THE KLEIN–GORDON EQUATION, THE HILBERT TRANSFORM AND GAUSS-TYPE MAPS: H^∞ APPROXIMATION

By

HAAKAN HEDENMALM*† AND ALFONSO MONTES-RODRÍGUEZ†

Abstract. The impetus to this work is the need to characterize when the system $\{\phi_1^m, \phi_2^n\}$ where $m, n = 0, 1, 2, \ldots$, is complete in the weak-star topology of H^∞ on the unit disk (or the half-plane). Here, ϕ_1 and ϕ_2 are two atomic inner functions, of the form

$$\phi_1(z) = \exp\left(\lambda_1 \frac{z+1}{z-1}\right) \quad \text{and} \quad \phi_2(z) = \exp\left(\lambda_2 \frac{z-1}{z+1}\right),$$

where λ_1, λ_2 are positive reals. Our main result asserts that the system of non-negative integral powers $\{\phi_1^m, \phi_2^n\}$ is weak-star dense in H^∞ of the unit disk if and only if $\lambda_1\lambda_2 \le \pi^2$. In earlier work in the L^∞ setting on the unit circle all the integer powers were considered, and the corresponding result was obtained (Hedenmalm and Montes-Rodríguez, 2011). The approach was to first transfer the completeness problem to the real line via the Cayley transform, and to then connect with the dynamics of Gauss-type transformations on the interval $[-1, 1]$. Indeed, the nonexistence of nontrivial finite absolutely continuous invariant measures for the Gauss-type map was the key ingredient of the analysis. Moreover, it was shown that the answer to the completeness problem has striking consequences for the Klein–Gordon equation. Here, the analysis is much more subtle as a result of the required finer topology. To appreciate the difference, we observe that the standard quotient space L^1/H^1 used as the predual of H^∞ is not appropriate for our purposes. Instead we model the predual in the real way, as L^1 plus the Hilbert transform of L^1, in analogy with the decomposition of BMO. The next step is to analyze carefully the iterates of the transfer operator applied to the Hilbert kernel. The approach involves a splitting of the Hilbert kernel which is induced by the transfer operator. The careful analysis of this splitting involves detours to the Hurwitz zeta function as well as to the theory of totally positive matrices.

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1 Introduction

1.1 Heisenberg uniqueness pairs. Let μ be a finite complex-valued Borel measure in the plane $\mathbb{R}^2$, and associate with it the Fourier transform

$$\hat{\mu}(\xi) := \int_{\mathbb{R}^2} \mathrm{e}^{\mathrm{i}\pi\langle x,\xi\rangle} \mathrm{d}\mu(x),$$

where $x = (x_1, x_2)$ and $\xi = (\xi_1, \xi_2)$, with inner product

$$\langle x, \xi\rangle = x_1\xi_1 + x_2\xi_2.$$

The Fourier transform $\hat{\mu}$ is a continuous and bounded function on $\mathbb{R}^2$. In [13], the concept of a Heisenberg uniqueness pair (HUP) was introduced. It is similar to the notion of weakly mutually annihilating pairs of Borel measurable sets having positive area measure, which appears, e.g., in the book by Havin and Jöricke [12]. For $\Gamma \subset \mathbb{R}^2$ which is a finite disjoint union of smooth curves in $\mathbb{R}^2$, let $\mathrm{M}(\Gamma)$ denote the Banach space of complex-valued finite Borel measures in $\mathbb{R}^2$, supported on Γ. Moreover, let $\mathrm{AC}(\Gamma)$ denote the closed subspace of $\mathrm{M}(\Gamma)$ consisting of the measures that are absolutely continuous with respect to arc length measure on Γ.

Definition 1.1.1. Let Γ be a finite disjoint union of smooth curves in $\mathbb{R}^2$. For a set $\Lambda \subset \mathbb{R}^2$, we say that (Γ, Λ) is a **Heisenberg uniqueness pair** provided that

$$\forall \mu \in \mathrm{AC}(\Gamma): \quad \hat{\mu}|_\Lambda = 0 \implies \mu = 0.$$

Heisenberg uniquess pairs in which Γ is a straight line or the union of two parallel lines were described in [13]. Later, Blasi [3] solved particular cases of the union of three parallel lines. The ellipse case was considered independently by Lev and Sjölin in [18] and [24]; Sjölin also considered the parabola in [25]. More recently, Jaming and Kellay in [16] devoloped new powerful tools to study Heisenberg uniqueness pairs for a variety of curves Γ, while Giri and Srivastava studied four parallel lines among other interesting curves [10]. As for higher dimensional analogues, in [11] Gröchenig and Jaming connected the topic with the Cramér–Wold theorem on quadratic surfaces. Recently, it was discovered that the method of Viazovska [30] for the sphere packing problem to construct new functions with interpolating properties (based on automorphic forms and Eichler cohomology) allows modification to the interpolation problem associated with the lattice-cross of critical density for the Klein–Gordon equation which in turn leads to an affirmative solution of the Goursat problem. This is the topic of forthcoming work [2].

1.2 The Zariski closure. We turn to the notion of the Zariski closure. Note that the Zariski topology (or hull-kernel topology) is a standard concept in, e.g., Algebraic Geometry, in the setting of spaces of polynomials. As for notation, we let $\mathrm{AC}(\Gamma;\Lambda)$ be the subspace of $\mathrm{AC}(\Gamma)$ consisting of those measures μ whose Fourier transform vanishes on Λ.

Definition 1.2.1. Let Γ be a finite disjoint union of smooth curves in $\mathbb{R}^2$, and let $\Lambda \subset \mathbb{R}^2$ be arbitrary. With respect to $\mathrm{AC}(\Gamma)$, the **Zariski closure of** Λ is the set

$$\mathrm{zclos}_\Gamma(\Lambda) := \{\xi \in \mathbb{R}^2 : [\forall \mu \in \mathrm{AC}(\Gamma;\Lambda) : \hat{\mu}(\xi) = 0]\}.$$

