infty.$$

3.3. **Existence and uniqueness of hyperbolic Fourier series expansion.** A first version of the main result on hyperbolic Fourier series runs as follows.

Theorem 3.3.1. *Suppose f is a distribution on the extended real line $\mathbb{R} \cup \{\infty\}$. Then f may be expanded in a hyperbolic Fourier series*

$$f(t) = a_0 + \sum_{n \in \mathbb{Z}_{\neq 0}} a_n \mathrm{e}^{\mathrm{i}\pi nt} + b_n \mathrm{e}^{-\mathrm{i}\pi n/t},$$

with coefficients that grow at most polynomially, i.e., with (3.2.4), so that the series converges in the sense of distribution theory. These coefficients are then uniquely determined by f: $a_0 = a_0(f)$, $a_n = a_n(f)$, $b_n = b_n(f)$.

The proof of Theorem 3.3.1 is supplied in Subsection 8.5 (see, e.g., Remark 8.5.3).

Remark 3.3.2. This theorem was anticipated in the 2011 work of Hedenmalm and Montes-Rodríguez [15], where it was shown that the system of unimodular functions $e_n(t) = \mathrm{e}^{\mathrm{i}\pi nt}$ and $\mathbf{J}_1^* e_n(t) = \mathrm{e}^{-\mathrm{i}\pi n/t}$ together spanned a weak-star dense subspace of $L^\infty(\mathbb{R})$ as n ranges over the integers. It was the additional insight of Viazovska et al which was based on the need to construct specific magic functions connected with optimal sphere packing problems in dimensions 8 and 24 which provided the framework to attain the result, see [27], [11]. We should mention that the work on Fourier interpolation by Radchenko and Viazovska [21] is quite close to the above theorem, see, for instance, Section 11 below.

More precise control may be obtained of the coefficients if the distribution we expand is a point mass.

Corollary 3.3.3. *Suppose δ_x is the associated unit point mass at a point $x \in \mathbb{R}$. This point mass may be expanded in a hyperbolic Fourier series*

$$\delta_x(t) = A_0(x) + \sum_{n \in \mathbb{Z}_{\neq 0}} \left(A_n(x) \mathrm{e}^{\mathrm{i}\pi nt} + B_n(x) \mathrm{e}^{-\mathrm{i}\pi n/t} \right),$$

which converges in the sense of distribution theory. The coefficient functions A_0, A_n, B_n for $n \in \mathbb{Z}_{\neq 0}$ are all in in the test function space $C_1^\infty(\mathbb{R} \cup \{\infty\})$, and enjoy the growth control $A_0(x) = \mathrm{O}(1 + x^2)^{-1}$ and

$$A_n(x), B_n(x) = \mathrm{O}\left(\frac{|n| \log^2(|n| + 1)}{1 + x^2} \right), \qquad n \neq 0,$$

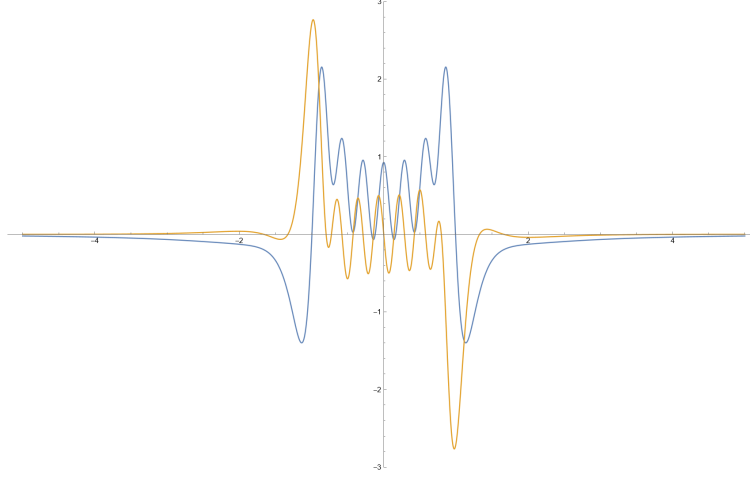*uniformly in n and $x \in \mathbb{R}$.*

FIGURE 3.3.1. Graph of the function $A_n(t)$ for $n = 7$, real part in blue, imaginary part in yellow.

The proof of Corollary 3.3.3 is supplied in Subsection 9.2.

As stated, the coefficients $A_n(x)$ and $B_n(x)$ are smooth functions of x, and we will study them in some detail later on. See Figure 3.3.1 for the graph of the function $t \mapsto A_n(t)$ with $n = 7$. The symmetry properties of the functions A_n, B_n are listed below.

Proposition 3.3.4. *The functions A_0 and A_n, B_n for $n \in \mathbb{Z}_{\neq 0}$ have the following symmetry properties:*

$$A_n(x) = \overline{A_{-n}(x)} = A_{-n}(-x), \qquad n \in \mathbb{Z},\ x \in \mathbb{R},$$

and

$$B_n(x) = \overline{B_{-n}(x)} = B_{-n}(-x), \qquad n \in \mathbb{Z}_{\neq 0},\ x \in \mathbb{R}.$$

They also enjoy the properties that

$$A_0(x) = \mathbf{J}_1 A_0(x) = x^{-2} A_0(-1/x), \quad B_n(x) = \mathbf{J}_1 A_n(x) = x^{-2} A_n(-1/x),$$

for all $x \in \mathbb{R}$ and all $n \in \mathbb{Z}_{\neq 0}$.

Proposition 3.3.4 gets restated in Subsection 9.1 in the form of Proposition 9.1.1. and the proof is supplied there.

In particular, it follows that once we find the functions A_n for $n \in \mathbb{Z}_{\geq 0}$, we automatically have the functions A_0 and A_n, B_n for all $n \in \mathbb{Z}_{\neq 0}$ as well. These functions form a biorthogonal system to the basis (3.2.3) associated with hyperbolic Fourier series.

Corollary 3.3.5. *The functions A_0, A_n, B_n for $n \in \mathbb{Z}_{\neq 0}$ have the following properties:*

$$\langle A_0, \mathbf{J}_1^* e_m \rangle_{\mathbb{R}} = \langle A_0, e_m \rangle_{\mathbb{R}} = \int_{\mathbb{R}} A_0(x) e_m(x) \mathrm{d}x = \delta_{m,0}, \qquad m \in \mathbb{Z},$$

while

$$\langle B_n, \mathbf{J}_1^* e_m \rangle_{\mathbb{R}} = \langle A_n, e_m \rangle_{\mathbb{R}} = \int_{\mathbb{R}} A_n(x) e_m(x) \mathrm{d}x = \delta_{m,n}, \qquad m \in \mathbb{Z}, n \in \mathbb{Z}_{\neq 0},$$

and

$$\langle A_n, \mathbf{J}_1^* e_m \rangle_{\mathbb{R}} = \langle B_n, e_m \rangle_{\mathbb{R}} = \int_{\mathbb{R}} B_n(x) e_m(x) \mathrm{d}x = 0, \qquad m \in \mathbb{Z}, n \in \mathbb{Z}_{\neq 0}.$$

Here, $\delta_{m,n}$ stands for the Kronecker delta, which equals 1 when $m = n$ and vanishes when $m \neq n$.

The proof of Corollary 3.3.5 is supplied in Subsection 9.2. Although the functions A_0 and A_n, B_n for $n \in \mathbb{Z}_{\neq 0}$ are rather mysterious, it is possible to evaluate them at the origin in terms of the counting functions $\mathrm{r}_4(n, \mathbb{Z})$ and $\mathrm{r}_4(n, \mathbb{Z} + \frac{1}{2})$. Here, $\mathbb{Z} + \frac{1}{2}$ stands for the set of all half-integers that are not integers, and, given an $x \in \mathbb{R}_{\geq 0}$ and a subset $A \subset \mathbb{R}$,

$$\mathrm{r}_4(x, A) = \#\big\{ (a_1, a_2, a_3, a_4) \in A^4 : a_1^2 + a_2^2 + a_3^2 + a_4^2 = x \big\}$$

where $\#$ counts the number of occurrences.

Proposition 3.3.6. *We have that*

$$A_0(0) = \frac{4}{\pi^2} \log 2$$

while for $n \in \mathbb{Z}_{\neq 0}$, it holds that

$$A_n(0) = \frac{\mathrm{r}_4(|n|, \mathbb{Z} + \frac{1}{2})}{2\pi^2 |n|}, \quad A_n(0) + B_n(0) = \frac{\mathrm{r}_4(|n|, \mathbb{Z})}{2\pi^2 |n|}.$$

The proof of Proposition 3.3.6 is deferred to a separate publication [14].

We shall need to take Theorem 3.3.1 (which says that distributions have unique hyperbolic Fourier series expansions) beyond the realm of distribution theory to a more general context (ultradistribution theory). To avoid excessive technicalities we shall present the expansions in terms of harmonic extensions to the upper half-plane instead of introducing narrower classes of test functions. This allows us a direct approach to hyperbolic Fourier series expansions, and the appropriate classes of test functions may be considered later on when necessary.

3.4. **Harmonic extension and distribution theory.** Let $P(\tau, t)$ denote the *Poisson kernel*

$$(3.4.1) \qquad P(\tau, t) := \pi^{-1} \frac{\mathrm{Im}\, \tau}{|\tau - t|^2}, \qquad \tau \in \mathbb{H},\ t \in \mathbb{R},$$

where $\mathbb{H}$ denotes the upper half-plane

$$\mathbb{H} := \{ \tau \in \mathbb{C} : \mathrm{Im}\, \tau > 0 \}.$$

We observe that for $\tau \in \mathbb{H}$, the function $P(\tau, \cdot)$ belongs to the space of test functions $C_1^\infty(\mathbb{R} \cup \{\infty\})$, and it depends harmonically on τ. It follows that given a distribution f on the extended real line $\mathbb{R} \cup \{\infty\}$, the formula

$$(3.4.2) \qquad \tilde{f}(\tau) := \langle P(\tau, \cdot), f \rangle_{\mathbb{R}}, \qquad \tau \in \mathbb{H},$$

defines a harmonic function, called the *Poisson extension*, with polynomial growth bound

$$(3.4.3) \qquad |\tilde{f}(\tau)| = \mathrm{O}\left(\frac{1 + |\tau|^2}{\mathrm{Im}\,\tau} \right)^k$$

with an implicit constant that is uniform in $\mathbb{H}$, for some finite k. By considering the distributional limit of $\tilde{f}(\tau)$ as τ approaches the real line, we recover f from $\tilde{f}$. Given a harmonic function $h : \mathbb{H} \to \mathbb{C}$ with the same growth bound as $\tilde{f}$, we get distributional boundary values on $\mathbb{R} \cup \{\infty\}$, and then the harmonic function h equals the Poisson extension of its boundary distribution. In other words, we have the following.

Proposition 3.4.1. *The distributions f on the extended real line are in a one-to-one correspondence with the harmonic functions $\tilde{f}$ subject to the polynomial growth bound* (3.4.3).

We believe this result goes back to the work of Laurent Schwartz. However, lacking a direct reference to Schwartz, we found instead the reference Straube [25]. If we use the Cayley transform to move the upper half-plane to the unit disk, Straube's result applied to the disk gives the assertion. We mention the paper by Korenblum [19] which shows that in the case of the disk the distributional interpretation of the boundary values was well understood in the holomorphic setting. These results are to be applied for the unit disk, and the Cayley transform then moves between the disk and the upper half-plane. In the sequel, we shall equate the harmonic extension $\tilde{f}$ with the distribution f on the extended real line, and hence suppress the tilde notation.

Remark 3.4.2. The connection between smoothness of test functions and the growth on the dual side of holomorphic or harmonic functions can be very detailed. See, e.g., [6] for a study of quasianalytic smoothness in terms of almost holomorphic extensions.

3.5. **Harmonic extension of hyperbolic Fourier series.** If we have a hyperbolic Fourier series expansion

$$f(t) = a_0 + \sum_{n \in \mathbb{Z}_{\neq 0}} \left(a_n e^{\mathrm{i}\pi n t} + b_n e^{-\mathrm{i}\pi n / t} \right) = a_0 + \sum_{n \in \mathbb{Z}_{\neq 0}} \left(a_n e_n(t) + b_n \mathbf{J}_1^* e_n(t) \right)$$

with at most polynomial growth of coefficients (3.2.4), then the series converges in the sense of distribution theory to the distribution f on the extended real line. The harmonic extension to the upper half-plane $\mathbb{H}$ of this equality then reads

$$(3.5.1) \qquad f(\tau) = a_0 + \sum_{n \in \mathbb{Z}_{\neq 0}} \left(a_n e_n(\tau) + b_n (\mathbf{J}_1^* e_n)(\tau) \right), \qquad \tau \in \mathbb{H},$$

where we omit the tildes of the Poisson extensions given by (3.4.2). In particular, we have that

$$(3.5.2) \qquad e_n(\tau) = \mathrm{e}^{\mathrm{i}\pi n \tau} \text{ and } (\mathbf{J}_1^* e_n)(\tau) = \mathrm{e}^{-\mathrm{i}\pi n/\tau}, \qquad n \in \mathbb{Z}_{\geq 0},$$

while

$$(3.5.3) \qquad e_n(\tau) = \mathrm{e}^{\mathrm{i}\pi n \bar{\tau}} \text{ and } (\mathbf{J}_1^* e_n)(\tau) = \mathrm{e}^{-\mathrm{i}\pi n/\bar{\tau}}, \qquad n \in \mathbb{Z}_{<0}.$$

The representation (3.5.1) then naturally splits as

$$(3.5.4) \qquad f(\tau) = a_0 + f_+(\tau) + f_-(\tau),$$

where

$$(3.5.5) \qquad f_+(\tau) := \sum_{n \in \mathbb{Z}_{>0}} \left(a_n \mathrm{e}^{\mathrm{i}\pi n \tau} + b_n \mathrm{e}^{-\mathrm{i}\pi n/\tau} \right), \qquad \tau \in \mathbb{H},$$

is holomorphic, whereas

$$(3.5.6) \qquad f_-(\tau) := \sum_{n \in \mathbb{Z}_{<0}} \left(a_n \mathrm{e}^{\mathrm{i}\pi n \bar{\tau}} + b_n \mathrm{e}^{-\mathrm{i}\pi n/\bar{\tau}} \right), \qquad \tau \in \mathbb{H}.$$

is conjugate-holomorphic. The imposed growth control on the coefficients (3.2.4) guarantees that each of f_+, f_- meets the growth condition (3.4.3). We call the expansion (3.5.1), or, which is the same, (3.5.4) combined with (3.5.5) and (3.5.6), the *harmonic hyperbolic Fourier series expansion* of f. The expansion (3.5.5) is referred to as a *holomorphic hyperbolic Fourier series* without constant term. Likewise, (3.5.6) is a *conjugate-holomorphic hyperbolic Fourier series* without constant term. We are free to add a constant term to the holomorphic hyperbolic Fourier and still call it that. The same goes for the conjugate-holomorphic hyperbolic Fourier series.

3.6. **Hyperbolic Fourier series expansion of a harmonic function.** We begin with the version of Theorem 3.3.1 which applies to harmonic functions in the sense of Proposition 3.4.1.

Theorem 3.6.1. *Suppose $f : \mathbb{H} \to \mathbb{C}$ is harmonic, subject to the growth bound*

$$\exists k : \ |f(\tau)| = \mathrm{O}\left(\frac{1 + |\tau|^2}{\mathrm{Im}\,\tau} \right)^k$$

with an implicit constant that is uniform in $\mathbb{H}$. Then f may be expanded in a harmonic hyperbolic Fourier series

$$f(\tau) = a_0 + \sum_{n \in \mathbb{Z}_{>0}} \left(a_n \mathrm{e}^{\mathrm{i}\pi n \tau} + b_n \mathrm{e}^{-\mathrm{i}\pi n/\tau} \right) + \sum_{n \in \mathbb{Z}_{<0}} \left(a_n \mathrm{e}^{\mathrm{i}\pi n \bar{\tau}} + b_n \mathrm{e}^{-\mathrm{i}\pi n/\bar{\tau}} \right), \qquad \tau \in \mathbb{H},$$

with coefficients that grow at most polynomially, i.e., with (3.2.4) . These coefficients are then uniquely determined by f: $a_0 = a_0(f)$, $a_n = a_n(f)$, $b_n = b_n(f)$.

The proof of Theorem 3.6.1 is supplied in Subsection 8.5.

Remark 3.6.2. The series can be seen to converge in the norm

$$\|f\|_k := \sup_{\tau \in \mathbb{H}} \left(\frac{\mathrm{Im}\,\tau}{1+|\tau|^2} \right)^k |f(\tau)|$$

for some finite k (not necessarily the same as the various parameters k in the theorem). Moreover, since convergence in this norm implies convergence of the boundary distribution, the assertions of Theorems 3.3.1 and 3.6.1 are equivalent.

A Taylor series

$$g(\zeta) = \sum_{n=0}^{+\infty} c_n \zeta^n$$

converges in the unit disk

$$\mathbb{D} := \{\zeta \in \mathbb{C} : |\zeta| < 1\}$$

if and only if the coefficients grow at most subexponentially:

(3.6.1) $\forall \varepsilon > 0 : |c_n| = \mathrm{O}(e^{\varepsilon n}) \quad \text{as } n \to +\infty.$

Writing $\zeta = e^{\mathrm{i}\pi\tau}$, it follows that the Fourier series

$$g(e^{\mathrm{i}\pi\tau}) = \sum_{n=0}^{+\infty} c_n e^{\mathrm{i}\pi n\tau}$$

converges on $\mathbb{H}$ if and only if (3.6.1) holds. Analogously, if we instead write $\zeta = e^{-\mathrm{i}\pi/\tau}$, we again see that

$$g(e^{-\mathrm{i}\pi/\tau}) = \sum_{n=0}^{+\infty} c_n e^{-\mathrm{i}\pi n/\tau}$$

converges on $\mathbb{H}$ if and only if (3.6.1) holds. From these considerations we easily obtain the following proposition.

Proposition 3.6.3. *Suppose that the coefficients a_n, b_n are subject to the subexponential growth bound*

$$\forall \varepsilon > 0 : |a_n|, |b_n| = \mathrm{O}(e^{\varepsilon|n|}) \quad \text{as } |n| \to +\infty.$$

Then the harmonic hyperbolic Fourier series

$$f(\tau) = a_0 + \sum_{n \in \mathbb{Z}_{>0}} \left(a_n e^{\mathrm{i}\pi n\tau} + b_n e^{-\mathrm{i}\pi n/\tau} \right) + \sum_{n \in \mathbb{Z}_{<0}} \left(a_n e^{\mathrm{i}\pi n\bar{\tau}} + b_n e^{-\mathrm{i}\pi n/\bar{\tau}} \right), \qquad \tau \in \mathbb{H},$$

converges absolutely and uniformly on compact subsets to a harmonic function $f : \mathbb{H} \to \mathbb{C}$.

The natural question pops up whether the harmonic hyperbolic Fourier series represent some small subclass of the harmonic functions. Pleasantly, it is not so.

Theorem 3.6.4. *Suppose that $f : \mathbb{H} \to \mathbb{C}$ is a harmonic function. Then there exist coefficients a_n, b_n with the growth bound*

$$\forall \varepsilon > 0 : \ |a_n|, |b_n| = \mathrm{O}(e^{\varepsilon|n|}) \quad as \ |n| \to +\infty,$$

such that f enjoys the harmonic hyperbolic Fourier series expansion

$$f(\tau) = a_0 + \sum_{n \in \mathbb{Z}_{>0}} \left(a_n\, e^{i\pi n\tau} + b_n\, e^{-i\pi n/\tau}\right) + \sum_{n \in \mathbb{Z}_{<0}} \left(a_n\, e^{i\pi n\bar{\tau}} + b_n\, e^{-i\pi n/\bar{\tau}}\right), \qquad \tau \in \mathbb{H}.$$

The proof of Theorem 3.6.4 is supplied in Subsection 8.2.

We note that when we allow for faster growing coefficients, uniqueness is generally speaking lost.

A general reference to elliptic function theory is Chandrasekharan's book [9].

Example 3.6.5. Let $\vartheta_{10}(\tau)$ and $\vartheta_{00}(\tau)$ denote the Jacobi theta functions (with first variable $z = 0$, omitted)

$$(3.6.2) \qquad \vartheta_{10}(\tau) := \sum_{n \in \mathbb{Z}} e^{i\pi(n+\frac{1}{2})^2 \tau}, \qquad \vartheta_{00}(\tau) := \sum_{n \in \mathbb{Z}} e^{i\pi n^2 \tau},$$

and put

$$(3.6.3) \qquad \lambda(\tau) := \frac{\vartheta_{10}(\tau)^4}{\vartheta_{00}(\tau)^4}, \qquad \tau \in \mathbb{H}.$$

This function is called the *modular lambda function*, and since ϑ_{00} is zero-free in $\mathbb{H}$, it is holomorphic in $\mathbb{H}$. It is 2-periodic and hence it has a Fourier expansion

$$(3.6.4) \qquad \lambda(\tau) = \sum_{n=1}^{+\infty} \widehat{\lambda}(n)\, e^{i\pi n\tau}, \qquad \tau \in \mathbb{H},$$

where the coefficients turn out to be integers, the first being $\widehat{\lambda}(1) = 16$, $\widehat{\lambda}(2) = -128$, and $\widehat{\lambda}(3) = 704$. Moreover, it is known that the coefficients have the asymptotics

$$\widehat{\lambda}(n) = (1 + \mathrm{o}(1)) \frac{(-1)^{n-1}}{32} n^{-\frac{3}{4}} \exp(2\pi\sqrt{n}) \quad \text{as } n \to +\infty.$$

The lambda function has the important functional properties

$$(3.6.5) \qquad \lambda(\tau+1) = \frac{\lambda(\tau)}{\lambda(\tau)-1}, \quad \lambda(-1/\tau) = 1 - \lambda(\tau).$$

In particular, the latter property entails that

$$-1 + \lambda(\tau) + \lambda(-1/\tau) = -1 + \sum_{n=1}^{+\infty} \widehat{\lambda}(n)\left(e^{in\tau} + e^{-i\pi n/\tau}\right) = 0, \qquad \tau \in \mathbb{H},$$

a nontrivial representation of the 0 function in a holomorphic hyperbolic Fourier series with coefficients $a_0 = -1$ and $a_n = b_n = \widehat{\lambda}(n)$ for $n \in \mathbb{Z}_{>0}$. This means that uniqueness fails in the harmonic hyperbolic series expansion if we allow the coefficients to grow as quickly as $\widehat{\lambda}(n)$.

Theorem 3.6.6. (uniqueness) *Suppose that the coefficients a_n, b_n are subject to the subexponential growth bound*

$$\forall \varepsilon > 0: \ |a_n|, |b_n| = \mathrm{O}(\mathrm{e}^{\varepsilon |n|}) \quad \text{as } |n| \to +\infty,$$

and that associated harmonic hyperbolic Fourier series is trivial:

$$a_0 + \sum_{n \in \mathbb{Z}_{>0}} \left(a_n \mathrm{e}^{\mathrm{i}\pi n \tau} + b_n \mathrm{e}^{-\mathrm{i}\pi n/\tau}\right) + \sum_{n \in \mathbb{Z}_{<0}} \left(a_n \mathrm{e}^{\mathrm{i}\pi n \bar{\tau}} + b_n \mathrm{e}^{-\mathrm{i}\pi n/\bar{\tau}}\right) = 0, \qquad \tau \in \mathbb{H}.$$

Suppose in addition that

$$|a_n| = \mathrm{o}(|n|^{-3/4} \exp\left(2\pi\sqrt{|n|}\right) \quad \text{as } |n| \to +\infty.$$

Then $a_0 = 0$ and $a_n = b_n = 0$ for all $n \in \mathbb{Z}_{\neq 0}$.

The proof of Theorem 3.6.6 is supplied in Subsection 7.1.

Remark 3.6.7. (a) Theorem 3.6.6 would remain true if we make the growth assumption on the coefficients b_n in place of the a_n. In addition we may replace the two-sided assumption on a_n (in the sense that the assumption concerns the limits as $n \to +\infty$ and as $n \to -\infty$) by a one-sided assumption on a_n, and a likewise one-sided assumption on b_n, as long as the two sides are in opposite directions.

(b) Note that the theorem is sharp in the sense that if the little "o" is replaced by big "O", the assertion fails, by Example 3.6.5.

3.7. **Beyond uniqueness of the harmonic hyperbolic Fourier series expansion.**
We now describe a more general situation where the harmonic hyperbolic Fourier series fails to be unique due to a finite-dimensional space of *exceptional coefficients*.

Definition 3.7.1. The sequence of coefficients $\{a_0, a_n, b_n\}_{n>0}$ is *exceptional of degree* $\leq k$ if there exists a polynomial of P of degree $\leq k$ with $P(0) = 0$, $P(1) = -a_0$, such that

$$\sum_{n=1}^{+\infty} a_n \mathrm{e}^{\mathrm{i}\pi n \tau} = P(\lambda(\tau)), \quad \sum_{n=1}^{+\infty} b_n \mathrm{e}^{\mathrm{i}\pi n \tau} = -a_0 - P(1 - \lambda(\tau)),$$

for all $\tau \in \mathbb{H}$. We write $\mathfrak{E}_k^{\mathrm{hol}}$ for the linear space of all exceptional coefficients of degree $\leq k$, which has dimension k.

We extend the definition to two-sided sequences.

Definition 3.7.2. The two-sided sequence of coefficients $\{a_0, a_n, b_n\}_{n \neq 0}$ is *exceptional of degree* $\leq k$ if for some parameter value $c_0 \in \mathbb{C}$, the two one-sided sequences $\{a_0 - c_0, a_n, b_n\}_{n>0}$ and $\{c_0, a_{-n}, b_{-n}\}_{n>0}$ are both exceptional of degree $\leq k$. We write $\mathfrak{E}_k^{\mathrm{harm}}$ for the linear space of all two-sided exceptional coefficients of degree $\leq k$, which has dimension $2k$.

When we let the coefficients grow faster than the range permitted in Theorem 3.6.6, we do get nonuniqueness, but at least for a while we can still control the loss of uniqueness and describe it in terms of the exceptional coefficient classes.

Theorem 3.7.3. (beyond uniqueness) *Suppose that the coefficients a_n, b_n are subject to the subexponential growth bound*

$$\forall \varepsilon > 0: \ |a_n|, |b_n| = \mathrm{O}(e^{\varepsilon |n|}) \quad as \ |n| \to +\infty,$$

and that associated harmonic hyperbolic Fourier series is trivial:

$$a_0 + \sum_{n \in \mathbb{Z}_{>0}} \left(a_n e^{i\pi n \tau} + b_n e^{-i\pi n/\tau}\right) + \sum_{n \in \mathbb{Z}_{<0}} \left(a_n e^{i\pi n \bar{\tau}} + b_n e^{-i\pi n/\bar{\tau}}\right) = 0, \qquad \tau \in \mathbb{H}.$$

Suppose, in addition, that for some integer $k = 1, 2, 3, \ldots,$

$$|a_n| = \mathrm{o}\big(|n|^{-3/4} \exp\big(2\pi \sqrt{k|n|}\big)\big) \quad as \ |n| \to +\infty.$$

Then $\{a_0, a_n, b_n\}_{n \neq 0}$ belongs to the space of two-sided exceptional sequences $\mathfrak{E}_{k-1}^{\mathrm{harm}}$.

The proof of Theorem 3.7.3 is supplied in Subsection 7.1.

Remark 3.7.4. As in Remark 3.6.7(a), Theorem 3.7.3 would remain true if we make the growth assumption on the coefficients b_n in place of the a_n. In addition we may replace the two-sided assumption on a_n (in the sense that the assumption concerns the limits as $n \to +\infty$ and as $n \to -\infty$) by a one-sided assumption on a_n, and a likewise one-sided assumption on b_n, as long as the two sides are in opposite directions.

3.8. **Growth of coefficients and the hyperbolic Fourier series.** Let $\mathrm{M} : \mathbb{H} \to \mathbb{R}_{>0}$ denote the function

$$(3.8.1) \qquad\qquad\qquad \mathrm{M}(\tau) := \frac{\max\{1, |\tau|^2\}}{\mathrm{Im}\,\tau}.$$

This function enjoys the following symmetry properties:

$$(3.8.2) \qquad\qquad \mathrm{M}(-1/\tau) = \mathrm{M}(1/\bar{\tau}) = \mathrm{M}(\tau), \qquad \tau \in \mathbb{H}.$$

We first show that if we control the coefficients, we obtain growth control on the holomorphic hyperbolic Fourier series.

Proposition 3.8.1. *Suppose that for some fixed β with $0 < \beta < +\infty$, we have that the sequence $\{a_n, b_n\}_{n>0}$ has the growth control*

$$|a_n|, |b_n| = \mathrm{O}\big(n^{-3/4} \exp(2\pi \sqrt{\beta n})\big) \quad as \ n \to +\infty.$$

Then, if $f : \mathbb{H} \to \mathbb{C}$ is the associated holomorphic hyperbolic Fourier series

$$f(\tau) = \sum_{n=1}^{+\infty} \left(a_n e^{i\pi n \tau} + b_n e^{-i\pi n/\tau}\right),$$

it follows that

$$|f(\tau)| = \mathrm{O}\big(\exp(\beta \pi \mathrm{M}(\tau))\big)$$

holds uniformly in $\tau \in \mathbb{H}$.

Proposition 3.8.1 is obtained from a straightforward but rather tedious calculation, based on Lemma 5.2.4 which is formulated and proved in Subsection 5.2 below. The instance of harmonic Fourier series is stated in Proposition 5.2.3 below, and Proposition 3.8.1 then follows by adding up the two corresponding estimates. The rather trivial details are left to the reader.

What is less straightforward is that given a holomorphic function of the given growth, we may control the coefficients in the same way with only a slight loss.

Theorem 3.8.2. *Suppose* $f : \mathbb{H} \to \mathbb{C}$ *holomorphic with*
$$|f(\tau)| = \mathrm{O}\big(\exp(\beta\pi\mathrm{M}(\tau))\big)$$
with an implicit constant that is uniform on $\mathbb{H}$*. Here,* β *is a fixed parameter with* $0 < \beta < +\infty$*. Then* f *may be written as a holomorphic hyperbolic Fourier series*

$$f(\tau) = a_0 + \sum_{n=1}^{+\infty} \big(a_n \mathrm{e}^{\mathrm{i}\pi n\tau} + b_n \mathrm{e}^{-\mathrm{i}\pi n/\tau}\big), \qquad \tau \in \mathbb{H},$$

where if β *is not an integer, the coefficients satisfy the growth condition*

$$|a_n|, |b_n| = \mathrm{O}\big((\log^2 n)\exp(2\pi\sqrt{\beta n})\big) \quad \text{as } n \to +\infty.$$

On the other hand, if β *is an integer, we instead have the slightly weaker estimate*

$$|a_n|, |b_n| = \mathrm{O}\big(n^{1/2}\exp(2\pi\sqrt{\beta n})\big) \quad \text{as } n \to +\infty.$$

The proof of Theorem 3.8.2 is supplied in Subsection 8.7.

This is all well for holomorphic functions, but what about harmonic functions? By splitting the given harmonic hyperbolic Fourier series according to (3.5.4), we estimate each term in (3.5.5) and (3.5.6) in accordance with Proposition 5.2.3, which gives us the the harmonic hyperbolic Fourier series version of Proposition 5.2.3. On the other hand, if we have a harmonic function with the same growth bound as satisfied by the holomorphic function of Theorem 3.8.2, we would like to expand it as a harmonic hyperbolic Fourier series with control on the coefficients. We can achieve this by obtaining a splitting of the harmonic function in a holomorphic and a conjugate-holomorphic piece.

Proposition 3.8.3. *Suppose* $f : \mathbb{H} \to \mathbb{C}$ *harmonic with*
$$|f(\tau)| = \mathrm{O}\big(\exp(\beta\pi\mathrm{M}(\tau))\big)$$
with an implicit constant that is uniform on $\mathbb{H}$*. Here,* β *is a fixed parameter with* $0 < \beta < +\infty$*. Then* f *splits as*
$$f(\tau) = f(\mathrm{i}) + f_+(\tau) + f_-(\tau),$$
where $f_+(\tau)$ *is holomorphic in* τ*, while* $f_-(\tau)$ *is conjugate-holomorphic, with* $f_+(\mathrm{i}) = f_-(\mathrm{i}) = 0$*, each having the same growth bound:*
$$|f_+(\tau)|, |f_-(\tau)| = \mathrm{O}\big(\exp(\beta\pi\mathrm{M}(\tau))\big)$$
with an implicit constant that is uniform in $\mathbb{H}$*.*

The proof of Proposition 3.8.3 can be found in Subsection 5.1, and is restated in the form of Proposition 5.1.1.

3.9. **Power skewed holomorphic hyperbolic Fourier series.** If we begin with a holomorphic hyperbolic Fourier series

$$f(\tau) = a_0 + \sum_{n=1}^{+\infty} \left(a_n \, e^{i\pi n\tau} + b_n \, e^{-i\pi n/\tau} \right),$$

such that

(3.9.1) $$\forall \varepsilon > 0 : \ |a_n|, |b_n| = \mathrm{O}(e^{\varepsilon n}) \quad \text{as } n \to +\infty,$$

so that the series converges in $\mathbb{H}$, we may form the derivative:

$$f'(\tau) = \sum_{n=1}^{+\infty} \left(i\pi n \, a_n \, e^{i\pi n\tau} + i\pi n \tau^{-2} \, b_n \, e^{-i\pi n/\tau} \right),$$

which has a similar form to the original holomorphic hyperbolic Fourier series. The main difference is that the constant term disappeared and that we got the factor τ^{-2} in the second expression involving the coefficients b_n. This suggests the following more general type of series, which we may call *power skewed holomorphic hyperbolic Fourier series*:

(3.9.2) $$f(\tau) = a_0 + \sum_{n=1}^{+\infty} \left(a_n \, e^{i\pi n\tau} + b_n \, (\tau/i)^{-\beta} \, e^{-i\pi n/\tau} \right),$$

where β is a real parameter (called the *skew power*). The instance $\beta = 0$ is the holomorphic hyperbolic Fourier series we have discussed extensively already, whereas $\beta = 2$ corresponds to taking the derivative of holomorphic hyperbolic Fourier series (except that we allow the constant term to hang in there). For $\tau \in \mathbb{H}$, the ratio τ/i has positive real part, so we use the principal branch of the logarithm to define the power

$$(\tau/i)^{-\beta} = \exp\left(-\beta \log(\tau/i) \right), \qquad \tau \in \mathbb{H}.$$

The power skewed holomorphic hyperbolic Fourier series (3.9.2) appears naturally in the context of Fourier interpolation of radial functions in $\mathbb{R}^d$, with parameter $\beta = d/2$. More about this later on. We shall now see how to relate power skewed holomorphic hyperbolic Fourier series to the ordinary holomorphic hyperbolic Fourier series. This is achieved by invoking the Jacobi theta function ϑ_{00} given by (3.6.2), that is,

$$\vartheta_{00}(\tau) := \sum_{n \in \mathbb{Z}} e^{i\pi n^2 \tau}, \qquad \tau \in \mathbb{H}.$$

The important fact here is that it is zero-free and enjoys the functional identity

$$\vartheta_{00}(\tau) = (\tau/i)^{-1/2} \vartheta_{00}(-1/\tau), \qquad \tau \in \mathbb{H}.$$

Since $\vartheta_{00}(iy) > 1$ for $y \in \mathbb{R}_{>0}$, and $\vartheta_{00}(\tau) \neq 0$ for all $\tau \in \mathbb{H}$, we may choose a suitable holomorphic logarithm $\log \vartheta_{00}(\tau)$ which assumes positive values on $i\mathbb{R}_{>0}$. This

logarithm is then also 2-periodic: $\log \vartheta_{00}(\tau+2) = \log \vartheta_{00}(\tau)$. Using this logarithm, we define the power

$$\big(\vartheta_{00}(\tau)\big)^\gamma = \exp\big(\gamma \log \vartheta_{00}(\tau)\big), \qquad \tau \in \mathbb{H},$$

for any $\gamma \in \mathbb{C}$. The following proposition explains the algebraic relationship between power skewed hyperbolic Fourier series and ordinary hyperbolic Fourier series.

Proposition 3.9.1. *Suppose $f, g : \mathbb{H} \to \mathbb{C}$ are holomorphic and connected via the formula*

$$g(\tau) = \vartheta_{00}(\tau)^{-2\beta} f(\tau),$$

for some $\beta \in \mathbb{R}$. Then, if g has a hyperbolic Fourier series expansion

$$g(\tau) = \tilde{a}_0 + g_1(\tau) + g_2(-1/\tau),$$

where

$$g_1(\tau) = \sum_{n=1}^{+\infty} \tilde{a}_n \, e^{i\pi n\tau}, \quad g_2(\tau) = \sum_{n=1}^{+\infty} \tilde{b}_n \, e^{i\pi n\tau},$$

it follows that f has a power skewed expansion

$$f(\tau) = a_0 + f_1(\tau) + (\tau/i)^{-\beta} f_2(-1/\tau)$$

where

$$f_1(\tau) = \sum_{n=1}^{+\infty} a_n \, e^{i\pi n\tau}, \quad f_2(\tau) = \sum_{n=1}^{+\infty} b_n \, e^{i\pi n\tau},$$

and vice versa. Here, the coefficient sequences a_n, b_n and $\tilde{a}_n, \tilde{b}_n$ meet the growth bound (3.9.1) *simultaneously. Moreover, the relationship between the two expansions is given by $\tilde{a}_0 = a_0$ and*

$$f_1(\tau) = \big(\vartheta_{00}(\tau)^{2\beta} - 1\big)\tilde{a}_0 + \vartheta_{00}(\tau)^{2\beta} g_1(\tau), \quad f_2(\tau) = \vartheta_{00}(\tau)^{2\beta} g_2(\tau).$$

The proof of Proposition 3.9.1 is supplied in Subsection 10.1.

