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On Hörmander's solution of the $\bar{\partial}$ -equation. I

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Abstract We explain how Hörmander's classical solution of the $\bar{\partial}$ -equation in the plane with a weight which permits growth near infinity carries over to the rather opposite situation when we ask for decay near infinity. Here, however, a natural condition on the datum needs to be imposed. The condition is not only natural but also necessary to have the result at least in the Fock weight case. The norm identity which leads to the estimate is related to general area-type results in the theory of conformal mappings.

Keywords $\bar{\partial}$ -Equation · Weighted estimates $\bar{\partial}$ -equation · Hörmander's theorem · Growing weight · Existence · Uniqueness of the solution

Mathematics Subject Classification 35A05 · 32W05 · 30H20

1 Introduction

1.1 Basic notation

Let

$$\Delta := \frac{1}{4} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right), \quad dA(z) := dx dy,$$

denote the normalized Laplacian and the area element, respectively. Here, $z = x + iy$ is the standard decomposition into real and imaginary parts. We let \mathbb{C} denote the complex plane.

In memory of Lars Hörmander.

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We also need the standard complex differential operators

$$\bar{\partial}_z := \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), \quad \partial_z := \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right),$$

so that Δ factors as $\Delta = \partial_z \bar{\partial}_z$. We sometimes drop indication of the differentiation variable z .

1.2 Hörmander's solution of the $\bar{\partial}$ -problem

We present Hörmander's theorem [12, 13] in the simplest possible case, when the domain is the entire complex plane and the weight $\phi : \mathbb{C} \rightarrow \mathbb{R}$ is C^2 -smooth with $\Delta\phi > 0$ everywhere.

Theorem 1.1 (Hörmander) *If the complex-valued function f is locally area L^2 -integrable in the plane \mathbb{C} , then there exists a solution to the $\bar{\partial}$ -equation $\bar{\partial}u = f$ with*

$$\int_{\mathbb{C}} |u|^2 e^{-2\phi} dA \leq \frac{1}{2} \int_{\mathbb{C}} |f|^2 \frac{e^{-2\phi}}{\Delta\phi} dA.$$

Here, we remark that the assertion of the theorem is void unless the integral on the right hand side is finite.

1.3 The $\bar{\partial}$ -equation with growing weights

While Theorem 1.1 essentially deals with decaying weights $e^{-2\phi}$, it is natural to ask what happens if we were to consider the growing weights $e^{2\phi}$ in place of $e^{-2\phi}$. So when could we say that there exists a solution to $\bar{\partial}u = f$ with

$$\int_{\mathbb{C}} |u|^2 e^{2\phi} dA \lesssim \int_{\mathbb{C}} |f|^2 e^{2\phi} dA, \tag{1.1}$$

where the symbol “ \lesssim ” is understood liberally? Here, we should have the (Fock weight) example $\phi(z) = \frac{1}{2}|z|^2$ in mind. It is rather clear that we cannot hope to have an estimate of the type (1.1) without an additional condition on the datum f . For instance, if $\phi(z) = \frac{1}{2}|z|^2$ and $f(z) = e^{-|z|^2}$, it is not possible to find such a fast-decaying function u with $\bar{\partial}u = f$ (cf. Sect. 3 below). It is natural to look for a class of data f that would come from functions u with compact support. Let $C_c^\infty(\mathbb{C})$ denote the standard space of infinitely differentiable compactly supported test functions. The calculation

$$\int_{\mathbb{C}} fg dA = \int_{\mathbb{C}} g \bar{\partial} u dA = - \int_{\mathbb{C}} u \bar{\partial} g dA, \quad u \in C_c^\infty(\mathbb{C}),$$

shows that the datum $f = \bar{\partial}u$ with $u \in C_c^\infty(\mathbb{C})$ must satisfy, for entire functions g ,

$$\int_{\mathbb{C}} fg dA = 0. \tag{1.2}$$

We remark here that the calculation (1.2) is the basis for what is known as Havin's lemma [7] (see also, e.g., [8]).

We let $L^2(\mathbb{C}, e^{-2\phi})$ and $L^2(\mathbb{C}, e^{2\phi})$ be the weighted area L^2 -spaces with the indicated weights. The corresponding norms are

$$\|g\|_{L^2(\mathbb{C}, e^{-2\phi})}^2 = \int_{\mathbb{C}} |g|^2 e^{-2\phi} dA, \quad \|g\|_{L^2(\mathbb{C}, e^{2\phi})}^2 = \int_{\mathbb{C}} |g|^2 e^{2\phi} dA.$$

The space of entire functions in $L^2(\mathbb{C}, e^{-2\phi})$ is denoted by $A^2(\mathbb{C}, e^{-2\phi})$. We recall the standing assumption that ϕ be C^2 -smooth with $\Delta\phi > 0$ everywhere.