Less formally, the Zariski closure (or hull) is the set where the Fourier transform of a measure $\mu \in \mathrm{AC}(\Gamma)$ must vanish given that it already vanishes on Λ. Now, as the Fourier image of $\mathrm{AC}(\Gamma)$ does not form an algebra with respect to pointwise multiplication of functions, we cannot expect the Zariski closure to correspond to a topology. This means that the union of two Zariski closures need not be a closure itself. However, it is easy to see that the closure operation is idempotent, that is: $\mathrm{zclos}_\Gamma^2 = \mathrm{zclos}_\Gamma$. The Zariski closure allows us the following convenient description: (Γ, Λ) is a Heisenberg uniqueness pair if and only if

$$\mathrm{zclos}_\Gamma(\Lambda) = \mathbb{R}^2.$$

1.3 The Klein–Gordon equation. In natural units, the Klein–Gordon equation in one spatial dimension reads

$$\partial_t^2 u - \partial_x^2 u + M^2 u = 0,$$

where $M > 0$ corresponds to the mass. In terms of the (preferred) coordinates

$$\xi_1 := t + x, \quad \xi_2 := t - x,$$

the Klein–Gordon equation becomes

$$\partial_{\xi_1}\partial_{\xi_2} u + \frac{M^2}{4} u = 0. \tag{1.3.1}$$

Remark 1.3.1. Since $t^2 - x^2 = \xi_1\xi_2$, the **time-like vectors** (those vectors $(t,x) \in \mathbb{R}^2$ with $t^2 - x^2 > 0$) correspond to the union of the first quadrant $\xi_1, \xi_2 > 0$ and the third quadrant $\xi_1, \xi_2 < 0$ in the (ξ, ξ_2)-plane. Likewise, the **space-like vectors** correspond to the union of the second quadrant $\xi_1 > 0$, $\xi_2 < 0$ and the fourth quadrant $\xi_1 < 0$, $\xi_2 > 0$.

1.4 Fourier analytic treatment of the Klein–Gordon equation. In the sequel, we will not need to talk about the time and space coordinates (t, x) as such. So, e.g., we are free to use the notation $x = (x_1, x_2)$ for the Fourier dual coordinate to $\xi = (\xi_1, \xi_2)$.

Let $\mathcal{M}(\mathbb{R}^2)$ denote the Banach space of all finite complex-valued Borel measures in $\mathbb{R}^2$. We suppose that u is the Fourier transform of a $\mu \in \mathcal{M}(\mathbb{R}^2)$:

$$u(\xi) = \hat{\mu}(\xi) := \int_{\mathbb{R}^2} \mathrm{e}^{\mathrm{i}\pi\langle x,\xi\rangle} \mathrm{d}\mu(x), \quad \xi \in \mathbb{R}^2. \tag{1.4.1}$$

As for the measure μ, the assumption that u solves the Klein–Gordon equation (1.3.1) asks that

$$\Big(x_1 x_2 - \frac{M^2}{4\pi^2}\Big)\mathrm{d}\mu(x) = 0$$

as a measure on $\mathbb{R}^2$, which we see is the same as a requirement on the support set of the measure

$$\operatorname{supp}\mu \subset \Gamma_M := \Big\{x \in \mathbb{R}^2 : \, x_1 x_2 = \frac{M^2}{4\pi^2}\Big\}. \tag{1.4.2}$$

The set Γ_M is a hyperbola. We may use the x_1-axis to supply a global coordinate for Γ_M, and define a complex-valued finite Borel measure $\pi_1\mu$ on $\mathbb{R}$ by setting

$$\pi_1\mu(E) = \int_E \mathrm{d}\pi\mu(x_1) := \mu(E \times \mathbb{R}) = \int_{E\times\mathbb{R}} \mathrm{d}\mu(x). \tag{1.4.3}$$

We shall at times refer to $\pi_1\mu$ as the *compression* of μ to the x_1-axis. It is easy to see that μ may be recovered from $\pi_1\mu$; indeed,

$$u(\xi) = \hat{\mu}(\xi) = \int_{\mathbb{R}^\times} \mathrm{e}^{\mathrm{i}\pi[\xi_1 t + M^2\xi_2/(4\pi^2 t)]} \mathrm{d}\pi_1\mu(t), \quad \xi \in \mathbb{R}^2. \tag{1.4.4}$$

Here, we use the standard notational convention $\mathbb{R}^\times := \mathbb{R} \setminus \{0\}$. We note that μ is absolutely continuous with respect to arc length measure on Γ_M if and only if $\pi_1\mu$ is absolutely continuous with respect to Lebesgue length measure on $\mathbb{R}^\times$.

1.5 The lattice-cross as a uniqueness set for solutions to the Klein–Gordon equation. For positive reals α, β, let $\Lambda_{\alpha,\beta}$ denote the lattice-cross

$$\Lambda_{\alpha,\beta} := (\alpha\mathbb{Z} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z}), \tag{1.5.1}$$

so that the spacing along the ξ_1-axis is α, and along the ξ_2-axis it is β. In the work [13], Hedenmalm and Montes-Rodríguez found the following.

Theorem 1.5.1 (Hedenmalm, Montes)**.** *If we fix positive reals M, α, β, then $(\Gamma_M, \Lambda_{\alpha,\beta})$ is a Heisenberg uniqueness pair if and only if $\alpha\beta M^2 \le 4\pi^2$.*

In terms of the Zariski closure, the theorem says that

$$\mathrm{zclos}_{\Gamma_M}(\Lambda_{\alpha,\beta}) = \mathbb{R}^2$$

holds if and only if $\alpha\beta M^2 \leq 4\pi^2$. By taking the relation (1.4.4) into account, and by reducing the redundancy of the constants (we may without loss of generality consider $M = 2\pi$ and $\alpha = 1$), Theorem 1.5.1 is equivalent to the following statement: the linear span of the functions

$$\mathrm{e}^{\mathrm{i}\pi mt}, \quad \mathrm{e}^{-\mathrm{i}\pi\beta n/t}, \quad m, n \in \mathbb{Z},$$

is weak-star dense in $L^\infty(\mathbb{R})$ if and only if $\beta \leq 1$. Here, we supply new and unexpected insight into the theory of Heisenberg uniqueness pairs, such as a new connection with the standard Gauss map (motivated by Theorem 1.6.1), and, more importantly, we uncover, in the framework of Fourier Analysis, profound connections between the Hilbert transform and the dynamics of transfer operators intimately related to Gauss-type maps leading up to Theorem 1.9.2.

1.6 Dynamic unique continuation from a branch of the hyperbola. Just looking at Theorem 1.5.1, we are immediately led to ask what happens if we replace the hyperbola Γ_M by one of its two branches, say

$$\Gamma_M^+ := \Gamma_M \cap (\mathbb{R}_+ \times \mathbb{R}_+) = \left\{x \in \mathbb{R}^2 : x_1 x_2 = \frac{M^2}{4\pi^2} \text{ and } x_1 > 0\right\}. \tag{1.6.1}$$

There is also a uniqueness theorem for the branch Γ_M^+ of the hyperbola Γ_M, which turns out to be closely related to the famous Gauss–Kuzmin–Wirsing operator and the Gauss map $x \mapsto 1/x \bmod \mathbb{Z}$.