Corollary 3.9.2. *Fix a parameter $\beta \in \mathbb{R}$, and suppose $f : \mathbb{H} \to \mathbb{C}$ is holomorphic. Then there exist complex coefficients a_n, b_n with the growth bound*

$$\forall \varepsilon > 0 : \ |a_n|, |b_n| = \mathrm{O}(e^{\varepsilon |n|}) \quad \text{as } |n| \to +\infty,$$

such that f has a β-power skewed holomorphic hyperbolic Fourier series expansion

$$f(\tau) = a_0 + \sum_{n=1}^{+\infty} \big(a_n \, e^{i\pi n\tau} + b_n \, (\tau/i)^{-\beta} e^{-i\pi n/\tau}\big), \qquad \tau \in \mathbb{H}.$$

Proof. The assertion is an immediate consequence of the holomorphic version of Theorem 3.6.4 combined with the algebraic correspondence of Proposition 3.9.1. □

Remark 3.9.3. Naturally, we would like to know when the coefficients a_n, b_n can be chosen to grow at most polynomially, for a given given holomorphic function f. As in the context of Theorem 3.6.1, the appropriate growth condition on f is

$$\exists k : \ |f(\tau)| = \mathrm{O}\left(\frac{1 + |\tau|^2}{\mathrm{Im}\,\tau}\right)^k$$

where the implicit constant is uniform in $\mathbb{H}$. This would require us to delve a little deeper into the methods that give Theorem 3.6.4.

3.10. **Exponentially skewed hyperbolic Fourier series.** Given two real parameters $\omega_1, \omega_2 \in \mathbb{R}$, we may consider the associated *exponentially skewed* holomorphic hyperbolic Fourier series

$$(3.10.1) \quad f(\tau) = f_{\omega_1, \omega_2}(\tau)$$
$$= a_0 e^{i\pi\omega_1\tau} + \sum_{n=1}^{+\infty} \left(a_n e^{i\pi(n+\omega_1)\tau} + b_n e^{-i\pi(n+\omega_2)/\tau} \right), \qquad \tau \in \mathbb{H},$$

where the coefficients are assumed to grow at most subexponentially (this means that they satisfy condition (3.9.1)). It is possible to view this as a shift in the "frequency domain" compared with the ordinary holomorphic hyperbolic Fourier series. Moreover, the shifts $(\omega_1, \omega_2) \mapsto (\omega_1 + k_1, \omega_2 + k_2)$ for $k_1, k_2 \in \mathbb{Z}_{\geq 0}$ give rise to

$$f_{\omega_1+k_1, \omega_2+k_2}(\tau) = \sum_{n=k_1}^{+\infty} a_{n-k_1} e^{i\pi(n+\omega_1)\tau} + \sum_{n=k_2}^{+\infty} b_{n+1-k_2} e^{-i\pi(n+\omega_2)/\tau}, \qquad \tau \in \mathbb{H},$$

which is of the same type as (3.10.1). The only difference is that the coefficients get shifted to the right, which means that some of the first coefficients get replaced by 0. This suggests that we should fix a fundamental domain $\mathcal{Q}$ of $\mathbb{R}^2/\mathbb{Z}^2$ and consider $(\omega_1, \Omega_2) \in \mathcal{Q}$. A natural first choice is the square

$$\mathcal{Q} := \{(\omega_1, \omega_2) : 0 \leq \omega_1 < 1, \ -1 < \omega_2 \leq 0\}$$

for then

$$\mathrm{Im}\big((n+\omega_1)\tau\big) \geq 0, \qquad \tau \in \mathbb{H}, \ n \in \mathbb{Z}_{\geq 0},$$

and

$$\mathrm{Im}\big((n+\omega_2)/\tau\big) \geq 0, \qquad \tau \in \mathbb{H}, \ n \in \mathbb{Z}_{>0},$$

both hold, which entails that each individual term in the series (3.10.1) is uniformly bounded in $\mathbb{H}$.

We would like, if at all possible, to relate the exponentially skewed series (3.10.1) with the ordinary holomorphic hyperbolic Fourier series that we have encountered so far. What saves the day here is the modular lambda function $\lambda(\tau)$, given by (3.6.3), with the expansion (3.6.4). Since $\widehat{\lambda}(1) = 16$, we may write (3.6.4) in the form

$$\lambda(\tau) = 16 e^{i\pi\tau} \left(1 + \frac{1}{16} \sum_{n=1}^{+\infty} \widehat{\lambda}(n+1) e^{i\pi n\tau} \right), \qquad \tau \in \mathbb{H}.$$

For a parameter $\alpha \in \mathbb{C}$, we may now define the power

$$(3.10.2) \qquad \lambda(\tau)^\alpha = 16^\alpha e^{i\pi\alpha\tau} \left(1 + \frac{1}{16} \sum_{n=1}^{+\infty} \widehat{\lambda}(n+1) e^{i\pi n\tau} \right)^\alpha, \qquad \tau \in \mathbb{H},$$

as a holomorphic function in $\mathbb{H}$. The constant is $16^{\alpha} = e^{\alpha \log 16}$, with the standard natural logarithm, and the power is given by the binomial expansion

$$(3.10.3) \qquad (1+z)^{\alpha} = 1 + \binom{\alpha}{1} z + \binom{\alpha}{2} z^2 + \cdots,$$

for $|z| < 1$. Since the expression raised to the α power in (3.10.2) is zero-free in $\mathbb{H}$, the power extends holomorphically to all of $\mathbb{H}$, and it follows that the function

$$e^{-i\pi\alpha\tau} \lambda(\tau)^{\alpha} = 16^{\alpha} \left(1 + \frac{1}{16} \sum_{n=1}^{+\infty} \widehat{\lambda}(n+1) e^{i\pi n\tau} \right)^{\alpha}, \qquad \tau \in \mathbb{H},$$

is a well-defined functions which may be expressed as a convergent power series in the variable $e^{i\pi\tau}$. Mutatis mutandis, the same applies if we replace $\tau \mapsto -1/\tau$:

$$e^{i\pi\alpha/\tau} \lambda(-1/\tau)^{\alpha} = 16^{\alpha} \left(1 + \frac{1}{16} \sum_{n=1}^{+\infty} \widehat{\lambda}(n+1) e^{-i\pi n/\tau} \right)^{\alpha}, \qquad \tau \in \mathbb{H},$$

is a well-defined functions which may be expressed as a convergent power series in the variable $e^{-i\pi/\tau}$. We need one more property of the modular lambda function. It does not assume the value 1, which means that $1 - \lambda(\tau)$ is zero-free. As such, we may define the associated power

$$(1 - \lambda(\tau))^{\alpha}$$

by the same binomial expansion (3.10.3) for large positive values of $\mathrm{Im}\tau$, and extend it holomorphically to all of $\mathbb{H}$, to obtain a convergent power series in the variable $e^{i\pi\tau}$. Likewise, the function

$$(1 - \lambda(-1/\tau))^{\alpha}$$

is a well-defined convergent power series in the variable $e^{-i\pi/\tau}$. The following proposition explains the algebraic relationship between exponentially skewed hyperbolic Fourier series and ordinary hyperbolic Fourier series.

Proposition 3.10.1. *Suppose $f, g : \mathbb{H} \to \mathbb{C}$ are holomorphic and connected via the relation*

$$g(\tau) = \lambda(\tau)^{-\omega_1} \lambda(-1/\tau)^{-\omega_2} f(\tau),$$

for some $\beta \in \mathbb{R}$. Then, if g has a hyperbolic Fourier series expansion

$$g(\tau) = \tilde{a}_0 + g_1(\tau) + g_2(-1/\tau),$$

where

$$g_1(\tau) = \sum_{n=1}^{+\infty} \tilde{a}_n e^{i\pi n\tau}, \quad g_2(\tau) = \sum_{n=1}^{+\infty} \tilde{b}_n e^{i\pi n\tau},$$

it follows that f has an exponentially skewed hyperbolic Fourier expansion

$$f(\tau) = e^{i\pi\omega_1\tau}(a_0 + f_1(\tau)) + e^{-i\pi\omega_2/\tau} f_2(-1/\tau)$$

where

$$f_1(\tau) = \sum_{n=1}^{+\infty} a_n e^{i\pi n\tau}, \quad f_2(\tau) = \sum_{n=1}^{+\infty} b_n e^{i\pi n\tau},$$

and vice versa. Here, the coefficient sequences a_n, b_n and $\tilde{a}_n, \tilde{b}_n$ meet the growth bound (3.9.1) *simultaneously. Moreover, the relationship between the two expansions is given by*

$$a_0 + f_1(\tau) = \left(e^{-i\pi\omega_1\tau}\lambda(\tau)^{\omega_1}\right)(1 - \lambda(\tau))^{\omega_2}(\tilde{a}_0 + g_1(\tau))$$

and

$$f_2(\tau) = \left(e^{-i\pi\omega_2\tau}\lambda(\tau)^{\omega_2}\right)(1 - \lambda(\tau))^{\omega_1}g_2(\tau).$$

In particular, the constants are related via $a_0 = 16^{\omega_1}\tilde{a}_0$.

The proof of Proposition 3.10.1 is supplied in Subsection 10.2.

Corollary 3.10.2. *Fix two parameter $\omega_1, \omega_2 \in \mathbb{R}$, and suppose $f : \mathbb{H} \to \mathbb{C}$ is holomorphic. Then there exist complex coefficients a_n, b_n with the growth bound*

$$\forall \varepsilon > 0 : \ |a_n|, |b_n| = O(e^{\varepsilon|n|}) \quad as \ |n| \to +\infty,$$

such that f has the exponentially skewed holomorphic hyperbolic Fourier series expansion

$$f(\tau) = a_0\, e^{i\pi\omega_1\tau} + \sum_{n=1}^{+\infty} \left(a_n\, e^{i\pi(n+\omega_1)\tau} + b_n\, e^{-i\pi(n+\omega_2)/\tau}\right), \qquad \tau \in \mathbb{H}.$$

Proof. The assertion is an immediate consequence of the holomorphic version of Theorem 3.6.4 combined with the algebraic correspondence of Proposition 3.10.1. $\qquad\square$

Remark 3.10.3. Naturally, we would like to know when the coefficients a_n, b_n can be chosen to grow at most polynomially, for a given given holomorphic function f with (cf. Theorem 3.6.1) the growth bound

$$(3.10.4) \qquad\qquad \exists k : \ |f(\tau)| = O\left(\frac{1 + |\tau|^2}{\mathrm{Im}\,\tau}\right)^k$$

where the implicit constant is uniform in $\mathbb{H}$. After all, this growth bound is associated with having boundary values in the sense of distribution theory. This will require us to delve a little deeper than the above Proposition 3.10.1. An immediate inspection suggests that it is natural to impose the conditions $\omega_1 > 0$, $\omega_2 > -1$ in this context, for otherwise, even if the functions f_1, f_2 meet (3.10.4) individually, the combined function

$$f(\tau) = e^{i\pi\omega_1\tau}(a_0 + f_1(\tau)) + e^{-i\pi\omega_2/\tau}f_2(-1/\tau)$$

need not.

4. APPLICATIONS TO THE KLEIN-GORDON EQUATION

4.1. **Oscillatory processes and the Klein-Gordon equation.** Little is known regarding the interpolation of oscillatory processes. We can visualize a typical problem as the study of pressure fluctuations inside a natural gas storage facility in terms of a finite number of strategically located pressure gauges. Unfortunately, only some numerical methods based on considering solutions with extreme value of entropy

are known (see, e.g., [20], [12]). Since oscillatory processes are associated with hyperbolic partial differential equations, it makes sense to begin with the most basic of such equations, which are the wave equation and the associated eigenvalue problem. The eigenvalue problem for the wave operator (for eigenvalues $\neq 0$) is called the *Klein-Gordon equation*. In one spatial and one temporal dimension, the Klein-Gordon equation may be expressed in suitable alternative coordinates as

$$(4.1.1) \qquad\qquad u''_{xy} + u = 0, \qquad (x,y) \in \mathbb{R}^2.$$

If we try for pure complex exponential solutions

$$(4.1.2) \qquad\qquad u(x,y) = e^{i\lambda x + i\mu y},$$

where $\lambda, \mu \in \mathbb{C}$, then

$$u''_{xy} + u = (1 - \lambda\mu)\, e^{i\lambda x + i\mu y} = 0$$

holds if and only if $\lambda\mu = 1$. The complex exponential u given by (4.1.2) is bounded in $\mathbb{R}^2$ if and only if $(\lambda, \mu) \in \mathbb{R}^2$, so that the bounded pure complex exponential solutions are parametrized by $(\lambda, \mu) \in \mathbb{R}^2$ with $\lambda\mu = 1$. Replacing λ by t and μ by $1/t$, such solutions are given by

$$u(x,y) = e^{itx + iy/t},$$

for $t \in \mathbb{R}_{\neq 0} = \mathbb{R} \setminus \{0\}$. Forming linear combinations of such pure solutions, we have again a bounded smooth solution. An L^1-*mixed bounded exponential solution* is a generalized linear combination of the form

$$(4.1.3) \qquad\qquad u(x,y) = U[\varphi](x,y) := \int_{\mathbb{R}} e^{ixt + iy/t} \varphi(t)\mathrm{d}t,$$

where $\varphi \in L^1(\mathbb{R})$. Such L^1-mixed bounded exponential solution are not so exotic after all, as can be seen from the following argument. First, we note that according to [15], we can think of (4.1.3) as saying that $u = U[\varphi]$ is the Fourier transform of a finite complex-valued Borel measure on $\mathbb{R}^2$ supported on a certain hyperbola (corresponding to $\{(\lambda, \mu) \in \mathbb{R}^2 : \lambda\mu = 1\}$ in the above context) which in addition is absolutely continuous with respect to arc length measure. On the other hand, if $u \in L^\infty(\mathbb{R}^2)$ is given, its inverse Fourier transform, thought of as a distribution, is a pseudomeasure on $\mathbb{R}^2$. Moreover, if u solves the Klein-Gordon equation (4.1.1) in the sense of distribution theory, then that pseudomeasure would be supported on the same hyperbola in $\mathbb{R}^2$. Expressed differently, u would be the Fourier transform of a pseudomeasure whose support is the given hyperbola. So the additional requirement in (4.1.3) is that the pseudomeasure be an actual finite Borel measure which is absolutely continuous with respect to arc length measure on the hyperbola. A further remark worth mentioning is that on a given compact subset, such L^1-mixed bounded exponential solutions can approximate any continuous solution of the Klein-Gordon equation.

4.2. **The discretized Goursat problem for the Klein-Gordon equation.** The lines $x = 0$ and $y = 0$ are characteristics for the equation $u_{xy} + u = 0$, and the Goursat problem would ask for a solution with given data e.g. along the semi-axes $\{(x,y) \in \mathbb{R}^2 : x = 0, y \leq 0\}$ and $\{(x,y) \in \mathbb{R}^2 : x \geq 0, y = 0\}$. This problem is concerned with the space-like quarter-plane $\{(x,y) \in \mathbb{R}^2 : x \geq 0, y \leq 0\}$, and it was investigated by Riemann. Riemann's solution is often unbounded in the plane. If we need a bounded solution, some kind of compatibility property needs to be met for the data along the two characteristics. For the L^1-mixed bounded exponential solutions, we can be more precise. In terms of notation, we write

$$J_1(x,y) := \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!(k+1)!} x^{k+1} y^k$$

for a two-variable version of the familiar Bessel function. The following result was mentioned in [3].

Proposition 4.2.1. *If $u(x,y) = U[\varphi](x,y)$ is given by (4.1.3) for some $\varphi \in L^1(\mathbb{R})$, then*

$$u(0,y) = u(0,0) - \int_0^{+\infty} J_1(-y,t) u(t,0) \, dt, \qquad y \leq 0.$$

In terms of the Goursat problem, this means that the boundary data on the semi-axis $\{(x,y) \in \mathbb{R}^2 : x \geq 0, y = 0\}$ necessarily induces boundary data on other semi-axis $\{(x,y) : x = 0, y \leq 0\}$ as well (and, in fact, vice versa). But we would prefer to work with independent data. One way to achieve this is to thin out the given data on the two semi-axes by discretizing the axes.

Given that we look for a bounded solution of the form (4.1.3), it makes sense to reduce the given Goursat boundary data along the two characteristics to the point that the data is no longer overdetermined. The question is how to thin out the data. If we consider the two-sided *discretized Goursat problem of critical density*, which instead prescribes the values along the discrete subset of the characteristics consisting of equidistant points $(\pi m, 0)$ and $(0, \pi n)$ where $m, n \in \mathbb{Z}$, it turns out that we have uniqueness (we write that it is two-sided because we consider the full lattice-cross, which is the union of the discretized boundaries of the two spacelike quarter-planes $\{(x,y) \in \mathbb{R}^2 : x \geq 0, y \leq 0\}$ and $\{(x,y) \in \mathbb{R}^2 : x \leq 0, y \geq 0\}$). In other words, for L^1-mixed bounded exponential solutions $u = U_\varphi$, the values $u(\pi m, 0)$ and $u(0, \pi n)$ determine u uniquely, provided that m, n range over the integers $\mathbb{Z}$. This is the main result of [15], which may be formulated in the following manner.

Theorem 4.2.2. (Hedenmalm, Montes-Rodríguez) *Suppose $\varphi \in L^1(\mathbb{R})$ and that*

$$\int_{\mathbb{R}} e^{i\pi nt} \varphi(t) dt = \int_{\mathbb{R}} e^{-i\pi n/t} \varphi(t) dt = 0, \quad n \in \mathbb{Z}.$$

Then $\varphi = 0$.

The corresponding result which applies to the space-like quarter-planes reads as follows (see [16], [17]).

Theorem 4.2.3. (Hedenmalm, Montes-Rodríguez) *Suppose $\varphi \in L^1(\mathbb{R})$ and that*

$$\int_{\mathbb{R}} e^{i\pi nt} \varphi(t) dt = \int_{\mathbb{R}} e^{-i\pi n/t} \varphi(t) dt = 0, \quad n \in \mathbb{Z}_{\geq 0}.$$

Then $\varphi \in H^1_+(\mathbb{R})$.

Here, $H^1_+(\mathbb{R})$ stands for the Hardy space of functions in $L^1(\mathbb{R})$ whose Poisson harmonic extensions (cf. equation (3.4.2)) to the upper half-plane $\mathbb{H}$ are holomorphic. We will use the notation $H^1_-(\mathbb{R})$ for for the analogous Hardy space where the harmonic extension is instead conjugate-holomorphic.

The method of proof of Theorems 4.2.2 and 4.2.3 is careful analysis of the transfer operator associated with the Gauss-type map $t \mapsto -1/t$ modulo 1 on the interval $[-1,1]$. In terms of the L^1-mixed bounded exponential solutions to the Klein-Gordon equation, these results entail the following.

Corollary 4.2.4. *Let $u = U[\varphi]$ be of the form (4.1.3) for some $\varphi \in L^1(\mathbb{R})$.*
(a) If $u(\pi n, 0) = u(0, -\pi n) = 0$ for each $n \in \mathbb{Z}_{\geq 0}$, then $\varphi \in H^1_+(\mathbb{R})$, and hence $u(x,y) = 0$ in the quarter-plane $\{(x,y) \in \mathbb{R}^2 : x \geq 0, y \leq 0\}$.
(b) If instead $u(-\pi n, 0) = u(0, \pi n) = 0$ for each $n \in \mathbb{Z}_{\geq 0}$, then $\varphi \in H^1_-(\mathbb{R})$, and hence $u(x,y) = 0$ in the quarter-plane $\{(x,y) \in \mathbb{R}^2 : x \leq 0, y \geq 0\}$.
(c) If $u(\pi n, 0) = u(0, -\pi n) = 0$ for each $n \in \mathbb{Z}$, then $\varphi = 0$, and hence $u(x,y) = 0$ on all of $\mathbb{R}^2$.

Remark 4.2.5. In Corollary 4.2.4, parts (a) and (b) are equivalent, in the sense that (b) follows from (a), and (a) follows from (b). Moreover, (c) is immediate from (a) and (b), since the assumptions imply that $\varphi \in H^1_+(\mathbb{R}) \cap H^1_-(\mathbb{R}) = \{0\}$.

Since according to Corollary 4.2.4(c), a solution $u = U[\varphi]$ (with $\varphi \in L^1(\mathbb{R})$) to the Klein-Gordon equation is uniquely determined by its values along the *lattice-cross*

$$\{(\pi m, \pi n) : m, n \in \mathbb{Z}, \ mn = 0\}.$$

it becomes natural to ask whether we can freely prescribe the values there. Pleasantly, this is answered in the affirmative by Theorem 4.2.8 below, which in turn builds on the following result. Note that the functions A_0 and A_n, B_n are all in $L^1(\mathbb{R})$ for $n \in \mathbb{Z}_{\neq 0}$. After all, in view of Corollary 3.3.3, they are test functions in $C_1^\infty(\mathbb{R} \cup \{\infty\})$.

Theorem 4.2.6. *For $n \in \mathbb{Z}$, we put*

$$u_{(n,0)}(x,y) := U[A_n](x,y) = \int_{\mathbb{R}} e^{ixt + iy/t} A_n(t) dt, \qquad (x,y) \in \mathbb{R}^2,$$

and declare, in addition, that, for $n \in \mathbb{Z}_{\neq 0}$,

$$u_{(0,n)}(x,y) := U[B_{-n}](x,y) = U[A_n](y,x) = \int_{\mathbb{R}} e^{iyt + ix/t} A_n(t) dt, \qquad (x,y) \in \mathbb{R}^2.$$

Then if we use Kronecker delta notation, we have

$$u_{(n,0)}(\pi m, 0) = \delta_{m,n}, \quad u_{(n,0)}(0, \pi m) = \delta_{0,m}\delta_{m,0},$$
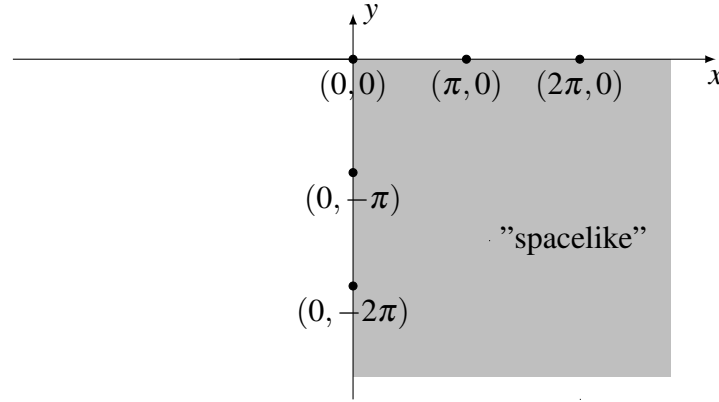
FIGURE 4.2.1. The discretized lattice-cross for a spacelike quarter-plane.

for each $m \in \mathbb{Z}$. Likewise, we find that for $n \in \mathbb{Z}_{\neq 0}$ and $m \in \mathbb{Z}$,

$$u_{(0,n)}(\pi m, 0) = 0, \quad u_{(0,n)}(0, \pi m) = \delta_{m,n}.$$

The proof of Theorem 4.2.6 is supplied in Subsection 9.2.

Remark 4.2.7. The functions $u_{(n,0)}$ and $u_{(0,n)}$ are the analogous functions in the context of the Klein-Gordon equation (4.1.1) to the classical cardinal sine functions for bandlimited signals, modelled by the Paley-Wiener space of entire functions (which we may think of as the Fourier transform applied to the Hilbert space $L^2([-1,1])$). Although defined in terms of the complex-valued functions A_n and B_n, the functions $u_{(n,0)}(x,y)$ and $u_{(0,n)}(x,y)$ are necessarily real-valued. This is related with the fact that the Klein-Gordon equation (4.1.1) which these functions solve has real coefficients. Another consequence of Theorem 4.2.6 and Corollary 4.2.4 is that $A_n, B_n \in H^1_-(\mathbb{R})$ for $n > 0$ while $A_n, B_n \in H^1_+(\mathbb{R})$ for $n < 0$.

Theorem 4.2.8. *Suppose the sequence of complex numbers α_0 and α_n, β_n is given for $n \in \mathbb{Z}_{\neq 0}$, with*

$$|\alpha_0| + \sum_{n \in \mathbb{Z}_{\neq 0}} \left(|\alpha_n| + |\beta_n| \right) \left(\log^3(|n| + 1) \right) < +\infty.$$

Then if φ is given by

$$\varphi(t) = \alpha_0 A_0(t) + \sum_{n \in \mathbb{Z}_{\neq 0}} \left(\alpha_n A_n(t) + \beta_n B_{-n}(t) \right), \qquad t \in \mathbb{R},$$
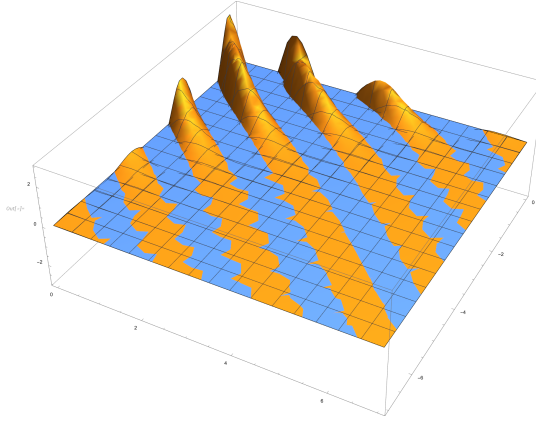
we have that $\varphi \in L^1(\mathbb{R})$, and, in addition, the function

$$u(x,y) = U[\varphi](x,y) = \alpha_0 u_{(0,0)}(x,y) + \sum_{n \in \mathbb{Z}_{\neq 0}} \left(\alpha_n u_{(n,0)}(x,y) + \beta_n u_{(0,n)}(x,y) \right)$$

solves the interpolation problem

$$u(\pi m, 0) = \alpha_m, \quad u(0, \pi n) = \beta_n,$$

for all $m \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\neq 0}$.

Working precision 60, n=5.

FIGURE 4.2.2. Graph of the positive part of the function $u_{(n,0)}(x,y)$ for $n = 5$ in the quarter-plane where $x \geq 0$ and $y \leq 0$. The computation is based on Corollary 9.2.6 below.

The proof of Theorem 4.2.8 is supplied in Subsection 9.2.

Remark 4.2.9. According to part (a) or (b) of Corollary 4.2.4, we may consider the interpolation problem along the discretized boundary of a spacelike quarterplane, and obtain a unique solution on the corresponding quarterplane. This interpolation problem may be solved in a completely analogous fashion to that of Theorem 4.2.8. We omit the necessary details.

4.3. **Alternative discretization variant: normal derivatives.** The vanishing of two-sided Goursat boundary data along the two axes amounts to having

(4.3.1) $u(x,0) = u(0,y) = 0, \qquad (x,y) \in \mathbb{R}^2.$

What happens if we replace one of the conditions, say the condition along the y-axis, by a condition on the derivative or even a higher derivative? For instance, if we consider the vanishing boundary data

(4.3.2) $u(x,0) = (\partial_x^k u)(0,y) = 0, \qquad (x,y) \in \mathbb{R}^2,$

it follows immediately from the Klein-Gordon equation that, at least if u is smooth, we have that

$$0 = \partial_y^k\big((\partial_x^k u)(0,y)\big) = (\partial_y^k \partial_x^k u)(0,y) = (-1)^k u(0,y), \qquad y \in \mathbb{R}.$$

In other words, the condition (4.3.2) appears to be slightly stronger than (4.3.1). As written, the two conditions apparently have inequivalent discretizations. We are already familiar with the discretization of (4.3.1). The corresponding discretization of (4.3.2) is

$$u(\pi m, 0) = (\partial_x^k u)(0, \pi n) = 0, \qquad m, n \in \mathbb{Z}.$$

If we as before take as our point of departure the formula (4.1.3) for $u = U[\varphi]$, we see that

$$\partial_x^k u(0,y) = \mathrm{i}^k \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}y/t} \varphi(t) t^k \, \mathrm{d}t,$$

provided that the integral makes sense. The condition on φ that

$$(4.3.3) \qquad \int_{\mathbb{R}} |\varphi(t)|(1+|t|^k)\mathrm{d}t < +\infty$$

will guarantee that $u(x,y)$ and $\partial_x^k u(x,y)$ are bounded and continuous functions on $\mathbb{R}^2$.

Proposition 4.3.1. *Suppose* $\varphi \in L^1(\mathbb{R})$ *has the extra integrability* (4.3.3) *for some* $k \in \mathbb{Z}_{\geq 0}$. *Then if* $u = U[\varphi]$ *is of the form* (4.1.3), *and*

$$\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\pi mt} \varphi(t) \, \mathrm{d}t = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\pi n/t} \varphi(t) t^k \, \mathrm{d}t = 0, \qquad m,n \in \mathbb{Z},$$

holds, it follows that $\varphi = 0$.

The proof of Proposition 4.3.1 is supplied in Subsection 6.2.

Corollary 4.3.2. *Suppose* $\varphi \in L^1(\mathbb{R})$ *has the extra integrability* (4.3.3) *for some* $k \in \mathbb{Z}_{\geq 0}$, *and let* $u = U[\varphi]$ *be given by* (4.1.3). *Suppose in addition that*

$$u(\pi m, 0) = \partial_x^k u(0, \pi n) = 0, \qquad m,n \in \mathbb{Z},$$

holds. It then follows that $\varphi = 0$ *and hence that* $u = 0$.

This connects with the notion of power skewed hyperbolic Fourier series. The interpolation problem

$$u(\pi m, 0) = \alpha_m, \quad \partial_y^k u(0, \pi n) = \beta_n,$$

for $m,n \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\neq 0}$ amounts to looking for a suitable φ such that

$$u(\pi m, 0) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\pi mt} \varphi(t)\mathrm{d}t = \alpha_n, \qquad m \in \mathbb{Z},$$

while at the same time

$$(\partial_x^k u)(0, \pi n) = \mathrm{i}^k \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\pi n/t} \varphi(t) t^k \, \mathrm{d}t = \beta_n, \qquad n \in \mathbb{Z}.$$

It is of course entirely possible that the coefficients α_n, β_n in the interpolation problem cannot be chosen completely freely, but it is very likely to be the case up to a finite-dimensional defect. Following the same scheme as in Theorem 4.2.8, which is based on Theorem 4.2.6, we should try to find, if at all possible, a biorthogonal system to the system of functions

$$\mathrm{e}^{\mathrm{i}\pi mt}, \quad t^k \mathrm{e}^{-\mathrm{i}\pi n/t},$$

for $m,n \in \mathbb{Z}$. This automatically hints at the the study of power skewed hyperbolic Fourier series, as defined in Subsection 3.9 with $\beta = -k$.

4.4. **Alternative discretization: translation along the axes.** An alternative app-
roach to the discretization of the two axes is that we consider, for $u = U[\varphi]$, a dis-
cretization of the two-sided Goursat boundary data (4.3.1) with equal density of the
form

(4.4.1) $u(\pi(m + \omega_1), 0) = u(0, \pi(n + \omega_2)) = 0, \qquad m, n \in \mathbb{Z},$

where $\omega_1, \omega_2 \in \mathbb{R}$ are frequency shifts.

Proposition 4.4.1. *For $\varphi \in L^1(\mathbb{R})$, let $u = U[\varphi]$ be given by (4.1.3). If (4.4.1) holds
for some $\omega_1, \omega_2 \in \mathbb{R}$, then $\varphi = 0$ and hence $u = 0$.*

  The proof of Proposition 4.4.1 is supplied in Subsection 6.1.
  The interpolation problem associated with (4.4.1) reads

$$u(\pi(m + \omega_1), 0) = \alpha_m, \quad u(0, \pi(n + \omega_2)) = \beta_n,$$

and in view of Proposition 4.4.1, the solution $u = U[\varphi]$ with $\varphi \in L^1(\mathbb{R})$, is necessarily
unique, provided that it exists. Since

$$u(\pi(m + \omega_1), 0) = \int_{\mathbb{R}} e^{i\pi(m+\omega_1)t} \varphi(t)\, dt$$

and

$$u(0, \pi(n + \omega_2)) = \int_{\mathbb{R}} e^{-i\pi(n+\omega_2)/t} \varphi(t)\, dt$$

we see the relevance of exponentially skewed hyperbolic Fourier series (see Subsec-
tion 3.10) to the study of the interpolation problem.

5. PREPARATORY WORK

5.1. **The splitting of a harmonic function.** It is well-known that a harmonic func-
tion in a simply connected domain is the sum of a holomorphic function and a
conjugate-holomorphic function. This decomposition is unique up to an additive
constant. Let us do this for a harmonic function $h : \mathbb{H} \to \mathbb{C}$ with the growth bound

(5.1.1) $|h(\tau)| \le C \exp(\beta \pi M(\tau)), \qquad \tau \in \mathbb{H},$

for a positive constant C, where we recall that $M(\tau)$ is given by (3.8.1). The parame-
ter β is assumed positive. The gradient of h can be estimated using the Poisson kernel
representation

$$h(\tau + \varepsilon\eta) = \int_{-\pi}^{\pi} \frac{1 - |\eta|^2}{|\eta - e^{it}|^2} h(\tau + \varepsilon e^{it}) \frac{dt}{2\pi},$$

valid for $\tau \in \mathbb{H}$ and ε with $0 < \varepsilon < \text{Im}\,\tau$. We obtain that

$$\partial_\tau h(\tau) = \frac{1}{\varepsilon} \int_{-\pi}^{\pi} e^{-it} h(\tau + \varepsilon e^{it}) \frac{dt}{2\pi}, \quad \bar{\partial}_\tau h(\tau) = \frac{1}{\varepsilon} \int_{-\pi}^{\pi} e^{it} h(\tau + \varepsilon e^{it}) \frac{dt}{2\pi},$$

and since

$$M(\tau + \varepsilon e^{it}) \le \frac{\max\{1, |\tau|^2 - \varepsilon^2\}}{\text{Im}\,\tau - \varepsilon} \le \frac{\max\{1, |\tau|^2\}}{\text{Im}\,\tau - \varepsilon}, \qquad t \in \mathbb{R},$$

it is an immediate consequence of (5.1.1) that

$$\max\left\{|\partial_\tau h(\tau)|,|\bar\partial_\tau h(\tau)|\right\} \le \frac{C}{\varepsilon}\exp\left(\beta\pi\frac{\max\{1,|\tau|^2\}}{\operatorname{Im}\tau-\varepsilon}\right)$$

holds. By suitably optimizing over ε with $0<\varepsilon<\operatorname{Im}\tau$, it can be shown that

$$(5.1.2)\quad \max\left\{|\partial_\tau h(\tau)|,|\bar\partial_\tau h(\tau)|\right\} \le \frac{C\mathrm{e}}{(\operatorname{Im}\tau)^2}\left(\beta\pi\max\{1,|\tau|^2\}+\operatorname{Im}\tau\right)\exp\left(\beta\mathrm{M}(\tau)\right)$$

holds. The function $\tau\mapsto\partial_\tau h(\tau)$ is holomorphic, whereas $\tau\mapsto\bar\partial_\tau h(\tau)$ is conjugate-holomorphic. If we put

$$h_+(\tau) := \int_{\mathrm{i}}^\tau \partial_\eta h(\eta)\,\mathrm{d}\eta, \quad h_-(\tau) := \int_{\mathrm{i}}^\tau \bar\partial_\eta h(\eta)\,\mathrm{d}\bar\eta,$$

we obtain a holomorphic function h_+ and a conjugate-holomorphic function h_- with $h_+(\mathrm{i}) = h_-(\mathrm{i}) = 0$ such that

$$h(\tau) = h_+(\tau)+h_-(\tau)+h(\mathrm{i}).$$

Using the estimate (5.1.2), we find that

$$(5.1.3)\quad |h_+(\tau)| \le \int_{\gamma(\mathrm{i},\tau)} |\partial_\eta h(\eta)|\,|\mathrm{d}\eta|$$

$$\le C\mathrm{e}\int_{\gamma(\mathrm{i},\tau)}\frac{1}{(\operatorname{Im}\eta)^2}\left(\beta\pi\max\{1,|\eta|^2\}+\operatorname{Im}\eta\right)\exp\left(\beta\pi\mathrm{M}(\eta)\right)|\mathrm{d}\eta|$$

where $\gamma(\mathrm{i},\tau)$ is a suitable path in $\mathbb{H}$ from i to τ. Moreover, if we apply the change-of-variables $\eta=1/\bar\xi$ (reflection in the unit circle) on the right-hand side integral in (5.1.3), it follows, after some simplification, that

$$(5.1.4)\quad |h_+(\tau)| \le \int_{\gamma'(\mathrm{i},1/\bar\tau)}\frac{C\mathrm{e}}{(\operatorname{Im}\xi)^2}\left(\beta\pi\max\{1,|\xi|^2\}+\operatorname{Im}\xi\right)\exp\left(\beta\pi\mathrm{M}(\xi)\right)|\mathrm{d}\xi|$$

where $\gamma'(\mathrm{i},1/\bar\tau)$ is any path in $\mathbb{H}$ which connects i to $1/\bar\tau$. This estimate is precisely the same as in (5.1.3), only the endpoint $\tau\in\mathbb{H}$ is replaced by $1/\bar\tau\in\mathbb{H}$. This means that it is enough to estimate $h_+(\tau)$ using (5.1.3) for $|\tau|\le1$, as the remaining case $|\tau|>1$ can be handled using (5.1.4), since then the endpoint has $|1/\bar\tau|<1$. So, we consider $\tau\in\mathbb{H}$ with $|\tau|\le1$, and by symmetry, we may assume that $\operatorname{Re}\tau>0$ as well. We choose the path $\gamma(\mathrm{i},\tau)$ which passes along the unit circle to the point with the same real part as τ, and then straight down to τ along the line segment parallel to the imaginary axis. As we implement suitable parametrizations, the integral on the right-hand side of (5.1.4) obtains the estimate

$$\int_{\gamma(\mathrm{i},\tau)}\frac{1}{(\operatorname{Im}\eta)^2}\left(\beta\pi\max\{1,|\eta|^2\}+\operatorname{Im}\eta\right)\exp\left(\beta\pi\mathrm{M}(\eta)\right)|\mathrm{d}\eta|$$

$$\le \int_\alpha^{\pi/2}\left(\frac{\beta\pi}{\sin^2 t}+\frac{1}{\sin t}\right)\exp\left(\frac{\beta\pi}{\sin t}\right)\mathrm{d}t + \int_{\operatorname{Im}\tau}^{\sqrt{1-(\operatorname{Re}\tau)^2}}\left(\frac{\beta\pi}{t^2}+\frac{1}{t}\right)\exp\left(\frac{\beta\pi}{t}\right)\mathrm{d}t$$

where $\cos\alpha=\operatorname{Re}\tau$. The first integral on the right-hand side is handled by by splitting the interval, from α to $\pi/4$, and then from $\pi/4$ to $\pi/2$. The second part is estimated

using a trivial estimate, while the first part is handled using integration by parts. The second integral is analogously analyzed using integration by parts. Omitting the necessary details, we find that

$$\int_{\gamma(\mathrm{i},\tau)} \frac{1}{(\operatorname{Im}\eta)^2} \left(\beta\pi \max\{1,|\eta|^2\} + \operatorname{Im}\eta\right) \exp\left(\beta\pi\pi\operatorname{M}(\eta)\right) |\mathrm{d}\eta|$$

$$\leq 2(1+\beta\pi)\exp(\beta\pi\sqrt{2}) + \left(\sqrt{2} + \frac{3}{\beta\pi}\right)\exp\left(\frac{\beta\pi}{\operatorname{Im}\tau}\right).$$

We now conclude from (5.1.3) and (5.1.4) that

$$|h_+(\tau)| \leq 2Ce\,(1+\beta)\exp(\beta\pi\sqrt{2}) + Ce\left(\sqrt{2} + \frac{3}{\beta\pi}\right)\exp\left(\beta\pi\operatorname{M}(\tau)\right).$$

The same kind of estimates applied to the conjugate-holomorphic function h_- as well. We formulate this splitting less precisely in the form of a proposition.