Theorem 1.2 Suppose $f \in L^2(\mathbb{C}, e^{2\phi})$ meets the condition (1.2) for all $g \in A^2(\mathbb{C}, e^{-2\phi})$. Then there exists a solution to the $\bar{\partial}$ -equation $\bar{\partial}u = f$ with

$$\int_{\mathbb{C}} |u|^2 e^{2\phi} \Delta\phi dA \leq \frac{1}{2} \int_{\mathbb{C}} |f|^2 e^{2\phi} dA.$$

Superficially, this theorem looks quite different than Hörmander's Theorem 1.1. However, it is in a sense which can be made precise dual to Theorem 1.1. In our presentation, we will derive it from the same rather elementary calculation which Hörmander uses in e.g. [13], p. 250; cf. also Section 2 of [5] on Carleman inequalities, where essentially the same calculation is made.

We might add that if the polynomials are dense in the space $A^2(\mathbb{C}, e^{-2\phi})$ of entire functions, as they are, e.g., for radial ϕ , then the condition (1.2) needs to be verified only for monomials $g(z) = z^j$, $j = 0, 1, 2, \dots$.

Remark 1.3 Naturally, Theorem 1.2 should generalize to the setting of several complex variables.

Under a simple condition on the weight, the solution supplied by Theorem 1.2 is unique.

Theorem 1.4 If, in addition, ϕ is C^4 -smooth and meets the curvature-type condition

$$\frac{1}{\Delta\phi} \Delta \log \Delta\phi \geq -2 \quad \text{on } \mathbb{C},$$

then the solution u in Theorem 1.2 is unique.

2 The proof of Theorem 1.2

2.1 A norm identity

We denote by $\|\cdot\|_{L^2}$ the norm in the space $L^2(\mathbb{C})$. For a function F , we let \mathbf{M}_F denote the operator of multiplication by F . The first step is the following norm identity for $v \in C_c^\infty(\mathbb{C})$:

$$\|\bar{\partial}v - v\bar{\partial}\phi\|_{L^2}^2 - \|\partial v + v\partial\phi\|_{L^2}^2 = 2 \int_{\mathbb{C}} |v|^2 \Delta\phi dA. \quad (2.1)$$

To arrive at (2.1), we do as follows. As for the first term, if we let $\langle \cdot, \cdot \rangle_{L^2}$ denote the standard sesquilinear inner product of $L^2(\mathbb{C})$, we see that

$$\begin{aligned} \|\bar{\partial}v - v\bar{\partial}\phi\|_{L^2}^2 &= \langle \bar{\partial}v - v\bar{\partial}\phi, \bar{\partial}v - v\bar{\partial}\phi \rangle_{L^2} = \langle (\bar{\partial} - \mathbf{M}_{\bar{\partial}\phi})v, (\bar{\partial} - \mathbf{M}_{\bar{\partial}\phi})v \rangle_{L^2} \\ &= \langle (\bar{\partial} - \mathbf{M}_{\bar{\partial}\phi})^*(\bar{\partial} - \mathbf{M}_{\bar{\partial}\phi})v, v \rangle_{L^2} \end{aligned}$$

and together with the corresponding calculation for the second term, we find that (2.1) expresses that

$$(\bar{\partial} - \mathbf{M}_{\bar{\partial}\phi})^*(\bar{\partial} - \mathbf{M}_{\bar{\partial}\phi}) - (\partial + \mathbf{M}_{\partial\phi})^*(\partial + \mathbf{M}_{\partial\phi}) = 2\mathbf{M}_{\Delta\phi}. \quad (2.2)$$

Here, the adjoints involved are readily expressed: $\partial^* = -\bar{\partial}$, $\bar{\partial}^* = -\partial$, and $\mathbf{M}_F^* = \bar{\mathbf{M}}_{\bar{F}}$. The product rule says that $\bar{\partial}\mathbf{M}_F = \mathbf{M}_{\bar{\partial}F} + \mathbf{M}_F\bar{\partial}$ and $\partial\mathbf{M}_F = \mathbf{M}_{\partial F} + \mathbf{M}_F\partial$. By identifying adjoints and carrying out the necessary algebraic manipulations, (2.2) is immediate. We note in passing that the left-hand side of (2.2) expresses a *self-commutator*.