Theorem 1.6.1. *Fix positive reals α, β, M. Then $(\Gamma_M^+, \Lambda_{\alpha,\beta})$ is a Heisenberg uniqueness pair if and only if $\alpha\beta M^2 < 16\pi^2$. Moreover, in the critical case $\alpha\beta M^2 = 16\pi^2$, the space $\mathrm{AC}(\Gamma_M^+, \Lambda_{\alpha,\beta})$ is the one-dimensional space spanned by the measure $\mu_0 \in \mathrm{AC}(\Gamma_M^+, \Lambda_{\alpha,\beta})$ whose x_1-compression is given by*

$$\mathrm{d}\pi_1\mu_0(t) := \left\{\frac{1_{[0,2/\alpha]}(t)}{2(2+\alpha t)} - \frac{1_{[2/\alpha,+\infty[}(t)}{\alpha t(2+\alpha t)}\right\}\mathrm{d}t.$$

The proof of Theorem 1.6.1 is presented in [14], where it is also shown that in the critical parameter regime $\alpha\beta = 16\pi^2$, $(\Gamma_M^+, \Lambda_{\alpha,\beta}^\star)$ is indeed a Heisenberg uniqueness pair, where $\Lambda_{\alpha,\beta}^\star := \Lambda_{\alpha,\beta} \cup \{\xi^\star\}$, and $\xi^\star \in (\{0\} \times \mathbb{R}) \cup (\mathbb{R} \times \{0\})$ is any point off the lattice-cross $\Lambda_{\alpha,\beta}$. The analysis of the proof of the latter result involves a geometric object known as the **Nielsen spiral**.

Again, by taking the relation (1.4.4) into account, and by reducing the redundancy of the constants (again we may without loss of generality consider $M = 2\pi$ and $\alpha = 1$), it is easy to see that Theorem 1.6.1 entails the following assertion: the restriction to $\mathbb{R}_+$ of the linear span of the functions

$$e^{i\pi m t}, \quad e^{-i\pi\beta n/t}, \quad m, n \in \mathbb{Z},$$

is weak-star dense in $L^\infty(\mathbb{R}_+)$ if and only if $\beta < 4$. Moreover, if $\beta = 4$ the weak-star closure of this linear span has codimension one in $L^\infty(\mathbb{R}_+)$.

Theorem 1.6.1 has the following consequence in terms of unique continuation from the branch Γ_M^+ (or, alternatively, the complementary branch $\Gamma_M^- := \Gamma_M \setminus \Gamma_M^+$), to the entire hyperbola Γ_M.

Corollary 1.6.2. *Fix positive reals α, β, M. Then $\mu \in \mathrm{AC}(\Gamma_M, \Lambda_{\alpha,\beta})$ is uniquely determined by its restriction to the hyperbola branch Γ_M^+ if and only if $\alpha\beta M^2 < 16\pi^2$. The same holds with Γ_M^+ replaced by Γ_M^- as well.*

1.7 The Zariski closure of the axes and half-axes. We first consider the Zariski closure of the two axes $\mathbb{R} \times \{0\}$ and $\{0\} \times \mathbb{R}$ with respect to the space $\mathrm{AC}(\Gamma_M)$ of absolutely continuous measures, with respect to arc length, on the hyperbola Γ_M.

Proposition 1.7.1. *Fix a positive real M. If $\mu \in \mathrm{AC}(\Gamma_M)$ is such that $\hat{\mu}$ vanishes on one of the axes, $\mathbb{R} \times \{0\}$ or $\{0\} \times \mathbb{R}$, then $\mu = 0$ identically. In terms of Zariski closures, this means that*

$$\mathrm{zclos}_{\Gamma_M}(\mathbb{R} \times \{0\}) = \mathrm{zclos}_{\Gamma_M}(\{0\} \times \mathbb{R}) = \mathbb{R}^2.$$

The next proposition will show the difference between time-like and space-like quarterplanes. First, we need some notation. Let $\mathbb{R}_+ := \{t \in \mathbb{R} : t > 0\}$ and $\mathbb{R}_- := \{t \in \mathbb{R} : t < 0\}$ be the positive and negative half-lines, respectively. We write

$$\bar{\mathbb{R}}_+ := \{t \in \mathbb{R} : t \geq 0\} \quad \text{and} \quad \bar{\mathbb{R}}_- := \{t \in \mathbb{R} : t \leq 0\}$$

for the corresponding closed half-lines.

Proposition 1.7.2. *Fix a positive real M. Then the Zariski closures of each of the four semi-axes $\mathbb{R}_+ \times \{0\}$, $\mathbb{R}_- \times \{0\}$, $\{0\} \times \mathbb{R}_+$, and $\{0\} \times \mathbb{R}_-$ are as follows:*

$$\mathrm{zclos}_{\Gamma_M}(\mathbb{R}_+ \times \{0\}) = \mathrm{zclos}_{\Gamma_M}(\{0\} \times \mathbb{R}_-) = \bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_-$$

and

$$\mathrm{zclos}_{\Gamma_M}(\mathbb{R}_- \times \{0\}) = \mathrm{zclos}_{\Gamma_M}(\{0\} \times \mathbb{R}_+) = \bar{\mathbb{R}}_- \times \bar{\mathbb{R}}_+.$$

Remark 1.7.3. In each of the instances in Proposition 1.7.2, we note that the Zariski closure of a semi-axis equals the topological closure of the adjacent quadrant of space-like vectors.