Proposition 5.1.1. *Fix a β with $0 < \beta < +\infty$. Suppose $h : \mathbb{H} \to \mathbb{C}$ is harmonic with*

$$|h(\tau)| = \mathrm{O}\big(\exp\big(\beta\pi\operatorname{M}(\tau)\big)\big)$$

uniformly in $\mathbb{H}$. Then there exist unique functions $h_+, h_- : \mathbb{H} \to \mathbb{C}$ such that h_+ is holomorphic, h_- is conjugate-holomorphic, with $h_+(\mathrm{i}) = h_-(\mathrm{i}) = 0$, and the splitting

$$h(\tau) = h_+(\tau) + h_-(\tau) + h(\mathrm{i}), \qquad \tau \in \mathbb{H},$$

holds. These functions have the same growth estimate:

$$|h_+(\tau)|, |h_-(\tau)| = \mathrm{O}\big(\exp\big(\beta\pi\operatorname{M}(\tau)\big)\big),$$

uniformly in $\mathbb{H}$.

The instance of $\beta = 0$ in Proposition 5.1.1 is a little different.

Proposition 5.1.2. *Suppose $h : \mathbb{H} \to \mathbb{C}$ is harmonic and bounded, i.e.,*

$$|h(\tau)| = \mathrm{O}(1)$$

holds uniformly in $\mathbb{H}$. Then there exist unique functions $h_+, h_- : \mathbb{H} \to \mathbb{C}$ such that h_+ is holomorphic, h_- is conjugate-holomorphic, with $h_+(\mathrm{i}) = h_-(\mathrm{i}) = 0$, and the splitting

$$h(\tau) = h_+(\tau) + h_-(\tau) + h(\mathrm{i}), \qquad \tau \in \mathbb{H},$$

holds. These functions enjoy the logarithmic growth estimate:

$$|h_+(\tau)|, |h_-(\tau)| = \mathrm{O}\left(\log\frac{1+|\tau|^2}{\operatorname{Im}\tau}\right),$$

uniformly in $\mathbb{H}$.

This statement is well-known, and indeed much more is known, since for bounded h the function h_+ is in BMOA of the upper half-plane (the bounded mean oscillation space on $\mathbb{R}$ whose Poisson extensions are holomorphic).

Another instance of a growth condition appeared earlier in the context of harmonic extension of distributions on the extended real line $\mathbb{R} \cup \{\infty\}$. A direct proof can be constructed along the lines of the proof of Proposition 5.1.1.

Proposition 5.1.3. *Fix a real α with $0 < \alpha < +\infty$. Suppose $h : \mathbb{H} \to \mathbb{C}$ is harmonic and that*

$$|h(\tau)| = O\left(\frac{1 + |\tau|^2}{\operatorname{Im}\tau}\right)^\alpha$$

holds uniformly in $\mathbb{H}$, Then there exist unique functions $h_+, h_- : \mathbb{H} \to \mathbb{C}$ such that h_+ is holomorphic, h_- is conjugate-holomorphic, with $h_+(\mathrm{i}) = h_-(\mathrm{i}) = 0$, and the splitting

$$h(\tau) = h_+(\tau) + h_-(\tau) + h(\mathrm{i}), \qquad \tau \in \mathbb{H},$$

holds. These functions enjoy the same growth estimate:

$$|h_+(\tau)|, |h_-(\tau)| = O\left(\frac{1 + |\tau|^2}{\operatorname{Im}\tau}\right)^\alpha,$$

uniformly in $\mathbb{H}$.

5.2. **Coefficient growth and function growth in harmonic Fourier series.** If we have a distribution h on the circle $\mathbb{R}/2\mathbb{Z}$, i.e., a 2-periodic distribution, it has a Fourier series expansion

$$h(t) = \sum_{n \in \mathbb{Z}} c_n e_n(t), \quad \text{where} \quad e_n(t) = e^{\mathrm{i}\pi nt},$$

and the coefficients grow at most polynomially: $|c_n| = O(|n|^k)$ as $|n| \to +\infty$ for some finite k. The convergence of the series is not understood pointwise on $\mathbb{R}$, but only in the sense of distribution theory with convergence against a test function (cf. Definition 3.2.1). We extend h to a harmonic function in $\mathbb{H}$ via the Poisson kernel as in (3.4.2) and call the extension $h = \tilde{h}$ as well. Then h meets the growth bound (3.4.3), and since h is 2-periodic, we can do better. Indeed, by applying the Poisson extension term by term, we find that

$$(5.2.1) \qquad h(\tau) = \sum_{n \in \mathbb{Z}} c_n e_n(\tau), \qquad \tau \in \mathbb{H},$$

where the functions $e_n(\tau)$ are given by (3.5.2) and (3.5.3), depending on whether $n \geq 0$ or $n < 0$. This gives the estimate

$$|h(\tau)| \leq \sum_{n \in \mathbb{Z}} |c_n| |e_n(\tau)| = O\left(\sum_{n \in \mathbb{Z}} (1 + |n|)^k e^{-\pi |n| \operatorname{Im}\tau}\right) = O\left(1 + \frac{1}{(\operatorname{Im}\tau)^{k+1}}\right)$$

uniformly in $\tau \in \mathbb{H}$. We write this down as a proposition.

Proposition 5.2.1. *Fix k with $0 < k < +\infty$. If the coefficients $c_n \in \mathbb{C}$ have $|c_n| = O(|n|^k)$ as $|n| \to +\infty$, it follows that the harmonically extended Fourier series (5.2.1) has the growth bound*

$$|h(\tau)| \leq \sum_{n \in \mathbb{Z}} |c_n| |e_n(\tau)| = O\left(1 + \frac{1}{(\operatorname{Im}\tau)^{k+1}}\right)$$

uniformly in $\tau \in \mathbb{H}$.

This proposition has a converse.

Proposition 5.2.2. *Fix k with $0 < k < +\infty$. Suppose $h : \mathbb{H} \to \mathbb{C}$ is harmonic and that*

$$|h(\tau)| = O\left(1 + \frac{1}{(\mathrm{Im}\,\tau)^k}\right)$$

holds uniformly in $\tau \in \mathbb{H}$. Then if h is 2-periodic, i.e., $h(\tau+2) = h(\tau)$ holds, it follows that h is uniquely represented by a harmonic Fourier series of the form (5.2.1), where the coefficients have the growth $|c_n| = O(|n|^k)$ as $|n| \to +\infty$.

Proof. Given that f is 2-periodic, we can use standard Fourier expansion on the slice function $h(\cdot + iy)$ for fixed $y > 0$ to the effect that

$$(5.2.2) \qquad h(\tau + iy) = \sum_{n \in \mathbb{Z}} c_n(y)\, e_n(\tau),$$

holds first for $\tau \in \mathbb{R}$ with the coefficients $c_n(y)$ in ℓ^2, and then in a second step for $\tau \in \mathbb{H}$, by the generalized maximum principle. The coefficients are then given by the formula

$$c_n(y) = \frac{1}{2} \int_{-1}^{1} e^{-i\pi nt} h(t + iy)\, dt,$$

so that in view of the growth bound on h,

$$(5.2.3) \qquad |c_n(y)| = O(1 + y^{-k})$$

holds uniformly in $y > 0$. It remains to analyze the dependence of the coefficients $c_n(y)$ on the parameter $y > 0$. If we apply (5.2.2) with $\tau = t + is$ for $t \in \mathbb{R}$ and some $s > 0$, it is immediate that, since for $n \geq 0$,

$$e_n(t + is) = e^{i\pi n(t+is)} = e^{-\pi ns} e^{i\pi nt} = e^{-\pi ns} e_n(t),$$

whereas for $n < 0$,

$$e_n(t + is) = e^{i\pi n(t-is)} = e^{\pi ns} e^{i\pi nt} = e^{\pi ns} e_n(t),$$

We derive that

$$h(t + is + iy) = \sum_{n \in \mathbb{Z}} c_n(s+y)\, e_n(t) = \sum_{n \in \mathbb{Z}} c_n(y)\, e_n(t+is) = \sum_{n \in \mathbb{Z}} c_n(y)\, e^{-\pi|n|s}\, e_n(t),$$

and equating coefficients, that

$$c_n(s+y) = e^{-\pi|n|s} c_n(y).$$

As this holds for all combinations of $s, y > 0$, we conclude that the limit

$$c_n := \lim_{y \to 0^+} c_n(y)$$

exists, and in the process also that the $c_n(y)$ may be recovered from c_n:

$$c_n(y) = e^{-\pi|n|y} c_n.$$

In view of (5.2.3), we find that

$$|c_n| = O\big((1 + y^{-k}) e^{\pi|n|y}\big)$$

and we are free to optimize over $y > 0$. For $n \neq 0$, we choose $y = 1/|n|$, which gives the desired estimate. $\qquad\square$

We need to consider also wilder growth of the coefficients or of the periodic harmonic function.

Proposition 5.2.3. *Suppose, for $\alpha \in \mathbb{R}$ and $0 < \beta < +\infty$, that the coefficients c_n have the growth control*

$$|c_n| = O\big(|n|^{-\frac{3}{4}} \exp\big(2\pi\sqrt{\beta|n|}\big)\big) \quad as \quad |n| \to +\infty.$$

Then the associated harmonic Fourier series (5.2.1) *enjoys the growth bound*

$$|h(\tau)| \le \sum_{n\in\mathbb{Z}} |c_n|\, e^{-\pi|n|\mathrm{Im}\tau} = O(\exp(\pi\beta/\mathrm{Im}\tau))$$

uniformly in $\mathbb{H}$. Moreover, under the corresponding little "o" condition on the coefficients, the conclusion holds with little "o" in the limit as $\mathrm{Im}\,\tau \to 0^+$.

The proposition is immediate from the $\alpha = \frac{3}{2}$ instance of the following lemma.

Lemma 5.2.4. *For positive α, β, y, and an integer $N > \alpha^2/(4\pi^2\beta)$, we have that*

$$\sum_{n\ge N} n^{-\alpha/2} \exp\big(2\pi\sqrt{\beta n} - \pi n y\big) = O\big(y^{\alpha-\frac{3}{2}}\, e^{\pi\beta/y}\big)$$

as $y \to 0^+$.

Proof. We consider the function

$$\varphi(t) := -\pi y t^2 + 2\pi\sqrt{\beta}\,t - \alpha\log t,$$

for positive real parameters α, β, y, and observe that

(5.2.4) $$n^{-\alpha/2}\exp(2\pi\sqrt{\beta n} - \pi n y) = \exp(\varphi(\sqrt{n})).$$

As the derivate equals

$$\varphi'(t) = -2\pi y t + 2\pi\sqrt{\beta} - \frac{\alpha}{t},$$

we obtain two critical points,

$$t_0(y) = \frac{\sqrt{\beta}}{2y}\left(1 - \sqrt{1 - \frac{2\alpha y}{\pi\beta}}\right) = \frac{\alpha}{2\pi\sqrt{\beta}} + O(y)$$

and

$$t_1(y) = \frac{\sqrt{\beta}}{2y}\left(1 + \sqrt{1 - \frac{2\alpha y}{\pi\beta}}\right) = \frac{\sqrt{\beta}}{y} - \frac{\alpha}{2\pi\sqrt{\beta}} + O(y)$$

as $y \to 0^+$. For small $y > 0$, the first critical point $t_0(y)$ is smaller than $\sqrt{N}$, so it lies outside of the given interval, and we may disregard it. On the interval $[\sqrt{N}, t_1(y)]$, the function φ is strictly increasing, whereas on $[t_1(y), +\infty[$, φ is instead strictly decreasing. The value $\varphi(t_1(y))$ is then the unique absolute maximum on $[\sqrt{N}, +\infty[$, and we calculate that

$$\varphi(t_1(y)) = \frac{\pi\beta}{y} - \alpha\log\frac{\sqrt{\beta}}{y} + O(y)$$

as $y \to 0^+$. The second derivative equals

$$\varphi''(t) = -2\pi y + \frac{\alpha}{t^2}.$$

Let γ_0 be the positive real parameter given by

$$\gamma_0^2 = \frac{\alpha}{2(\pi - \eta_0)},$$

where $0 < \eta_0 < \pi$, and observe that

$$\varphi''(t) = -2\pi y + \frac{\alpha}{t^2} \le \left(-2\pi + \frac{\alpha}{\gamma_0^2} \right) y = -2\eta_0 y, \qquad t \in [\gamma_0 y^{-\frac{1}{2}}, +\infty[.$$

By Taylor's formula, we have that

$$\varphi(t) = \varphi(t_1(y)) + \int_{t_1(y)}^{t} (t-s)\varphi''(s)ds = \varphi(t_1(y)) + \frac{1}{2}(t - t_1(y))^2 \varphi''(\xi),$$

for some point ξ between $t_1(y)$ and t, and consequently, we obtain

$$(5.2.5) \quad \varphi(t) = \varphi(t_1(y)) + \frac{1}{2}(t-t_1(y))^2 \varphi''(\xi) \le \varphi(t_1(y)) - \eta_0 y(t-t_1(y))^2$$

$$= \frac{\pi\beta}{y} - \alpha \log \frac{\sqrt{\beta}}{y} - \eta_0 y(t - t_1(y))^2 + O(y), \qquad t \in [\gamma_0 y^{-\frac{1}{2}}, +\infty[,$$

since $t_1(y) \ge \gamma_0 y^{-\frac{1}{2}}$ holds as $y \to 0^+$. Note that in (5.2.5), the $O(y)$ term is independent of t. As for the remaining interval, we use monotonicity:

$$(5.2.6) \quad \varphi(t) \le \varphi(\gamma_0 y^{-\frac{1}{2}})$$

$$= -\pi\gamma_0^2 - \alpha \log \gamma_0 + \frac{\alpha}{2} \log y + 2\pi\sqrt{\beta}\gamma_0 y^{-\frac{1}{2}}, \qquad t \in [\sqrt{N}, \gamma_0 y^{-\frac{1}{2}}].$$

Using the monotonicity properties of φ, we find that

$$\sum_{N \le n < t_1(y)^2} \exp(\varphi(\sqrt{n})) \le \int_{N}^{t_1(y)^2} \exp(\varphi(\sqrt{s}))ds + \exp(\varphi(t_1(y)))$$

and

$$\sum_{n \ge t_1(y)^2} \exp(\varphi(\sqrt{n})) \le \int_{t_1(y)^2}^{+\infty} \exp(\varphi(\sqrt{s}))ds + \exp(\varphi(t_1(y))).$$

By adding up these estimates, it follows that

$$(5.2.7) \quad \sum_{n \ge N} \exp(\varphi(\sqrt{n})) \le \int_{N}^{+\infty} \exp(\varphi(\sqrt{s}))ds + 2\exp(\varphi(t_1(y)))$$

$$= 2 \int_{\sqrt{N}}^{+\infty} \exp(\varphi(t))t\,dt + 2\exp(\varphi(t_1(y))).$$

Next, from the estimate (5.2.6), it follows that

$$2\int_{\sqrt{N}}^{\gamma_0 y^{-\frac{1}{2}}} \exp(\varphi(t))\, t\, dt \le \gamma_0^2 y^{-1}\exp(\varphi(\gamma_0 y^{-\frac{1}{2}})) = \gamma_0^{2-\alpha} e^{-\pi\gamma_0^2} y^{-1+\frac{\alpha}{2}}\exp(2\pi\sqrt{\beta}\gamma_0 y^{-\frac{1}{2}}).$$

On the other hand, from (5.2.5), we find that

$$2\int_{\gamma_0 y^{-\frac{1}{2}}}^{+\infty}\exp(\varphi(t))t\, dt \le (1+\mathrm{O}(y))\beta^{-\alpha/2}y^\alpha e^{\pi\beta/y}\int_{\gamma_0 y^{-1/2}}^{+\infty}\exp(-\eta_0 y(t-t_1(y))^2)2t\, dt$$

$$\le (1+\mathrm{O}(y))\beta^{-\alpha/2}y^\alpha e^{\pi\beta/y}\int_0^{+\infty}\exp(-\eta_0 y(t-t_1(y))^2)2t\, dt$$

$$\le (1+\mathrm{O}(y))\beta^{-\alpha/2}y^\alpha e^{\pi\beta/y}\big(\eta_0^{-1}y^{-1}e^{-\eta_0 y t_1(y)^2}+2\pi^{\frac{1}{2}}\eta_0^{-\frac{1}{2}}y^{-\frac{1}{2}}t_1(y)\big)$$

$$\le 2\sqrt{\pi}\eta_0^{-\frac{1}{2}}\beta^{(1-\alpha)/2}y^{\alpha-\frac{3}{2}}(1+\mathrm{O}(y))\big(1+\mathrm{O}(\eta_0^{-\frac{1}{2}}y^{\frac{1}{2}})\big)e^{\pi\beta/y}$$

as $y\to 0^+$. We may now fix $\eta_0 := \pi/2$ and hence $\gamma_0^2 = \frac{\alpha}{\pi}$. This permits us to not worry about the dependence on η_0 and γ_0 in the above estimates. Putting things together, we find that the inequality (5.2.7) leads to

$$\sum_{n\ge N}\exp(\varphi(\sqrt{n})) \le \int_N^{+\infty}\exp(\varphi(\sqrt{s}))\, ds + 2\exp(\varphi(t_1(y)))$$

$$= \mathrm{O}\Big(y^{-1+\frac{\alpha}{2}}\exp\big(2\sqrt{\pi\alpha\beta}y^{-\frac{1}{2}}\big)\Big)$$

$$+ \sqrt{2}\beta^{(1-\alpha)/2}y^{\alpha-\frac{3}{2}}(1+\mathrm{O}(y^{\frac{1}{2}}))e^{\pi\beta/y} + 2y^\alpha e^{\pi\beta/y}(1+\mathrm{O}(y))$$

$$= \sqrt{2}\beta^{(1-\alpha)/2}(1+\mathrm{O}(y^{\frac{1}{2}}))y^{\alpha-\frac{3}{2}}e^{\pi\beta/y}$$

as $y\to 0^+$. In view of the identity (5.2.4), the claimed assertion now follows.   □

The converse to Proposition 5.2.3 reads as follows.

Proposition 5.2.5. *Suppose, for $0 < \beta < +\infty$, $h : \mathbb{H}\to\mathbb{C}$ is a harmonic function which is 2-periodic, i.e., $h(\tau+2) = h(\tau)$ holds, and enjoys the estimate*

$$|h(\tau)| = \mathrm{O}\big(\exp(\pi\beta/\mathrm{Im}\tau)\big)$$

uniformly in $\tau \in \mathbb{H}$. It then follows that h is uniquely represented by a harmonic Fourier series of the form (5.2.1)*, where the coefficients have the growth bound*

$$|c_n| = \mathrm{O}\big(\exp\big(2\pi\sqrt{\beta|n|}\big)\big)$$

as $|n|\to+\infty$.

Proof sketch. Following along the same lines as the proof of Proposition 5.2.2, we may use the following formula to recover the Fourier coefficients:

$$(5.2.8)\qquad\qquad c_n = \frac{e^{\pi|n|s}}{2}\int_{-1}^1 e^{-i\pi nt}h(t+is)\, dt,$$

provided that $s > 0$. The growth estimate of the function f now gives that

$$|c_n| \leq C \exp(\pi|n|s) \exp(\pi\beta/s),$$

where C is the implied constant. Now optimizing over $s > 0$ gives the indicated coefficient estimate. □

5.3. **Solving the Dirichlet problem in the half-plane with possibly rapid growth.**
We consider the Dirichlet problem of finding a harmonic function $u : \mathbb{H} \to \mathbb{C}$ which extends continuously to the closure $\mathbb{H} \cup \mathbb{R}$, with $u = \varphi$ on $\mathbb{R}$, where $\varphi : \mathbb{R} \to \mathbb{C}$ is a given continuous function. If φ has finite integral

$$(5.3.1) \qquad \int_{\mathbb{R}} \frac{|\varphi(t)|}{1+t^2}\, dt < +\infty,$$

we can use the Poisson extension (3.4.2) to define such a solution

$$(5.3.2) \qquad u(\tau) = \int_{\mathbb{R}} P(\tau,t)\, \varphi(t)\, dt = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\operatorname{Im}\tau}{|t-\tau|^2}\, \varphi(t)\, dt, \qquad \tau \in \mathbb{H}.$$

This is the case in the following instance.

Proposition 5.3.1. *Suppose that $\varphi : \mathbb{R} \to \mathbb{C}$ is continuous with logarithmic power growth:*

$$|\varphi(t)| = \mathrm{O}(\log(2+|t|))^k \quad \text{as } |t| \to +\infty,$$

for some integer k, $0 < k < +\infty$. Then the Poisson extension u given by (5.3.2) has the same growth rate

$$|u(\tau)| = \mathrm{O}(\log(2+|\tau|))^k,$$

uniformly in $\mathbb{H}$.

Proof. The growth bound (5.3.1) on φ holds. We consider the holomorphic function

$$V(\tau) = \log(\tau+2\mathrm{i}) = \log|\tau+2\mathrm{i}| + \mathrm{i}\arg(\tau+2\mathrm{i}), \qquad \tau \in \mathbb{H},$$

where the argument is chosen in the interval from 0 to π. The k-power of V has

$$V(\tau)^k = \log^k|\tau+2\mathrm{i}| + \mathrm{O}(\log^{k-1}|\tau+2\mathrm{i}|), \qquad \tau \in \mathbb{H},$$

so that the same applies to its real part:

$$\operatorname{Re} V(\tau)^k = \log^k|\tau+2\mathrm{i}| + \mathrm{O}(\log^{k-1}|\tau+2\mathrm{i}|), \qquad \tau \in \mathbb{H}.$$

Next, we observe that without loss of generality we may assume that φ is real-valued. Indeed, if the assertion of the lemma holds for the real and imaginary parts of φ separately, then it holds also for φ. In this case, the Poisson extension u is harmonic and real-valued, and

$$\varphi(t) - A\operatorname{Re} V(t)^k \leq B, \qquad t \in \mathbb{R},$$

holds for some positive constants A, B, by the growth assumption on φ. This then carries over to the Poisson extension of the left-hand side:

$$u(\tau) - A\operatorname{Re} V(\tau)^k \leq B, \qquad \tau \in \mathbb{H}.$$

This establishes the estimate from above on u. The analogous estimate from below follows by the same argument applied to $-u$ in place of u and $-\varphi$ in place of φ. In view of this, the assertion now follows. $\qquad\square$

The maximum principle can also help in cases of slower rates of growth.

Proposition 5.3.2. *Suppose that $\varphi : \mathbb{R} \to \mathbb{C}$ is continuous with double logarithmic growth:*

$$|\varphi(t)| = O(\log\log(10 + |t|)) \quad as \ |t| \to +\infty.$$

Then the Poisson extension u given by (5.3.2) *has the same growth rate*

$$|u(\tau)| = O(\log\log(10 + |\tau|)),$$

uniformly in $\mathbb{H}$.

Proof sketch. If we let V be the holomorphic function

$$V(\tau) = \log\log(\tau + 10i), \qquad \tau \in \mathbb{H},$$

we may use its real part as a comparison function, as in the proof of Proposition 5.3.1. The details of the argument are left to the reader. $\qquad\square$

Although the Poisson extension works very well under the integrability condition (5.3.1), we are left to wonder about the uniqueness. For instance, the function $\tau \mapsto \operatorname{Im}\tau$ is harmonic in $\mathbb{H}$ and vanishes on the line $\mathbb{R}$, so we may add it to any given solution u to the Dirichlet problem and still solve the boundary value problem. The following proposition explains the degree of nonuniquess if we do not impose any growth condition on the solution u. The result is well-known, but the proof is short and we include it for completeness.

Proposition 5.3.3. *Suppose that $u : \mathbb{H} \to \mathbb{C}$ is harmonic and extends continuously to the closed half-plane $\mathbb{H} \cup \mathbb{R}$, and that $u(t) = 0$ holds for each $t \in \mathbb{R}$. Then there exist two entire functions $F_1, F_2 : \mathbb{C} \to \mathbb{C}$ with real Taylor coefficients at the point 0, such that*

$$u(\tau) = \operatorname{Im}F_1(\tau) + i\operatorname{Im}F_2(\tau), \qquad \tau \in \mathbb{H}.$$

Proof. By the Schwarz reflection principle [22] (Theorem 11.17), if we extend u to the lower half-plane by setting

$$u(\bar{\tau}) := -u(\tau), \qquad \tau \in \mathbb{H},$$

we obtain a harmonic function in the whole plane $\mathbb{C}$ which vanishes on $\mathbb{R}$. Considering the real and imaginary parts of u separately, we find entire functions F_1, F_2 such that $\operatorname{Re}u = \operatorname{Im}F_1$ and $\operatorname{Im}u = \operatorname{Im}F_2$. These two entire functions take real values along $\mathbb{R}$, i.e., they have real Taylor coefficients at 0. $\qquad\square$

As for the solvability of the of the Dirichlet problem for general continuous $\varphi : \mathbb{R} \to \mathbb{C}$, it has been considered and the answer is affirmative (see the work of Finkelstein and Scheinberg [13]). We have found an alternative short proof which we also supply.

Proposition 5.3.4. *Suppose $\varphi : \mathbb{R} \to \mathbb{C}$ is continuous. Then there exists a harmonic function $u : \mathbb{H} \to \mathbb{C}$ which extends continuously to the closed half-plane $\mathbb{H} \cup \mathbb{R}$, such that $u(t) = \varphi(t)$ holds for all $t \in \mathbb{R}$.*

Proof. It is an exercise to find an even entire function Ψ with real Taylor coefficients at the point 0, which grows at least as quickly along the real line as any given real-valued continuous function. In particular, it is possible to find such a Ψ with

$$\int_{\mathbb{R}} |\varphi(t)| \exp(-\operatorname{Re}\Psi(t)) \, dt < +\infty.$$

Then it can be shown that the formula

$$u(\tau) = \frac{1}{\pi} \int_{\mathbb{R}} \operatorname{Im}\left(\frac{\exp(\Psi(\tau) - \Psi(t))}{t - \tau} \right) \varphi(t) \, dt, \qquad \tau \in \mathbb{H},$$

defines a harmonic function u whose boundary values equal φ.   □

This settles the matter of the Dirichlet problem with arbitrary continuous boundary values, at the cost of allowing arbitrary growth of the solution u. But what about an intermediate situation, when we have growth control on φ, can we get analogous growth control of the solution u?

Proposition 5.3.5. *Fix an $\alpha \in \mathbb{R}_{>0}$, and suppose $\varphi : \mathbb{R} \to \mathbb{C}$ is continuous with the growth bound*

$$|\varphi(t)| = \mathrm{O}\big((1 + |t|)^{\alpha}\big) \quad \text{as } |t| \to +\infty.$$

Then if $\alpha \notin \mathbb{Z}$, there exists a harmonic function $u = u_{\alpha} : \mathbb{H} \to \mathbb{C}$ which extends continuously to the closed half-plane $\mathbb{H} \cup \mathbb{R}$ with $u = \varphi$ on the line $\mathbb{R}$, such that

$$|u(\tau)| = \mathrm{O}\big((1 + |\tau|)^{\alpha}\big)$$

holds uniformly in $\tau \in \mathbb{H}$. Moreover, if $\alpha = k \in \mathbb{Z}_{>0}$, there still is such a solution u, but the growth bound is slightly worse:

$$|u(\tau)| = \mathrm{O}\big((1 + |\tau|)^{k} \log(2 + |\tau|)\big),$$

uniformly in $\tau \in \mathbb{H}$.

The proof of the proposition is deferred to the separate publication [18].

Remark 5.3.6. (a) For $0 < \alpha < 1$, the solution u offered by Proposition 5.3.5 coincides with the Poisson extension. However, for $\alpha \geq 1$ alternative harmonic extension kernels are needed.

(b) If we consider for $k \in \mathbb{Z}_{>0}$ the harmonic function

$$u(\tau) = \operatorname{Im}(\tau^{k} \log \tau), \qquad \tau \in \mathbb{H},$$

where we use the principal branch of the logarithm, we find that the boundary values are $u(t) = \pi t^{k}$ for $t < 0$ while $u(t) = 0$ for $t \geq 0$. In particular, $u(t) = \mathrm{O}(|t|^{k})$ while the growth of $u(\tau)$ in the upper half-plane $\mathbb{H}$ involves a logarithm. This explains why the integer case $\alpha = k \in \mathbb{Z}_{>0}$ of Proposition 5.3.5 is different from the non-integer case.

6. UNIQUENESS ISSUES FOR THE KLEIN-GORDON EQUATION

6.1. **Translated lattice-crosses.** The following lemma uses the idea of periodization in Fourier analysis. It was key in, e.g., [15], [16].

Lemma 6.1.1. *Let* $\psi \in L^1(\mathbb{R})$. *Then*

$$\forall n \in \mathbb{Z} : \quad \int_{\mathbb{R}} e^{i\pi nt} \psi(t) \, dt = 0$$

holds if and only if

$$\sum_{j \in \mathbb{Z}} \psi(t + 2j) = 0 \quad a.e. \ on \quad \mathbb{R}.$$

Proof of Lemma 6.1.1. The Fourier coefficients of the 2-periodic periodization

$$\Psi(t) := \sum_{j \in \mathbb{Z}} \psi(t + 2j)$$

are given by

$$\frac{1}{2} \int_{[-1,1]} e^{-i\pi mt} \Psi(t) \, dt = \frac{1}{2} \int_{[-1,1]} e^{-i\pi mt} \sum_{j \in \mathbb{Z}} \psi(t+2j) \, dt = \frac{1}{2} \int_{\mathbb{R}} e^{-i\pi mt} \psi(t) \, dt$$

for $m \in \mathbb{Z}$. If the Fourier coefficients all vanish, then the function Ψ must vanish as well. On the other hand, if Ψ vanishes almost everywhere, then the Fourier coefficients must vanish too. $\qquad\square$

Proof of Proposition 4.4.1. We begin with the setting of Proposition 4.4.1. This means that $\varphi \in L^1(\mathbb{R})$ is supposed to have the following two properties, for some fixed $\omega_1, \omega_2 \in \mathbb{R}$:

$$(6.1.1) \qquad\qquad \forall m \in \mathbb{Z} : \quad \int_{\mathbb{R}} e^{i\pi(m+\omega_1)t} \varphi(t) \, dt = 0$$

and

$$(6.1.2) \qquad \forall n \in \mathbb{Z} : \quad \int_{\mathbb{R}} e^{i\pi(n+\omega_2)/t} \varphi(t) \, dt = \int_{\mathbb{R}} e^{-i\pi(n+\omega_2)s} \varphi(-1/s) \, s^{-2} ds = 0.$$

We apply Lemma 6.1.1 to the functions

$$\psi_1(t) := e^{i\pi\omega_1 t} \varphi(t), \quad \psi_2(t) := t^{-2} e^{-i\pi\omega_2 t} \varphi(-1/t),$$

and find that the conditions (6.1.1) and (6.1.2) amount to having

$$\sum_{j \in \mathbb{Z}} \psi_l(t + 2j) = 0, \qquad l = 1, 2,$$

almost everywhere on $\mathbb{R}$, respectively. The condition for $l = 1$ means that

$$(6.1.3) \qquad\qquad \varphi(t) = - \sum_{j \in \mathbb{Z}_{\neq 0}} e^{i2\pi j\omega_1} \varphi(t + 2j)$$

almost everywhere on $\mathbb{R}$, while the condition for $l = 2$ is the same as

$$(6.1.4) \qquad\qquad \varphi(t) = - \sum_{j \in \mathbb{Z}_{\neq 0}} \frac{e^{-i2\pi j\omega_2}}{(1 - 2jt)^2} \varphi\left(\frac{t}{1 - 2jt}\right),$$

again almost everywhere on $\mathbb{R}$. We may combine the conditions (6.1.3) and (6.1.4) in two different ways, by inserting (6.1.3) into the right-hand side of (6.1.4) or, alternatively, by inserting (6.1.4) into the right-hand side of (6.1.3). We choose the second alternative, and obtain

$$(6.1.5) \qquad \varphi(t) = \sum_{j,k \in \mathbb{Z}_{\neq 0}} \frac{e^{i2\pi(k\omega_1 - j\omega_2)}}{(1 - 2j(t + 2k))^2} \varphi\left(\frac{t + 2k}{1 - 2j(t + 2k)}\right), \quad \text{a.e. } t \in \mathbb{R}.$$

If we put, for a given $\omega \in \mathbb{R}$,

$$(6.1.6) \qquad \mathbf{T}_\omega f(t) := \sum_{j \in \mathbb{Z}_{\neq 0}} \frac{e^{i2\pi j\omega}}{(2j - t)^2} f\left(\frac{1}{2j - t}\right),$$

the equation (6.1.5) may be expressed in the form

$$(6.1.7) \qquad \varphi(t) = \mathbf{T}_{\omega_2} \mathbf{T}_{-\omega_1} \varphi(t), \quad \text{a.e. } t \in \mathbb{R}.$$

In view of the triangle inequality, we find that

$$|\mathbf{T}_\omega f(t)| = \left| \sum_{j \in \mathbb{Z}_{\neq 0}} \frac{e^{i2\pi j\omega}}{(2j - t)^2} f\left(\frac{1}{2j - t}\right) \right|$$

$$\leq \sum_{j \in \mathbb{Z}_{\neq 0}} \frac{1}{(2j - t)^2} |f|\left(\frac{1}{2j - t}\right) = \mathbf{T}_0 |f|(t), \qquad t \in \mathbb{R}.$$

In particular, by iteration, if φ solves (6.1.7), then, by taking absolute values, we find that

$$(6.1.8) \qquad |\varphi(t)| = |\mathbf{T}_{\omega_2} \mathbf{T}_{-\omega_1} \varphi(t)| \leq \mathbf{T}_0(|\mathbf{T}_{-\omega_1} \varphi|) \leq \mathbf{T}_0^2 |\varphi|(t), \quad \text{a.e. } t \in \mathbb{R}.$$

Here, we used the rather obvious property that $\mathbf{T}_0$ is positive, in the sense that if $f, g : \mathbb{R} \to \mathbb{R}_{\geq 0}$ and $f \leq g$ holds holds almost everywhere, then $\mathbf{T}_0 f \leq \mathbf{T}_0 g$ holds almost everywhere as well. Another important relevant observation is that in view of the definition (6.1.6) of $\mathbf{T}_0$, we have the following *localization principle*: the restriction of f to $[-1, 1]$ determines the restriction of $\mathbf{T}_0 f$ to $[-1, 1]$. Next, according to Proposition 3.4.1 in [16],

$$\int_{[-1,1]} \mathbf{T}_0^2 |\varphi|(t) \, dt = \int_{[-1,1]} \mathbf{T}_0 |\varphi|(t) \, dt = \int_{[-1,1]} |\varphi(t)| \, dt,$$

which means that (6.1.8) is only possible if in fact

$$(6.1.9) \qquad |\varphi(t)| = \mathbf{T}_0^2 |\varphi|(t), \quad \text{a.e. } t \in [-1, 1].$$

This can be thought of as asking that $|\varphi(t)| dt$ be an absolutely continuous invariant measure for the Gauss-type map $t \mapsto -1/t \pmod 1$ on the interval $[-1, 1]$, which is not possible unless $\varphi(t) = 0$ almost everywhere on $[-1, 1]$. This was shown using the Birkhoff ergodic theorem in the infinite ergodic setting in [15]. A more direct proof follows from Proposition 3.13.3 in [16], which asserts that the iterates $\mathbf{T}_0^{2n} |\varphi|$ tend to

0 in the L^1 norm on subintervals $[-1+\varepsilon, 1-\varepsilon]$, for fixed ε with $0 < \varepsilon < 1$. Since by the localization principle applied to (6.1.9), we must have that

$$|\varphi(t)| = \mathbf{T}_0^{2n}|\varphi|(t), \quad \text{a.e. } t \in [-1,1],$$

for $n = 1, 2, 3, \ldots$, it is immediate that $\varphi(t) = 0$ for almost every $t \in [-1,1]$. It remains to show that $\varphi(t) = 0$ holds for almost every $t \in \mathbb{R}$. To this end, we can use the identity (6.1.4), since we already know that $\varphi(t) = 0$ almost everywhere on $[-1,1]$. For $t \in \mathbb{R}$ with $|t| > 1$ and $j \in \mathbb{Z}_{\neq 0}$,

$$\frac{t}{1-2jt} \in [-1,1],$$

so that in view of (6.1.4), the information about φ on $[-1,1]$ carries over to the complement $\mathbb{R} \setminus [-1,1]$, and we conclude that $\varphi = 0$ as an element of $L^1(\mathbb{R})$. $\square$

6.2. **Normal derivatives on the lattice-cross.**
We turn to the proof of Proposition 4.3.1.