2.2 Reduction to a norm inequality

While the norm identity (2.1) is interesting by itself, we observe here that it has the consequence that

$$2 \int_{\mathbb{C}} |v|^2 \Delta \phi dA \leq \|\bar{\partial} v - v \bar{\partial} \phi\|_{L^2}^2, \quad v \in C_c^\infty(\mathbb{C}). \quad (2.3)$$

We write $\mathbf{T} := \bar{\partial} - \mathbf{M}_{\bar{\partial}\phi}$, and express (2.3) again:

$$2 \int_{\mathbb{C}} |v|^2 \Delta \phi dA \leq \|\mathbf{T}v\|_{L^2}^2, \quad v \in C_c^\infty(\mathbb{C}). \quad (2.4)$$

If v_j is a sequence of functions such that $\mathbf{T}v_j$ converges in $L^2(\mathbb{C})$, then by (2.4) the functions $v_j \sqrt{\Delta \phi}$ converge in $L^2(\mathbb{C})$ as well, and in particular, v_j converges locally as an L^2 -function. It follows that if $h \in L^2(\mathbb{C})$ is in to the $L^2(\mathbb{C})$ -closure of $\mathbf{T}C_c^\infty(\mathbb{C})$, then there exists a function $v \in L^2(\mathbb{C}, \Delta \phi)$ such that $\mathbf{T}v = h$ in the sense of distribution theory, with

$$2 \int_{\mathbb{C}} |v|^2 \Delta \phi dA \leq \|h\|_{L^2}^2. \quad (2.5)$$

It remains to identify the $L^2(\mathbb{C})$ -closure of $\mathbf{T}C_c^\infty(\mathbb{C})$. To this end, we identify the orthogonal complement of $\mathbf{T}C_c^\infty(\mathbb{C})$. So, let $k \in L^2(\mathbb{C})$ be such that

$$\langle k, \mathbf{T}v \rangle_{L^2} = 0, \quad v \in C_c^\infty(\mathbb{C}). \quad (2.6)$$

If we write $\mathbf{T}^* := -\partial - \mathbf{M}_{\partial\phi}$, distribution theory gives that (2.6) is the same as

$$\langle \mathbf{T}^* k, v \rangle_{L^2} = 0, \quad v \in C_c^\infty(\mathbb{C}),$$

which in its turn expresses that $\mathbf{T}^* k = 0$ holds in the sense of distributions. Let us write $\ker \mathbf{T}^*$ for the space of all $k \in L^2(\mathbb{C})$ with $\mathbf{T}^* k = 0$. We have arrived at the following result.

Theorem 2.1 Suppose $h \in L^2(\mathbb{C}) \ominus \ker \mathbf{T}^*$. Then there exists a function $v \in L^2(\mathbb{C}, \Delta \phi)$ such that $\mathbf{T}v = h$ with

$$2 \int_{\mathbb{C}} |v|^2 \Delta \phi dA \leq \|h\|_{L^2}^2.$$

Since $\mathbf{T}^* = -\mathbf{M}_{e^{-\phi}} \partial \mathbf{M}_{e^\phi}$, $k \in \ker \mathbf{T}^*$ holds if and only if $e^\phi \bar{k} \in A^2(\mathbb{C}, e^{-2\phi})$.

Proof of Theorem 1.2 We put $h := e^\phi f \in L^2(\mathbb{C})$. By condition (1.2) for all $g \in A^2(\mathbb{C}, e^{-2\phi})$, we have—if we write $k = e^{-\phi} \bar{g}$ —that

$$0 = \int_{\mathbb{C}} f g dA = \int_{\mathbb{C}} h \bar{k} dA.$$

As the function g runs over $A^2(\mathbb{C}, e^{-2\phi})$, k runs over all elements of $\ker \mathbf{T}^*$. We conclude that $h \in L^2(\mathbb{C}) \ominus \ker \mathbf{T}^*$, so that Theorem 2.1 applies, and gives a $v \in L^2(\mathbb{C}, \Delta \phi)$ with $\mathbf{T}v = h$ with the norm control