1.8 The Gauss-type maps on the symmetric unit interval. The Gauss-type map τ_β acting on the symmetric interval $I_1 :=]-1,1[$ is defined in the following fashion. First, we let $\{x\}_2$ denote the **even-fractional part** of x, by which we mean the unique number in the half-open interval $\tilde{I}_1 :=]-1,1]$ with $x - \{x\}_2 \in 2\mathbb{Z}$. The Gauss-type map $\tau_\beta : \tilde{I}_1 \to \tilde{I}_1$ is given by the expression

$$\tau_\beta(x) := \left\{ -\frac{\beta}{x} \right\}_2.$$

Here and in the sequel, β is assumed real with $0 < \beta \le 1$. The basic properties of τ_β are well-known; see, e.g., [14]. We outline the basic aspects below. For $0 < \beta < 1$, the set $I_1 \setminus \bar{I}_\beta$ acts as an attractor for the iterates under τ_β, and inside the attractor $I_1 \setminus \bar{I}_\beta$, the orbits form 2-cycles. Here, $\bar{I}_\beta$ denotes the symmetric interval $\bar{I}_\beta := [-\beta, \beta]$. For $\beta = \frac{3}{4}$, we show the graph of τ_β in Figure 1.8.1. For $\beta = 1$, on the other hand, we have the invariant measure $(1-x^2)^{-1}\mathrm{d}x$. The reason is that the endpoints ± 1 are only weakly repelling.

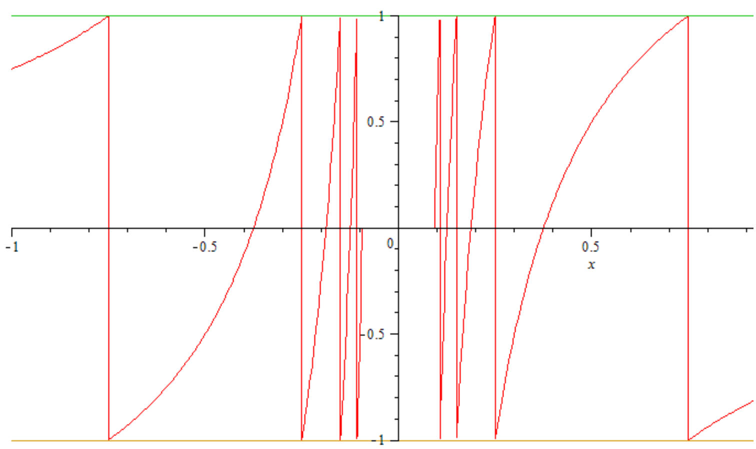

Figure 1.8.1. Illustration of the Gauss-type map τ_β for $\beta = \frac{3}{4}$. The vertical lines indicate where the graph has jumps.

The **transfer operator** $\mathfrak{T}_\beta$ linked with the map τ_β is the operator which can be understood as taking the unit point mass δ_x at a point $x \in \tilde{I}_1$ to the unit point mass $\delta_{\tau_\beta(x)}$ at the point $\tau_\beta(x)$. To be more definitive, for a function $f \in L^1(I_1)$, we write f as an integral of point masses,

$$f(x) = \int_{I_1} f(t)\, \delta_x(t)\, \mathrm{d}t = \int_{I_1} f(t)\, \delta_t(x)\, \mathrm{d}t, \tag{1.8.1}$$

understood in the sense of distribution theory, and say that

$$\mathbf{\mathfrak{T}}_\beta f(x) := \int_{I_1} f(t)\, \mathbf{\mathfrak{T}}_\beta \delta_t(x)\,\mathrm{d}t = \int_{I_1} f(t)\, \delta_{\tau_\beta(t)}(x)\,\mathrm{d}t, \qquad x \in I_1, \tag{1.8.2}$$

which is seen to be the same as the more explicit formula

$$\mathbf{\mathfrak{T}}_\beta f(x) = \begin{cases} \sum_{j\in\mathbb{Z}} \frac{\beta}{(x+2j)^2} f(-\frac{\beta}{x+2j}), & x \in I_1, \\ 0, & x \in \mathbb{R}\setminus I_1, \end{cases} \tag{1.8.3}$$

which has the added advantage that the values off the interval I_1 are declared to vanish. The behavior of τ_β is rather uninteresting on the attractor $I_1 \setminus \bar{I}_\beta$, and for this reason, we introduce the **subtransfer operator** $\mathbf{T}_\beta$ which discards the point masses from the attractor. In other words, we set

$$\mathbf{T}_\beta f(x) := \mathbf{\mathfrak{T}}(1_{\bar{I}_\beta} f)(x) = \int_{\bar{I}_\beta} f(t)\, \mathbf{\mathfrak{T}}_\beta \delta_t(x)\,\mathrm{d}t = \int_{\bar{I}_\beta} f(t)\, \delta_{\tau_\beta(t)}(x)\,\mathrm{d}t, \qquad x \in I_1. \tag{1.8.4}$$

In more direct terms, this is the same as

$$\mathbf{T}_\beta f(x) := \sum_{j\in\mathbb{Z}^\times} \frac{\beta}{(2j+x)^2} f\Big(-\frac{\beta}{2j+x}\Big), \qquad x \in I_1, \tag{1.8.5}$$

which we see from (1.8.3). Here, $\mathbb{Z}^\times = \mathbb{Z}\setminus\{0\}$, as before. For $0 < \beta < 1$, the τ_β-orbit of a point $x \in I_1$ falls into the attractor $I_1 \setminus \bar{I}_\beta$ almost surely. In terms of the subtransfer operator $\mathbf{T}_\beta$, this means that

$$\forall f \in L^1(I_1): \quad \mathbf{T}_\beta^n f \to 0 \quad \text{in } L^1(I_1), \qquad \text{if } 0 < \beta < 1. \tag{1.8.6}$$

For $\beta = 1$, things are a little more subtle. Nevertheless, it can be shown (see [19], for instance) that

$$\forall f \in L^1(I_1): \quad 1_{I_\eta} \mathbf{T}_1^n f \to 0 \quad \text{in } L^1(I_1), \tag{1.8.7}$$

for every fixed real η with $0 < \eta < 1$. Here, as expected, I_η is the symmetric interval $I_\eta :=]-\eta, \eta[$. In particular, there is no nontrivial function $f \in L^1(I_1)$ with $\mathbf{T}_\beta f = \lambda f$ for any $\lambda \in \mathbb{C}$ with $|\lambda| \geq 1$ and any β with $0 < \beta \leq 1$.