Proof of Proposition 4.3.1. If $k = 0$, the assertion coincides with that of Theorem 4.2.2, so we may assume in the rest of the proof that $k > 0$. It is given that $\varphi \in L^1(\mathbb{R})$ has the extra integrability (4.3.3), and that

$$\int_{\mathbb{R}} e^{i\pi mt} \varphi(t)\,dt = \int_{\mathbb{R}} e^{-i\pi n/t} \varphi(t)\, t^k\,dt = 0, \qquad m, n \in \mathbb{Z},$$

holds. After a change of variables in the latter integral, the conditions read

$$(6.2.1) \qquad \int_{\mathbb{R}} e^{i\pi mt} \varphi(t)\,dt = \int_{\mathbb{R}} e^{i\pi nt} \varphi(-1/t)\, t^{-2-k}\,dt = 0, \qquad m, n \in \mathbb{Z},$$

We apply Lemma 6.1.1 to the functions

$$\psi_1(t) := \varphi(t), \quad \psi_2(t) := t^{-2-k}\varphi(-1/t),$$

which are in $L^1(\mathbb{R})$ because of (4.3.3), and find that the condition (6.2.1) amounts to having

$$\sum_{j \in \mathbb{Z}} \psi_l(t+2j) = 0, \qquad l = 1, 2,$$

almost everywhere on $\mathbb{R}$, respectively. The condition for $l = 1$ is

$$(6.2.2) \qquad\qquad \varphi(t) = -\sum_{j \in \mathbb{Z}_{\neq 0}} \varphi(t+2j)$$

almost everywhere on $\mathbb{R}$, while the condition for $l = 2$ may be expressed in the form

$$(6.2.3) \qquad\qquad \varphi(t) = -\sum_{j \in \mathbb{Z}_{\neq 0}} (1-2jt)^{-2-k}\varphi\left(\frac{t}{1-2jt}\right),$$

again almost everywhere on $\mathbb{R}$. We may combine the conditions (6.2.2) and (6.2.3) in two different ways, by inserting (6.2.2) into the right-hand side of (6.2.3) or, alternatively, by inserting (6.2.3) into the right-hand side of (6.2.2). We choose the second

alternative, and obtain

$$(6.2.4) \qquad \varphi(t) = \sum_{j,j' \in \mathbb{Z}_{\neq 0}} (1 - 2j(t + 2j'))^{-2-k} \, \varphi\left(\frac{t + 2j'}{1 - 2j(t + 2j')}\right), \quad \text{a.e. } t \in \mathbb{R}.$$

If we consider the "transfer operator"

$$\mathbf{T}_k f(t) := \sum_{j \in \mathbb{Z}_{\neq 0}} (2j - t)^{-2-k} f\left(\frac{1}{2j - t}\right),$$

the equation (6.2.4) may be expressed in the form

$$(6.2.5) \qquad\qquad \varphi(t) = (-1)^k \mathbf{T}_k^2 \varphi(t), \quad \text{a.e. } t \in \mathbb{R}.$$

In view of the triangle inequality, we find that

$$|\mathbf{T}_k f(t)| = \left| \sum_{j \in \mathbb{Z}_{\neq 0}} (2j - t)^{-2-k} f\left(\frac{1}{2j - t}\right) \right|$$

$$\leq \sum_{j \in \mathbb{Z}_{\neq 0}} |2j - t|^{-2-k} |f|\left(\frac{1}{2j - t}\right) = \mathbf{T}_{\langle k \rangle} |f|(t), \qquad t \in \mathbb{R},$$

where $\mathbf{T}_{\langle k \rangle}$ denotes the operator

$$\mathbf{T}_{\langle k \rangle} f(t) := \sum_{j \in \mathbb{Z}_{\neq 0}} |2j - t|^{-2-k} f\left(\frac{1}{2j - t}\right).$$

In particular, by iteration and by monotonicity, if φ solves (6.2.2) and (6.2.3), then, by taking absolute values, we find that

$$|\varphi(t)| = |\mathbf{T}_k^2 \varphi(t)| \leq \mathbf{T}_{\langle k \rangle}(|\mathbf{T}_k \varphi|) \leq \mathbf{T}_{\langle k \rangle}^2 |\varphi|(t), \quad \text{a.e. } t \in \mathbb{R}.$$

The operators $\mathbf{T}_k$ and $\mathbf{T}_{\langle k \rangle}$ meet the following *localization principle*: the restriction of f to $[-1, 1]$ determines the restriction of both $\mathbf{T}_k f$ and $\mathbf{T}_{\langle k \rangle} f$ to $[-1, 1]$. We find that for measurable f, a suitable change of variables yields

$$(6.2.6) \quad \int_{[-1,1]} |\mathbf{T}_k f(t)| \, dt \leq \int_{[-1,1]} \mathbf{T}_{\langle k \rangle} |f|(t) \, dt$$

$$= \sum_{j \in \mathbb{Z}_{\neq 0}} \int_{[-1,1]} |2j - t|^{-2-k} |f|\left(\frac{1}{2j - t}\right) dt = \sum_{j \in \mathbb{Z}_{\neq 0}} \int_{[1/(2j+1), 1/(2j-1)]} |f(t)| \, |t|^k dt$$

$$= \int_{[-1,1]} |f(t)| \, |t|^k dt \leq \int_{[-1,1]} |f(t)| \, dt,$$

provided that the right-hand side integral is finite. Next, by iteration of (6.2.6),

$$(6.2.7) \quad \int_{[-1,1]} |\mathbf{T}_k^2 \varphi(t)| \, dt \leq \int_{[-1,1]} |\mathbf{T}_k \varphi(t)| \, dt$$

$$\leq \int_{[-1,1]} |\varphi(t)| \, |t|^k dt \leq \int_{[-1,1]} |\varphi(t)| \, dt.$$

We proceed to show that for $\varphi \in L^1(\mathbb{R})$, the only solution to (6.2.5) is $\varphi = 0$. By (6.2.5) and (6.2.7),

$$(6.2.8) \quad \int_{[-1,1]} |\varphi(t)| \mathrm{d}t = \int_{[-1,1]} |\mathbf{T}_k^2 \varphi(t)| \, \mathrm{d}t \leq \int_{[-1,1]} |\varphi(t)| \, |t|^k \mathrm{d}t \leq \int_{[-1,1]} |\varphi(t)| \, \mathrm{d}t.$$

The left-hand and right-hand sides of (6.2.8) coincide, so we must have equality throughout in (6.2.8). But since $k > 0$, it follows that $|t|^k < 1$ for $|t| < 1$, which entails that the last inequality in (6.2.7) is strict unless $\varphi = 0$ almost everywhere on $[-1,1]$. Hence we conclude from (6.2.7) that $\varphi(t) = 0$ holds almost everywhere on $[-1,1]$. It remains to show that φ vanishes almost everywhere on $\mathbb{R} \setminus [-1,1]$ as well. To this end we can use (6.2.3) with $C = 0$, since the fact that

$$\frac{t}{1 - 2jt} \in [-1,1] \quad \text{for} \quad t \in \mathbb{R} \setminus [-1,1], \ j \in \mathbb{Z}_{\neq 0}$$

holds shows that

$$\varphi(t) = - \sum_{j \in \mathbb{Z}_{\neq 0}} (1 - 2jt)^{-2-k} \varphi\left(\frac{t}{1 - 2jt}\right) = 0, \qquad \text{a.e. } t \in \mathbb{R} \setminus [-1,1].$$

The conclusion that $\varphi(t) = 0$ almost everywhere on $\mathbb{R}$ is now immediate. $\qquad \square$

7. UNIQUENESS AND BEYOND FOR HYPERBOLIC FOURIER SERIES

7.1. **Proofs of the theorems on uniqueness.** We turn to the uniqueness of the harmonic hyperbolic Fourier expansion.

Proof of Theorems 3.6.6 and 3.7.3. We first note that Theorem 3.6.6 may be viewed as the special case of $k = 1$ in Theorem 3.7.3, so we proceed to obtain Theorem 3.7.3 for general $k = 1, 2, 3, \ldots$, with the understanding that $\mathfrak{E}_{k-1}^{\mathrm{harm}} = \{0\}$ for $k = 1$. We note that under the given growth control on the coefficients, the series (3.5.1) converges absolutely and uniformly on compact subsets of $\mathbb{H}$. In terms of the functions e_n given by (3.5.2) (for $n \geq 0$) and (3.5.3) (for $n < 0$), the vanishing condition reads

$$a_0 + \sum_{n \neq 0} \left(a_n e_n(\tau) + b_n e_n(-1/\tau)\right) = 0, \qquad \tau \in \mathbb{H},$$

which we rewrite as

$$(7.1.1) \quad a_0 + \sum_{n > 0} \left(a_n e_n(\tau) + b_n e_n(-1/\tau)\right)$$
$$= - \sum_{n < 0} \left(a_n e_n(\tau) + b_n e_n(-1/\tau)\right), \qquad \tau \in \mathbb{H}.$$

We observe that the left-hand side is holomorphic, whereas the right-hand side is conjugate-holomorphic. This is possible only if each of the sides is constant:

$$(7.1.2) \qquad \sum_{n \geq 0} \left(a_n e_n(\tau) + b_n e_n(-1/\tau)\right) = C, \qquad \tau \in \mathbb{H}.$$

Our next task is to combine this with the growth control on the coefficients a_n to show that the combined sequence of coefficients $\{a_0, a_n, b_n\}_{n>0}$ is exceptional of degree $\leq k$ in the sense of Definition 3.7.1. To this end, we introduce the functions

$$F_+(\tau) =: a_0 + \sum_{n>0} a_n e_n(\tau), \qquad G_+(\tau) := \sum_{n>0} b_n e_n(\tau),$$

which are holomorphic functions in $\mathbb{H}$ with $F_+(+i\infty) = a_0$ and $G_+(+i\infty) = 0$. In view of our growth assumption on the coefficients a_n, the little "o" statement of Proposition 5.2.3 gives that

(7.1.3)
$$\limsup_{\mathrm{Im}\,\tau \to 0^+} e^{-\pi k/\mathrm{Im}\,\tau} |F_+(\tau)| = 0.$$

Moreover, the equality (7.1.2) asserts that

(7.1.4)
$$F_+(\tau) + G_+(-1/\tau) = C, \qquad \tau \in \mathbb{H},$$

which we write in the form

(7.1.5)
$$F_+(\tau) = C - G_+(-1/\tau), \qquad \tau \in \mathbb{H}.$$

The functions F_+, G_+ are both 2-periodic: $F_+(\tau+2) = F_+(\tau)$ and $G_+(\tau+2) = G_+(\tau)$, which we can express as $F_+ \circ \mathrm{T}^2 = F_+$ and $G_+ \circ \mathrm{T}^2 = G_+$, where $\mathrm{T}(\tau) := \tau + 1$. If we write $\mathrm{S}(\tau) := -1/\tau$, we find that according to (7.1.5), $F_+(\tau)$ is must be both 2-periodic and invariant under the transformation $\mathrm{ST}^2\mathrm{S} : \tau \mapsto \tau/(1-2\tau)$. This puts us in the setting of Lemma 1.1 of [2]. Informally speaking, that lemma is concerned with the possible poles of meromorphic functions on a (punctured) compact Riemann surface. With the parameter values $n_0 = n_\infty = 0$ and $n_1 = k-1$, the conditions (1), (2), and (3) of Lemma 1.1 [2] are all met. This follows from (7.1.3) and the fact that $F_+(\tau)$ has limit 0 at $+i\infty$. It is now a consequence of the assertion of Lemma 1.1 of [2] that $F_+(\tau)$ is of the form

(7.1.6)
$$F_+(\tau) = a_0 + P(\lambda(\tau)), \qquad \tau \in \mathbb{H},$$

where P stands for a polynomial of degree $\leq k-1$ with $P(0) = 0$. But then (7.1.4) tells us that G_+ takes the form

(7.1.7) $\quad G_+(\tau) = C - F_+(-1/\tau) = C - a_0 - P(\lambda(-1/\tau)) = C - a_0 - P(1-\lambda(\tau)),$

where we used the functional properties (3.6.5). We note that by plugging in the value $\tau = +i\infty$, while using that $G_+(+i\infty) = \lambda(+i\infty) = 0$, we obtain from (7.1.7) that

$$P(1) = C - a_0.$$

We note that (7.1.6) and (7.1.7) assert that *the sequence $\{a_0 - C, a_n, b_n\}_{n>0}$ is exceptional of degree $\leq k-1$.* From (7.1.1) and (7.1.2) we conclude that

(7.1.8)
$$\sum_{n<0} \big(a_n e_n(\tau) + b_n e_n(-1/\tau)\big) = -C, \qquad \tau \in \mathbb{H}.$$

Since for $n > 0$, $e_{-n}(\tau) = e_n(-\bar{\tau})$ holds, it is natural to introduce the holomorphic functions

$$F_-(\tau) := \sum_{n>0} a_{-n} e_n(\tau), \qquad G_-(\tau) := \sum_{n>0} b_{-n} e_n(\tau),$$

and to observe that (7.1.8) asserts that

$$F_-(\tau) + G_-(-1/\tau) = -C, \qquad \tau \in \mathbb{H}.$$

We are now in an analogous situation as we were in with F_+, G_+, and may conclude from a repetition of the argument that F_- and G_- take the form

(7.1.9) $$F_-(\tau) = Q(\lambda(\tau)), \quad G_-(\tau) = -C - Q(1 - \lambda(\tau)),$$

where Q is a polynomial of degree $\leq k - 1$, with $Q(0) = 0$. Moreover, from the second relation in (7.1.9) we find that $Q(1) = -\bar{C}$. It follows from the relation (7.1.9) that *the sequence $\{C, a_{-n}, b_{-n}\}_{n>0}$ is an exceptional sequence of degree $\leq k - 1$.* We have now established that the two-sided sequence $\{a_0, a_n, b_n\}_{n \neq 0}$ is exceptional of degree $\leq k$, as claimed. $\qquad \square$

7.2. Mapping properties of the modular λ function.

The proof of Proposition 8.1.3 would get more streamlined if we would use the mapping properties of the modular λ function. Let us first get acquainted with the following transformations on the upper half-plane $\mathbb{H}$:

$$\mathrm{T}(\tau) := \tau + 1, \quad \mathrm{S}(\tau) := -1/\tau, \quad \mathrm{S}^* := 1/\bar{\tau}.$$

Of these, S and T are holomorphic automorphisms, while S^* is the conjugate-holomorphic automorphism corresponding to reflection in the circle $\mathbb{T}$. Starting from the fundamental domain $\mathscr{D}_\Theta$ given by

$$\mathscr{D}_\Theta := \{\tau \in \mathbb{H} : |\operatorname{Re}\tau| < 1, \ |\tau| > 1\},$$

we may form its "double copy"

$$\mathscr{D}_{2\Theta} := \mathscr{D}_\Theta \cup \mathrm{S}^*(\mathscr{D}_\Theta) \cup \mathbb{T}_+ = \mathscr{D}_\Theta \cup \mathrm{S}(\mathscr{D}_\Theta) \cup \mathbb{T}_+,$$

which is a fundamental domain in its own right. It can be described by the formula

$$\mathscr{D}_{2\Theta} = \left\{\tau \in \mathbb{H} : |\operatorname{Re}\tau| < 1, \ |\tau + \tfrac{1}{2}| > \tfrac{1}{2}, \ |\tau - \tfrac{1}{2}| > \tfrac{1}{2}\right\}.$$

We introduce the notation

$$\mathbb{C}_{\operatorname{Re}<\frac{1}{2}} := \{z \in \mathbb{C} : \operatorname{Re}z < \tfrac{1}{2}\}, \quad \mathbb{C}_{\operatorname{Re}>\frac{1}{2}} := \{z \in \mathbb{C} : \operatorname{Re}z > \tfrac{1}{2}\},$$

and, for a given real number α, we put

(7.2.1) $$\mathscr{L}_\alpha := \{z \in \mathbb{C} : \operatorname{Re}z = \alpha\}, \quad \mathscr{L}_\alpha^+ := \{z \in \mathbb{H} : \operatorname{Re}z = \alpha\},$$

for the associated vertical line and half-line, respectively.

Proposition 7.2.1. *The modular λ function maps conformally the domain $\mathscr{D}_{2\Theta}$ onto the slit plane $\mathbb{C} \setminus (]-\infty, 0] \cup [1, +\infty[)$. In addition, it maps the subdomain $\mathscr{D}_\Theta$ onto the slit half-plane $\mathbb{C}_{\mathrm{Re} < \frac{1}{2}} \setminus]-\infty, 0]$, while it maps $\mathrm{S}(\mathscr{D}_\Theta)$ onto the slit half-plane $\mathbb{C}_{\mathrm{Re} > \frac{1}{2}} \setminus [0, +\infty[$. The semicircle $\mathbb{T}_+$ is mapped onto the line $\mathscr{L}_{\frac{1}{2}}$ in such a way that as $\tau \in \mathbb{T}_+$ increases its argument, the point $\lambda(\tau) \in \mathscr{L}_{\frac{1}{2}}$ moves downward in the negative imaginary direction. The half-lines $\mathscr{L}_{-1}^+$ and $\mathscr{L}_1^+$ are both mapped onto the slit $]-\infty, 0[$, while the edges ± 1 are mapped to infinity and the point $+\mathrm{i}\infty$ is mapped to 0. Similarly, the semicircles $\mathrm{S}(\mathscr{L}_{-1}^+)$ and $\mathrm{S}(\mathscr{L}_1^+)$ are both mapped onto the slit $]1, +\infty[$. Finally, the imaginary half-line $\mathscr{L}_0^+$ is mapped by λ onto the interval $]0, 1[$, where the endpoints $+\mathrm{i}\infty$ and 0 are mapped onto 0 and 1, respectively.*

The mapping properties of the modular λ function presented in Proposition 7.2.1 are classical. An elementary exposition may be found in [2].

We shall also need to approximate $\lambda(\tau)$ for $\tau \in \mathscr{D}_{2\Theta}$ near the cusps at ± 1 and $+\mathrm{i}\infty$. We recall the Fourier representation formula (3.6.4)

$$\lambda(\tau) = \sum_{n=1}^{+\infty} \widehat{\lambda}(n) \, e^{\mathrm{i}\pi n \tau}, \qquad \tau \in \mathbb{H},$$

where the coefficients are integers, the first three being $\widehat{\lambda}(1) = 16$, $\widehat{\lambda}(2) = -128$, and $\widehat{\lambda}(3) = 704$. In particular, we have that

$$(7.2.2) \qquad \lambda(\tau) = 16 \, e^{\mathrm{i}\pi\tau} - 128 \, e^{\mathrm{i}2\pi\tau} + \mathrm{O}(e^{-3\pi \mathrm{Im}\,\tau})$$

holds uniformly provided that $\mathrm{Im}\,\tau \geq 1$. In view of the 2-periodicity and the functional properties (3.6.5), it follows that

$$(7.2.3) \qquad \lambda(\tau) = 1 - \frac{1}{\lambda(-1/(\tau+1))} = 1 - \frac{1}{\lambda(-1/(\tau-1))}, \qquad \tau \in \mathbb{H}.$$

In view of (7.2.2), the functional property (7.2.3) gives that

$$(7.2.4) \quad \lambda(\tau) = 1 - \frac{1}{\lambda(-1/(\tau+1))}$$

$$= 1 - \frac{1}{16 e^{-\mathrm{i}\pi/(\tau+1)} - 128 e^{-\mathrm{i}2\pi/(\tau+1)} + \mathrm{O}(e^{-3\pi \mathrm{Im}\,\tau/|\tau+1|^2})}$$

$$= 1 - \frac{e^{\mathrm{i}\pi/(\tau+1)}}{16(1 - 8 e^{-\mathrm{i}\pi/(\tau+1)} + \mathrm{O}(e^{-2\pi \mathrm{Im}\,\tau/|\tau+1|^2}))}$$

$$= -\frac{1}{16} e^{\mathrm{i}\pi/(\tau+1)} + \frac{1}{2} + \mathrm{O}(e^{-\pi \mathrm{Im}\,\tau/|\tau+1|^2}),$$

uniformly provided that $\tau \in \mathbb{H}$ lies in the horocyclic disk $|\tau + 1 - \frac{i}{2}| \leq \frac{1}{2}$, which amounts to the condition $\operatorname{Im}\tau \geq |\tau + 1|^2$. Moreover, by 2-periodicity, the same applies near the cusp at 1:

$$(7.2.5) \quad \lambda(\tau) = \lambda(\tau - 2) = 1 - \frac{1}{\lambda(-1/(\tau - 1))}$$
$$= -\frac{1}{16}e^{i\pi/(\tau-1)} + \frac{1}{2} + O(e^{-\pi\operatorname{Im}\tau/|\tau-1|^2}),$$

uniformly provided that $\tau \in \mathbb{H}$ lies in the horocyclic disk $|\tau - 1 - \frac{i}{2}| \leq \frac{1}{2}$. A similar argument applies to the derivative λ', and gives that

$$(7.2.6) \quad \lambda'(\tau) = \frac{\lambda'(-1/(\tau + 1))}{(\tau + 1)^2(\lambda(-1/(\tau + 1)))^2}$$
$$= \frac{i\pi}{16(\tau + 1)^2}e^{i\pi/(\tau+1)} + O(|\tau + 1|^{-2}e^{-\pi\operatorname{Im}\tau/|\tau+1|^2}),$$

again uniformly provided that $\tau \in \mathbb{H}$ lies in the horocyclic disk $|\tau + 1 - \frac{i}{2}| \leq \frac{1}{2}$, and this extends by 2-periodicity to the translated horocyclic disk $|\tau - 1 - \frac{i}{2}| \leq \frac{1}{2}$ as well.

Since the modular λ function maps conformally $\mathcal{D}_{2\Theta}$ onto the slit plane $\mathbb{C} \setminus (]-\infty, 0] \cup [1, +\infty[)$, we can use the approximations (7.2.4) and (7.2.5) to estimate the inverse mapping

$$\lambda^{-1} : \mathbb{C} \setminus (]-\infty, 0] \cup [1, +\infty[) \to \mathcal{D}_{2\Theta}.$$

If $\zeta \in \mathbb{C} \setminus (]-\infty, 0] \cup [1, +\infty[)$ has large modulus $|\zeta|$, then we try to solve the equation $\lambda(\tau) = \zeta$ with $\tau \in \mathcal{D}_{2\Theta}$, and realize that only τ near the cusps at ± 1 are possible. The inverse formulae are, for $\zeta \in \mathbb{C} \setminus (]-\infty, 0] \cup [1, +\infty[)$ with big $|\zeta|$,

$$(7.2.7) \qquad \lambda^{-1}(\zeta) = -1 + \frac{i\pi}{\log(8 - 16\zeta) + O(|\zeta|^{-1})}, \qquad \operatorname{Im}\zeta \ll -1,$$

and

$$(7.2.8) \qquad \lambda^{-1}(\zeta) = 1 + \frac{i\pi}{\log(8 - 16\zeta) + O(|\zeta|^{-1})}, \qquad \operatorname{Im}\zeta \gg 1,$$

respectively.

8. THE SCHWARZ TRANSFORM AND ITS CONSEQUENCES

8.1. **The Schwarz transform based on reflection in the circle.** We introduce a mapping from the 2-periodic harmonic functions in $\mathbb{H}$ to the holomorphic functions in $\mathbb{H}$, which we will call the *Schwarz transform* $\mathfrak{w}$ as it is based on reflection in the semi-circle

$$\mathbb{T}_+ := \{\tau \in \mathbb{H} : |\tau| = 1\}.$$

Definition 8.1.1. Suppose $h : \mathbb{H} \to \mathbb{C}$ is harmonic, so that the function can be split uniquely as
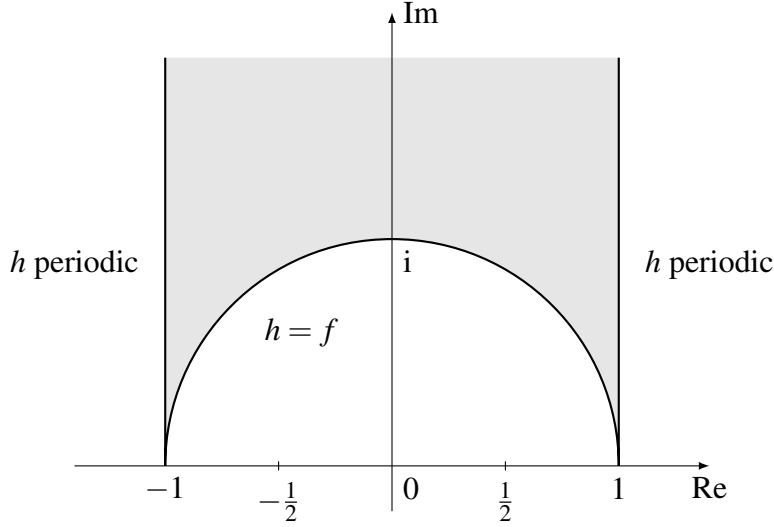
$$h(\tau) = h(i) + h_+(\tau) + h_-(\tau),$$

FIGURE 8.1.1. The domain $\mathscr{D}_\Theta$ (in gray) and the corresponding Dirichlet problem.

where $h_+ : \mathbb{H} \to \mathbb{C}$ is holomorphic with $h_+(\mathrm{i}) = 0$ and $h_- : \mathbb{H} \to \mathbb{C}$ is conjugate-holomorphic with $h_-(\mathrm{i}) = 0$. The *Schwarz transform* of h, written $\mathbf{w}h$, is the holomorphic function

$$\mathbf{w}h(\tau) := h(\mathrm{i}) + h_+(\tau) + h_-\left(\frac{1}{\bar\tau}\right), \qquad \tau \in \mathbb{H}.$$

It is natural to ask when we can recover the harmonic function h from knowing its Schwarz transform. We may immediately recover partial information about h.

Proposition 8.1.2. *If $h : \mathbb{H} \to \mathbb{C}$ is harmonic, then the Schwarz transform $\mathbf{w}h$ is a well-defined holomorphic function on $\mathbb{H}$, and*

$$\mathbf{w}h(\tau) = h(\tau), \qquad \tau \in \mathbb{T}_+.$$

Proof. The mapping $\tau \mapsto 1/\bar\tau$ is a conjugate-holomorphic automorphism of $\mathbb{H}$, which makes $\mathbf{w}h$ well-defined and holomorphic, since the composition of two conjugate-holomorphic functions is holomorphic. Moreover, since $1/\bar\tau = \tau$ holds for $\tau \in \mathbb{T}_+$, $\mathbf{w}h$ and h coincide on $\mathbb{T}_+$. $\qquad\square$

Clearly some additional information is needed, as can be seen from some elementary examples. The following proposition gives appropriate conditions for uniqueness. The formulation involves the domain $\mathscr{D}_\Theta$ which may be considered to be the fundamental domain for the so-called Theta group. See Figure 8.1.1 for an illustration.

Proposition 8.1.3. *Suppose $h : \mathbb{H} \to \mathbb{C}$ is harmonic and 2-periodic, so that $h(\tau+2) = h(\tau)$ holds. If in addition*

$$|h(\tau)| = \mathrm{o}(\mathrm{Im}\tau) \quad \text{as } \mathrm{Im}\tau \to +\infty,$$

and if

$$|h(\tau)| = \mathrm{o}\big(\exp(\pi/\mathrm{Im}\,\tau)\big) \quad as \ \mathscr{D}_\Theta \ni \tau \to \pm 1,$$

the function h is uniquely determined by its Schwarz transform $\mathfrak{w}\,h$.

Proof. We consider the Riemann surface $\mathscr{S}_\Theta$ obtained by gluing together the vertical linear boundary segments of the fundamental domain $\mathscr{D}_\Theta$ to form a cylindrical surface whose boundary consists of the circle $\mathbb{T}_+$, the points ± 1 which are identified with one another, and the isolated boundary point $+\mathrm{i}\infty$. The assumptions on h entail that it is a harmonic function on $\mathscr{S}_\Theta$. The growth condition $|h(\tau)| = \mathrm{o}(\mathrm{Im}\,\tau)$ as τ approaches $+\mathrm{i}\infty$ means that h has a removable singularity at $+\mathrm{i}\infty$, and hence it extends harmonically across that point.

We need to show that h is uniquely determined by its Schwarz transform $\mathfrak{w}\,h$. By linearity, it is enough to show that if $\mathfrak{w}\,h = 0$ holds, then we must have $h = 0$ as well. By Proposition 8.1.2, the assumption $\mathfrak{w}\,h = 0$ entails that h vanishes on $\mathbb{T}_+$. On the Riemann surface $\mathscr{S}_\Theta \cup \{+\mathrm{i}\infty\}$, we then know that h is harmonic and extends continuously to vanish along $\mathbb{T}_+$, which constitutes most of the boundary. The only boundary point not included is the cusp point ± 1. In order to model the cusp in $\mathscr{S}_\Theta$ better, we consider the alternative realization of in the form of

$$\mathscr{D}_\Theta^+ := \big\{\tau \in \mathbb{H} : 0 < \mathrm{Re}\,\tau < 2, \ |\tau| > 1, \ |\tau - 2| > 1\big\},$$

where the surface $\mathscr{S}_\Theta$ again results from gluing together the vertical linear segments. On $\mathscr{D}_\Theta^+$, the cusp is located at the boundary point $\tau = 1$. Let $\psi : \mathbb{H} \to \mathbb{H}$ be the involutive Möbius mapping

$$\psi(\tau) := \frac{\tau - 2}{\tau - 1},$$

which maps $\mathscr{D}_\Theta^+$ onto itself, and the cusp at 1 gets sent to $+\mathrm{i}\infty$, while $+\mathrm{i}\infty$ gets mapped to 1. We note in addition that the circular parts where $|\tau| = 1$ and where $|\tau - 2| = 1$ get sent to the vertical linear segments of the boundary of $\mathscr{D}_\Theta^+$. This means that the vanishing of h on $\mathbb{T}_+$ entails that the harmonic function $h \circ \psi(\tau)$ vanishes for $\tau \in \partial \mathscr{D}_\Theta^+$ with $\mathrm{Re}\,\tau \in \{0, 2\}$. Moreover, the growth bound on h near the cusp amounts to having

$$h \circ \psi(\tau) = \mathrm{o}\big(\exp(\pi \mathrm{Im}\,\tau)\big) \quad \mathscr{D}_\Theta^+ \ni \tau \to +\mathrm{i}\infty.$$

By a version of the classical Phragmén-Lindelöf principle, the permitted growth in the strip is insufficient to allow the function to be unbounded near the boundary point $+\mathrm{i}\infty$. In other words, h must be bounded in a neighborhood of the cusp at 1 within the domain $\mathscr{D}_\Theta^+$. By periodicity, then, h is bounded in a neighborhood of the cusp at ± 1 inside $\mathscr{D}_\Theta$. Since h is continuous at all other boundary points of $\mathscr{S}_\Theta$, including the removable singularity at $+\infty$, it follows that we may apply the maximum principle to the real and imaginary parts of h separately (see, e.g., Tsuji [26]). The result is that $h = 0$ on the surface $\mathscr{S}_\Theta \cup \{+\mathrm{i}\infty\}$, in particular on $\mathscr{D}_\Theta$. But h is assumed harmonic in $\mathbb{H}$, and vanishing on the open set $\mathscr{D}_\Theta$ forces h to vanish throughout $\mathbb{H}$. $\quad\square$

The harmonic 2-periodic functions on $\mathbb{H}$ have a characterization in terms of harmonic Fourier series, provided we have the appropriate growth bound in the imaginary direction. With some rather specific growth bounds, this topic has been covered back in Propositions 5.2.2 and 5.2.5.

Proposition 8.1.4. *Suppose $h : \mathbb{H} \to \mathbb{C}$ is harmonic and 2-periodic, i.e., $h(\tau+2) = h(\tau)$ holds, and enjoys the growth estimate*

$$|h(\tau)| = \mathrm{o}(\mathrm{Im}\,\tau) \quad \textit{as } \mathrm{Im}\,\tau \to +\infty.$$

Then h has a harmonic Fourier series expansion

$$h(\tau) = \sum_{n \in \mathbb{Z}} c_n e_n(\tau), \qquad \tau \in \mathbb{H},$$

where the functions e_n are given by (3.5.2) for $n \geq 0$ and (3.5.3) for $n < 0$, and the coefficients grow subexponentially: $|c_n| = \mathrm{O}(\exp(\varepsilon |n|))$ as $|n| \to +\infty$.

Proof. The periodicity and the growth bound on h entail that $h(\tau) = \mathrm{O}(1)$ as $\mathrm{Im}\,\tau \to +\infty$, by the Phragmén-Lindelöf principle for harmonic functions. But the by periodicity and compactness, $h(\cdot + \mathrm{i}\varepsilon)$ is uniformly bounded in $\mathbb{H}$ for any $\varepsilon > 0$. Proposition 5.2.2 gives that there are coefficients $c_n(\varepsilon)$ with $c_n(\varepsilon) = \mathrm{O}(1)$ as $|n| \to +\infty$ such that

$$h(\tau + \mathrm{i}\varepsilon) = \sum_{n \in \mathbb{Z}} c_n(\varepsilon) e_n(\tau), \qquad \tau \in \mathbb{H},$$

and the proof of Proposition 5.2.2 also shows that $c_n(\varepsilon) = \mathrm{e}^{-\pi\varepsilon |n|} c_n$ holds, where c_n are the coefficients in the expansion of h. Since $c_n(\varepsilon)$ is bounded for each $\varepsilon > 0$, it now follows that the coefficients c_n grow subexponentially. $\square$

Proposition 8.1.5. *Suppose $h : \mathbb{H} \to \mathbb{C}$ is harmonic and represented by a harmonic Fourier series*

$$h(\tau) = \sum_{n \in \mathbb{Z}} c_n e_n(\tau), \qquad \tau \in \mathbb{H},$$

where the coefficients c_n grow subexponentially. Then the Schwarz transform of h is a holomorphic hyperbolic Fourier series given by

$$\mathbf{w}\,h(\tau) = \sum_{n \in \mathbb{Z}_{\geq 0}} c_n e_n(\tau) + \sum_{n \in \mathbb{Z}_{< 0}} c_n e_n(1/\bar{\tau}), \qquad \tau \in \mathbb{H}.$$

Proof. We put

$$h_+(\tau) = \sum_{n \in \mathbb{Z}_{\geq 0}} c_n e_n(\tau) - \sum_{n \in \mathbb{Z}_{\geq 0}} c_n e_n(\mathrm{i})$$

and

$$h_-(\tau) = \sum_{n \in \mathbb{Z}_{< 0}} c_n e_n(\tau) - \sum_{n \in \mathbb{Z}_{< 0}} c_n e_n(\mathrm{i}),$$

so that $h_+ : \mathbb{H} \to \mathbb{C}$ is holomorphic while $h_- : \mathbb{H} \to \mathbb{C}$ is conjugate-holomorphic, with $h_+(\mathrm{i}) = h_-(\mathrm{i}) = 0$. The decomposition

$$h(\tau) = h(\mathrm{i}) + h_+(\tau) + h_-(\tau), \qquad \tau \in \mathbb{H},$$

then holds, and by the definition of the Schwarz transform, we have

$$\mathbf{w}h(\tau) = h(\mathrm{i}) + h_+(\tau) + h_-(1/\bar{\tau}) = \sum_{n\in\mathbb{Z}_{\geq 0}} c_n e_n(\tau) + \sum_{n\in\mathbb{Z}_{<0}} c_n e_n(1/\bar{\tau}), \qquad \tau \in \mathbb{H},$$

as claimed. □

8.2. **Solution scheme for the inverse Schwarz transform problem.** It remains to
decide when we can solve the inverse problem: *To find for a given holomorphic
function $f : \mathbb{H} \to \mathbb{C}$ an associated harmonic function $h : \mathbb{H} \to \mathbb{C}$ which is 2-periodic
and has $\mathbf{w}h = f$.* As it turns out, we can always do this. But it is not always possible
to do this in the uniqueness regime of Proposition 8.1.3, simply because any solution
must have $h = f$ on $\mathbb{T}_+$, and the given f need not obey the boundary growth bound
at ± 1, which means that we cannot expect h to do so either. We follow an algorithm
to define h, where the first step is the only one where there is some freedom. We call
it the *harmonic extension algorithm.*

STEP I: *The Dirichlet problem in $\mathscr{D}_\Theta$.* We find a harmonic function $h_0 : \mathscr{D}_\Theta \to \mathbb{C}$
which solves a Dirichlet boundary problem along $\mathbb{T}_+$, namely $h_0 = f$ on $\mathbb{T}_+$, and has
periodic boundary conditions along the linear boundary of $\mathscr{D}_\Theta$:

$$\forall y > 0 : \ h_0(-1+\mathrm{i}y) = h_0(1+\mathrm{i}y) \quad \text{and} \quad \partial_x h_0(-1+\mathrm{i}y) = \partial_x h_0(1+\mathrm{i}y).$$

In addition, $h_0(\tau) = \mathrm{O}(1)$ as $\mathrm{Im}\,\tau \to +\infty$ is required. In terms of the alternative nome
coordinate $q = \mathrm{e}^{\mathrm{i}\pi\tau}$, this amounts to an ordinary Dirichlet problem for a bounded
simply connected domain $\widehat{\mathscr{D}}_\Theta$ with real-analytic Jordan boundary curve, except for
a single cusp at $q = -1$ corresponding to the points $\tau = \pm 1$. After all, the periodic
boundary condition means that in the q-domain $\widehat{\mathscr{D}}_\Theta$, h_0 extends harmonically across
the segment $]-1,0[$, so we may as well include the segment in $\widehat{\mathscr{D}}_\Theta$. Being simply
connected, the domain $\widehat{\mathscr{D}}_\Theta$ is conformally equivalent to the upper half-plane $\mathbb{H}$, and
the cusp point $q = -1$ may be sent to the boundary point at infinity of $\mathbb{H}$. *Proposition
5.3.4 now guarantees that such a harmonic function h_0 exists, no matter how quickly
the function f grows near the cusp ± 1 along $\mathbb{T}_+$.* If we want growth control on h_0
for some type of growth bound on f, we can use for instance Propositions 5.3.1 and
5.3.5.

STEP II: *Periodic extension.* We extend h_0 periodically from $\mathscr{D}_\Theta$ to the domain

$$(8.2.1) \qquad\qquad \Omega_0 := \mathbb{H} \setminus \bigcup_{j\in\mathbb{Z}} \bar{\mathbb{D}}(2j,1).$$

Then $h_0(\tau + 2) = h_0(\tau)$ holds for all $\tau \in \Omega_0$, and the extended function, still called
h_0, is harmonic in Ω_0 and bounded at $+\mathrm{i}\infty$. These properties are immediate if we
express the Dirichlet problem in terms of the nome variable q.