$$2 \int_{\mathbb{C}} |v|^2 \Delta \phi dA \leq \|h\|_{L^2}^2. \quad (2.7)$$

We put $u := e^{-\phi} v$. The operator \mathbf{T} factors $\mathbf{T} = \mathbf{M}_{e^\phi} \bar{\partial} \mathbf{M}_{e^{-\phi}}$, which means that the equation $\mathbf{T}v = h$ is equivalent to $\bar{\partial} u = f$. Finally, the estimate (2.7) is equivalent to the estimate

$$2 \int_{\mathbb{C}} |u|^2 e^{2\phi} \Delta \phi dA \leq \int_{\mathbb{C}} |f|^2 e^{2\phi} dA,$$

which concludes the proof of the theorem. \square

Proof of Theorem 1.4 We first observe that any two solutions of the $\bar{\partial}$ -equation differ by an entire function. Moreover, under the given curvature-type condition, an entire function $F \in L^2(\mathbb{C}, e^{2\phi} \Delta \phi)$ necessarily must vanish everywhere, by the following argument. The function $|F|^2 e^{2\phi} \Delta \phi$ is clearly nonnegative and it is also *subharmonic* in \mathbb{C} . Indeed, if F is nontrivial, we have that [in the sense of distribution theory] $\Delta \log |F| \geq 0$, and consequently

$$\Delta \log [|F|^2 e^{2\phi} \Delta \phi] = 2\Delta \log |F| + 2\Delta \phi + \Delta \log \Delta \phi \geq 0,$$

which gives that the exponentiated function $|F|^2 e^{2\phi} \Delta \phi$ is subharmonic, as claimed. In the remaining case when F vanishes identically the claim is trivial. Next, by the estimate of each solution of the $\bar{\partial}$ -problem supplied by Theorem 1.2, it is given that the function $|F|^2 e^{2\phi} \Delta \phi$ is in $L^2(\mathbb{C})$. If $\mathbb{D}(z_0, r)$ denotes the open disk of radius r about z_0 , the sub-mean value property of subharmonic functions gives that

$$|F(z_0)|^2 e^{2\phi(z_0)} \Delta \phi(z_0) \leq \frac{1}{\pi r^2} \int_{\mathbb{D}(z_0, r)} |F|^2 e^{2\phi} \Delta \phi dA \leq \frac{1}{\pi r^2} \int_{\mathbb{C}} |F|^2 e^{2\phi} \Delta \phi dA.$$

Letting $r \rightarrow +\infty$, we see that the left hand side vanishes. As z_0 is arbitrary, it follows that $F(z) \equiv 0$. This completes the proof. \square

Remark 2.2 In Lemma 3.1 of [2] appears a condition which is analogous to (1.2) in the context of distributions with compact support. Also, in the related paper [15], the Fock weight case $\phi(z) = \frac{1}{2}|z|^2$ is considered in the soft-topology setting of solutions which are distributions.

Remark 2.3 With $\phi = 0$, the norm identity (2.1) expresses that the Beurling operator is an isometry on $L^2(\mathbb{C})$, which implies the Grunsky inequalities in the theory of conformal mapping (see, e.g., [3]; cf. [4]). In the (somewhat singular) case when $\phi(z) = \theta \log |z|$, (2.1) provides the main norm identity of [9], which leads to a Prawitz–Grunsky type inequality for conformal maps. We remark here that the Prawitz inequality was the key input estimate in the analysis of the universal integral means spectrum in [10, 11]. The most general Prawitz–Grunsky type inequality (with multiple “branch points”) for conformal maps was obtained by Shimorin [16] (see also [1]). It would be of interest to see if the results of Shimorin may be obtained from the general norm identity (2.1).

3 Discussion of the necessity of the orthogonality condition on the datum

3.1 The Fock weight case

We now narrow down the discussion to the Fock weight $\phi(z) = \frac{1}{2}|z|^2$. Since the weight is radial, the polynomials are dense in $A^2(\mathbb{C}, e^{-|z|^2})$. Also, the curvature-type condition of Theorem 1.4 is readily checked. It follows that Theorems 1.2 and 1.4 combine to give the following result.

Theorem 3.1 Suppose $f \in L^2(\mathbb{C}, e^{|z|^2})$. If the datum f satisfies the moment condition

$$\int_{\mathbb{C}} z^j f(z) dA(z) = 0, \quad j = 0, 1, 2, \dots,$$

then there exists a unique solution u to the equation $\bar{\partial}u = f$ with

$$\int_{\mathbb{C}} |u(z)|^2 e^{|z|^2} dA(z) \leq \int_{\mathbb{C}} |f(z)|^2 e^{|z|^2} dA(z).$$

We may ask what would happen if the orthogonality condition is not satisfied. Maybe there still exists some rapidly decaying solution u anyway? The answer is definitely no.