Let $\mathbf{H}$ stand for the **Hilbert transform**, given by the principal value integral

$$\mathbf{H}g(x) := \frac{1}{\pi}\mathrm{pv}\int_{\mathbb{R}} g(t)\frac{\mathrm{d}t}{x-t} = \lim_{\epsilon\to 0^+} \frac{1}{\pi}\int_{\mathbb{R}\setminus[x-\epsilon,x+\epsilon]} g(t)\frac{\mathrm{d}t}{x-t},$$

and $L^1_0(\mathbb{R})$ is the codimension 1 subspace

$$L^1_0(\mathbb{R}) := \Big\{ g \in L^1(\mathbb{R}) : \int_{\mathbb{R}} g(t)\mathrm{d}t = 0 \Big\}.$$

In [14], the subtransfer operator $\mathbf{T}_\beta$ was shown to extend to a bounded operator on the space $\mathfrak{L}(I_1)$, whose elements are distributions on I_1. The space $\mathfrak{L}(I_1)$ consists of the restrictions to the open interval I_1 of the distributions in the space

$$\mathfrak{L}(\mathbb{R}) := L^1(\mathbb{R}) + \mathbf{H}L^1_0(\mathbb{R}),$$

supplied with the induced quotient norm, as we mod out with respect to all the distributions whose support is contained in $\mathbb{R} \setminus I_1$. The quotient norm comes from the norm on the space $\mathfrak{L}(\mathbb{R})$, which is given by

$$\|u\|_{\mathfrak{L}(\mathbb{R})} := \inf\{\|f\|_{L^1(\mathbb{R})} + \|g\|_{L^1(\mathbb{R})} : u = f + \mathbf{H}g,\ f \in L^1(\mathbb{R}),\ g \in L^1_0(\mathbb{R})\}, \tag{1.8.8}$$

and we should mention that the $\mathfrak{L}(\mathbb{R})$ is in the natural sense the predual of the real H^∞-space on the line, denoted by $H^\infty_\circledast(\mathbb{R})$, which consists of all the functions in $L^\infty(\mathbb{R})$ whose modified Hilbert transform also is in $L^\infty(\mathbb{R})$.

By a theorem of Kolmogorov, the Hilbert transform of an $L^1(\mathbb{R})$ function is well-defined pointwise almost everywhere as a function in the quasi-Banach space $L^{1,\infty}(\mathbb{R})$ of weak-L^1 functions. More generally, if $E \subset \mathbb{R}$ is Lebesgue measurable with positive length, the **weak-L^1 space** $L^{1,\infty}(E)$ consists of all measurable functions $f : E \to \mathbb{C}$ with finite quasinorm

$$\|f\|_{L^{1,\infty}(E)} := \sup\{\lambda |N_f(\lambda)| : \lambda > 0\}, \tag{1.8.9}$$

where $N_f(\lambda)$ denotes the set

$$N_f(\lambda) := \{t \in E : |f(t)| > \lambda\},$$

and the absolute value sign in (1.8.9) assigns the linear length to a given set. Kolmogorov's theorem allows us to think of the distributions (or pseudomeasures) in $\mathfrak{L}(\mathbb{R})$ as elements of $L^{1,\infty}(\mathbb{R})$, so that in particular, $\mathfrak{L}(I_1)$ can be identified with a subspace of $L^{1,\infty}(I_1)$, the corresponding weak-L^1 space on the interval I_1. For the pointwise interpretation, the formula (1.8.5) for the operator $\mathbf{T}_\beta$ remains valid. We will work mainly in the setting of distribution theory. When we need to speak of the pointwise function rather than the distribution u, we write $\mathrm{vap}(u)$ in place of u, and call it the **valeur au point**. So "vap" maps from distributions to functions. Note that there is no canonical interpretation of functions in $L^{1,\infty}$ as distributions, so going back from the function to the distribution is nontrivial.

On the space $L^1(I_1)$, the subtransfer operators $\mathbf{T}_\beta$ all act contractively. This is not the case with the extension to $\mathfrak{L}(I_1)$.

Theorem 1.8.1. *Fix $0 < \beta \leq 1$. Then the operator $\mathbf{T}_\beta : \mathfrak{L}(I_1) \to \mathfrak{L}(I_1)$ is bounded but not contractive.*

The proof of Theorem 1.8.1 is supplied in Subsection 9.7.

A decomposition analogous to (1.8.1) holds for distributions $u \in \mathfrak{L}(I_1)$ as well, only we would need two integrals, one with $\delta_t(x)$ and the other with $\mathbf{H}\delta_t(x)$ (and the latter integral should be taken over a bigger interval, e.g. $I_2 =]-2, 2[$ to allow for tails). Thinking physically, we allow for tw,o kinds of "phases of matter", focused particles δ_t as well as spread-out phases $\mathbf{H}\delta_t$. Then $\mathfrak{L}(I_1)$ is a space of "extended" observables, and $\mathbf{T}_\beta$ acts on this space. It is then natural to ask whether there is a nontrivial invariant extended observable under $\mathbf{T}_\beta$. More generally, we would ask whether there exists a $u \in \mathfrak{L}(I_1)$ with $\mathbf{T}_\beta u = \lambda u$ for any scalar $\lambda \in \mathbb{C}$ with $|\lambda| \geq 1$. To appreciate the subtlety of this question, we note that in the slightly larger space $L^{1,\infty}(I_1)$, there are plenty of invariant states $u \in L^{1,\infty}(I_1)$ with $\mathbf{T}_\beta u = u$; see the example provided in Remark 11.2.1. That example is constructed as the Hilbert transform of the difference of two Dirac point masses, with one point inside I_1 and the other point outside $\bar{I}_1$. The example in fact suggests that within the space of Hilbert transforms of finite Borel measures, the invariant states might possess an intricate and interesting structure. In the space $\mathfrak{L}(\mathbb{R})$, which contains the Hilbert transforms of the absolutely continuous measures, this is however not the case.

Theorem 1.8.2. *Fix $0 < \beta < 1$. For $u_0 \in \mathfrak{L}(I_1)$, we have the asymptotic decay* $\mathrm{vap}(\mathbf{T}_\beta^N u_0) \to 0$ *in $L^{1,\infty}(I_1)$ as $N \to +\infty$.*

So, although $\mathbf{T}_\beta$ has norm that exceeds 1 on $\mathfrak{L}(I_1)$, the orbit of a given $u \in \mathfrak{L}(I_1)$ converges to 0 in the weaker sense of the quasinorm in $L^{1,\infty}(I_1)$. In other words, the $L^{1,\infty}$-quasinorm serves as a Lyapunov energy for the asymptotic stability of the $\mathbf{T}_\beta$-orbits. In the setting of the smaller space $L^1(I_1)$, this convergence amounts to the statement that the basin of attraction of the attractor $I_1 \setminus \bar{I}_\beta$ contains almost every point of the interval I_1. Apparently, this property extends to the larger space $\mathfrak{L}(I_1)$, but not to, e.g., $L^{1,\infty}(I_1)$ (see Remark 11.2.1). The proof of Theorem 1.8.2 is supplied in Subsection 11.2.