STEP III: *Schwarz reflection.* We now use the fact that the Dirichlet boundary data
on $\mathbb{T}_+$ comes from a holomorphic function $f : \mathbb{H} \to \mathbb{C}$. Indeed, we consider the
function $g_0 := h_0 - f$ which by the above Step II is harmonic in the periodic domain
Ω_0. The periodicity of h_0 does not carry over to g_0, because the function f need not
be periodic, but a good thing is that $g_0 = 0$ holds on $\mathbb{T}_+$. This is good because it puts

us in the setting of the Schwarz reflection principle (see, e.g., [1], pp. 172-173, in the instance of a linear boundary), which asserts that g_0 extends harmonically across $\mathbb{T}_+$, and the extension has the form

$$(8.2.2) \qquad\qquad\qquad g_0(\tau) = -g_0(1/\bar{\tau}),$$

provided that one of the sides makes sense. So, if $1/\bar{\tau} \in \Omega_0$, the left-hand side of (8.2.2) gets defined by the right-hand side. If we put (note that $\Omega_0 \cap \mathbb{D}_e = \Omega_0$ trivially)

$$(8.2.3) \qquad\qquad \Omega_1' := (\Omega_0 \cap \mathbb{D}_e) \cup \mathbb{T}_+ \cup S^*(\Omega_0 \cap \mathbb{D}_e),$$

where we recall the notation $S^*(\tau) = 1/\bar{\tau}$, it is clear that Ω_1' is open and connected, and that $\Omega_0 \subset \Omega_1'$. Moreover, the function g_0 extends harmonically to Ω_1'. But then $h_0 = g_0 + f$ gets extended as a harmonic function to Ω_1' as well.

STEP IV: *Iterative application of the steps II and III.* The domain Ω_0 is periodic, and it was extended by forming Ω_1'. We note that the enlargement from Ω_0 to Ω_1' was localized to the semidisk $\bar{\mathbb{D}} \cap \mathbb{H}$. We may now use the periodicity of h_0 to further extend the function h_0 to the periodized domain

$$\Omega_1 := \big\{ \tau \in \mathbb{H} : \ \tau - 2k \in \Omega_1' \ \text{holds for some} \ k \in \mathbb{Z} \big\}.$$

Then clearly $\Omega_0 \subset \Omega_1' \subset \Omega_1$ holds, and we let h_1 denote the resulting 2-periodic harmonic function $h_1 : \Omega_1 \to \mathbb{C}$ with restriction $h_1 = h_0$ on Ω_1'. After that, we again apply Step III based on Schwarz reflection to extend h_1 harmonically to the bigger domain

$$\Omega_2' := (\Omega_1 \cap \mathbb{D}_e) \cup \mathbb{T}_+ \cup S^*(\Omega_1 \cap \mathbb{D}_e),$$

After all, we proceed as before and write $g_1 := h_1 - f$, which is harmonic in Ω_1 since both h_1 and f are, and the Schwarz reflection formula (8.2.2) remains valid:

$$(8.2.4) \qquad\qquad\qquad g_1(\tau) = -g_1(1/\bar{\tau}),$$

In view of (8.2.4), g_1 extends harmonically to Ω_2', and hence $h_1 = g_1 + f$ extends harmonically to Ω_2' as well. At the next level, we use the periodicity of h_1 to further extend the function h_1 to the periodized domain

$$\Omega_2 := \big\{ \tau \in \mathbb{H} : \ \tau - 2k \in \Omega_2' \ \text{holds for some} \ k \in \mathbb{Z} \big\}.$$

We let h_2 denote the resulting 2-periodic function $h_2 : \Omega_2 \to \mathbb{C}$ with restriction $h_2 = h_1$ on Ω_2'. These are just the first few iteration steps of a general iteration procedure. Suppose that we have, for a given $j = 1, 2, 3, \ldots$, a 2-periodic harmonic function h_j on a 2-periodic domain Ω_j. We then extend it to a 2-periodic harmonic function h_{j+1} on a larger 2-periodic domain Ω_{j+1} in the following two-step fashion. We form $g_j := h_j - f$, which is harmonic in Ω_j. This function vanishes on the semicircle $\mathbb{T}_+$, so that by Schwarz reflection,

$$g_j(\tau) = -g_j(1/\bar{\tau}),$$

which extends g_j harmonically to the domain

$$(8.2.5) \qquad\qquad \Omega_{j+1}' := (\Omega_j \cap \mathbb{D}_e) \cup \mathbb{T}_+ \cup S^*(\Omega_j \cap \mathbb{D}_e).$$
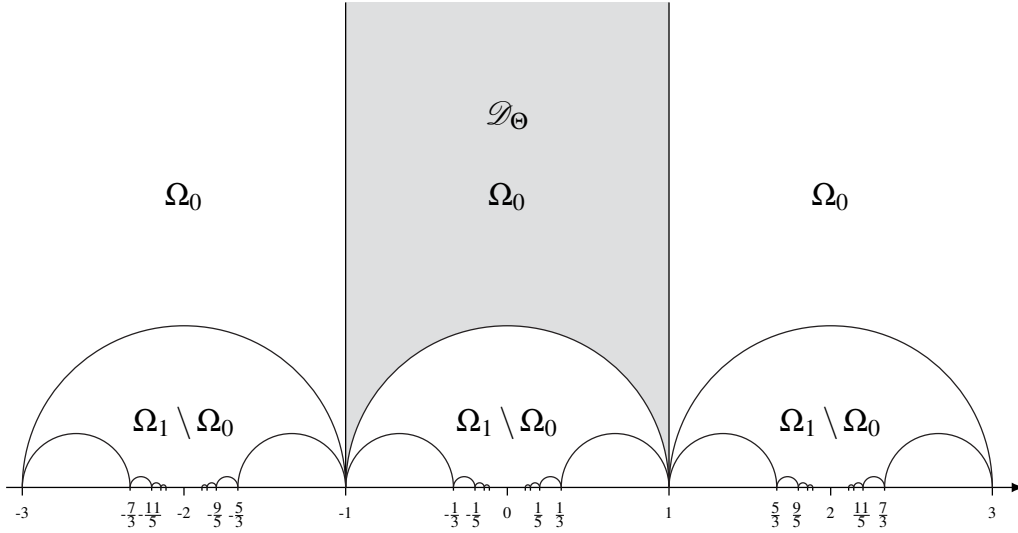
FIGURE 8.2.1. The domain Ω_0 is marked as well as $\Omega_1 \setminus \Omega_0$. The fundamental domain $\mathscr{D}_\Theta$ is shaded in gray.

Consequently, $h_j = g_j + f$ extends harmonically to Ω'_{j+1} as well. In a second step, the harmonic function h_j gets extended by 2-periodicity to the larger domain

$$(8.2.6) \qquad \Omega_{j+1} := \left\{ \tau \in \mathbb{H} : \ \tau - 2k \in \Omega'_{j+1} \ \text{holds for some} \ k \in \mathbb{Z} \right\},$$

and the extension gets to be called h_{j+1}, and it is harmonic and 2-periodic on Ω_{j+1}.

We need to make some remarks concerning the structure of the sets Ω_j for $j = 0, 1, 2, \ldots$. The statement involves the concept of a cuspidal hyperbolic polygon.

Definition 8.2.1. An open subset Ω of the upper half-plane $\mathbb{H}$ is a *cuspidal hyperbolic polygon* if the complement $\mathbb{H} \setminus \Omega$ is a finite or countable union of disjoint relatively closed disks or half-planes, provided that the boundary of each such disk or half-plane is hyperbolic geodesic.

We remark that the hyperbolic geodesics are either semicircles centered along the real line, or vertical half-lines.

Proposition 8.2.2. *For each $j = 0, 1, 2, \ldots$, the domains Ω_j and Ω'_{j+1} are cuspidal hyperbolic polygons, and we have the containments*

$$\Omega_j \subset \Omega'_{j+1} \subset \Omega_{j+1}.$$

Proof. We read off from the definition of Ω_{j+1} that the containment $\Omega'_{j+1} \subset \Omega_{j+1}$ holds, so we need only be concerned with the remaining containment $\Omega_j \subset \Omega'_{j+1}$. We resort to induction to deal with the outstanding issues. We first devote attention to the containment $\Omega_j \subset \Omega'_{j+1}$. For $j = 0$ the assertion is clearly true. Indeed, inspection of (8.2.3) gives that $\Omega'_1 \cap \mathbb{D}_e = \Omega_0 \cap \mathbb{D}_e$, while clearly $\Omega_0 \cap \mathbb{D} = \emptyset$. To proceed with

the induction, we assume that the containment $\Omega_j \subset \Omega'_{j+1}$ holds for all $j = 0, \ldots, j_0$, and aim to show that the containment holds for $j = j_0 + 1$ as well. We show that $\Omega_{j_0+1} \subset \Omega'_{j_0+2}$ holds by establishing separately that (a) $\Omega_{j_0+1} \cap \mathbb{D}_{\mathrm{e}} \subset \Omega'_{j_0+2} \cap \mathbb{D}_{\mathrm{e}}$, (b) $\Omega_{j_0+1} \cap \mathbb{D} \subset \Omega'_{j_0+2} \cap \mathbb{D}$, and that (c) $\Omega_{j_0+1} \cap \mathbb{T}_+ \subset \Omega'_{j_0+2} \cap \mathbb{T}_+$.

To arrive at the inclusion (a), we assume that $\tau \in \Omega_{j_0+1} \cap \mathbb{D}_{\mathrm{e}}$. By (8.2.5) for $j = j_0 + 1$, it follows that $\tau \in \Omega'_{j_0+2} \cap \mathbb{D}_{\mathrm{e}}$, so that the containment holds within the exterior disk $\mathbb{D}_{\mathrm{e}}$. This finishes the verification of part (a).

To arrive at the inclusion (b), we assume that $\tau \in \Omega_{j_0+1} \cap \mathbb{D}$. Then in view of (8.2.6) with $j = j_0$, we have that there exists a $k = k(\tau) \in \mathbb{Z}$ such that $\tau - 2k \in \Omega'_{j_0+1}$. If $k = 0$, then $\tau \in \Omega'_{j_0+1} \cap \mathbb{D}$ and by (8.2.5) with $j = j_0$ it follows that

$$\Omega'_{j_0+1} \cap \mathbb{D} = \mathrm{S}^*(\Omega_{j_0} \cap \mathbb{D}_{\mathrm{e}}) \subset \mathrm{S}^*(\Omega'_{j_0+1} \cap \mathbb{D}_{\mathrm{e}}) \subset \mathrm{S}^*(\Omega_{j_0+1} \cap \mathbb{D}_{\mathrm{e}}),$$

where the first containment is a consequence of the induction hypothesis. In particular, it follows that if $\tau \in \Omega_{j_0+1} \cap \mathbb{D}$ with associated $k(\tau) = 0$, then $\tau \in \Omega'_{j_0+2} \cap \mathbb{D}$, by (8.2.5) for $j = j_0 + 1$. As for the remaining case when $k \neq 0$, then by the triangle inequality,

$$|\tau - 2k| \geq 2|k| - |\tau| > 2k - 1 \geq 1,$$

so that by (8.2.5) with $j = j_0$,

$$\tau - 2k \in \Omega'_{j_0+1} \cap \mathbb{D}_{\mathrm{e}} = \Omega_{j_0} \cap \mathbb{D}_{\mathrm{e}}.$$

By the 2-periodicity of Ω_{j_0}, it follows that $\tau \in \Omega_{j_0}$, and since $\tau \in \mathbb{D}$ was assumed, we obtain that

$$\tau \in \Omega_{j_0} \cap \mathbb{D} \subset \Omega'_{j_0+1} \cap \mathbb{D} \subset \Omega_{j_0+1} \cap \mathbb{D},$$

where the first containment follows from the induction hypothesis. In view of the definition (8.2.5) with $j = j_0 + 1$, we find that $\tau \in \Omega'_{j_0+2} \cap \mathbb{D}$, as claimed. This finishes the verification of (b).

Finally, we turn to the inclusion (c). To this end, we assume that $\tau \in \Omega_{j_0+1} \cap \mathbb{T}_+$. Since by (8.2.5) with $j = j_0 + 1$ we know that $\Omega'_{j_0+1} \cap \mathbb{T}_+ = \mathbb{T}_+$, the conclusion that $\tau \in \Omega_{j_0+1} \cap \mathbb{T}_+$ is immediate. This finishes the verification of part (c).

We turn to the remaining assertion that the domains Ω_j and Ω'_{j+1} are cuspidal hyperbolic polygons. First, this is true about Ω_0 by inspection, and about Ω'_1 by reflection. The general case can be obtained by another induction argument, which is left to the interested reader. The important matter is that each of the two domains (Ω_j and Ω'_{j+1}) has the points 1 and -1 as endpoints of geodesic complementary disks, and this property survives along the induction process, since all we do is Schwarz reflection in $\mathbb{T}_+$ and exploitation of the 2-periodicity. $\square$

We need to understand some symmetry properties of the domains Ω_j, and how they grow with j. Let $\mathrm{R}^*(\tau) := -\bar{\tau}$ denote the reflection in the imaginary axis.

Proposition 8.2.3. *For $j = 0, 1, 2, \ldots$, the domains Ω_j are 2-periodic and symmetric with respect to the imaginary axis:* $\mathrm{R}^*(\Omega_j) = \Omega_j$ *and* $\mathrm{T}^2(\Omega_j) = \Omega_j$. *Moreover, we*

have that we have that

$$\sup\{\operatorname{Im}\tau:\ \tau\in\mathbb{H}\setminus\Omega_j\}=\frac{1}{2j+1},$$

so that in particular, the union of all the domains Ω_j over $j\in\mathbb{Z}_{\geq 0}$ equals all of $\mathbb{H}$.

Proof. The domain Ω_0 is symmetric with respect to the imaginary axis. This property gets inherited by Ω_1', since $S^*(\Omega_0\cap\mathbb{D}_e)$ is symmetric too, which we see from the commutation relation $R^*S^*=S^*R^*$. The periodicity operation which produces Ω_1 must then also preserve the symmetry, since $R^*T^2=T^2R^*$. The same arguments apply under the two-step iteration which gives Ω_{j+1} from the smaller domain Ω_j, and we find that all the domains Ω_j are symmetric under reflection in the imaginary axis. The fact that they are all 2-periodic follows from the definition of (8.2.6).

We turn to the assertion about the maximal imaginary part of a point from the complement $\mathbb{H}\setminus\Omega_j$. For $j=0$, the complement $\mathbb{H}\setminus\Omega_0$ contains the half-disk $\bar{\mathbb{D}}\cap\mathbb{H}$ which contains the point i with imaginary part 1. For $j>0$, $\mathbb{H}\setminus\Omega_j$ contains a relatively closed half-disk whose boundary semicircle is a hyperbolic geodesic and passes through the points 1 and $(2j-1)/(2j+1)$. It is a circle centered at $2j/(2j+1)$ of radius $1/(2j+1)$, and it contains the point

$$\frac{2j}{2j+1}+\frac{i}{2j+1}.$$

This point has the claimed maximal imaginary part $1/(2j+1)$. The fact that this is the largest possible imaginary part can be obtained as a consequence of Lemma 2 on p. 66 of [21]. □

Corollary 8.2.4. *The harmonic function h_0 which is defined in Step I as one solution to the Dirichlet problem on $\mathscr{D}_\Theta$ with boundary datum f on $\mathbb{T}_+$ and periodic conditions along the linear segments, extends harmonically to a function $h_\infty:\mathbb{H}\to\mathbb{C}$, which is 2-periodic and has $h_\infty=f$ on $\mathbb{T}_+$. Moreover, $h_\infty(\tau)=O(1)$ as $\operatorname{Im}\tau\to+\infty$.*

Proof. According to the iteration in Step IV, h_0 extends harmonically to a function $h_j:\Omega_j\to\mathbb{C}$. Letting j tend to infinity we find that h_j extends to a function h_∞ which is defined and harmonic in $\mathbb{H}$, by Proposition 8.2.3. □

Corollary 8.2.5. *The harmonic function $h_\infty:\mathbb{H}\to\mathbb{C}$ of Corollary 8.2.4 has a harmonic Fourier series expansion*

$$h_\infty(\tau)=\sum_{n\in\mathbb{Z}}c_n e_n(\tau),\qquad \tau\in\mathbb{H},$$

where the coefficients c_n grow subexponentially. Moreover, its Schwarz transform is $\textit{w}\,h_\infty=f$, and consequently, the holomorphic function $f:\mathbb{H}\to\mathbb{C}$ has the holomorphic Fourier series expansion

$$f(\tau)=\textit{w}\,h_\infty(\tau)=c_0+\sum_{n\in\mathbb{Z}_{>0}}c_n e_n(\tau)+\sum_{n\in\mathbb{Z}_{<0}}c_n e_n(1/\bar{\tau}),\qquad \tau\in\mathbb{H}.$$

Proof. In view of Corollary 8.2.4 and Proposition 8.1.4, the function h_∞ has claimed harmonic Fourier series expansion. The Schwarz transform $\textit{w}\,h_\infty$ is automatically

holomorphic on $\mathbb{H}$, and by Proposition 8.1.2 and Corollary 8.2.4, we obtain that $\mathpzc{w} h_\infty = h_\infty = f$ holds on $\mathbb{T}_+$. But then the difference $\mathpzc{w} h_\infty - f$ is holomorphic on $\mathbb{H}$ and vanishes along the semicircle $\mathbb{T}_+$, so by the uniqueness theorem for holomorphic functions, $\mathpzc{w} h_\infty - f$ vanishes throughout $\mathbb{H}$. The claimed hyperbolic Fourier series expansion now follows from Proposition 8.1.5. $\square$

We may now write down the proof of Theorem 3.6.4.

Proof of Theorem 3.6.4. We first split the harmonic function $f : \mathbb{H} \to \mathbb{C}$ as

$$f(\tau) = f(\mathrm{i}) + f_+(\tau) + f_-(\tau), \qquad \tau \in \mathbb{H},$$

where $f_+ : \mathbb{H} \to \mathbb{C}$ is holomorphic while $f_- : \mathbb{H} \to \mathbb{C}$ is conjugate-holomorphic, both with $f_+(\mathrm{i}) = f_-(\mathrm{i}) = 0$. We then apply the holomorphic hyperbolic Fourier series expansion of Corollary 8.2.5 first to $f_+(\tau)$ and then to $f_- \circ \mathrm{R}^*(\tau) = f_-(-\bar{\tau})$. The result is the claimed harmonic hyperbolic Fourier series expansion of f. $\square$

In association with the solution algorithm which extends the harmonic function h_0 uniquely to a harmonic function $h_\infty : \mathbb{H} \to \mathbb{C}$, there are the domains Ω_j, which in a natural fashion suggests the introduction of the concept of the height (or level) of a point $\tau \in \mathbb{H}$.

Definition 8.2.6. The *height* $\mathrm{n}(\tau)$ of a point $\tau \in \mathbb{H}$ is the smallest value $j \in \mathbb{Z}_{\geq 0}$ such that $\tau \in \Omega_j$.

Corollary 8.2.7. *We have that*

$$\mathrm{n}(\tau) \leq \frac{1}{2} + \frac{1}{2\,\mathrm{Im}\,\tau}, \qquad \tau \in \mathbb{H}.$$

Proof. This is an immediate consequence of Proposition 8.2.3. $\square$

The average height is, however, much smaller. The following asymptotics was found by Bondarenko, Radchenko, and Seip in Proposition 6.7 of [5].

Proposition 8.2.8. *We have that*

$$\frac{1}{2} \int_{-1}^{1} \mathrm{n}(t + \mathrm{i}y)\,\mathrm{d}t = \frac{1}{\pi^2} \log^2 \frac{1}{y} + \mathrm{O}\left(\log \frac{1}{y}\right),$$

as $y \to 0^+$.

8.3. **The even Gauss map lifted to the upper half-plane.** In the algorithm presented in Subsection 8.2, the following sequences of domains appear, Ω_j and Ω'_{j+1}, for $j = 0, 1, 2, \ldots$, nested according to

$$\Omega_0 \subset \Omega'_1 \subset \Omega_1 \subset \Omega'_2 \subset \Omega_2 \subset \cdots.$$

We recall the transformations $\mathrm{S}, \mathrm{S}^* : \mathbb{H} \to \mathbb{H}$, inversion and reflection in $\mathbb{T}_+$, respectively, given by $\mathrm{S}(\tau) = -1/\tau$ and $\mathrm{S}^*(\tau) = 1/\bar{\tau}$. We shall also need the mapping mod_2, given by

$$\mathrm{mod}_2(\tau) := \tau - 2k, \qquad \tau \in \mathbb{C},$$

where the integer $k \in \mathbb{Z}$ is chosen such that $\tau' = \tau - 2k$ lies in the closed vertical strip

$$\bar{\mathcal{V}} := \{\tau' \in \mathbb{C} : -1 \leq \operatorname{Re}\tau' \leq 1\}.$$

This defines the value of k uniquely unless $\operatorname{Re}\tau' = \pm 1$. In the exceptional case that $\operatorname{Re}\tau' = \pm 1$, we are free to choose $k \in \mathbb{Z}$ such that, e.g., $\operatorname{Re}\tau' = -1$. We observe that then

$$\tau \in \Omega_0 \iff \operatorname{mod}_2(\tau) \in \mathscr{D}_\Theta \cup \mathscr{L}_{-1}^+,$$

where we recall the half-line notation $\mathscr{L}_{-1}^+$ from (7.2.1). The periodic extension (8.2.6) which defines Ω_{j+1} for $j = 0, 1, 2, \ldots$ amounts to

(8.3.1) $$\tau \in \Omega_{j+1} \iff \operatorname{mod}_2(\tau) \in \Omega_{j+1}'.$$

At the same time, it is clear from (8.2.5) that for $j = 0, 1, 2, \ldots$,

(8.3.2) $$\tau \in \Omega_{j+1}' \setminus \Omega_j \iff \tau \in (\mathbb{H} \cap \bar{\mathbb{D}}) \setminus \Omega_j \text{ and } \mathrm{S}^*(\tau) \in (\Omega_j \cap \mathbb{D}_e) \cup \mathbb{T}_+.$$

Moreover, by the reflection symmetry of Ω_j stated in Proposition 8.2.3, we know that

$$\mathrm{S}^*(\tau) \in (\Omega_j \cap \mathbb{D}_e) \cup \mathbb{T}_+ \iff \mathrm{S}(\tau) \in (\Omega_j \cap \mathbb{D}_e) \cup \mathbb{T}_+.$$

If we view the 2-periodicity of the domain Ω_j in the form

$$\tau \in \Omega_j \iff \operatorname{mod}_2(\tau) \in \Omega_j,$$

we may combine (8.3.1) with (8.3.2) and arrive at

(8.3.3) $$\tau \in \Omega_{j+1} \setminus \Omega_j \iff \operatorname{mod}_2(\tau) \in (\mathbb{H} \cap \bar{\mathbb{D}}) \setminus \Omega_j$$
$$\text{and } \mathrm{S}^*\big(\operatorname{mod}_2(\tau)\big) \in (\Omega_j \cap \mathbb{D}_e) \cup \mathbb{T}_+.$$

If we write

$$\mathscr{V}_+ := \{\tau \in \mathbb{H} : -1 < \operatorname{Re}\tau < 1\}$$

for the open vertical half-strip, then $\mathbb{H} \cap \bar{\mathbb{D}} = \mathscr{V}_+ \setminus \mathscr{D}_\Theta$, and this allows us to express the equivalence (8.3.3) in the form

$$\tau \in \Omega_{j+1} \setminus \Omega_j \iff \operatorname{mod}_2(\tau) \in \mathscr{V}_+ \setminus \Omega_j \text{ and } \mathrm{S}^*\big(\operatorname{mod}_2(\tau)\big) \in (\Omega_j \cap \mathbb{D}_e) \cup \mathbb{T}_+.$$

We now connect with the *hyperbolically lifted conjugate Gauss-type map* $\mathrm{g}_2^* : \mathbb{H} \to \mathscr{V}_+ \cup \mathscr{L}_{-1}^+ \subset \mathbb{H}$ given by

$$\mathrm{g}_2^*(\tau) = \operatorname{mod}_2 \circ \mathrm{S}^*(\tau), \qquad \tau \in \mathbb{H}.$$

Given a point $\tau \in \mathbb{H}$, we consider the following *fly-catcher algorithm*, which associates an algorithmic height number $\mathrm{n}_{\mathrm{alg}}^*(\tau) \in \mathbb{Z}_{\geq 0}$ with each point $\tau \in \mathbb{H}$. As for notation, we write $\bar{\mathscr{D}}_\Theta$ for the closure of the fundamental domain $\mathscr{D}_\Theta$.

Definition 8.3.1. (The fly-catcher algorithm) We begin with a point $\tau \in \mathbb{H}$.
STEP I: *Apply* mod_2. We form the point $\tau_0 := \operatorname{mod}_2(\tau) \in \mathscr{V}_+ \cup \mathscr{L}_{-1}^+$, and if $\tau_0 \in \bar{\mathscr{D}}_\Theta$, we declare that $\mathrm{n}_{\mathrm{alg}}^*(\tau) = 0$, and we stop the algorithm. On the other hand, in the

remaining case $\tau_0 \in (\mathscr{V}_+ \cup \mathscr{L}_{-1}^+) \setminus \bar{\mathscr{D}}_\Theta = \mathbb{H} \cap \mathbb{D}$, we declare that $\mathrm{n}_{\mathrm{alg}}^*(\tau) > 0$ and proceed to evaluate this number by passing to the next step.

STEP II: *Apply* g_2^*. We form the point $\tau_1 := \mathrm{g}_2^*(\tau_0) \in \mathscr{V}_+ \cup \mathscr{L}_{-1}^+$, and if $\tau_1 \in \bar{\mathscr{D}}_\Theta$, we declare that $\mathrm{n}_{\mathrm{alg}}^*(\tau) = 1$ and stop the algorithm. In the remaining case $\tau_1 \in (\mathscr{V}_+ \cup \mathscr{L}_{-1}^+) \setminus \bar{\mathscr{D}}_\Theta = \mathbb{H} \cap \mathbb{D}$, we declare that $\mathrm{n}_{\mathrm{alg}}^*(\tau) > 1$ and proceed to evaluate this number by passing to the next general iteration step.

STEP III: *Apply* g_2^* *iteratively*. For $j \geq 1$ we suppose that we have points $\tau_0, \ldots, \tau_j \in \mathbb{H} \cap \mathbb{D}$ which we obtained earlier from $\tau \in \mathbb{H}$, and that it is determined that $\mathrm{n}_{\mathrm{alg}}^*(\tau) > j$ holds. We then proceed to form the point $\tau_{j+1} := \mathrm{g}_2^*(\tau_j) \in \mathscr{V}_+ \cup \mathscr{L}_{-1}^+$. Now, if $\tau_{j+1} \in \bar{\mathscr{D}}_\Theta$ holds, we declare that $\mathrm{n}_{\mathrm{alg}}^*(\tau) = j+1$, and stop the algorithm. In the remaining case we must have instead $\tau_{j+1} \in (\mathscr{V}_+ \cup \mathscr{L}_{-1}^+) \setminus \bar{\mathscr{D}}_\Theta = \mathbb{H} \cap \mathbb{D}$, and we then declare that $\mathrm{n}_{\mathrm{alg}}^*(\tau) > j+1$ must hold. In this instance, we run Step III again. Finally, if Step III never terminates, we declare that $\mathrm{n}_{\mathrm{alg}}^*(\tau) = +\infty$.

Remark 8.3.2. We note that in each iteration in Step III, we have that the points move ever further away from the real line:

$$\mathrm{Im}\,\tau_{j+1} = \mathrm{Im}\,\mathrm{g}_2^*(\tau_j) = \mathrm{Im}\,\mathrm{mod}_2 \circ \mathrm{S}^*(\tau_j) = \mathrm{Im}\,\mathrm{S}^*(\tau_j) = \mathrm{Im}\,\frac{1}{\bar{\tau}_j} = \frac{\mathrm{Im}\,\tau_j}{|\tau_j|^2} > \mathrm{Im}\,\tau_j,$$

since $\tau_j \in \mathbb{H} \cap \mathbb{D}$ holds as long as the algorithm has not terminated.

Definition 8.3.3. In the context of the above algorithm, suppose that for some $j \geq 0$, $\mathrm{n}_{\mathrm{alg}}^*(\tau) = j$ holds for a point $\tau \in \mathbb{H}$, which means that $\tau_k \in \mathbb{H} \cap \mathbb{D}$ for all $k < j$ while $\tau_j \in \bar{\mathscr{D}}_\Theta$ holds, we say that τ is a *mesh point* if $\tau_j \in \partial\mathscr{D}_\Theta$, where the boundary is understood in the relative sense (inside $\mathbb{H}$).

Remark 8.3.4. If in Definition 8.3.1 we have $j = 0$, the requirement for τ to be a mesh point is just that $\tau_0 \in \partial\mathscr{D}_\Theta$.

We need to compare the concepts of height of Definition 8.2.6 with the algorithmic height of the above fly-catcher algorithm.

Proposition 8.3.5. *Unless $\tau \in \mathbb{H}$ is a mesh point, we have that the equality of height and algorithmic height $\mathrm{n}(\tau) = \mathrm{n}_{\mathrm{alg}}^*(\tau)$. Moreover, at a mesh point $\tau \in \mathbb{H}$, we have the inequality $\mathrm{n}_{\mathrm{alg}}^*(\tau) \leq \mathrm{n}(\tau)$. As a consequence, the fly-catcher algorithm always terminates: $\mathrm{n}_{\mathrm{alg}}^*(\tau) < +\infty$ at all $\tau \in \mathbb{H}$.*

Proof. This follows from a comparison of the procedures which define the two height concepts, if we take into account the relation (8.3.3). $\square$

We also need the *hyperbolically lifted Gauss-type map* $\mathrm{g}_2 : \mathbb{H} \to \mathscr{V}_+ \cup \mathscr{L}_{-1}^+$ defined by

$$\mathrm{g}_2(\tau) = \mathrm{mod}_2 \circ \mathrm{S}(\tau), \qquad \tau \in \mathbb{H}.$$

The fly-catcher algorithm of Definition 8.3.1 can be modified so as to involve the mapping g_2 in place of g_2^*. This then results in another height number $n_{\mathrm{alg}}(\tau)$. Pleasantly, these concepts coincide:

$$(8.3.4) \qquad\qquad n_{\mathrm{alg}}(\tau) = n_{\mathrm{alg}}^*(\tau), \qquad \tau \in \mathbb{H}.$$

The reason is that closed fundamental domain $\bar{\mathscr{D}}_\Theta$ is R*-invariant, and the mapping properties $S = S^* \circ R^*$ as well as

$$R^* \circ \mathrm{mod}_2 = \mathrm{mod}_2' \circ R^*,$$

if $\mathrm{mod}_2'(\tau) = \tau - 2k$ where $k \in \mathbb{Z}$ is determined such that $-1 < \mathrm{Re}(\tau - 2k) \le 1$.

Definition 8.3.6. The Theta group Γ_Θ consists of all Möbius automorphisms $\gamma : \mathbb{H} \to \mathbb{H}$ generated by the elements $S(\tau) = -1/\tau$ and $T^2(\tau) = \tau + 2$.

This means that a group element $\gamma \in \Gamma_\Theta$ may be written in the form

$$\gamma = T^{2k_N} \circ S^{\sigma_N} \circ T^{2k_{N-1}} \circ \cdots \circ S^{\sigma_2} \circ T^{2k_1},$$

where the powers have $k_j, \sigma_j \in \mathbb{Z}$. Since S involutive, that is, $S^2(\tau) = \tau$ holds, we may assume $\sigma_j \in \{0, 1\}$. Moreover, instances of $\sigma_0 = 0$ may be skipped, and the same if $k_j = 0$, unless they are first or last (when they are kept for formal reasons). So we assume $\sigma_j = 1$ for all j, and $k_j \neq 0$ for $2 \le j \le N - 1$. The Theta group may also be identified as consisting of the Möbius automorphisms of the form

$$\gamma(\tau) = \frac{a\tau + b}{c\tau + d}, \qquad a, b, c, d \in \mathbb{Z} \text{ with } ad - bc = 1,$$

where in addition

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pmod{2}.$$

Definition 8.3.7. The sets $\gamma(\mathscr{D}_\Theta)$, where $\gamma \in \Gamma_\Theta$, are called *tiles*. Their union

$$\Gamma_\Theta(\mathscr{D}_\Theta) := \cup\{\gamma(\mathscr{D}_\Theta) : \gamma \in \Gamma_\Theta\}$$

is called the *union of tiles*.

Proposition 8.3.8. *The mesh points form the set*

$$\Gamma_\Theta(\partial\mathscr{D}_\Theta) = \cup\{\gamma(\partial\mathscr{D}_\Theta) : \gamma \in \Gamma_\Theta\},$$

which is a σ-finite set with respect to linear measure. It is disjoint from the union of tiles $\Gamma_\Theta(\mathscr{D}_\Theta)$, and together these two sets cover the hyperbolic plane: $\mathbb{H} = \Gamma_\Theta(\mathscr{D}_\Theta) \cup \Gamma_\Theta(\partial\mathscr{D}_\Theta)$.

Proposition 8.3.9. *Suppose $\gamma \in \Gamma_\Theta$ is of the form*

$$\gamma = T^{2k_N} \circ S \circ T^{2k_{N-1}} \circ \cdots \circ S \circ T^{2k_1},$$

where $k_j \in \mathbb{Z}_{\neq 0}$ for all j with $2 \le j \le N - 1$. If $\tau \in \gamma(\mathscr{D}_\Theta)$, we find that $g_2^l(\mathrm{mod}_2(\tau)) \in \mathbb{H} \cap \mathbb{D}$ for $0 \le l \le N - 2$ while $g_2^{N-1}(\mathrm{mod}_2(\tau)) = \gamma^{-1}(\tau) \in \mathscr{D}_\Theta$. In particular, the level of the point is given by $n(\tau) = n_{\mathrm{alg}}(\tau) = n_{\mathrm{alg}}^*(\tau) = N - 1$.