Theorem 3.2 Suppose $f \in L^2(\mathbb{C}, e^{|z|^2})$, and that u solves the equation $\bar{\partial}u = f$ while $u \in L^2(\mathbb{C}, e^{\epsilon|z|^2})$ for some positive real ϵ . Then the datum f has

$$\int_{\mathbb{C}} z^j f(z) dA(z) = 0, \quad j = 0, 1, 2, \dots$$

Before we turn to the proof, we observe that if $f \in L^2(\mathbb{R})$ with

$$\int_{\mathbb{R}} |f(x)| e^{x^2/\beta} dx < +\infty$$

for some real $\beta > 0$, then its Fourier transform

$$\hat{f}(\xi) := \int_{\mathbb{R}} e^{-i\xi x} f(x) dx$$

extends to an entire function with

$$\int_{\mathbb{C}} |\hat{f}(\xi)|^2 e^{-\beta|\text{Im } \xi|^2} dA(\xi) = \frac{2\pi^{3/2}}{\sqrt{\beta}} \int_{\mathbb{R}} |f(x)| e^{x^2/\beta} dx < +\infty.$$

In fact, the standard Bargmann transform theory asserts that this integrability condition characterizes the Fourier image of this weighted L^2 space on \mathbb{R} (cf. [6]). The two-variable extension of the above-mentioned result maintains that if $f \in L^2(\mathbb{C}, e^{|z|^2/\beta})$, then its Fourier transform

$$\hat{f}(\xi, \eta) := \int_{\mathbb{C}} e^{-i(\xi x + \eta y)} f(x + iy) dx dy$$

is an entire function of two variables, with

$$\iint_{\mathbb{C} \times \mathbb{C}} |\hat{f}(\xi, \eta)|^2 e^{-\beta(|\text{Im } \xi|^2 + |\text{Im } \eta|^2)} dA(\xi) dA(\eta) = \frac{4\pi^3}{\beta} \int_{\mathbb{C}} |f(z)| e^{|z|^2/\beta} dA(z) < +\infty. \quad (3.1)$$

Proof of Theorem 3.2 Since $f \in L^2(\mathbb{C}, e^{|z|^2})$, the function \hat{f} is an entire function of two variables with the norm bound (3.1) with $\beta = 1$. Likewise, as $u \in L^2(\mathbb{C}, e^{\epsilon|z|^2})$ for some positive ϵ , \hat{u} is an entire function of two variables. After Fourier transformation, the relation $\bar{\partial}u = f$ reads

$$(i\xi - \eta)\hat{u}(\xi, \eta) = 2\hat{f}(\xi, \eta).$$

This is only possible if $\hat{f}(\xi, \eta)$ vanishes when $i\xi - \eta = 0$, i.e., $\hat{f}(\xi, i\xi) \equiv 0$. By the definition of the Fourier transform, this means that

$$\begin{aligned} 0 = \hat{f}(\xi, i\xi) &= \int_{\mathbb{C}} e^{-i\xi(x+iy)} f(x+iy) dA(x+iy) = \int_{\mathbb{C}} e^{-i\xi z} f(z) dA(z) \\ &= \sum_{j=0}^{+\infty} \frac{(-i\xi)^j}{j!} \int_{\mathbb{C}} z^j f(z) dA(z). \end{aligned}$$

By Taylor's formula, then, this implies that

$$\int_{\mathbb{C}} z^j f(z) dA(z) = 0, \quad j = 0, 1, 2, \dots,$$

as needed. \square

Remark 3.3 The constant in Theorem 3.1 is sharp. Indeed, we may consider the datum $f(z) = -ze^{-|z|^2}$ for which the solution is $u(z) = e^{-|z|^2}$. We calculate that

$$\int_{\mathbb{C}} |u(z)|^2 e^{|z|^2} dA(z) = \int_{\mathbb{C}} e^{-|z|^2} dA(z) = \pi$$

while

$$\int_{\mathbb{C}} |f(z)|^2 e^{|z|^2} dA(z) = \int_{\mathbb{C}} |z|^2 e^{-|z|^2} dA(z) = \pi,$$

which gives the desired sharpness.

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