Corollary 1.8.3. *Fix $0 < \beta < 1$. If $\mathbf{T}_\beta u = \lambda u$ for some $u \in \mathfrak{L}(I_1)$ and some scalar $\lambda \in \mathbb{C}$ with $|\lambda| \geq 1$, then $u = 0$.*

In other words, for $0 < \beta < 1$, the point spectrum of the operator

$$\mathbf{T}_\beta : \mathfrak{L}(I_1) \to \mathfrak{L}(I_1)$$

is contained in the open unit disk $\mathbb{D}$. It is clear that Corollary 1.8.3 follows from Theorem 1.8.2.

For $\beta = 1$ we will separate the analysis according to symmetry. We recall that a distribution, defined on a symmetric interval about 0, is odd if its action on the even test functions equals 0.

Theorem 1.8.4 ($\beta = 1$)**.** *For odd $u_0 \in \mathfrak{L}(I_1)$, we have the asymptotic decay $1_{I_\eta}\mathrm{vap}(\mathbf{T}_1^N u_0) \to 0$ in $L^{1,\infty}(I_1)$ as $N \to +\infty$ for each η with $0 < \eta < 1$.*

The proof, which is supplied in Subsection 14.2, is much more sophisticated than that of Theorem 1.8.2. It uses the full strength of the machinery developed around a subtle dynamical decomposition of the odd part of the Hilbert kernel. A similar dynamical decomposition is available for the even part of the Hilbert kernel as well, but the estimates take a different form (compare with the remarks following the formulation of Theorem 1.8.6 below).

At the critical point $\beta = 1$, the proof is based on analyzing the iterates of the transfer operator on the sum space consisting of $L^1(I_1)$ plus its Hilbert transform with a suitable topology. For the details see, e.g., Theorem 12.4.1 below. After finding the commutant of the Hilbert transform and the transfer operator, we are led to analyze very precisely a series which involves the iterates of the transfer operator acting on the odd Hilbert kernel. Here, we refer to the odd Hilbert kernel in the context of splitting the Hilbert transform $\mathbf{H} = \mathbf{H}^I + \mathbf{H}^{II}$, where the even Hilbert kernel $\mathbf{H}^I$ maps to even functions and the odd Hilbert transform $\mathbf{H}^{II}$ maps to odd functions. To perform this analysis, it is necessary to consider a family of infinite matrices expressed in terms of the polygamma function. Each such matrix is a Hankel matrix which is strictly totally positive. Via the Variation Diminishing Property Theorem, this positivity permits us to study in great detail the sign changes of the Taylor coefficients of a system of kernels that are derived from the odd Hilbert kernel. A key step to control things is to estimate the Hurwitz zeta function. Put together, these ingredients allow us to control the series of the transfer operator acting on the odd Hilbert kernel.

In the setting of the smaller space $L^1(I_1)$, the corresponding statement is based on the fact that the dynamics of τ_1 has ± 1 as a weakly repelling fixed point, so that the ergodic invariant measure for $L^1(I_1)$ gets infinite mass and cannot be in $L^1(I_1)$. It follows immediately from Theorem 1.8.4 that the point spectrum of the operator

$$\mathbf{T}_1 : \mathfrak{L}_{\mathrm{odd}}(I_1) \to \mathfrak{L}_{\mathrm{odd}}(I_1)$$

is contained in the open unit disk $\mathbb{D}$. In particular, there is no $\mathbf{T}_1$-invariant element of $\mathfrak{L}_{\mathrm{odd}}(I_1)$, the subspace of the odd distributions in $\mathfrak{L}(I_1)$. It turns out that the oddness assumption is actually superfluous (see Corollary 1.8.7 below).

Corollary 1.8.5 ($\beta = 1$)**.** *If $\mathbf{T}_1 u = \lambda u$ for some odd $u \in \mathfrak{L}(I_1)$ and some scalar $\lambda \in \mathbb{C}$ with $|\lambda| \geq 1$, then $u = 0$.*

As already mentioned, this corollary is an immediate consequence of Theorem 1.8.4.

From a dynamical perspective, it is quite natural to introduce the odd-even symmetry, as the transformation τ_β itself is odd: $\tau_\beta(-x) = -\tau_\beta(x)$ (except possibly at the endpoints ± 1). For example, in connection with the partial fraction expansions with even partial quotients, it is standard to keep track of only the orbit of the absolute values on the interval I_1^+. Note that clearly, the subtransfer operators $\mathbf{T}_\beta$ preserve odd-even symmetry. As for the remaining even symmetry case, we observe that $\mathbf{H}(\delta_{-1} - \delta_1) = \frac{2}{\pi}(1 - x^2)^{-1}$ which is even and equals (a constant multiple of) the density of the ergodic invariant measure. In other words, the infinite ergodic invariant measure is the Hilbert transform of a finite measure consisting of two point masses. In the appropriate weak sense, such point masses are well approximated by functions in $L^1(I_1)$, and hence the invariant density is well approximated by elements of $\mathfrak{L}(I_1)$. Since the invariant measure is an even function, the even analogue of Theorem 1.8.4 could possibly be even more challenging. Pleasantly, we can also control the orbits of the even distributions in $\mathfrak{L}(I_1)$ in terms of another Lyapunov energy, but the energy functional is very weak. To formulate properly the assertion, we need the Volterra operator

$$\mathbf{V}f(x) := \int_0^x f(t)\mathrm{d}t, \qquad x \in I_1.$$

If the function f is even, then the Volterra operator can be expressed in the form

$$\mathbf{V}f(x) = \frac{1}{2}\int_{-x}^x f(t)\mathrm{d}t, \qquad x \in I_1,$$

and the resulting function $\mathbf{V}f$ is then odd. The latter formulation is suitable for even distributions $u \in \mathfrak{L}(I_1)$ (we write $\mathfrak{L}_{\mathrm{even}}(I_1)$ for the subspace of such even distributions in $\mathfrak{L}(I_1)$), and it is possible to show, e.g., that $\mathbf{V}u \in L^2(I_1)$ for $u \in \mathfrak{L}_{\mathrm{even}}(I_1)$.