Proof. We agree to write $\tau_\gamma := \gamma^{-1}(\tau) \in \mathscr{D}_\Theta$. Then for $k_1 \in \mathbb{Z}$, no matter whether $k_1 = 0$ or not, $S \circ T^{2k_1}(\tau_\gamma) \in \mathbb{H} \cap \mathbb{D}$ holds. Next, since $k_2 \in \mathbb{Z}_{\neq 0}$, we find that $S \circ T^{2k_2} \circ S \circ T^{2k_1}(\tau_\gamma) \in \mathbb{H} \cap \mathbb{D}$. As we keep going, we find that

$$S \circ T^{2k_{N-1}} \circ \cdots \circ S \circ T^{2k_1}(\tau_\gamma) \in \mathbb{H} \cap \mathbb{D}.$$

Then

$$\begin{aligned}
\mathrm{mod}_2(\tau) = \mathrm{mod}_2(\gamma(\tau_\gamma)) &= \mathrm{mod}_2(\gamma(\tau_\gamma)) \\
&= \mathrm{mod}_2(T^{2k_N} \circ S \circ T^{2k_{N-1}} \circ \cdots \circ S \circ T^{2k_1}(\tau_\gamma)) \\
&= S \circ T^{2k_{N-1}} \circ \cdots \circ S \circ T^{2k_1}(\tau_\gamma) \in \mathbb{H} \cap \mathbb{D},
\end{aligned}$$

and an application of the lifted Gauss-type map gives that

$$\begin{aligned}
\mathrm{g}_2 \circ \mathrm{mod}_2(\tau) &= \mathrm{g}_2 \circ S \circ T^{2k_{N-1}} \circ \cdots \circ S \circ T^{2k_1}(\tau_\gamma) \\
&= S \circ T^{2k_{N-2}} \circ \cdots \circ S \circ T^{2k_1}(\tau_\gamma) \in \mathbb{H} \cap \mathbb{D}.
\end{aligned}$$

By repeated application of the lifted Gauss-map, we find that

$$\mathrm{g}_2^l \circ \mathrm{mod}_2(\tau) = S \circ T^{2k_{N-l-1}} \circ \cdots \circ S \circ T^{2k_1}(\tau_\gamma) \in \mathbb{H} \cap \mathbb{D}, \qquad l = 1, \ldots, N-2.$$

It now follows that

$$\mathrm{g}_2^{N-1} \circ \mathrm{mod}_2(\tau) = \mathrm{g}_2 \circ S \circ T^{2k_1}(\tau_\gamma) = \mathrm{mod}_2 \circ T^{2k_1}(\tau_\gamma) = \tau_\gamma \in \mathscr{D}_\Theta,$$

as claimed. This means that the fly-catcher algorithm of Definition 8.3.1 applied to g_2 in place of g_2^* ends with a point $\tau_{N-1} = \mathrm{g}_2^{N-1}(\tau) \notin \partial \mathscr{D}_\Theta$, in which case $\tau \in \gamma(\mathscr{D}_\Theta)$ cannot be a mesh point. Moreover, the claim that $\mathrm{n}_{\mathrm{alg}}(\tau) = N-1$ now follows from reading the algorithm. Moreover, by (8.3.4) and the observation that τ is a non-mesh point, Proposition 8.3.5 gives that $\mathrm{n}(\tau) = \mathrm{n}_{\mathrm{alg}}^*(\tau) = \mathrm{n}_{\mathrm{alg}}(\tau) = N-1$, as claimed. $\qquad\square$

Proof of Proposition 8.3.8. It is well-known that $\mathscr{D}_\Theta$ is the fundamental domain for $\mathbb{H}$ modulo the Theta group Γ_Θ. This entails that the union of the tiles fills the hyperbolic plane $\mathbb{H}$, except for the boundaries of the tiles. The union of the boundaries of the tiles is given by the set of mesh points, which we may see from applying the argument of the proof of Proposition 8.3.9 to the case of $\tau \in \gamma(\partial \mathscr{D}_\Theta)$. $\qquad\square$

8.4. The harmonic extension algorithm in terms of the Theta group and the extended Gauss-type map.

We follow the solution scheme in Subsection 8.2, where a 2-periodic harmonic function $h = h_\infty : \mathbb{H} \to \mathbb{C}$ is found with $\mathfrak{w} h = f$ for a given holomorphic function $f : \mathbb{H} \to \mathbb{C}$. We note that the only freedom to choose the harmonic function h emerges in Step I of the algorithm in Subsection 8.2, where a solution h_0 is found to a given Dirichlet problem on the fundamental domain $\mathscr{D}_\Theta$. The harmonic extension algorithm outlined in Subsection 8.2 then provides the harmonic extension $h_\infty : \mathbb{H} \to \mathbb{C}$ of the given initial solution h_0, which is then uniquely determined by the choice of h_0. Our aim here is to represent the extension $h = h_\infty$ in a fashion which allows for convenient estimation. We begin with the two basic properties of h, that

$h = f$ on $\mathbb{T}_+$, and that by Steps II, III, and IV in the harmonic extension algorithm, we have that

$$h \circ T^2 = h, \quad g \circ S^* = -g,$$

where $g := h - f$ is harmonic in $\mathbb{H}$. Inserting $h - f$ in place of g the second equality, we find that

(8.4.1) $$h \circ T^2 = h, \quad h = -h \circ S^* + f + f \circ S^*.$$

It will be convenient to introduce the notation

$$f_{\text{sym}} := f + f \circ S^*,$$

which is a harmonic function in $\mathbb{H}$ with the symmetry property $f_{\text{sym}} \circ S^* = f_{\text{sym}}$. In terms of f_{sym}, the equations (8.4.1) take the simplified form

$$h \circ T^2 = h, \quad h = -h \circ S^* + f_{\text{sym}}.$$

These two relations may be combined to form, in a first step,

$$h = -h \circ T^{2k} \circ S^* + f_{\text{sym}}, \qquad k \in \mathbb{Z},$$

and, by iteration,

(8.4.2) $$h = (-1)^N h \circ T^{2k_N} \circ S^* \circ \cdots \circ T^{2k_1} \circ S^* + f_{\text{sym}}$$
$$+ \sum_{j=1}^{N-1} (-1)^j f_{\text{sym}} \circ T^{2k_j} \circ S^* \circ \cdots \circ T^{2k_1} \circ S^*,$$

where $k_1, \ldots, k_N \in \mathbb{Z}$ and $N = 1, 2, 3, \ldots$. We apply (8.4.2) at a point $\tau \in \mathcal{V}_+ \cup \mathcal{L}_{-1}^+$, in order to estimate the value $h(\tau)$ using the right-hand side. Here, we recall that $\tau \in \mathcal{V}_+ \cup \mathcal{L}_{-1}^+$ means that $\tau \in \mathbb{H}$ and that $-1 \leq \operatorname{Re} \tau < 1$. Note that by the 2-periodicity of h, it is enough to consider such τ. The result is

(8.4.3) $$h(\tau) = (-1)^N h \circ T^{2k_N} \circ S^* \circ \cdots \circ T^{2k_1} \circ S^*(\tau) + f_{\text{sym}}(\tau)$$
$$+ \sum_{j=1}^{N-1} (-1)^j f_{\text{sym}} \circ T^{2k_j} \circ S^* \circ \cdots \circ T^{2k_1} \circ S^*(\tau).$$

We let the integer $k_j = k_j(\tau) \in \mathbb{Z}$ depend on τ in such a fashion that

$$g_2^*(\tau) = \text{mod}_2 \circ S^*(\tau) = T^{2k_1} \circ S^*(\tau),$$

and, more generally,

$$g_2^*(\tau_{j-1}) = \text{mod}_2 \circ S^*(\tau_j) = T^{2k_j} \circ S^*(\tau_{j-1}), \quad \text{where} \ \ \tau_{j-1} = (g_2^*)^{j-1}(\tau).$$

It then follows from (8.4.3) that

$$h(\tau) = (-1)^N h \circ (g_2^*)^N(\tau) + \sum_{j=0}^{N-1} (-1)^j f_{\text{sym}} \circ (g_2^*)^j(\tau),$$

for $\tau \in \mathbb{H}$ with $-1 \leq \operatorname{Re}\tau < 1$. This fits in nicely with the fly-catcher algorithm of Definition 8.3.1 if we pick $N = \mathrm{n}^*_{\mathrm{alg}}(\tau)$:

$$h(\tau) = (-1)^{\mathrm{n}^*_{\mathrm{alg}}(\tau)} h \circ (\mathrm{g}^*_2)^{\mathrm{n}^*_{\mathrm{alg}}(\tau)}(\tau) + \sum_{j=0}^{\mathrm{n}^*_{\mathrm{alg}}(\tau)-1} (-1)^j f_{\mathrm{sym}} \circ (\mathrm{g}^*_2)^j(\tau),$$

again for $\tau \in \mathbb{H}$ with $-1 \leq \operatorname{Re}\tau < 1$. *Indeed, for $N = \mathrm{n}^*_{\mathrm{alg}}(\tau)$ we know that $(\mathrm{g}^*_2)^N(\tau) \in \bar{\mathscr{D}}_\Theta$ (this is the stopping condition for the algorithm) while $(\mathrm{g}^*_2)^j(\tau) \in \mathbb{H} \cap \mathbb{D}$ for $0 \leq j < N$. In addition, the imaginary parts of the iterates $j \mapsto \operatorname{Im}(\mathrm{g}^*_2)^j(\tau)$ form a strictly increasing sequence over the integer interval $0 \leq j \leq N$, by Remark 8.3.2.* In terms of the family of norms

$$\|F\|_{(\varepsilon)} := \sup\big\{|F(\tau)| : \; \tau \in \mathbb{H} \cap \mathbb{D} \text{ with } \operatorname{Im}\tau \geq \varepsilon\big\}, \qquad 0 < \varepsilon < 1,$$

we obtain the following key lemma.

Lemma 8.4.1. *If $\tau \in \mathbb{H}$ with $-1 \leq \operatorname{Re}\tau < 1$, we choose $N = \mathrm{n}^*_{\mathrm{alg}}(\tau)$, which if $N \geq 1$ gives the estimate*

$$|h(\tau)| \leq |h \circ (\mathrm{g}^*_2)^N(\tau)| + N \|f_{\mathrm{sym}}\|_{(\operatorname{Im}\tau)}.$$

Remark 8.4.2. In view of Corollary 8.2.7 and Proposition 8.3.5, we have that

$$0 \leq \mathrm{n}^*_{\mathrm{alg}}(\tau) \leq \mathrm{n}(\tau) \leq \frac{1}{2} + \frac{1}{2\operatorname{Im}\tau}, \qquad \tau \in \mathbb{H}.$$

8.5. Hyperbolic Fourier series for distributions on the extended real line. We implement the estimate of Lemma 8.4.1, while assuming that $f : \mathbb{H} \to \mathbb{C}$ is holomorphic with

$$(8.5.1) \qquad\qquad |f(\tau)| = \mathrm{O}\left(\frac{1+|\tau|^2}{\operatorname{Im}\tau}\right)^k$$

uniformly in $\tau \in \mathbb{H}$, for some k with $0 < k < +\infty$. It is immediate from (8.5.1) that the estimate carries over to f_{sym}:

$$(8.5.2) \qquad\qquad |f_{\mathrm{sym}}(\tau)| = |f(\tau) + f(1/\bar{\tau})| = \mathrm{O}\left(\frac{1+|\tau|^2}{\operatorname{Im}\tau}\right)^k,$$

again uniformly in $\tau \in \mathbb{H}$. As a next step, we find a solution h_0 to the Dirichlet problem outlined in Step I of the harmonic extension algorithm in Subsection 8.2. As outlined in Proposition 7.2.1, the modular lambda function maps the fundamental domain $\mathscr{D}_\Theta$ conformally onto the half-plane $\mathbb{C}_{\operatorname{Re}<\frac{1}{2}}$ with a slit removed along the negative real axis. Let us write λ^{-1} for the inverse conformal mapping from the slit half-plane onto $\mathscr{D}_\Theta$, which then maps the boundary line $\mathscr{L}_{\frac{1}{2}} = \{\zeta \in \mathbb{C} : \operatorname{Re}\zeta = \frac{1}{2}\}$ onto the semicircle $\mathbb{T}_+$. Let H_0 be a harmonic function in the half-plane $\mathbb{C}_{\operatorname{Re}<\frac{1}{2}}$ which

equals $F := f \circ \lambda^{-1}$ along $\mathscr{L}_{\frac{1}{2}}$. We choose the Poisson extension given by (3.4.2), with the obvious modifications required by rotating and shifting the half-plane:

$$H_0(\zeta) := \frac{1}{\pi} \int_{\mathbb{R}} \frac{\frac{1}{2} - \operatorname{Re}\zeta}{|\zeta - \frac{1}{2} - it|^2} F(\tfrac{1}{2} + it)\,dt, \qquad \zeta \in \mathbb{C}_{\operatorname{Re} < \frac{1}{2}}.$$

The growth condition (8.5.1) entails that the estimate

$$|f(\tau)| = O(\operatorname{Im}\tau)^{-k}, \qquad \tau \in \mathbb{T}_+,$$

holds uniformly, and in view of (7.2.7) and (7.2.8), we know that

$$(8.5.3) \qquad\qquad \frac{1}{\operatorname{Im}\lambda^{-1}(\frac{1}{2} + it)} = \pi^{-1}\log(16|t|) + O(1)$$

uniformly for real t with $|t| \gg 1$, it follows from this that $F = f \circ \lambda^{-1}$ has

$$F(\tfrac{1}{2} + it) = O(\log^k(2 + |t|))$$

uniformly in $t \in \mathbb{R}$. We can now appeal to Proposition 5.3.1, which tells us that the solution H_0 to the Dirichlet problem has the growth bound

$$H_0(\zeta) = O(\log^k(2 + |\zeta|)),$$

uniformly in $\mathbb{C}_{\operatorname{Re} < \frac{1}{2}}$. The induced function $h_0 := H_0 \circ \lambda$ is 2-periodic and harmonic in Ω_0, with boundary values $h_0 = f$ along $\mathbb{T}_+$. Near the cusp at -1, we can use the approximation (7.2.4) and the observation that $\operatorname{Im}\tau \sim |\tau + 1|$ for $\tau \in \Omega_0$ near -1, together with 2-periodicity to obtain that

$$(8.5.4) \quad h_0(\tau) = H_0 \circ \lambda(\tau) = O(\log^k(2 + |\lambda(\tau)|)) = O\big(1 + (\operatorname{Im}\tau)^{-k}\big), \qquad \tau \in \bar{\Omega}_0,$$

with a uniform implicit constant. Here, $\bar{\Omega}_0$ stands for the relative closure of the domain Ω_0 in $\mathbb{H}$. This means that we have an effective estimate of the harmonic function $h = h_\infty : \mathbb{H} \to \mathbb{C}$ in the region $\bar{\Omega}_0$. We want to extend this estimate to all of $\mathbb{H}$. To this end, Lemma 8.4.1 comes in handy. In that context, we use periodicity to reduce to $-1 \leq \operatorname{Re}\tau < 1$, and if addition, $\tau \notin \bar{\Omega}_0$, we have that for $N = \mathrm{n}^*_{\mathrm{alg}}(\tau)$, the iterate $(\mathrm{g}_2^*)^N(\tau)$ is in $\bar{\mathscr{D}}_\Theta \subset \bar{\Omega}_0$, so that we may apply the estimate (8.5.4) at that point. Moreover, by the monotonicity of the imaginary parts of the iterates, we have

$$\operatorname{Im}(\mathrm{g}_2^*)^N(\tau) > \operatorname{Im}(\mathrm{g}_2^*)^N - 1(\tau) > \cdots > \operatorname{Im}\tau,$$

so that, consequently, the estimate (8.5.4) entails that uniformly

$$(8.5.5) \quad h_0 \circ (\mathrm{g}_2^*)^N(\tau) = O\big(1 + (\operatorname{Im}(\mathrm{g}_2^*)^N(\tau))^{-k}\big)$$

$$= O\big(1 + (\operatorname{Im}\tau)^{-k}\big), \qquad \tau \in \mathscr{V} \setminus \bar{\Omega}_0.$$

We arrive at the following proposition.

Proposition 8.5.1. *Suppose $f : \mathbb{H} \to \mathbb{C}$ is holomorphic, subject to the growth bound* (8.5.1) *for some k with $0 < k < +\infty$. Then there exists a 2-periodic harmonic function $h : \mathbb{H} \to \mathbb{C}$ such that $h = f$ along the semicircle $\mathbb{T}_+$, with*

$$|h(\tau)| = O\big(1 + (1 + \mathrm{n}^*_{\mathrm{alg}}(\tau))(\operatorname{Im}\tau)^{-k}\big) = O\big(1 + (\operatorname{Im}\tau)^{-k-1}\big),$$

with implied constants that are uniform in $\tau \in \mathbb{H}$.

Proof. By the harmonic extension algorithm in Subsection 8.2, the function h_0 extends to harmonic function $h = h_\infty : \mathbb{H} \to \mathbb{C}$ that is 2-periodic and solves $h = f$ along $\mathbb{T}_+$ (see Corollary 8.2.4). To obtain the claimed estimate, we observe first that it holds on $\bar{\Omega}_0$, by (8.5.4). In the remainder $\mathbb{H} \setminus \bar{\Omega}_0$, we use the 2-periodicity of h to restrict to the half-strip with $-1 \leq \operatorname{Re}\tau < 1$ and then apply Lemma 8.4.1 to get the asserted estimate from (8.5.2) combined with Remark 8.4.2, as well as the estimates (8.5.4) and (8.5.5). □

We obtain a more precise version of Theorem 3.6.1.

Corollary 8.5.2. *Suppose $f : \mathbb{H} \to \mathbb{C}$ is harmonic, subject to the growth bound* (8.5.1) *for some k with $0 < k < +\infty$. Then there exist complex coefficients a_0 and a_n, b_n for $n \in \mathbb{Z}_{\neq 0}$, with growth*

$$a_n, b_n = \mathrm{O}\big((|n|+1)^k \log^2(2+|n|)\big) \quad \text{as } |n| \to +\infty,$$

such that f has the convergent hyperbolic Fourier series expansion

$$f(\tau) = a_0 + \sum_{n \in \mathbb{Z}_{>0}} \big(a_n \mathrm{e}^{\mathrm{i}\pi n\tau} + b_n \mathrm{e}^{-\mathrm{i}\pi n/\tau}\big) + \sum_{n \in \mathbb{Z}_{<0}} \big(a_n \mathrm{e}^{\mathrm{i}\pi n\bar{\tau}} + b_n \mathrm{e}^{-\mathrm{i}\pi n/\bar{\tau}}\big), \qquad \tau \in \mathbb{H}.$$

As such, the coefficients are uniquely determined by f.

Proof. The uniqueness part follows from Theorem 3.6.6. We proceed to show that the coefficient sequence exists. We apply Proposition 5.1.3 with $\alpha = k$, which gives the unique splitting of the harmonic function f in three parts:

$$f(\tau) = f_+(\tau) + f_-(\tau) + f(\mathrm{i}),$$

where f_+ is holomorphic with $f_+(\mathrm{i}) = 0$, f_- is conjugate-holomorphic with $f_-(\mathrm{i}) = 0$, and $f(\mathrm{i})$ is constant. The functions f_+ and f_- enjoy the same growth bound as f itself. We apply Proposition 8.5.1 to the holomorphic function f_+, which gives us a 2-periodic harmonic function $h_+ : \mathbb{H} \to \mathbb{C}$ with $h_+ = f_+$ on $\mathbb{T}_+$ subject to the growth bound

$$(8.5.6) \qquad |h_+(\tau)| = \mathrm{O}\big(1 + (1 + \mathrm{m}^*_{\mathrm{alg}}(\tau))(\operatorname{Im}\tau)^{-k}\big) = \mathrm{O}\big(1 + (\operatorname{Im}\tau)^{-k-1}\big),$$

uniformly in $\mathbb{H}$. Then, by Proposition 8.1.4, h_+ has a convergent harmonic Fourier series expansion

$$h_+(\tau) = \sum_{n \in \mathbb{Z}} c_n e_n(\tau), \qquad \tau \in \mathbb{H},$$

and the Schwarz transform of h_+ gives back f_+: $\mathbf{v} h_+ = f_+$. Moreover, Proposition 5.2.2 together with the growth control on h_+ ensures that the Fourier coefficients have the growth bound $c_n = \mathrm{O}(|n|^{k+1})$ as $|n| \to +\infty$. We can, however, do slightly better if we use the intermediate bound in (8.5.6). Indeed, by (5.2.8), we may represent the coefficients in the form

$$c_n = \frac{\mathrm{e}^{\pi|n|s}}{2} \int_{-1}^{1} \mathrm{e}^{-\mathrm{i}\pi n t} h_+(t + \mathrm{i}s)\, \mathrm{d}t,$$

provided that $s > 0$. By the triangle inequality for integrals, this leads to

$$|c_n| \leq \frac{\mathrm{e}^{\pi|n|s}}{2} \int_{-1}^{1} |h_+(t+\mathrm{i}s)|\,\mathrm{d}t,$$

and, as we implement the first estimate in (8.5.6), we find that if $C > 0$ is the implied constant,

$$|c_n| \leq C\mathrm{e}^{\pi|n|s}\left(1 + s^{-k}\int_{-1}^{1}\left(1 + \mathrm{n}_{\mathrm{alg}}^*(t+\mathrm{i}s)\right)\frac{\mathrm{d}t}{2}\right)$$
$$\leq C\mathrm{e}^{\pi|n|s}\left(1 + s^{-k}\frac{1}{\pi^2}\log^2\frac{1}{s} + \mathrm{O}\left(s^{-k}\log\frac{1}{s}\right)\right),$$

if $s > 0$ is small. Here, we used the comparison of height functions in Proposition 8.3.5 as well as the asymptotic average height calculation of Proposition 8.2.8. For large $|n|$ we plug in $s = 1/|n|$, which gives

$$|c_n| \leq C\mathrm{e}^{\pi}\left(\pi^{-2}|n|^k\log^2|n| + \mathrm{O}(|n|^k\log|n|)\right) = \mathrm{O}(|n|^k\log^2|n|).$$

From Proposition 8.1.5, we know that $\mathbf{w}h_+ = f_+$ is represented by a holomorphic hyperbolic Fourier series. In an analogous fashion, we deal with the conjugate-holomorphic part. For instance, we can proceed as follows. We take the complex conjugate $\bar{f}_-$ of f_- to get a holomorphic function, and find a 2-periodic harmonic function h_- with $h_- = \bar{f}_-$ on $\mathbb{T}_+$ with the same type of growth bound as for h_+. Hence the function h_- has a harmonic Fourier series expansion, with the same kind of coefficient bound, and if we take its Schwarz transform we recover $\bar{f}_-$: $\mathbf{w}h_- = \bar{f}_-$. This supplies a conjugate-holomorphic hyperbolic Fourier series expansion for f_-, and together, the holomorphic and conjugate-holomorphic hyperbolic Fourier series give a harmonic hyperbolic Fourier series expansion of f. $\qquad\square$

Remark 8.5.3. As we discussed back in Subsections 3.4 and 3.5, Theorems 3.3.1 and 3.6.1 are equivalent. For this reason, we will not supply a separate proof of Theorem 3.3.1.

8.6. Hyperbolic Fourier series for bounded harmonic functions. The instance of $k = 0$ in Proposition 8.5.2 is of particular interest. We cannot plug in $k = 0$ directly in Corollary 8.5.2, but we could of course settle for any $k > 0$ instead. We prefer to derive a more precise result. Let us begin with a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ subject to the logarithmic growth bound

$$(8.6.1) \qquad\qquad |f(\tau)| = \mathrm{O}\left(\log\frac{1+|\tau|^2}{\mathrm{Im}\,\tau}\right),$$

uniformly in $\tau \in \mathbb{H}$. It is immediate that (8.6.1) carries over to f_{sym} as well:

$$|f_{\mathrm{sym}}(\tau)| = |f(\tau) + f(1/\bar{\tau})| = \mathrm{O}\left(\log\frac{1+|\tau|^2}{\mathrm{Im}\,\tau}\right),$$

again uniformly in $\tau \in \mathbb{H}$. Next, we find the solution h_0 to the Dirichlet problem outlined in Step I of the harmonic extension algorithm in Subsection 8.2. As in

Subsection 8.5, we let H_0 be a harmonic function in the half-plane $\mathbb{C}_{\mathrm{Re}<\frac{1}{2}}$ which equals $F := f \circ \lambda^{-1}$ along the boundary line $\mathscr{L}_{\frac{1}{2}}$. We choose the Poisson extension given by (3.4.2), with the obvious modifications required by rotating and shifting the half-plane:

$$H_0(\zeta) := \frac{1}{\pi} \int_{\mathbb{R}} \frac{\frac{1}{2} - \mathrm{Re}\,\zeta}{|\zeta - \frac{1}{2} - \mathrm{i}t|^2} F(\tfrac{1}{2} + \mathrm{i}t)\, \mathrm{d}t, \qquad \zeta \in \mathbb{C}_{\mathrm{Re}<\frac{1}{2}}.$$

The growth condition (8.6.1) entails that the estimate

$$|f(\tau)| = \mathrm{O}\Big(\log \frac{2}{\mathrm{Im}\,\tau} \Big), \qquad \tau \in \mathbb{T}_+,$$

holds uniformly, and in view of (8.5.3), it follows from this that $F = f \circ \lambda^{-1}$ has

$$F(\tfrac{1}{2} + \mathrm{i}t) = \mathrm{O}(\log\log(10 + |t|))$$

uniformly in $t \in \mathbb{R}$. We can now appeal to Proposition 5.3.2, which tells us that the solution H_0 to the Dirichlet problem has the growth bound

$$H_0(\zeta) = \mathrm{O}(\log\log(10 + |\zeta|)),$$

uniformly in $\mathbb{C}_{\mathrm{Re}<\frac{1}{2}}$. The induced function $h_0 := H_0 \circ \lambda$ is 2-periodic and harmonic in Ω_0, with boundary values $h_0 = f$ along $\mathbb{T}_+$. Near the cusp at -1, we can use the approximation (7.2.4) and the observation that $\mathrm{Im}\,\tau \sim |\tau + 1|$ for $\tau \in \Omega_0$ near -1, together with 2-periodicity to obtain that

$$(8.6.2) \quad h_0(\tau) = H_0 \circ \lambda(\tau) = \mathrm{O}(\log\log(10 + |\lambda(\tau)|))$$

$$= \mathrm{O}\Big(\log\Big(2 + \frac{1}{\mathrm{Im}\,\tau} \Big) \Big), \qquad \tau \in \bar{\Omega}_0,$$

with a uniform implicit constant. This means that we have an effective estimate of the harmonic function $h = h_\infty : \mathbb{H} \to \mathbb{C}$ in the region $\bar{\Omega}_0$. We want to extend this estimate to all of $\mathbb{H}$. To this end, Lemma 8.4.1 comes in handy. In that context, we use periodicity to reduce to $-1 \leq \mathrm{Re}\,\tau < 1$, and note that if $\tau \notin \bar{\Omega}_0$, we find that for $N = \mathrm{m}_{\mathrm{alg}}^*(\tau)$, the iterate $(\mathrm{g}_2^*)^N(\tau)$ is in $\bar{\mathscr{D}}_\Theta \subset \bar{\Omega}_0$, so that we may apply the estimate (8.6.2) at that point. Moreover, by the monotonicity of the imaginary parts of the iterates, we have

$$\mathrm{Im}\,(\mathrm{g}_2^*)^N(\tau) > \mathrm{Im}\,(\mathrm{g}_2^*)^N - 1(\tau) > \cdots > \mathrm{Im}\,\tau,$$

so that, consequently, the estimate (8.6.2) entails that uniformly

$$(8.6.3) \quad h_0 \circ (\mathrm{g}_2^*)^N(\tau) = \mathrm{O}\Big(\log\Big(2 + \frac{1}{\mathrm{Im}\,(\mathrm{g}_2^*)^N(\tau)} \Big) \Big)$$

$$= \mathrm{O}\Big(\log\Big(2 + \frac{1}{\mathrm{Im}\,\tau} \Big) \Big), \qquad \tau \in \mathscr{V}_+ \setminus \bar{\Omega}_0.$$

We arrive at the following proposition.

Proposition 8.6.1. *Suppose* $f : \mathbb{H} \to \mathbb{C}$ *is holomorphic, subject to the logarithmic growth bound* (8.6.1). *Then there exists a 2-periodic harmonic function* $h : \mathbb{H} \to \mathbb{C}$ *such that* $h = f$ *along the semicircle* $\mathbb{T}_+$, *with*

$$|h(\tau)| = O\Big((1 + \mathrm{n}^*_{\mathrm{alg}}(\tau)) \log\Big(2 + \frac{1}{\mathrm{Im}\,\tau}\Big)\Big),$$

with an implied constant that is uniform in $\tau \in \mathbb{H}$.

Proof. By the harmonic extension algorithm in Subsection 8.2, the function h_0 extends to harmonic function $h = h_\infty : \mathbb{H} \to \mathbb{C}$ that is 2-periodic and solves $h = f$ along $\mathbb{T}_+$ (see Corollary 8.2.4). To obtain the claimed estimate, we observe first that it holds on $\bar{\Omega}_0$, by (8.6.2). In the remainder $\mathbb{H} \setminus \bar{\Omega}_0$, we use the 2-periodicity of h to restrict to the half-strip with $-1 \leq \mathrm{Re}\,\tau < 1$ and then apply Lemma 8.4.1 to get the asserted estimate from (8.6.1) as well as the estimates (8.6.2) and (8.6.3). □

We obtain a slow growth bound on the coefficients in the hyperbolic Fourier series expansion for a given bounded function.

Corollary 8.6.2. *Suppose* $f : \mathbb{H} \to \mathbb{C}$ *is harmonic and bounded. Then there exist complex coefficients* a_0 *and* a_n, b_n *for* $n \in \mathbb{Z}_{\neq 0}$, *with growth*

$$a_n, b_n = O\big(\log^3(2 + |n|)\big) \quad \text{as } |n| \to +\infty,$$

such that f *has the convergent hyperbolic Fourier series expansion*

$$f(\tau) = a_0 + \sum_{n \in \mathbb{Z}_{>0}} \big(a_n e^{i\pi n\tau} + b_n e^{-i\pi n/\tau}\big) + \sum_{n \in \mathbb{Z}_{<0}} \big(a_n e^{i\pi n\bar{\tau}} + b_n e^{-i\pi n/\bar{\tau}}\big), \qquad \tau \in \mathbb{H}.$$

As such, the coefficients are uniquely determined by f.

Remark 8.6.3. The estimate is uniform in f, provided that the L^∞-norm of f is bounded by a given constant.

Proof of Corollary 8.6.2. The uniqueness part follows from Theorem 3.6.6. We proceed to show that the coefficient sequence exists. We apply Proposition 5.1.2, which gives the unique splitting of the harmonic function f in three parts:

$$f(\tau) = f_+(\tau) + f_-(\tau) + f(i),$$

where f_+ is holomorphic with $f_+(i) = 0$, f_- is conjugate-holomorphic with $f_-(i) = 0$, and $f(i)$ is constant. The functions f_+ and f_- then enjoy the logarithmic growth bound (8.6.1). We apply Proposition 8.6.1 to the holomorphic function f_+, which gives us a 2-periodic harmonic function $h_+ : \mathbb{H} \to \mathbb{C}$ with $h_+ = f_+$ on $\mathbb{T}_+$ and the growth bound

$$(8.6.4) \qquad |h_+(\tau)| = O\Big((1 + \mathrm{n}^*_{\mathrm{alg}}(\tau)) \log\Big(2 + \frac{1}{\mathrm{Im}\,\tau}\Big)\Big),$$

uniformly in $\mathbb{H}$. Then, by Proposition 8.1.4, h_+ has a convergent harmonic Fourier series expansion

$$h_+(\tau) = \sum_{n \in \mathbb{Z}} c_n e_n(\tau), \qquad \tau \in \mathbb{H},$$

and the Schwarz transform of h_+ gives back f_+: $\mathbf{w} h_+ = f_+$. By (5.2.8), we may represent the coefficients in the form

$$c_n = \frac{e^{\pi |n| s}}{2} \int_{-1}^{1} e^{-i\pi nt} h_+(t + is) \, dt,$$

provided that $s > 0$. By the triangle inequality for integrals, this leads to

$$|c_n| \leq e^{\pi |n| s} \int_{-1}^{1} |h_+(t + is)| \frac{dt}{2},$$

and, as we implement the estimate in (8.6.4), we find that if $C > 0$ is the implied constant,

$$|c_n| \leq C e^{\pi |n| s} \left(\int_{-1}^{1} \left(1 + \mathrm{n}_{\mathrm{alg}}^*(t + is) \right) \frac{dt}{2} \, \log \left(2 + \frac{1}{s} \right) \right)$$

$$\leq C e^{\pi |n| s} \left(\frac{1}{\pi^2} \log^2 \frac{1}{s} + \mathrm{O}\left(\log \frac{1}{s} \right) \right) \log \left(2 + \frac{1}{s} \right),$$

provided that $s > 0$ is small. Here, we used as before the comparison of height functions in Proposition 8.3.5 as well as the asymptotic average height calculation of Proposition 8.2.8. For large $|n|$, we plug in $s = 1/|n|$, which gives

$$|c_n| \leq C e^{\pi} \left(\pi^{-2} |n|^k \log^2 |n| + \mathrm{O}(\log |n|) \right) \log(2 + |n|) = \mathrm{O}(|\log^3 |n|).$$

From Proposition 8.1.5, we know that $\mathbf{w} h_+ = f_+$ is represented by a holomorphic hyperbolic Fourier series. In an analogous fashion, we deal with the conjugate-holomorphic part. For instance, we can proceed as follows. We take the complex conjugate $\bar{f}_-$ of f_- to get a holomorphic function, and find a 2-periodic harmonic function h_- with $h_- = \bar{f}_-$ on $\mathbb{T}_+$ with the same type of growth bound as for h_+. Hence the function h_- has a harmonic Fourier series expansion, with the same kind of coefficient bound, and if we take its Schwarz transform we recover $\bar{f}_-$: $\mathbf{w} h_- = \bar{f}_-$. This supplies a conjugate-holomorphic hyperbolic Fourier series expansion for f_-, and together, the holomorphic and conjugate-holomorphic hyperbolic Fourier series give a harmonic hyperbolic Fourier series expansion of f. $\qquad\square$

8.7. **Harmonic hyperbolic Fourier series representation of exponentially growing functions.** Here, we consider a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ whose growth is allowed to be substantially faster:

(8.7.1) $$|f(\tau)| = \mathrm{O}\big(\exp(\beta \pi \mathrm{M}(\tau)) \big)$$

with an implicit constant that is uniform in $\tau \in \mathbb{H}$. The parameter β is positive: $0 < \beta < +\infty$. Moreover, we recall that the function $\mathrm{M}(\tau)$ is given by (3.8.1), that is,

$$\mathrm{M}(\tau) = \frac{\max\{1, |\tau|^2\}}{\mathrm{Im}\, \tau}.$$

It is immediate from the symmetry of M expressed in (3.8.2) that the symmetrized function $f_{\mathrm{sym}}(\tau) = f(\tau) + f(1/\bar{\tau})$ has the same estimate (8.7.1):

(8.7.2) $$|f_{\mathrm{sym}}(\tau)| = \mathrm{O}\big(\exp(\beta\pi\mathrm{M}(\tau))\big)$$

uniformly in $\tau \in \mathbb{H}$. As we restrict the estimate (8.7.1) to the semi-circle $\mathbb{T}_+$, we find that

(8.7.3) $$|f(\tau)| = \mathrm{O}\big(\exp(\beta\pi/\mathrm{Im}\,\tau)\big), \qquad \tau \in \mathbb{T}_+,$$

holds uniformly. We now follow the same scheme as outlined in Subsections 8.5 and 8.6. We let H_0 be a harmonic function in the half-plane $\mathbb{C}_{\mathrm{Re}<\frac{1}{2}}$ which equals the function $F = f \circ \lambda^{-1}$ along the boundary line $\mathscr{L}_{\frac{1}{2}}$. In view of (8.5.3), the growth condition (8.7.3) entails that

$$|F(\tfrac{1}{2} + \mathrm{i}t)| = \mathrm{O}\big((1 + |t|)^{\beta}\big), \qquad t \in \mathbb{R},$$

uniformly in t. We may now appeal to Proposition 5.3.5, which guarantees that if $\beta \notin \mathbb{Z}_{>0}$, then such a harmonic function H_0 may be found with the same order of growth:

(8.7.4) $$|H_0(\zeta)| = \mathrm{O}\big((1 + |\zeta|)^{\beta}\big), \qquad \zeta \in \mathbb{C}_{\mathrm{Re}<\frac{1}{2}},$$

with a uniform implicit constant. Moreover, if instead $\beta \in \mathbb{Z}_{>0}$, the estimate gets slightly worse,

$$|H_0(\zeta)| = \mathrm{O}\big((1 + |\zeta|)^{\beta} \log(2 + |\zeta|)\big), \qquad \zeta \in \mathbb{C}_{\mathrm{Re}<\frac{1}{2}},$$

again with a uniform implicit constant. We put as before $h_0 := H_0 \circ \lambda$, which defines a 2-periodic harmonic function in Ω_0, which extends continuously up to the relative boundary $\partial\Omega_0$, and as such has $h_0 = f$ on $\mathbb{T}_+$. For $\tau \in \mathscr{D}_{\Theta}$ close to the cusp point -1, we calculate that

$$\mathrm{Re}\,\frac{\mathrm{i}\pi}{\tau+1} = \frac{\pi}{\mathrm{Im}\,\tau} + \mathrm{O}(\mathrm{Im}\,\tau),$$

so that by (7.2.4),

(8.7.5) $$|\lambda(\tau)| = \frac{1}{16}\big(1 + \mathrm{O}(\mathrm{Im}\,\tau)\big)\, e^{\pi/\mathrm{Im}\,\tau}.$$

Moreover, by 2-periodicity, the estimate (8.7.5) extends to $\bar{\Omega}_0$, in the limit as $\mathrm{Im}\,\tau \to 0^+$. It now follows from the estimate (8.7.4) that if $\beta \notin \mathbb{Z}_{>0}$,

(8.7.6) $$|h_0(\tau)| = \mathrm{O}\big((1 + |\lambda(\tau)|)^{\beta}\big) = \mathrm{O}\big(\exp(\beta\pi/\mathrm{Im}\,\tau)\big), \qquad \tau \in \bar{\Omega}_0,$$

whereas if $\beta \in \mathbb{Z}_{>0}$,

(8.7.7) $$|h_0(\tau)| = \mathrm{O}\big((1 + |\lambda(\tau)|)^{\beta} \log(2 + |\lambda(\tau)|)\big)$$

$$= \mathrm{O}\Big(\Big(1 + \frac{1}{\mathrm{Im}\,\tau}\Big)\exp(\beta\pi/\mathrm{Im}\,\tau)\Big), \qquad \tau \in \bar{\Omega}_0,$$

Both estimates (8.7.6) and (8.7.7) hold uniformly, for fixed β. This means that we have an effective estimate of the function $h = h_\infty$ in the region $\bar{\Omega}_0$. In that context, we

use periodicity to reduce to $-1 \leq \operatorname{Re}\tau < 1$, and note that if $\tau \notin \bar{\Omega}_0$, we find that for $N = \mathrm{n}^*_{\mathrm{alg}}(\tau)$, the iterate $(\mathrm{g}_2^*)^N(\tau)$ is in $\bar{\mathscr{D}}_\Theta \subset \bar{\Omega}_0$, so that we may apply the estimate (8.7.6) at that point. Moreover, by the monotonicity of the imaginary parts of the iterates, we have

$$\operatorname{Im}(\mathrm{g}_2^*)^N(\tau) > \operatorname{Im}(\mathrm{g}_2^*)^N - 1(\tau) > \cdots > \operatorname{Im}\tau,$$

and, consequently, the estimate (8.7.6) entails that

$$(8.7.8) \qquad |h_0 \circ (\mathrm{g}_2^*)^N(\tau)| = \mathrm{O}\Big(\exp\big(\beta\pi/\operatorname{Im}(\mathrm{g}_2^*)^N(\tau)\big)\Big) = \mathrm{O}\big(\exp(\beta\pi/\operatorname{Im}\tau)\big)$$

holds uniformly for $\tau \in \mathbb{H} \setminus \bar{\Omega}_0$ with $-1 \leq \operatorname{Re}\tau < 1$, provided that $\beta \notin \mathbb{Z}_{>0}$. For $\beta \in \mathbb{Z}_{>0}$, we instead have the estimate

$$(8.7.9) \qquad |h_0 \circ (\mathrm{g}_2^*)^N(\tau)| = \mathrm{O}\Big(\Big(1 + \frac{1}{\operatorname{Im}\tau}\Big)\exp(\beta\pi/\operatorname{Im}\tau)\Big),$$

uniformly in $\tau \in \mathbb{H} \setminus \bar{\Omega}_0$. We obtain the following proposition.

Proposition 8.7.1. *Suppose $f : \mathbb{H} \to \mathbb{C}$ is holomorphic and meets the exponential growth bound* (8.7.1) *for a real parameter β with $0 < \beta < +\infty$. Then, if β is not an integer, there exists a 2-periodic harmonic function $h : \mathbb{H} \to \mathbb{C}$ such that $h = f$ along the semicircle $\mathbb{T}_+$, with*

$$|h(\tau)| = \mathrm{O}\big((1 + \mathrm{n}^*_{\mathrm{alg}}(\tau))\big(\exp(\beta\pi/\operatorname{Im}\tau)\big)\big),$$

with an implied constant that is uniform in $\tau \in \mathbb{H}$. On the other hand, if β is an integer, then there is still such a harmonic function, but the growth estimate gets slightly worse:

$$|h(\tau)| = \mathrm{O}\Big(\Big(1 + \frac{1}{\operatorname{Im}\tau}\Big)\big(\exp(\beta\pi/\operatorname{Im}\tau)\big)\Big),$$

uniformly in $\tau \in \mathbb{H}$.