Theorem 1.8.6 ($\beta = 1$). *For even $u_0 \in \mathfrak{L}(I_1)$, we have the asymptotic decay $1_{I_\eta}\mathbf{V}\mathbf{T}_1^N u_0 \to 0$ in $L^2(I_1)$ as $N \to +\infty$ for each η with $0 < \eta < 1$.*

While the proof of Theorem 1.8.6 is to some extent analogous to that of Theorem 1.8.4, it deviates in that we analyze the Volterra operator applied to the even summands in the "dynamical decomposition lemma" (cf. Proposition 12.3.1), and that a weaker control of the corresponding terms is obtained in the "uniform control of summands" (cf. Theorem 12.4.1). We defer the details of the proof of Theorem 1.8.6 to a separate publication [15].

A combination of Theorems 1.8.4 and 1.8.6 allows us to remove the oddness assumption in Corollary 1.8.5.

Corollary 1.8.7 ($\beta = 1$)**.** *If* $\mathbf{T}_1 u = \lambda u$ *for some* $u \in \mathfrak{L}(I_1)$ *and some scalar* $\lambda \in \mathbb{C}$ *with* $|\lambda| \geq 1$*, then* $u = 0$.

Remark 1.8.8. Corollaries 1.8.3 and 1.8.7 go beyond the standard notion of ergodicity. The main point is that we insert distribution theory in place of measure theory, although the distributions we work with can be thought of as functions (but as such not locally integrable, so it is not obvious how to get the distribution from the function).

1.9 The Zariski closure of the lattice-cross restricted to a time-like or space-like quadrant and the completeness of a system of unimodular functions. Denote by

$$\mathbb{Z}_+ := \{1, 2, 3, \ldots\}, \qquad \mathbb{Z}_- := \{-1, -2, -3, \ldots\},$$
$$\mathbb{Z}_{+,0} := \{0, 1, 2, \ldots\}, \qquad \mathbb{Z}_{-,0} := \{0, -1, -2, \ldots\}$$

the sets of positive, negative, nonnegative, and nonpositive integers, respectively. We consider the following four portions of the lattice-cross $\Lambda_{\alpha,\beta}$ given by (1.5.1):

$$\Lambda^{++}_{\alpha,\beta} := (\alpha\mathbb{Z}_{+,0} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z}_+), \qquad \Lambda^{+-}_{\alpha,\beta} := (\alpha\mathbb{Z}_{+,0} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z}_-),$$

and

$$\Lambda^{-+}_{\alpha,\beta} := (\alpha\mathbb{Z}_{-,0} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z}_+), \qquad \Lambda^{--}_{\alpha,\beta} := (\alpha\mathbb{Z}_{-,0} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z}_-).$$

We first calculate the Zariski closure of the sets lying in the first and third quadrants, which are both time-like.

Theorem 1.9.1 (time-like)**.** *Fix positive reals* α, β, M*. Then for each point* $\xi^\star \in \mathbb{R}^2 \setminus \Lambda^{++}_{\alpha,\beta}$*, there exists a measure* $\mu \in \mathrm{AC}(\Gamma_M)$ *such that* $\hat{\mu} = 0$ *on* $\Lambda^{++}_{\alpha,\beta}$*, while at the same time* $\hat{\mu}(\xi^\star) \neq 0$*. Moreover, the same assertion holds provided that* $\Lambda^{++}_{\alpha,\beta}$ *is replaced by* $\Lambda^{--}_{\alpha,\beta}$*. In terms of Zariski closures, this means that*

$$\mathrm{zclos}_{\Gamma_M}(\Lambda^{++}_{\alpha,\beta}) = \Lambda^{++}_{\alpha,\beta},$$
$$\mathrm{zclos}_{\Gamma_M}(\Lambda^{--}_{\alpha,\beta}) = \Lambda^{--}_{\alpha,\beta}.$$

The proof of this theorem, which is presented in [14], requires careful handling of the H^1-BMO duality and the explicit calculation of the Fourier transform of the unimodular function $t \mapsto \mathrm{e}^{\mathrm{i}/t}$ as a tempered distribution.

We turn to the Zariski closures of the remaining two portions of the lattice-cross. We first write down the statement in terms of the weak-star closure of the linear span of a sequence of unimodular functions, and then explain what it means for the Zariski closure in the form of a corollary.

As for notation, let $H^\infty_+(\mathbb{R})$ denote the weak-star closed subspace of $L^\infty(\mathbb{R})$ that consists of those functions whose Poisson extension to the upper half-plane is holomorphic. It is perhaps not obvious that Theorems 1.8.2 and 1.8.6 entail the following result on the completeness of a system of inner functions.

Theorem 1.9.2. *Fix positive reals α, β. Then the functions*

$$\mathrm{e}^{\mathrm{i}\pi\alpha m t}, \quad \mathrm{e}^{-\mathrm{i}\pi\beta n/t}, \quad m, n = 0, 1, 2, \ldots,$$

which are elements of $H^\infty_+(\mathbb{R})$, span together a weak-star dense subspace of $H^\infty_+(\mathbb{R})$ if and only if $\alpha\beta \le 1$.

Note that the "only if" part of Theorem 1.9.2 is quite simple, as for instance the work in [4] shows that in case $\alpha\beta > 1$, the weak-star closure of the linear span in question has infinite codimension in $H^\infty_+(\mathbb{R})$. Hence the main thrust of the theorem is the "if" part. The proof of Theorem 1.9.2 is supplied in two instalments: for $\alpha\beta < 1$ in Subsection 11.1, and for $\alpha\beta = 1$ in Subsection 14.1.

The Cayley transform brings the upper half-plane to the unit disk $\mathbb{D}$, and identifies the space $H^\infty_+(\mathbb{R})$ with $H^\infty(\mathbb{D})$, the space of all bounded holomorphic functions on $\mathbb{D}$. For this reason, Theorem 1.9.2 is equivalent to the following assertion, which we state as a corollary.

Corollary 1.9.3. *Fix two positive reals λ_1, λ_2. Then the linear span of the functions*

$$\phi_1(z)^m = \exp\left(m\lambda_1\frac{z+1}{z-1}\right) \quad \text{and} \quad \phi_2(z)^n = \exp\left(n\lambda_2\frac{z-1}{z+1}\right), \quad m, n = 0, 1, 2, \ldots,$$

is weak-star dense in $H^\infty(\mathbb{D})$ if and only if $\lambda_1\lambda_2 \le \pi^2$.

We omit the trivial proof of the corollary.