Proof. By the harmonic extension algorithm in Subsection 8.2, the function h_0 extends to harmonic function $h = h_\infty : \mathbb{H} \to \mathbb{C}$ that is 2-periodic and solves $h = f$ along $\mathbb{T}_+$ (see Corollary 8.2.4). To obtain the claimed estimate, we observe first that it holds on $\bar{\Omega}_0$, by (8.6.2). In the remainder $\mathbb{H} \setminus \bar{\Omega}_0$, we use the 2-periodicity of h to restrict to the half-strip with $-1 \leq \operatorname{Re}\tau < 1$, and then we apply Lemma 8.4.1 to get the asserted estimates from (8.7.2) as well as from the estimates (8.7.8) and (8.7.9) combined with Remark 8.4.2. $\qquad\square$

We now obtain the holomorphic hyperbolic Fourier series expansion claimed in Theorem 3.8.2.

Proof of Theorem 3.8.2. If $h : \mathbb{H}$ is the harmonic function of Proposition 8.7.1, we may appeal to Proposition 5.2.5 with a slightly bigger value of β. This guarantees that h has a harmonic Fourier series expansion (5.2.1):

$$h(\tau) = \sum_{n \in \mathbb{Z}} c_n e_n(\tau), \qquad \tau \in \mathbb{H},$$

where the coefficients are recovered by the formula (5.2.8), that is, for any $s > 0$, we have

$$c_n = \frac{e^{\pi|n|s}}{2} \int_{-1}^{1} e^{-i\pi nt} h(t + is)\, dt,$$

so that

(8.7.10)
$$|c_n| \leq \frac{e^{\pi|n|s}}{2} \int_{-1}^{1} |h(t + is)|\, dt.$$

In case that β is not an integer, we have from Proposition 8.7.1 applied to (8.7.10) that if $C > 0$ is the implicit constant in the estimate of Proposition 8.7.1,

$$|c_n| \leq C e^{\pi|n|s} \exp(\beta\pi/s)\left(1 + \int_{-1}^{1} \mathrm{m}^*_{\mathrm{alg}}(t + is)\, \frac{dt}{2}\right)$$

$$= C e^{\pi|n|s} \exp(\beta\pi/s)\left(\frac{1}{\pi^2} \log^2 \frac{1}{s} + O\left(\log \frac{1}{s}\right)\right),$$

provided $s > 0$ is small enough. Here, we used the comparison of height functions in Proposition 8.3.5 as well as the asymptotic average height calculation of Proposition 8.2.8. For large $|n|$, we plug in $s = \beta^{\frac{1}{2}}|n|^{-\frac{1}{2}}$, which gives that

$$|c_n| = O\left((\log^2 |n|) \exp\left(2\pi\sqrt{\beta|n|}\right)\right)$$

as $|n| \to +\infty$. In the remaining case when β is an integer, the analogous argument gives

$$|c_n| = O\left(|n|^{\frac{1}{2}} \exp\left(2\pi\sqrt{\beta|n|}\right)\right),$$

again as $|n| \to +\infty$. Since $h = f$ holds on the semicircle $\mathbb{T}_+$, the Schwarz transform recovers f: $\mathfrak{w}h = f$. Moreover, the harmonic Fourier series coefficients of h become the holomorphic hyperbolic Fourier series coefficients of f in a natural fashion: $a_n = c_n$ for $n \geq 0$, while $b_n = c_{-n}$ for $n > 0$. The proof is now complete. $\square$

9. HYPERBOLIC FOURIER SERIES EXPANSION OF A POINT MASS

9.1. **The Cauchy and Poisson kernels.** For a given point $x \in \mathbb{R}$, we consider the associated *Cauchy kernel*

(9.1.1)
$$f_x(\tau) := \frac{1}{i2\pi} \frac{1}{x - \tau}, \qquad \tau \in \mathbb{H},$$

whose boundary values on $\mathbb{R}$ in the sense of distribution theory are given by

$$2f_x(t) = \delta_x(t) + \frac{i}{\pi} \mathrm{pv}\, \frac{1}{t - x},$$

where "pv" indicates the *principal value* interpretation of the given function. In particular, a direct calculation gives that

$$2\mathrm{Re} f_x(\tau) = P(\tau, x) = \pi^{-1} \frac{\mathrm{Im}\, \tau}{|x - \tau|^2}, \qquad \tau \in \mathbb{H},$$

which is the Poisson kernel, and hence the harmonic extension of the unit point mass at x. Now, according to Theorem 3.6.1 and its improvement Corollary 8.5.2, there exist coefficients $a_n = a_n(f_x)$ and $b_n = b_n(f_x)$ with growth

$$a_n, b_n = \mathrm{O}\big((n+1)\log^2(2+n)\big)$$

as $n \to +\infty$, such that

$$f_x(\tau) = a_0 + \sum_{n \in \mathbb{Z}_{>0}} \big(a_n \, \mathrm{e}^{\mathrm{i}\pi n \tau} + b_n \, \mathrm{e}^{-\mathrm{i}\pi n/\tau}\big), \qquad \tau \in \mathbb{H}.$$

Moreover, as such, these coefficients are unique. Taking real parts on both sides, we see that

$$\pi^{-1} \frac{\mathrm{Im}\,\tau}{|x-\tau|^2} = 2\mathrm{Re}\, f_x(\tau) = A_0(x) + \sum_{n \in \mathbb{Z}_{>0}} \big(A_n(x)\, \mathrm{e}^{\mathrm{i}\pi n \tau} + B_n(x)\, \mathrm{e}^{-\mathrm{i}\pi n/\tau}\big)$$
$$+ \sum_{n \in \mathbb{Z}_{<0}} \big(A_n(x)\, \mathrm{e}^{-\mathrm{i}\pi n \bar{\tau}} + B_n(x)\, \mathrm{e}^{\mathrm{i}\pi n/\bar{\tau}}\big), \qquad \tau \in \mathbb{H},$$

where we write

(9.1.2) $A_0(x) := 2\mathrm{Re}\, a_0(f_x), \quad \forall n \in \mathbb{Z}_{>0}: \ A_n(x) := a_n(f_x) \text{ and } B_n(x) := b_n(f_x),$

and the coefficient functions get extended to negative indices via the symmetry property

(9.1.3) $A_n(x) = \bar{A}_{-n}(x), \quad B_n(x) = \bar{B}_{-n}(x).$

Here, the "bar" indicates complex conjugation of the whole expression. The Cauchy kernel has the functional properties

(9.1.4) $\bar{f}_{-x} \circ \mathrm{R}^*(\tau) = \bar{f}_{-x}(-\bar{\tau}) = f_x(\tau), \qquad \tau \in \mathbb{H},$

and, for $x \neq 0$,

(9.1.5) $f_x \circ \mathrm{S}(\tau) = f_x(-1/\tau) = \dfrac{1}{\mathrm{i}2\pi x} + x^{-2} f_{-1/x}(\tau), \qquad \tau \in \mathbb{H}.$

As we plug in the hyperbolic Fourier series expansion on both sides of (9.1.4), we find that

(9.1.6) $\bar{f}_{-x}(-\bar{\tau}) = \bar{a}_0(f_{-x}) + \sum_{n \in \mathbb{Z}_{>0}} \big(\bar{a}_n(f_{-x})\, \mathrm{e}^{\mathrm{i}\pi n \tau} + \bar{b}_n(f_{-x})\, \mathrm{e}^{-\mathrm{i}\pi n/\tau}\big)$

$$= a_0(f_x) + \sum_{n \in \mathbb{Z}_{>0}} \big(a_n(f_x)\, \mathrm{e}^{\mathrm{i}\pi n \tau} + b_n(f_x)\, \mathrm{e}^{-\mathrm{i}\pi n/\tau}\big) = f_x(\tau), \qquad \tau \in \mathbb{H}.$$

On the other hand, if we instead work with the identity (9.1.5) we obtain that

$$(9.1.7) \quad f_x(-1/\tau) = a_0(f_x) + \sum_{n \in \mathbb{Z}_{>0}} \left(a_n(f_x) e^{-i\pi n/\tau} + b_n(f_x) e^{i\pi n\tau} \right)$$

$$= \frac{1}{i2\pi x} + x^{-2} a_0(f_{-1/x}) + x^{-2} \sum_{n \in \mathbb{Z}_{>0}} \left(a_n(f_{-1/x}) e^{i\pi n\tau} + b_n(f_{-1/x}) e^{-i\pi n/\tau} \right)$$

$$= \frac{1}{i2\pi x} + x^{-2} f_{-1/x}(\tau), \qquad \tau \in \mathbb{H}.$$

Both sides of (9.1.6) and (9.1.7) are hyperbolic Fourier series, so in view of Theorem 3.6.6, the corresponding coefficients must coincide, given that the coefficient growth is rather modest. These two identities lead to symmetry properties of the functions A_n and B_n.

Proposition 9.1.1. (symmetry properties)
(a) *We have that $A_0(x)$ is real-valued and even, so that $A_0(-x) = A_0(x)$ holds for each $x \in \mathbb{R}$. Moreover, for $x \in \mathbb{R}_{\neq 0}$, we have that $A_0(x) = x^{-2} A_0(-1/x)$.*
(b) *For $n \in \mathbb{Z}_{\neq 0}$, the functions $A_n(x), B_n(x)$ have the symmetry properties that for each $x \in \mathbb{R}$,*

$$A_n(x) = \bar{A}_{-n}(x) = A_{-n}(-x), \quad B_n(x) = \bar{B}_{-n}(x) = B_{-n}(-x),$$

and, moreover, for $x \in \mathbb{R}_{\neq 0}$, they are connected via

$$B_n(x) = x^{-2} A_n(-1/x).$$

Remark 9.1.2. It is a consequence of the relations (9.1.6) and (9.1.7) that

$$\bar{a}_0(f_{-x}) = a_0(f_x), \quad a_0(f_x) = \frac{1}{i2\pi x} + x^{-2} a_0(f_{-1/x}),$$

which at least looks superficially stronger than the corresponding assertion of part (a) to the effect that $A_0(-x) = a_0(x)$ and $A_x(t) = x^{-2} A_0(-1/x)$.

Proof of Proposition 9.1.1. The assertions are immediate consequences of (9.1.3) and the identities (9.1.6), (9.1.7), by identifying the coefficients in the two holomorphic Fourier series expansions. $\qquad \square$

9.2. **Hyperbolic Fourier series expansion of the Cauchy kernel.** For $x \in \mathbb{R}$, we let h_x denote the function

$$(9.2.1) \qquad h_x(\tau) := \frac{1}{\pi} \int_{\mathbb{T}_+} \frac{\frac{1}{2} - \operatorname{Re} \lambda(\tau)}{|\lambda(\tau) - \lambda(\eta)|^2} f_x(\eta) |\lambda'(\eta)| |d\eta|, \qquad \tau \in \mathscr{D}_\Theta.$$

Here, f_x denotes the Cauchy kernel given by (9.1.1). Any possible worries about the convergence of the integral (9.2.1) are dispelled by the relations (7.2.4) and (7.2.5) combined with (7.2.6) and the comment thereafter. From the properties of the Poisson kernel in the upper half-plane (3.4.1) and the mapping properties of λ outlined in Proposition 7.2.1, it follows that h_x is harmonic in the fundamental domain $\mathscr{D}_\Theta$, and that its boundary values on the semicircle $\mathbb{T}_+$ equal those of f_x, so that $h_x = f_x$ on $\mathbb{T}_+$.

Moreover, as the function h_x may be interpreted as the composition of the harmonic function

$$H_x(\zeta) := \frac{1}{\pi} \int_{\mathbb{T}_+} \frac{\frac{1}{2} - \operatorname{Re}\zeta}{|\zeta - \lambda(\eta)|^2} f_x(\eta) |\lambda'(\eta)| |d\eta|, \qquad \zeta \in \mathbb{C}_{\operatorname{Re} < \frac{1}{2}},$$

with the modular lambda function, that is, $h_x = H_x \circ \lambda$, we conclude from the mapping properties and the 2-periodicity of λ that h_x extends to a 2-periodic harmonic function in the domain Ω_0 given by (8.2.1). This puts us in the setting of the algorithm of Subsection 8.2, with f_x as f and h_x as h_0. By Corollary 8.2.4, h_x extends harmonically to all of $\mathbb{H}$, and by Corollary 8.2.5, the fact that $h_x = f_x$ holds on $\mathbb{T}_+$ tells us that $f_x = \mathpzc{w} h_x$ is the holomorphic hyperbolic Fourier series expansion of f_x. Moreover, the harmonic Fourier series coefficients of h_x give us the holomorphic hyperbolic Fourier series coefficients of f_x. We proceed to obtain expressions for the harmonic Fourier coefficients $c_n = c_n(h_x)$ associated with the expansion of h_x:

$$h_x(\tau) = \sum_{n \in \mathbb{Z}} c_n e_n(\tau) = c_0 + \sum_{n \in \mathbb{Z}_{>0}} c_n e^{i\pi n \tau} + \sum_{n \in \mathbb{Z}_{<0}} c_n e^{i\pi n \bar{\tau}}, \qquad \tau \in \mathbb{H}.$$

The corresponding holomorphic hyperbolic Fourier series expansion of the Cauchy kernel f_x is then

$$f_x(\tau) = \mathpzc{w} h_x(\tau) = c_0 + \sum_{n \in \mathbb{Z}_{>0}} c_n e^{i\pi n \tau} + \sum_{n \in \mathbb{Z}_{<0}} c_n e^{i\pi n/\tau}, \qquad \tau \in \mathbb{H},$$

which means that the coefficients $a_n(f_x)$ and $b_n(f_x)$ are given by

(9.2.2) $\qquad a_0(f_x) = c_0(h_x), \ \forall n \in \mathbb{Z}_{>0}: \ a_n(f_x) = c_n(h_x), \ b_n(f_x) = c_{-n}(h_x).$

Clearly, the function f_x meets the uniform growth bound

$$|f_x(\tau)| = O(\operatorname{Im}\tau)^{-1}, \qquad \tau \in \mathbb{H},$$

which is uniform not only in τ but also in $x \in \mathbb{R}$. In particular, the growth condition (8.5.1) holds with parameter value $k = 1$, so that by Corollary 8.5.2, we know that

(9.2.3) $\qquad\qquad a_n(f_x), b_n(f_x) = O\big((n+1)\log^2(2+n)\big)$

holds uniformly in n and $x \in \mathbb{R}$. We can improve this estimate for large values of $|x|$.

Proposition 9.2.1. *We have the growth control*

$$A_n(x), B_n(x) = O\Big(\frac{(|n|+1)\log^2(2+|n|)}{1+x^2}\Big),$$

uniformly in $n \in \mathbb{Z}$ and $x \in \mathbb{R}$.

Proof. The assertion is immediate from (9.2.3) combined with the symmetry properties of Proposition 9.1.1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We now attempt to calculate the coefficient functions $A_n(x)$ and $B_n(x)$. To obtain these coefficients, we appeal to formula (5.2.8), which asserts that for any $s > 0$, we have

(9.2.4) $\qquad\qquad c_n(h_x) = \frac{e^{\pi|n|s}}{2} \int_{-1}^{1} e^{-i\pi n t} h_x(t+is) \, dt, \qquad n \in \mathbb{Z}.$

As we implement the formula (9.2.1) which defines h_x in $\mathscr{D}_\Theta$, the formula (9.2.4) becomes, for $s > 1$,

$$c_n(h_x) = \frac{e^{\pi|n|s}}{2\pi} \int_{-1}^{1} \int_{\mathbb{T}_+} e^{-i\pi nt} \frac{\frac{1}{2} - \operatorname{Re}\lambda(t+is)}{|\lambda(t+is) - \lambda(\eta)|^2} f_x(\eta) |\lambda'(\eta)| |d\eta| \, dt, \qquad n \in \mathbb{Z}.$$

Fubini's theorem allows us to interchange the integrations, which leaves us with the formula (for $n \in \mathbb{Z}$)

$$(9.2.5) \qquad c_n(h_x) = \frac{e^{\pi|n|s}}{2\pi} \int_{\mathbb{T}_+} \int_{-1}^{1} e^{-i\pi nt} \frac{\frac{1}{2} - \operatorname{Re}\lambda(t+is)}{|\lambda(t+is) - \lambda(\eta)|^2} \, dt \, f_x(\eta) |\lambda'(\eta)| |d\eta|.$$

The instance $n = 0$ is of particular interest:

$$(9.2.6) \qquad c_0(h_x) = \frac{1}{2\pi} \int_{\mathbb{T}_+} \int_{-1}^{1} \frac{\frac{1}{2} - \operatorname{Re}\lambda(t+is)}{|\lambda(t+is) - \lambda(\eta)|^2} \, dt \, f_x(\eta) |\lambda'(\eta)| |d\eta|$$

In view of the asymptotics (7.2.2), it follows that $\lambda(t+is) \to 0$ uniformly in $t \in \mathbb{R}$ as $s \to +\infty$, so that by letting $s \to +\infty$ in (9.2.6), we find that

$$(9.2.7) \qquad a_0(f_x) = c_0(h_x) = \frac{1}{2\pi} \int_{\mathbb{T}_+} \frac{|\lambda'(\eta)|}{|\lambda(\eta)|^2} f_x(\eta) |d\eta|.$$

As we take real parts in (9.2.7), we obtain an expression for A_0.

Proposition 9.2.2. *We have the formula for A_0:*

$$A_0(x) = 2\operatorname{Re} a_0(f_x) = \frac{1}{2\pi} \int_{\mathbb{T}_+} \frac{|\lambda'(\eta)|}{|\lambda(\eta)|^2} P(\eta,x) |d\eta|$$

$$= \frac{1}{2\pi^2} \int_{\mathbb{T}_+} \frac{|\lambda'(\eta)|}{|\lambda(\eta)|^2} \frac{\operatorname{Im}\eta}{|x-\eta|^2} |d\eta|, \qquad x \in \mathbb{R},$$

so that $A_0(x) > 0$ for each $x \in \mathbb{R}$. Moreover, the function A_0 is C^∞-smooth on $\mathbb{R}$.

Proof. Taking real parts in (9.2.7) gives us the indicated formula for A_0, and the pointwise positivity is immediate from the formula. As for the smoothness, we calculate successive derivatives in (9.2.7):

$$\partial_x^j a_0(f_x) = \frac{1}{2\pi} \int_{\mathbb{T}_+} \frac{|\lambda'(\eta)|}{|\lambda(\eta)|^2} \partial_x^j f_x(\eta) |d\eta| = \frac{(-1)^j}{i4\pi^2} \int_{\mathbb{T}_+} \frac{|\lambda'(\eta)|}{|\lambda(\eta)|^2} (x-\eta)^{-j-1} |d\eta|$$

which integrals converge nicely by the known asymptotics of λ and λ' near the cusp points at ± 1 (see (7.2.4), (7.2.5), (7.2.6)). Taking real parts, this smoothness carries over to A_0. $\qquad\square$

We proceed to evaluate A_n for $n > 0$, since the symmetry properties of Proposition 9.1.1 then give A_n and B_n for all $n \neq 0$. In view of (9.1.2) and (9.2.2) in combination

with (9.2.5), we have, for $s > 1$,

(9.2.8) $\quad A_n(x) = a_n(f_x) = c_n(h_x)$

$$= \frac{1}{2\pi} \int_{\mathbb{T}_+} \int_{-1}^{1} e^{-i\pi n(t+is)} \frac{\frac{1}{2} - \operatorname{Re} \lambda(t+is)}{|\lambda(t+is) - \lambda(\eta)|^2} \, dt \, f_x(\eta) |\lambda'(\eta)| |d\eta|$$

$$= \frac{1}{2\pi} \int_{\mathbb{T}_+} \int_{\gamma_s} e^{-i\pi n\tau} \frac{\frac{1}{2} - \operatorname{Re} \lambda(\tau)}{|\lambda(\tau) - \lambda(\eta)|^2} \, d\tau \, f_x(\eta) |\lambda'(\eta)| |d\eta|, \qquad n \in \mathbb{Z}_{>0},$$

provided that the line segment $\gamma_s := [-1,1] + is$ oriented from left to right. The mapping properties of the modular lambda function presented in Proposition 7.2.1 entail that $\operatorname{Re} \lambda(\eta) = \frac{1}{2}$ for $\eta \in \mathbb{T}_+$, and we calculate that

(9.2.9) $\quad \dfrac{\frac{1}{2} - \operatorname{Re} \lambda(\tau)}{|\lambda(\tau) - \lambda(\eta)|^2} = \dfrac{1 - \lambda(\tau)}{2(1 - \lambda(\eta))(\lambda(\eta) - \lambda(\tau))} + \dfrac{\bar{\lambda}(\tau)}{2\bar{\lambda}(\eta)(\bar{\lambda}(\eta) - \bar{\lambda}(\tau))}.$

The mapping properties of λ also show that as η moves along the semicircle $\mathbb{T}_+$ in the counterclockwise direction, we have the equality of forms

(9.2.10) $\qquad\qquad i\lambda'(\eta) \, d\eta = -i\bar{\lambda}'(\eta) \, d\bar{\eta} = |\lambda'(\eta)| \, |d\eta|.$

It now follows from (9.2.9) that

(9.2.11) $\quad \displaystyle\int_{\gamma_s} e^{-i\pi n\tau} \frac{\frac{1}{2} - \operatorname{Re} \lambda(\tau)}{|\lambda(\tau) - \lambda(\eta)|^2} \, d\tau$

$$= \frac{1}{2} \int_{\gamma_s} e^{-i\pi n\tau} \left(\frac{1 - \lambda(\tau)}{(1 - \lambda(\eta))(\lambda(\eta) - \lambda(\tau))} + \frac{\bar{\lambda}(\tau)}{\bar{\lambda}(\eta)(\bar{\lambda}(\eta) - \bar{\lambda}(\tau))} \right) d\tau$$

$$= \frac{1}{2(1 - \lambda(\eta))} \int_{\gamma_s} e^{-i\pi n\tau} \frac{1 - \lambda(\tau)}{\lambda(\eta) - \lambda(\tau)} \, d\tau, \qquad n \in \mathbb{Z}_{>0}.$$

The omitted term in the last step of (9.2.11) vanishes, as it is the complex conjugate of

$$\frac{1}{2} \int_{\gamma_s} e^{i\pi n\bar{\tau}} \frac{\lambda(\tau)}{\lambda(\eta)(\lambda(\eta) - \lambda(\tau))} \, d\tau = \frac{e^{2\pi ns}}{2\lambda(\eta)} \int_{\gamma_s} e^{i\pi n\tau} \frac{\lambda(\tau)}{\lambda(\eta) - \lambda(\tau)} \, d\tau = 0,$$

where the latter equality holds because the integral

$$\int_{\gamma_s} e^{i\pi n\tau} \frac{\lambda(\tau)}{\lambda(\eta) - \lambda(\tau)} \, d\tau$$

is independent of $s > 1$ (by deformation of countours), and at the same time it tends to 0 as $s \to +\infty$, for each $n > 0$. Moreover, a calculation gives that

$$\partial_\tau \frac{1 - \lambda(\tau)}{\lambda(\eta) - \lambda(\tau)} = (1 - \lambda(\eta)) \frac{\lambda'(\tau)}{(\lambda(\eta) - \lambda(\tau))^2},$$

so that by integration by parts and periodicity,

$$(9.2.12) \qquad \int_{\gamma_s} e^{-i\pi n\tau} \frac{1-\lambda(\tau)}{\lambda(\eta)-\lambda(\tau)} \, d\tau = \frac{1-\lambda(\eta)}{i\pi n} \int_{\gamma_s} e^{-i\pi n\tau} \frac{\lambda'(\tau)}{(\lambda(\eta)-\lambda(\tau))^2} \, d\tau.$$

If we combine (9.2.8) with (9.2.10), (9.2.11), and (9.2.12), we arrive at the formula

$$(9.2.13) \quad A_n(x) = \frac{1}{2\pi} \int_{\mathbb{T}_+} \int_{\gamma_s} e^{-i\pi n\tau} \frac{\frac{1}{2}-\operatorname{Re}\lambda(\tau)}{|\lambda(\tau)-\lambda(\eta)|^2} \, d\tau \, f_x(\eta) \, |\lambda'(\eta)| \, |d\eta|$$

$$= \frac{1}{4\pi^2 n} \int_{\mathbb{T}_+} \int_{\gamma_s} e^{-i\pi n\tau} \frac{\lambda'(\tau)}{(\lambda(\eta)-\lambda(\tau))^2} d\tau \, f_x(\eta) \lambda'(\eta) \, d\eta, \qquad n \in \mathbb{Z}_{>0},$$

for any $s > 1$.

We shall need the polynomials S_n defined by the following property.

Proposition 9.2.3. *For each $n \in \mathbb{Z}_{>0}$, there exists a unique polynomial S_n of degree n with $S_n(0) = 0$, such that*

$$e^{-i\pi n\tau} - S_n(1/\lambda(\tau)) = \mathrm{O}(1)$$

holds as $\operatorname{Im}\tau \to +\infty$.

Proof. In terms of the nome variable $q = e^{i\pi\tau}$, the lambda function is given by

$$\lambda(\tau) = \sum_{n=1}^{+\infty} \widehat{\lambda}(n) \, q^n,$$

where $\widehat{\lambda}(1) = 16$, and the series converges absolutely for $|q| < 1$, with a simple zero at $q = 0$ and no other zeros in the disk $|q| < 1$. As a consequence, for $k \in \mathbb{Z}_{>0}$, the function

$$\frac{1}{(\lambda(\tau))^k}$$

is holomorphic in $q \in \mathbb{D} \setminus \{0\}$ with a pole of order k at $q = 0$. Consequently, modulo the holomorphic functions in the variable $q \in \mathbb{D}$, the functions $1/(\lambda(\tau))^k, k = 1, \ldots, n$ span the same vector space as q^{-j} for $j = 1, \ldots, n$. In particular, there exists a unique polynomial S_n of degree n with $S_n(0) = 0$ such that

$$q^{-n} - S_n(1/\lambda(\tau))$$

is holomorphic at the point $q = 0$. Finally, for holomorphic functions in a punctured disk, boundedness at the puncture is the same as extending holomorphically across it. $\qquad\square$

In terms of the polynomial S_n, we obtain expressions for A_n and B_n, for $n > 0$.

Proposition 9.2.4. *For $n \in \mathbb{Z}_{>0}$, the function A_n is given by*

$$A_n(x) = -\frac{1}{4\pi^2 n} \int_{\mathbb{T}_+} \frac{S_n(1/\lambda(\eta))}{(x-\eta)^2} \, d\eta, \qquad x \in \mathbb{R},$$

while the function B_n is given by

$$B_n(x) = -\frac{1}{4\pi^2 n} \int_{\mathbb{T}_+} \frac{S_n(1/\lambda(\eta))}{(1+x\eta)^2} \, d\eta, \qquad x \in \mathbb{R}.$$

As a consequence, the functions A_n and B_n are both C^∞-smooth on $\mathbb{R}$.

Proof. We split in accordance with Proposition 9.2.3 the function

$$e^{-i\pi n\tau} = S_n(1/\lambda(\tau)) + R_n(\lambda(\tau)),$$

or, equivalently,

(9.2.14) $$e^{-i\pi n\lambda^{-1}(\zeta)} = S_n(1/\zeta) + R_n(\zeta),$$

where $R_n(\zeta)$ is holomorphic and bounded in a neighborhood of 0. The reason that we may represent the remainder as a function of $\lambda(\tau)$ is that $e^{-i\pi n\tau}$ is 2-periodic, while $\lambda(\tau)$ is 2-periodic too and has the mapping properties outlined in Proposition 7.2.1. Indeed, we see that the function $R_n(\zeta)$ extends holomorphically to the slit plane $\mathbb{C} \setminus [1, +\infty[$. As we apply the change-of-variables formula to the inner integral in (9.2.13), we find that

(9.2.15) $$\int_{\gamma_s} e^{-i\pi n\tau} \frac{\lambda'(\tau)}{(\lambda(\eta) - \lambda(\tau))^2} d\tau = \int_{\lambda(\gamma_s)} e^{-i\pi n\lambda^{-1}(\zeta)} \frac{d\zeta}{(\lambda(\eta) - \zeta)^2}$$

where the contour $\lambda(\gamma_s)$ is a closed loop in the half-plane $\mathbb{C}_{\mathrm{Re} < \frac{1}{2}}$ which goes around the origin once in the positive direction (counter-clockwise). In view of (9.2.14), we can handle the expressions $S_n(1/\zeta)$ and $R_n(\zeta)$ separately. By Cauchy's theorem,

$$\int_{\lambda(\gamma_s)} R_n(\zeta) \frac{d\zeta}{(\lambda(\eta) - \zeta)^2} = 0, \qquad \eta \in \mathbb{T}_+,$$

since $\lambda(\eta) \in \frac{1}{2} + i\mathbb{R}$ is exterior to the loop $\lambda(\gamma_s)$. On the other hand, Cauchy's theorem for the domain exterior to the loop $\lambda(\gamma_s)$ gives that

$$\int_{\lambda(\gamma_s)} S_n(1/\zeta) \frac{d\zeta}{(\lambda(\eta) - \zeta)^2} = -i2\pi \, \partial_\zeta S_n(1/\zeta)\Big|_{\zeta := \lambda(\eta)}, \qquad \eta \in \mathbb{T}_+.$$

It now follows from (9.2.15) and (9.2.15) together with the above two computations that

$$\int_{\gamma_s} e^{-i\pi n\tau} \frac{\lambda'(\tau)}{(\lambda(\eta) - \lambda(\tau))^2} d\tau = -i2\pi \, \partial_\zeta S_n(1/\zeta)\Big|_{\zeta := \lambda(\eta)} = i2\pi \lambda(\eta)^{-2} S_n'(1/\lambda(\eta)),$$

so that (9.2.13) gets simplified to (recall the definition (9.1.1) of f_x)

$$A_n(x) = \frac{1}{4\pi^2 n} \int_{\mathbb{T}_+} \int_{\gamma_s} e^{-i\pi n\tau} \frac{\lambda'(\tau)}{(\lambda(\eta) - \lambda(\tau))^2} d\tau \, f_x(\eta) \lambda'(\eta) \, d\eta$$

$$= \frac{1}{4\pi^2 n} \int_{\mathbb{T}_+} \frac{\lambda'(\eta) S_n'(1/\lambda(\eta))}{\lambda(\eta)^2} \frac{d\eta}{x - \eta}, \qquad n \in \mathbb{Z}_{>0},$$

where $s > 1$ in the middle expression. Finally, by integration by parts, it follows that

$$A_n(x) = \frac{1}{4\pi^2 n} \int_{\mathbb{T}_+} \frac{\lambda'(\eta) S_n'(1/\lambda(\eta))}{\lambda(\eta)^2} \frac{d\eta}{x - \eta}$$

$$= -\frac{1}{4\pi^2 n} \int_{\mathbb{T}_+} S_n(1/\lambda(\eta)) \frac{d\eta}{(x - \eta)^2}, \qquad n \in \mathbb{Z}_{>0},$$

as claimed, but we need to motivate why there is no contributions from the endpoints. The reason is that according to (7.2.4) and (7.2.5), the function $1/\lambda(\eta)$ decays rapidly to 0 as $\eta \in \mathbb{T}_+$ approaches the endpoints ± 1 while the polynomial S_n vanishes at the origin, and the function $(x - \eta)^{-1}$ cannot perturb much this rapid decay to 0. This establishes the formula for A_n, and the formula for $B_n(x)$ follows from Proposition 9.1.1 for $x \in \mathbb{R}_{\neq 0}$, and for $x = 0$ by continuity in the parameter x. As for the smoothness, we may represent successive derivatives of A_n and B_n as absolutely convergent integrals:

$$A_n^{(j)}(x) = \frac{(-1)^{j+1}(j+1)!}{4\pi^2 n} \int_{\mathbb{T}_+} S_n(1/\lambda(\eta)) \frac{d\eta}{(x - \eta)^{j+2}}, \qquad n \in \mathbb{Z}_{>0},$$

for $j = 0, 1, 2, \ldots$, and likewise for B_n. This settles the smoothness issue and completes the proof. □

Remark 9.2.5. We note that if λ^{-1} is (as in Subsection 7.2) the local inverse mapping to λ, we may apply the change-of-variables $\zeta = \lambda(\eta)$ in Proposition 9.2.4 and arrive at the representation (the contour is assumed directed upwards)

$$A_n(x) = \frac{1}{4\pi^2 n} \int_{\frac{1}{2}+i\mathbb{R}} \frac{S_n(1/\zeta)}{(x - \lambda^{-1}(\zeta))^2} (\lambda^{-1})'(\zeta) d\zeta$$

$$= \frac{1}{i4\pi^3 n} \int_{\frac{1}{2}+i\mathbb{R}} \frac{S_n(1/\zeta)}{\zeta(1 - \zeta)\left(x F_{\frac{1}{2},\frac{1}{2};1}(\zeta) - i F_{\frac{1}{2},\frac{1}{2};1}(1 - \zeta)\right)} d\zeta, \qquad x \in \mathbb{R},$$

with an analogous formula for B_n as well. Here, the Gauss hypergeometric function appears, as it is well-known that the local inverse λ^{-1} may be expressed in terms of a ratio of hypergeometric functions (see, e.g., [2]).

The explicit representation of the functions A_n, B_n has leads to a simplified formula for interpolating basis $u_{(n,0)}$ and $u_{(0,n)}$ in spacelike quarterplanes. We formulate the statement for $u_{(n,0)}$ with $n \geq 0$ only. As for the notation, see Theorem 4.2.6.

Corollary 9.2.6. *The function $u_{(0,0)}$ is real-valued and has the symmetry property $u_{(0,0)}(x,y) = u_{(0,0)}(-x,-y) = u_{(0,0)}(y,x)$, and moreover, it is expressed by the formula*

$$u_{(0,0)}(x,y) = \frac{1}{2\pi} \int_{\mathbb{T}_+} \frac{|\lambda'(\eta)|}{|\lambda(\eta)|^2} e^{ix\eta + iy/\eta} |d\eta| \quad if \quad x > 0, y < 0.$$

Furthermore, for $n \in \mathbb{Z}_{>0}$, the function $u_{(n,0)}$ is real-valued, and in the spacelike quarter-planes we have that

$$u_{(n,0)}(x,y) = 0 \quad if \quad x < 0, y > 0,$$

while

$$u_{(n,0)}(x,y) = \frac{1}{2\pi n} \int_{\mathbb{T}_+} S_n(1/\lambda(\eta))(x - y\eta^{-2}) e^{ix\eta + iy/\eta} \, d\eta \quad if \quad x > 0, \, y < 0.$$

Proof. We focus our attention to $n > 0$, the case $n = 0$ being analogous. The real-valuedness is a consequence of the symmetry properties of A_n as presented in Proposition 3.3.4. In view of Proposition 9.2.4 combined with the definition of $u_{(n,0)}$ in Theorem 4.2.6, we have that

$$u_{(n,0)}(x,y) \;=\; \int_{\mathbb{R}} e^{ixt + iy/t} A_n(t) \, dt \;=\; -\frac{1}{4\pi^2 n} \int_{\mathbb{T}_+} S_n(1/\lambda(\eta)) \int_{\mathbb{R}} \frac{e^{ixt + iy/t}}{(t - \eta)^2} \, dt \, d\eta,$$

where in the last step we used Fubini's theorem to interchange the order of integration. When $x < 0$, $y > 0$, the function $t \mapsto e^{ixt + iy/t}$ extends to a bounded holomorphic function in the lower half-plane, and we may deform the contour to see that

$$\int_{\mathbb{R}} \frac{e^{ixt + iy/t}}{(t - \eta)^2} \, dt = 0, \qquad \eta \in \mathbb{H},$$

from which $u_{(n,0)}(x,y) = 0$ is immediate. On the other hand, when $x > 0$, $y > 0$, we have instead that

$$\int_{\mathbb{R}} \frac{e^{ixt + iy/t}}{(t - \eta)^2} \, dt = -2\pi (x - y\eta^{-2}) e^{ix\eta + iy/\eta}, \qquad \eta \in \mathbb{H},$$

by deforming the contour in the upper half-plane, which gives the corresponding expression for $u_{(n,0)}(x,y)$. $\qquad \square$

Proof of Corollary 3.3.3. The fact that the functions A_0 and A_n, B_n for $n \in \mathbb{Z}_{\neq 0}$ belong to the test function space $C_1^\infty(\mathbb{R} \cup \{\infty\})$ follows from Propositions 9.2.2 and 9.2.4 combined with the symmetry properties of Proposition 9.1.1. The growth control on the coefficient functions can be found in Proposition 9.2.1. The expansion of the Dirac point mass δ_x is a direct consequence of (9.1.2) if we take into account the correspondence between distributions on the extended real line and harmonic functions in $\mathbb{H}$ outlined in Subsection 3.4. $\qquad \square$

Proof of Corollary 3.3.5. Since Proposition 9.1.1 asserts that $\mathbf{J}_1 A_0 = A_0$ holds, while $\mathbf{J}_1 A_n = B_n$ and consequently $\mathbf{J}_1 B_n = A_n$ holds for $n \in \mathbb{Z}_{\neq 0}$, we may focus our attention to the assertions that

$$\langle A_0, e_m \rangle_{\mathbb{R}} = \delta_{m,0}, \qquad m \in \mathbb{Z},$$

while

$$\langle A_n, e_m \rangle_{\mathbb{R}} = \delta_{m,n}, \qquad m \in \mathbb{Z}, \, n \in \mathbb{Z}_{\neq 0},$$

and

$$\langle B_n, e_m \rangle_{\mathbb{R}} = 0, \qquad m \in \mathbb{Z}, \, n \in \mathbb{Z}_{\neq 0}.$$

We note that the quantities $\langle A_n, e_m \rangle_{\mathbb{R}}$ and $\langle B_n, e_m \rangle_{\mathbb{R}}$ are well-defined in the usual integral sense by the estimate in 3.3.3, and that we in fact have

$$(9.2.16) \qquad \langle A_n, e_m \rangle_{\mathbb{R}}, \, \langle B_n, e_m \rangle_{\mathbb{R}} = \mathrm{O}\big((|n| + 1)\log^2(2 + |n|)\big)$$

uniformly in n. Next, we observe that in the sense of distribution theory, we may represent

$$e_m(t) = \int_{\mathbb{R}} e_m(x)\,\delta_x(t)\,\mathrm{d}x,$$

so by the expansion of the Dirac point mass in Corollary 3.3.3 we have the hyperbolic Fourier series expansion

$$e_m(t) = \mathrm{e}^{\mathrm{i}\pi m t} = \langle A_0, e_m\rangle_{\mathbb{R}} + \sum_{n\in\mathbb{Z}_{\neq 0}} \left(\langle A_n, e_m\rangle_{\mathbb{R}}\,\mathrm{e}^{\mathrm{i}\pi n t} + \langle B_n, e_m\rangle_{\mathbb{R}}\,\mathrm{e}^{-\mathrm{i}\pi n/t}\right),$$

understood as an equality of distributions on the extended real line. In view of the modest growth of the coefficients established in (9.2.16), the uniqueness aspect of Theorem 3.3.1 allows us to equate the coefficients. After all, the left-hand side is a hyperbolic Fourier series too. This gives rise to all the claims regarding $\langle A_n, e_m\rangle_{\mathbb{R}}$ $\langle B_n, e_m\rangle_{\mathbb{R}}$. $\qquad\square$

Proof of Theorem 4.2.6. The assertion is an immediate consequence of the biorthogonality properties stated in Corollary 3.3.5. $\qquad\square$

Theorem 9.2.7. *The $L^1(\mathbb{R})$-norms of A_n, B_n obey the growth bound*

$$\int_{\mathbb{R}} |A_n(x)|\,\mathrm{d}x = \int_{\mathbb{R}} |B_n(x)|\,\mathrm{d}x = \mathrm{O}(\log^3 |n|)$$

as $|n| \to +\infty$.

Proof. The equality of norms holds for $n \neq 0$ by the change-of-variables formula in the integral if we take into account the symmetry property of Proposition 9.1.1. Next, by the argument employed in the proof of Corollary 3.3.5, we have that the coefficients of the hyperbolic Fourier series expansion of a bounded measurable function $g \in L^\infty(\mathbb{R})$ are given by

$$a_n(g) = \langle A_n, g\rangle_{\mathbb{R}} = \int_{\mathbb{R}} A_n(x)g(x)\,\mathrm{d}x, \quad b_n(g) = \langle B_n, g\rangle_{\mathbb{R}} = \int_{\mathbb{R}} B_n(x)g(x)\,\mathrm{d}x.$$

We now estimate the $L^1(\mathbb{R})$-norm of A_n, since the $L^1(\mathbb{R})$-norm of B_n is just the same. In view of Corollary 8.6.2, we have the growth bound

$$a_n(g) = \mathrm{O}(\log^3 |n|) \quad \text{as } |n| \to +\infty,$$

and the estimate is uniform in the unit ball of $L^\infty(\mathbb{R})$, by Remark 8.6.3. So, by choosing $g = g_n$ with $|g_n| \leq 1$ and $A_n g_n = |A_n|$, the assertion is immediate. $\qquad\square$

Proof of Theorem 4.2.8. The assertion is an immediate consequence of Corollary 3.3.5 and the estimate of Theorem 9.2.7. $\qquad\square$

9.3. **Summation properties of the biorthogonal system.** We may now identify the periodizations of the functions A_n and B_n.

Theorem 9.3.1. *The function A_0 has the symmetry $A_0(x) = x^{-2}A_0(-1/x)$ and the summation property*

$$\sum_{j\in\mathbb{Z}} A_0(x+2j) = \frac{1}{2}, \qquad x \in \mathbb{R}.$$

Moreover, for each $n \in \mathbb{Z}_{\neq 0}$, we have the symmetry property

$$B_n(x) = x^{-2} A_n(-1/x), \qquad x \in \mathbb{R}_{\neq 0},$$

and the summation properties

$$\sum_{j \in \mathbb{Z}} A_n(x + 2j) = \frac{1}{2} e^{-i\pi nx}, \quad \sum_{j \in \mathbb{Z}} B_n(x + 2j) = 0,$$

hold for each $x \in \mathbb{R}$.

Proof. The symmetry properties were established in Proposition 9.1.1. By Corollary 3.3.3, the functions A_n, B_n are C^∞-smooth and satisfy the estimate

$$|A_0(x)| \leq \frac{C_0}{1 + x^2}, \qquad x \in \mathbb{R},$$

and for $n \in \mathbb{Z}_{\neq 0}$, the further estimates

$$|A_n(x)|, |B_n(x)| \leq \frac{C_n}{1 + x^2}, \qquad x \in \mathbb{R},$$

for a sequence of positive constants C_n with

$$C_n = \mathrm{O}\big((|n| + 1) \log^2(|n| + 2)\big)$$

as $|n| \to +\infty$. Since n is fixed here, we need not worry about the growth of C_n. It is now easy to see that the two periodization series

$$\mathbf{\Sigma}[A_n](x) := \sum_{j \in \mathbb{Z}} A_n(x + 2j), \quad \mathbf{\Sigma}[B_n](x) = \sum_{j \in \mathbb{Z}} B_n(x + 2j)$$

converge absolutely and uniformly on $\mathbb{R}$, and the two sums $\mathbf{\Sigma}[A_n](x)$ and $\mathbf{\Sigma}[B_n](x)$ are then 2-periodic and continuous. Moreover, we calculate that for $m, n \in \mathbb{Z}$,

$$\int_{[-1,1]} \mathbf{\Sigma}[A_n](x) e^{i\pi mx} dx = \int_{[-1,1]} \sum_{j \in \mathbb{Z}} A_n(x + 2j) e^{i\pi m(x + 2j)} dx$$

$$= \int_{\mathbb{R}} A_n(x) e^{i\pi mx} dx = \delta_{m,n}$$

where the last delta is in Kronecker's sense, and the right-most equality relies on Corollary 3.3.5. Similarly, for $n \in \mathbb{Z}_{\neq 0}$, we obtain from Corollary 3.3.5 that for $m \in \mathbb{Z}$,

$$\int_{[-1,1]} \mathbf{\Sigma}[B_n](x) e^{i\pi mx} dx = \int_{[-1,1]} \sum_{j \in \mathbb{Z}} B_n(x + 2j) e^{i\pi m(x + 2j)} dx$$

$$= \int_{\mathbb{R}} B_n(x) e^{i\pi mx} dx = 0.$$

It is now an immediate consequence of the completeness of the Fourier system that $\mathbf{\Sigma}[B_n](x) = 0$ holds almost everywhere, and then, by continuity, we conclude that

$\Sigma[B_n](x) = 0$ holds everywhere. Similarly, to identify $\Sigma[A_n]$, we consider the 2-periodic function

$$\Psi_n(x) = \Sigma[A_n](x) - \frac{1}{2} e^{-i\pi nx},$$

and see all its Fourier coefficients vanish. Consequently, then, Ψ_n vanishes almost everywhere, and hence everywhere, by continuity. $\qquad\square$

It is a natural question to ask if it is possible to express the properties of Theorem 9.3.1 in a way that produces more straightforward equations. One such approach is explored below.

Corollary 9.3.2. *For $n \in \mathbb{Z}_{\neq 0}$, we put*

$$A_n^+(x) := A_n(x) + B_n(x), \quad A_n^-(x) := A_n(x) - B_n(x).$$

These functions have the symmetries

$$A_n^+(x) = x^{-2} A_n^+(-1/x), \quad A_n^-(x) = -x^{-2} A_n^-(x),$$

and the summation properties

$$\sum_{j \in \mathbb{Z}} A_n^+(x+2j) = \frac{1}{2} e^{-i\pi nx}, \quad \sum_{j \in \mathbb{Z}} A_n^-(x+2j) = \frac{1}{2} e^{-i\pi nx}.$$

hold for all $x \in \mathbb{R}$.

Proof. The assertions are immediate consequences of Theorem 9.3.1. $\qquad\square$

We recall the notation

$$\mathbf{T}_0 f(x) = \sum_{j \in \mathbb{Z}_{\neq 0}} \frac{1}{(2j-x)^2} f\left(\frac{1}{2j-x}\right)$$

from (6.1.6).

Corollary 9.3.3. *For $n \in \mathbb{Z}_{\neq 0}$, the functions A_n^+ and A_n^- solve the equations*

$$(\mathbf{I} + \mathbf{T}_0)[A_n^+](x) = \frac{1}{2} e^{-i\pi nx}, \quad (\mathbf{I} - \mathbf{T}_0)[A_n^-](x) = \frac{1}{2} e^{-i\pi nx},$$

for all $x \in \mathbb{R} \setminus 2\mathbb{Z}$, where $\mathbf{I}$ denotes the identity operator.

Proof. By the symmetry and summation property of A_n^+ stated in Corollary 9.3.2,

$$\frac{1}{2} e^{-i\pi nx} = \sum_{j \in \mathbb{Z}} A_n^+(x-2j) = \sum_{j \in \mathbb{Z}} \frac{1}{(2j-x)^2} A_n^+\left(\frac{1}{2j-x}\right)$$
$$= x^{-2} A_n^+(-1/x) + \mathbf{T}_0 A_n^+(x) = A_n^+(x) + \mathbf{T}_0 A_n^+(x),$$

as claimed. This establishes the equation for A_n^+, and the analogous argument applied to A_n^- gives the remaining equation. $\qquad\square$

10. SKEWED HYPERBOLIC FOURIER SERIES

10.1. **Power skewed hyperbolic Fourier series.** It is well-known that the theta function ϑ_{00} is zero-free in the upper half-plane:

$$\vartheta_{00}(\tau) = \sum_{n \in \mathbb{Z}} e^{i\pi n^2 \tau} \neq 0, \qquad \tau \in \mathbb{H}.$$

Moreover, along the imaginary semi-axis $i\mathbb{R}_{>0}$, we have that

$$\vartheta_{00}(iy) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 y} = 1 + \sum_{n \in \mathbb{Z}_{\neq 0}} e^{-\pi n^2 y} > 1.$$

From these two properties, it follows that there exists a unique holomorphic function $L_{00}(\tau)$ such that

$$\exp(L_{00}(\tau)) = \vartheta_{00}(\tau), \qquad \tau \in \mathbb{H},$$

while at the same time $L_{00}(iy) > 0$ for $y \in i\mathbb{R}_{>0}$. This function has limit $L_{00}(+i\infty) = 0$, so that $L_{00}(\tau) = \log \theta_{00}(\tau)$ with the principal branch of the logarithm, provided that $\mathrm{Im}\,\tau$ is big enough. But then L_{00} is 2-periodic far up, and consequently it must be 2-periodic throughout:

$$L_{00}(\tau + 2) = L_{00}(\tau), \qquad \tau \in \mathbb{H}.$$

The functional equation

$$\vartheta_{00}(\tau) = (\tau/i)^{-1/2} \vartheta_{00}(-1/\tau)$$

tells us that

$$L_{00}(\tau) = -\frac{1}{2} \log \frac{\tau}{i} + L_{00}(-1/\tau), \qquad \tau \in \mathbb{H},$$

for some branch of the logarithm. By inspection of the two sides, we realize that the logarithm $\log(\tau/i)$ must be holomorphic in $\mathbb{H}$, and that should be real-valued along $i\mathbb{R}_{>0}$. The principal branch of the logarithm has these properties, and there is no other possibility. This permits us to form powers

$$(\vartheta_{00}(\tau))^{2\beta} = \exp(2\beta\, L_{00}(\tau))$$

and the functional relation becomes

$$(10.1.1) \quad (\vartheta_{00}(\tau))^{2\beta} = \exp(2\beta\, L_{00}(\tau)) = \exp\left(2\beta\left(-\frac{1}{2}\log\frac{\tau}{i} + L_{00}(-1/\tau)\right)\right)$$
$$= \exp\left(-\beta \log\frac{\tau}{i}\right) \exp\left(2\beta\, L_{00}(-1/\tau)\right) = (\tau/i)^{-\beta}(\vartheta_{00}(-1/\tau))^{2\beta},$$

Proof of Proposition 3.9.1. The result is an immediate consequence of the functional relationship (10.1.1). $\qquad\square$

10.2. **Exponentially skewed hyperbolic Fourier series.** Since the modular lambda function has the form

$$(10.2.1) \qquad \lambda(\tau) = \frac{\vartheta_{10}(\tau)^4}{\vartheta_{00}(\tau)^4}, \qquad \tau \in \mathbb{H},$$

and both ϑ_{00} and ϑ_{10} lack zeros in $\mathbb{H}$, $\lambda(\tau) \neq 0$ holds for all $\tau \in \mathbb{H}$. Moreover, since λ has the functional identity

$$(10.2.2) \qquad \lambda(-1/\tau) = 1 - \lambda(\tau),$$

it follows that $\lambda(\tau) \neq 1$ as well for $\tau \in \mathbb{H}$. Taking logarithms in (10.2.1), we obtain that

$$\log \lambda(\tau) = 4 \log \vartheta_{10}(\tau) - 4 \log \vartheta_{00}(\tau), \qquad \tau \in \mathbb{H},$$

where the right-hand side can be thought of as defining the left-hand side. On the right-hand side, the choice of one of the logarithms is the function $\log \vartheta_{00} = L_{00}$ discussed in Subsection 10.1. As for the other logarithm, we know that

$$\vartheta_{10}(\mathrm{i}y) = \sum_{n \in \mathbb{Z}} e^{-\pi(n+\frac{1}{2})^2 y} > 2 e^{-\frac{1}{4}\pi y} > 0, \qquad y \in \mathbb{R}_{>0},$$

so there is a unique holomorphic logarithm function $L_{10}(\tau)$ with $\exp(L_{10}(\tau)) = \vartheta_{10}(\tau)$, which is in addition real-valued along $\mathrm{i}\mathbb{R}_{>0}$. Since we may write ϑ_{10} in the form

$$\vartheta_{10}(\tau) = \sum_{n \in \mathbb{Z}} e^{\mathrm{i}\pi(n+\frac{1}{2})^2 \tau} = 2 e^{\mathrm{i}\pi\tau/4} \left(1 + \sum_{n=1}^{+\infty} e^{\mathrm{i}\pi n(n+1)\tau}\right),$$

we see that

$$L_{10}(\tau) = \log 2 + \frac{\mathrm{i}\pi\tau}{4} + \log\left(1 + \sum_{n=1}^{+\infty} e^{\mathrm{i}\pi n(n+1)\tau}\right).$$

For $\mathrm{Im}\,\tau \gg 1$, Taylor's formula shows the function

$$\tilde{L}_{10}(\tau) := \log\left(1 + \sum_{n=1}^{+\infty} e^{\mathrm{i}\pi n(n+1)\tau}\right)$$

can be expanded in a convergent power series in $e^{\mathrm{i}2\pi\tau}$, and this property then extends to all of $\mathbb{H}$. It follows that

$$L_{10}(\tau) = \log 2 + \frac{\mathrm{i}\pi\tau}{4} + \tilde{L}_{10}(\tau),$$

so that in particular, L_{10} has the functional property

$$L_{10}(\tau+1) = L_{10}(\tau) + \frac{\mathrm{i}\pi}{4}.$$

Note that $\tilde{L}_{10}(\tau)$ is holomorphic in $\mathbb{H}$, and has period 1, that is, $\tilde{L}_{10}(\tau+1) = \tilde{L}_{10}(\tau)$ holds. In addition, $\tilde{L}_{10}(\mathrm{i}y) > 0$ holds for all $y \in \mathbb{R}_{>0}$. In terms of these functions, we use as logarithm

$$(10.2.3) \qquad \log \lambda(\tau) = \Lambda(\tau) := 4 \log 2 + \mathrm{i}\pi\tau + 4\tilde{L}_{10}(\tau) - 4 L_{00}(\tau), \qquad \tau \in \mathbb{H}.$$

The periodicity property of the logarithm $\Lambda(\tau)$ we read off from (10.2.3) is that

$$(10.2.4) \qquad\qquad \Lambda(\tau+2) = \Lambda(\tau) + \mathrm{i}2\pi,$$

while the functional identity (10.2.2) takes the form

$$\exp(\Lambda(-1/\tau)) = 1 - \exp(\Lambda(\tau)).$$

The fact that $\lambda(\tau) \neq 1$ corresponds to having

$$\Lambda(\tau) \in \mathbb{C} \setminus \mathrm{i}2\pi\mathbb{Z}, \qquad \tau \in \mathbb{H}.$$

For $\operatorname{Im}\tau \gg 1$, we can define a holomorphic logarithm of $\lambda(-1/\tau)$ via

$$(10.2.5) \qquad \log\lambda(-1/\tau) = \log(1 - \lambda(\tau)) = -\sum_{k=1}^{+\infty} \frac{1}{k}\lambda(\tau)^k,$$

which takes real values along the imaginary semi-axis $\mathrm{i}\mathbb{R}_{>0}$. It must then necessarily coincide with the function $\Lambda(\tau)$ defined by (10.2.3), and since the right-hand side of (10.2.5) is 2-periodic, we find that

$$(10.2.6) \qquad \Lambda(-1/\tau) = -\sum_{k=1}^{+\infty} \frac{1}{k}\lambda(\tau)^k, \qquad \operatorname{Im}\tau \gg 1,$$

and that in addition to (10.2.4), Λ enjoys the periodicity property

$$\Lambda\left(-\frac{1}{\tau+2}\right) = \Lambda\left(-\frac{1}{\tau}\right), \qquad \tau \in \mathbb{H}.$$

For $\omega \in \mathbb{R}$, we use the logarithm to define the power

$$\lambda(\tau)^\omega = \exp(\omega\Lambda(\tau)), \qquad \tau \in \mathbb{H},$$

and then, according to (10.2.6),

$$\lambda(-1/\tau)^\omega = \exp(\omega\Lambda(-1/\tau)) = \exp(\omega\log(1 - \lambda(\tau)))$$
$$= (1 - \lambda(\tau))^\omega, \qquad \tau \in \mathbb{H},$$

where the right-hand side may be expanded first as a binomial series in $\lambda(\tau)$, and hence as a power series in the nome $q = \mathrm{e}^{\mathrm{i}\pi\tau}$. On the other hand, we also have

$$(10.2.7) \quad \lambda(\tau)^\omega = \exp(\omega\Lambda(\tau)) = 16^\omega\, \mathrm{e}^{\mathrm{i}\pi\omega\tau} \exp\left(4\omega(\tilde{L}_{10}(\tau) - L_{00}(\tau))\right), \qquad \tau \in \mathbb{H}.$$

Proof of Proposition 3.10.1. We see from (10.2.7) that

$$\mathrm{e}^{-\mathrm{i}\pi\omega\tau}\lambda(\tau)^\omega = 16^\omega \exp\left(4\omega(\tilde{L}_{10}(\tau) - L_{00}(\tau))\right), \qquad \tau \in \mathbb{H},$$

which we may express as convergent Taylor series in the nome $q = \mathrm{e}^{\mathrm{i}\pi\tau}$. As we apply this formula to $\omega = \omega_j$, for $j = 1,2$, with τ possibly replaced by $-1/\tau$, the correspondence between the holomorphic hyperbolic Fourier series and the exponentially skewed holomorphic hyperbolic Fourier series follows. $\qquad\square$

11. RELATION WITH RADIAL FOURIER INTERPOLATION

11.1. **Heat and Schrödinger evolution.** The *heat equation* in $\mathbb{R}^d$, where $d \in \mathbb{Z}_{>0}$, reads

$$\partial_s u(x,s) = \Delta_x u(x,s), \tag{11.1.1}$$

where $s \in \mathbb{R}$ is the temporal coordinate while $x \in \mathbb{R}^d$ is the spacial coordinate. The Laplacian is defined in the standard fashion:

$$\Delta_x := \partial_{x_1}^2 + \cdots + \partial_{x_d}^2, \qquad x = (x_1, \ldots, x_d) \in \mathbb{R}^d.$$

If we endow the heat equation with an initial datum (at time $s = 0$)

$$u(x,0) = \varphi(x), \qquad x \in \mathbb{R}^d,$$

the corresponding solution $u(x,s)$ may be written formally as

$$u(\cdot,s) = e^{s\Delta}\varphi. \tag{11.1.2}$$

For $s > 0$, it may be expressed in integral form

$$u(x,s) = \int_{\mathbb{R}^d} H_d(x - y, s)\, \varphi(y)\, \mathrm{dvol}_d(y), \tag{11.1.3}$$

where vol_d denotes volume measure in $\mathbb{R}^d$, and $H_d(x,s)$ denotes the *heat kernel*

$$H_d(x,s) := (4\pi s)^{-d/2} \exp\left(-\frac{|x|^2}{4s}\right). \tag{11.1.4}$$

We think of (11.1.2) and (11.1.3) as expressing the same solution, with some caveats. The solution (11.1.2) is well-defined for $s = 0$, since $\exp(0)$ is interpreted as the identity operator, whereas the integral (11.1.3) is undefined for $s = 0$. Moreover, for $s < 0$ the operator $\exp(s\Delta)$ would make sense as the densely defined operator inverse of $\exp(-s\Delta)$, while the integral (11.1.3) does not make sense for any $s < 0$ given the singularity of the kernel $H(x,s)$ for $s < 0$.

Some comments are in order here. For the solution given by (11.1.3) to be well-defined, the integral must of course make sense. Moreover, it is a natural question to ask whether the solution given by (11.1.3) for $s > 0$ is unique. As a matter of fact, this is not the case in general. If we consider the homogeneous initial boundary value problem

$$\partial_s v(x,s) = \Delta_x v(x,s), \quad v(x,0) = 0$$

for time $s \geq 0$, the solution is $v = 0$ provided that the following rather weak growth assumption is made:

$$\forall \varepsilon > 0: \ v(x,s) = \mathrm{O}\big(\exp(\varepsilon|x|^2)\big)$$

holds uniformly over x and $0 \leq s \leq s_0$ for each $s_0 > 0$. For the necessary details in dimension $d = 1$, see, e.g., [7], while [28] covers $d > 1$ as well.

We may extend the sense of the heat kernel $H_d(x,s)$ to complex values of s using the same formula (11.1.4). For odd dimensions, this would involve a choice of the

power $(4\pi s)^{-d/2}$, which we do by using the principal branch of the logarithm, which a branch cut along the negative real axis. The modulus of the heat kernel is

$$|H_d(x,s)| = (4\pi|s|)^{-d/2}\exp\left(-\operatorname{Re}\frac{|x|^2}{4s}\right) = (4\pi|s|)^{-d/2}\exp\left(-\frac{|x|^2\operatorname{Re}s}{4|s|^2}\right),$$

which has a nasty singularity when $\operatorname{Re}s < 0$, but has Gaussian decay for $\operatorname{Re}s > 0$. The imaginary axis is clearly special:

$$|H_d(x,s)| = (4\pi|s|)^{-d/2}, \qquad \operatorname{Re}s = 0.$$

This means that the heat evolution (11.1.2) makes sense in terms of (11.1.3) for $\operatorname{Re}s > 0$.

Given that we like to work with the upper half-plane $\mathbb{H}$ in place of the right half-plane, we apply the coordinate change $s\tau = \mathrm{i}s$, since then $\operatorname{Re}s > 0$ holds if and only if $\operatorname{Im}\tau > 0$, i.e. if $\tau \in \mathbb{H}$. If we agree to write $\tilde{u}(x,\tau) = u(x,\tau/\mathrm{i})$, the heat equation (11.1.1) becomes

$$(11.1.5) \qquad\qquad \mathrm{i}\partial_\tau\tilde{u}(x,\tau) = \Delta_x\tilde{u}(x,\tau),$$

which we recognize as the *Schrödinger equation* with trivial potential. The solution to the Schrödinger equation (11.1.5) with inital datum

$$\tilde{u}(x,0) = \varphi(x), \qquad x \in \mathbb{R}^d,$$

is given by the analogues of (11.1.2) and (11.1.3),

$$\tilde{u}(\cdot,\tau) = \mathrm{e}^{-\mathrm{i}\tau\Delta}\varphi$$

and, more explicitly,

$$(11.1.6) \qquad\qquad \tilde{u}(x,\tau) = \int_{\mathbb{R}^d} H_d(x-y,\tau/\mathrm{i})\,\varphi(y)\,\mathrm{dvol}_d(y),$$

for $\tau \in \mathbb{H}$.

Definition 11.1.1. The *Schrödinger transform* of the function φ is given by

$$\mathfrak{S}\varphi(\tau) = \tilde{u}(0,\tau) = \left(\mathrm{e}^{-\mathrm{i}\tau\Delta_x}\right)(0) = \int_{\mathbb{R}^d} H_d(y,\tau/\mathrm{i})\,\varphi(y)\,\mathrm{dvol}_d(y)$$

$$= (4\pi\tau/\mathrm{i})^{-d/2}\int_{\mathbb{R}^d}\exp\left(-\mathrm{i}\frac{|y|^2}{4\tau}\right)\varphi(y)\,\mathrm{dvol}_d(y), \qquad \tau \in \mathbb{H}.$$

Remark 11.1.2. (a) A sufficient condition on $\varphi : \mathbb{R}^d \to \mathbb{C}$ for the Schrödinger transform $\mathfrak{S}\varphi$ to be well-defined as a function on $\mathbb{H}$ is that φ is Lebesgue measurable, and that

$$\forall\varepsilon > 0: \int_{\mathbb{R}^d}|\varphi(x)|\mathrm{e}^{-\varepsilon|x|^2}\mathrm{dvol}_d(x) < +\infty.$$

(b) Given the shape of the heat kernel (11.1.4), the Schrödinger transform only takes into account the radial part φ_{rad} of φ, that is, $\mathfrak{S}\varphi = \mathfrak{S}\varphi_{\mathrm{rad}}$ holds. Here, φ_{rad} is the unique radial function whose integrals along spherical shells coincide with those of φ.

For a more restrictive collection of functions φ the Schrödinger transform extends continuously to $\mathbb{R}_{\neq 0}$, and the formula gives $\mathfrak{S}\varphi(\tau)$ also for $\tau \in \mathbb{R}_{\neq 0}$. This is the case if $\varphi \in L^1(\mathbb{R}^d)$. Indeed, for $\tau \in \mathbb{H} \cup \mathbb{R}_{\neq 0}$, we have the estimate

$$|\mathfrak{S}\varphi(\tau)| \leq \int_{\mathbb{R}^d} |H_d(y, \tau/\mathrm{i})| \, |\varphi(y)| \, \mathrm{dvol}_d(y)$$

$$= (4\pi|\tau|)^{-d/2} \int_{\mathbb{R}^d} \exp\left(-\frac{|y|^2 \mathrm{Im}\tau}{4|\tau|^2}\right) |\varphi(y)| \, \mathrm{dvol}_d(y)$$

$$\leq (4\pi|\tau|)^{-d/2} \int_{\mathbb{R}^d} |\varphi(y)| \, \mathrm{dvol}_d(y),$$

and an approximation argument gives the continuity. Next, if $\varphi \in L^1(\mathbb{R}^d)$ and $\psi \in L^1(\mathbb{R}, \mu_d)$, where $\mathrm{d}\mu_d(t) := |t|^{-d/2}\mathrm{d}t$, then, by Fubini's theorem,

$$\int_{\mathbb{R}} \mathfrak{S}\varphi(t)\psi(t) \, \mathrm{d}t = \int_{\mathbb{R}} \int_{\mathbb{R}^d} H_d(y, t/\mathrm{i})\psi(t)\,\varphi(y) \, \mathrm{dvol}_d(y) \, \mathrm{d}t$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}} (4\pi t/\mathrm{i})^{-d/2} \exp\left(-\mathrm{i}\frac{|y|^2}{4t}\right) \psi(t) \, \mathrm{d}t \, \varphi(y) \, \mathrm{dvol}_d(y)$$

$$= \int_{\mathbb{R}^d} \mathfrak{S}^*\psi(y)\,\varphi(y) \, \mathrm{dvol}_d(y),$$

where we write

$$(11.1.7) \quad \mathfrak{S}^*\psi(y) := \int_{\mathbb{R}} (4\pi t/\mathrm{i})^{-d/2} \exp\left(-\mathrm{i}\frac{|y|^2}{4t}\right) \psi(t) \, \mathrm{d}t$$

$$= \int_{\mathbb{R}} (\mathrm{i}4\pi/t)^{-d/2} \exp\left(\mathrm{i}\frac{|y|^2 t}{4}\right) \mathbf{J}_1\psi(t) \, \mathrm{d}t, \qquad y \in \mathbb{R}^d,$$

which is a radial expression. The involutive operator $\mathbf{J}_1$ is as before given by equation (3.2.1). The expression (11.1.7) is of Fourier type, essentially the Fourier transform of $(t/\mathrm{i})^{d/2}\mathbf{J}_1\psi(t)$.

11.2. The Schrödinger transform of complex exponential waves and point masses.
If $\varphi = \delta_\xi$, the unit point mass at a point $\xi \in \mathbb{R}^d$, we see from the definition that the associated Schrödinger transform is

$$(11.2.1) \qquad \mathfrak{S}\delta_\xi(\tau) = H_d(\xi, \tau/\mathrm{i}) = (4\pi\tau/\mathrm{i})^{-d/2} \exp\left(-\mathrm{i}\frac{|\xi|^2}{4\tau}\right), \qquad \tau \in \mathbb{H},$$

which makes sense also if $\tau \in \mathbb{R}_{\neq 0}$. We shall also need the Schrödinger transform of a complex exponential wave

$$e_\eta(x) := \mathrm{e}^{\mathrm{i}\langle x, \eta\rangle},$$

where we write

$$\langle x, \eta\rangle = x_1\eta_1 + \ldots + x_d\eta_d, \qquad x, \eta \in \mathbb{R}^d,$$

and use the notational convention $x = (x_1, \ldots, x_d)$, $\eta = (\eta_1, \ldots, \eta_d)$. The function

$$\tilde{u}(x, \tau) = \exp\left(\mathrm{i}\langle x, \eta\rangle + \mathrm{i}|\eta|^2\tau\right)$$

is bounded in $\tau \in \mathbb{H} \cup \mathbb{R}$, and has initial datum $\tilde{u}(x,0) = e_\eta(x)$ for $x \in \mathbb{R}^d$. It moreover solves the Schrödinger equation (11.1.5), so that by the uniqueness theorem for the heat equation, this must be the same as the function given by (11.1.6). Consequently, the Schrödinger transform of the wave e_η is given by

$$(11.2.2) \qquad \mathfrak{S}e_\eta(\tau) = \tilde{u}(0,\tau) = \exp(\mathrm{i}|\eta|^2\tau), \qquad \tau \in \mathbb{H} \cup \mathbb{R}.$$

In view of (11.2.1) and (11.2.2), we recognize the basis functions of a power skewed hyperbolic Fourier series, provided that

$$\frac{|\eta|^2}{\pi} \in \mathbb{Z}_{\geq 0}, \qquad \frac{|\xi|^2}{4\pi} \in \mathbb{Z}_{>0}.$$

11.3. **The Fourier transform on radial functions.** We write

$$\widehat{f}_d(y) := \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}2\pi\langle x,y\rangle} f(x)\,\mathrm{dvol}_d(x), \qquad y \in \mathbb{R}^d,$$

for the d-dimensional Fourier transform. When f is radial, we may think of the function as defined on $\mathbb{R}_{\geq 0}$ via the identification $f(|x|) = f(x)$ for $x \in \mathbb{R}$, and then its Fourier transform $\widehat{f}_d$ is radial too, and given by the formula

$$\widehat{f}_d(s) = 2\pi s^{1-\frac{d}{2}} \int_0^{+\infty} J_{\frac{d}{2}-1}(2\pi st)\, f(t)\, t^{d/2}\mathrm{d}t, \qquad s \in \mathbb{R}_{\geq 0},$$

extended in the same fashion to $\mathbb{R}^d$ via $\widehat{f}_d(y) = \widehat{f}_d(|y|)$. Here, J_ν is the familiar Bessel function,

$$J_\nu(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n+\nu}.$$

This means that the radialization in $x \in \mathbb{R}^d$ of the complex exponential wave e_η is given by

$$(e_\eta)_{\mathrm{rad}}(x) = 2^{\frac{d}{2}-1}\Gamma(d/2)\,|x|^{1-\frac{d}{2}}|\eta|^{1-\frac{d}{2}} J_{\frac{d}{2}-1}(|x|\,|\eta|).$$

11.4. **Skewed hyperbolic Fourier series expansion of a Schrödinger transform.**
We begin with a measurable radial function $f : \mathbb{R}^d \to \mathbb{C}$, such that its Schrödinger transform

$$F(\tau) := \mathfrak{S}f(\tau), \qquad \tau \in \mathbb{H},$$

is a well-defined holomorphic function (see Remark 11.1.2). As such, it may be represented by a power skewed holomorphic hyperbolic Fourier series

$$(11.4.1) \qquad F(\tau) = a_0 + \sum_{n=1}^{+\infty} \left(a_n\, \mathrm{e}^{\mathrm{i}\pi n\tau} + b_n\,(\tau/\mathrm{i})^{-d/2}\mathrm{e}^{-\mathrm{i}\pi n/\tau}\right), \qquad \tau \in \mathbb{H}.$$

This fact can be seen by combining the holomorphic version of Theorem 3.6.4 with Proposition 3.9.1 (which connects power skewed hyperbolic Fourier series with ordinary hyperbolic Fourier series). Depending on the growth control on $F(\tau)$, it may be that we do not have uniqueness in the representation (11.4.1) (this issue is basically

governed by Theorem 3.7.3). If we have non-uniqueness even for slowly growing F, that is, for F with the estimate

$$(11.4.2) \qquad |F(\tau)| = \mathrm{O}\Big(\frac{1+|\tau|^2}{\mathrm{Im}\,\tau}\Big)^k,$$

for some finite k, the solution is to throw away some terms in the series until we get unique coefficients. Instead of just throwing these terms away we can say that we declare some of the coefficients to vanish for each such slowly growing F (alternatively, some finite linear combinations of coefficients would vanish). With such a choice of coefficients, we can get a unique representation of the form (11.4.1), where $a_n = a_n(F)$ and $b_n = b_n(F)$ are linear functionals. Suppose we are able to control the growth of the coefficients like

$$a_n(F), b_n(F) = \mathrm{O}(|n|+1)^{k'}$$

for some $k' = k'(k)$ for each F with the growth bound (11.4.2), and that this holds for all finite k, then it can be shown that there exist test functions $A_{n,d}$ and $B_{n,d}$ in the class $C_1^\infty(\mathbb{R} \cup \{\infty\})$, such that

$$a_n(F) = \langle A_{n,d}, F \rangle_{\mathbb{R}}, \quad b_n(F) = \langle B_{n,d}, F \rangle_{\mathbb{R}},$$

where F is understood a a distribution on the extended real line $\mathbb{R} \cup \{\infty\}$. Next, we pick points $\xi^{\langle n \rangle}, \eta^{\langle n \rangle} \in \mathbb{R}^d$ such that $|\xi^{\langle n \rangle}|^2 = 4\pi n$ and $|\eta^{\langle n \rangle}|^2 = \pi n$ and write the representation (11.4.1) in the form

$$\begin{aligned} \mathfrak{S}f(\tau) = F(\tau) &= \langle A_{0,d}, F \rangle_{\mathbb{R}} \\ &+ \sum_{n=1}^{+\infty} \big(\langle A_{n,d}, F \rangle_{\mathbb{R}} \, \mathfrak{S}e_{\eta_n}(\tau) + \langle B_{n,d}, F \rangle_{\mathbb{R}} (4\pi)^{d/2} \mathfrak{S}\delta_{\xi_n}(\tau) \big), \qquad \tau \in \mathbb{H}. \end{aligned}$$

We can remove the Schrödinger operator on both sides provided we have projected to the radial functions:

$$f(x) = \langle A_{0,d}, F \rangle_{\mathbb{R}} + \sum_{n=1}^{+\infty} \big(\langle A_{n,d}, F \rangle_{\mathbb{R}} (e_{\eta_n})_{\mathrm{rad}}(x) + \langle B_{n,d}, F \rangle_{\mathbb{R}} (4\pi)^{d/2} (\delta_{\xi_n})_{\mathrm{rad}}(x) \big),$$

where the identity is understood in the sense of distribution theory on $\mathbb{R}^d$. If g is a C^∞-smooth radial function with compact support, this would mean that

$$
\int_{\mathbb{R}^d} f(x)\,g(x)\,\mathrm{dvol}_d(x) = \langle A_{0,d}, F \rangle_{\mathbb{R}}
$$

$$
+ \sum_{n=1}^{+\infty} \left(\langle A_{n,d}, F \rangle_{\mathbb{R}} \int_{\mathbb{R}^d} e_{\eta_n}(x)\,g(x)\,\mathrm{dvol}_d(x) + \langle B_{n,d}, F \rangle_{\mathbb{R}}\,(4\pi)^{d/2} g(\xi_n) \right)
$$

$$
= \langle A_{0,d}, \mathfrak{S}f \rangle_{\mathbb{R}} + \sum_{n=1}^{+\infty} \left(\langle A_{n,d}, \mathfrak{S}f \rangle_{\mathbb{R}}\,\widehat{g}_d(-\eta_n/2\pi) \right.
$$

$$
\left. + \langle B_{n,d}, \mathfrak{S}f \rangle_{\mathbb{R}}\,(4\pi)^{d/2} g(\xi_n) \right)
$$

$$
= \langle \mathfrak{S}^* A_{0,d}, f \rangle_{\mathbb{R}^d} + \sum_{n=1}^{+\infty} \left(\langle \mathfrak{S}^* A_{n,d}, f \rangle_{\mathbb{R}^d}\,\widehat{g}_d(\tfrac{1}{2}\sqrt{n/\pi}) \right.
$$

$$
\left. + \langle \mathfrak{S}^* B_{n,d}, f \rangle_{\mathbb{R}^d}\,(4\pi)^{d/2} g(2\sqrt{\pi n}) \right).
$$

Finally, we change our perspective and think of f as a radial test function, and realize that since the functions $\mathfrak{S}^* A_{n,d}$ and $\mathfrak{S}^* B_{n,d}$ are all radial, we must have the Fourier interpolation formula

$$
g(x) = \mathfrak{S}^* A_{0,d} + \sum_{n=1}^{+\infty} \left(\widehat{g}_d(\tfrac{1}{2}\sqrt{n/\pi})\,\mathfrak{S}^* A_{n,d}(x) + (4\pi)^{d/2} g(2\sqrt{\pi n})\,\mathfrak{S}^* B_{n,d}(x) \right),
$$

for $x \in \mathbb{R}^d$, understood in the sense of distribution theory.

Remark 11.4.1. We should mention that Martin Stoller considers more general Fourier interpolation formulæ for non-radial Schwartzian test functions in [23], [24].

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