Remark 1.9.4. Clearly, Corollary 1.9.3 supplies a complete and affirmative answer to Problems 1 and 2 in [20]. We recall the question from [20]: the issue was raised whether the algebra generated by the two inner functions

$$\phi_1(z) = \exp\left(\lambda_1\frac{z+1}{z-1}\right) \quad \text{and} \quad \phi_2(z) = \exp\left(\lambda_2\frac{z-1}{z+1}\right)$$

for $0 < \lambda_1, \lambda_2 < +\infty$, is weak-star dense in $H^\infty(\mathbb{D})$ if and only if $\lambda_1\lambda_2 \le \pi^2$. The "only if" was understood already in [20]. As pointed out in [20], it is a consequence of Corollary 1.9.3 that for $\lambda_1\lambda_2 \le \pi^2$, the lattice of the closed subspaces invariant with respect to multiplication by the two inner functions ϕ_1, ϕ_2 coincides with the usual shift invariant subspaces in the Hardy space $H^p(\mathbb{D})$, where $1 < p < +\infty$. It is remarkable that we do not need the whole algebra generated by the two inner functions to span the space $H^\infty(\mathbb{D})$.

We should stress that it is not possible to derive the assertion of Theorem 1.9.2 from Theorem 1.5.1. This is connected with the fact that there are elements of $L^\infty(\mathbb{R})$ with unbounded Hilbert transform. What is however possible to obtain based on Theorem 1.5.1 is the weak-star completeness in $\mathrm{BMO}_+(\mathbb{R})$ (analytic BMO on the line) of the system appearing in Theorem 1.9.2 for $\alpha\beta \le 1$. The two weak-star topologies are genuinely different, as it is not difficult to exhibit a sequence of functions in $H^\infty_+(\mathbb{R})$ which is weak-star complete in $\mathrm{BMO}_+(\mathbb{R})$ but not in $H^\infty_+(\mathbb{R})$. As already mentioned, we develop several new methods to handle the much finer weak-star topology of $H^\infty_+(\mathbb{R})$, such as calculating the commutator of the Hilbert transform and iterates of the transfer operator, a dynamical Hilbert kernel decomposition, monotonicity methods, totally positive matrices, as well as an estimate of the Hurwitz zeta function.

Theorem 1.9.2 can be restated in terms of uniqueness properties of solutions to the Klein–Gordon equation. Note that in the statement below, the pair $(\Lambda^{+-}_{\alpha,\beta}, \bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_-)$ can be replaced by $(\Lambda^{-+}_{\alpha,\beta}, \bar{\mathbb{R}}_- \times \bar{\mathbb{R}}_+)$ without perturbing the validity of the result.

Corollary 1.9.5. *Fix positive reals α, β, M with $\alpha\beta M^2 \le 4\pi^2$. Suppose that $u = \hat{\mu}$ solves the Klein–Gordon equation* (1.3.1)*, where μ is finite complex Borel measure on $\mathbb{R}^2$, which is assumed absolutely continuous with respect to one-dimensional Hausdorff measure. Then the values of u on the space-like quarter-plane $\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_-$ are determined by the values of u on the set $\Lambda^{+-}_{\alpha,\beta}$, which is the portion of the lattice-cross in the given quarter-plane. This property does not hold for $\alpha\beta M^2 > 4\pi^2$.*

This formulation is actually a consequence of the Zariski closure result of Corollary iii below, so we refer to the explanatory remarks that follow right after it.

Corollary 1.9.6 (space-like)**.** *Fix positive reals α, β, M. The following assertions are equivalent:*

(i) $\mathrm{zclos}_{\Gamma_M}(\Lambda^{+-}_{\alpha,\beta}) = \bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_-$,

(ii) $\mathrm{zclos}_{\Gamma_M}(\Lambda^{-+}_{\alpha,\beta}) = \bar{\mathbb{R}}_- \times \bar{\mathbb{R}}_+$,

(iii) $\alpha\beta M^2 \le 4\pi^2$.

Here, the main part of the equivalence (i)⇔(iii) is the implication (iii)⇒(i'), where (i') is as follows:

(i') $\mathrm{zclos}_{\Gamma_M}(\Lambda^{+-}_{\alpha,\beta}) \supset \bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_-$.

The latter implication can be understood in the following terms. Under the density condition (iii), any measure $\mu \in \mathrm{AC}(\Gamma_M)$ whose Fourier transform $\hat{\mu}$ vanishes on $\Lambda^{+-}_{\alpha,\beta}$, has the property that $\hat{\mu}$ actually vanishes on the entire space-like adjacent quarter-plane $\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_-$. This assertion is seen to be equipotent with Theorem 1.9.2, after a scaling argument which permits us to assume that $M := 2\pi$. Finally, to obtain the equality (i) from the inclusion (i') which results from Theorem 1.9.2, we need the fact that the Zariski closure operation is idempotent, in the sense that $\mathrm{zclos}_\Gamma^2 = \mathrm{zclos}_\Gamma$, plus the fact that the indicated quarter-planes are themselves Zariski closures, which we have from Proposition 1.7.2. The remaining equivalence (ii)$\Leftrightarrow$(iii) is, by a symmetry argument, the same as the the equivalence (i)$\Leftrightarrow$(iii).

Remark 1.9.7. Let us now check how Theorem 1.5.1 is an immediate consequence of the much deeper result of Corollary iii. First, an elementary argument (see [13], [4]) shows that $\mathrm{zclos}_{\Gamma_M}(\Lambda_{\alpha,\beta}) \neq \mathbb{R}^2$ for $\alpha\beta M^2 > 4\pi^2$, so that we just need to obtain the implication

$$\alpha\beta M^2 \le 4\pi^2 \implies \mathrm{zclos}_{\Gamma_M}(\Lambda_{\alpha,\beta}) = \mathbb{R}^2.$$

In view of Theorem 1.9.2,

$$\begin{aligned}&\alpha\beta M^2 \le 4\pi^2\\ &\quad\implies \mathrm{zclos}_{\Gamma_M}(\Lambda_{\alpha,\beta}) = \mathrm{zclos}_{\Gamma_M}(\Lambda^{+-}_{\alpha,\beta} \cup \Lambda^{-+}_{\alpha,\beta}) \supset (\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_-) \cup (\bar{\mathbb{R}}_- \times \bar{\mathbb{R}}_+) \supset \mathbb{R} \times \{0\},\end{aligned}$$

and Theorem 1.5.1 becomes a consequence of Proposition 1.7.1 together with the above-mentioned idempotent property $\mathrm{zclos}_\Gamma^2 = \mathrm{zclos}_\Gamma$.

2 Basic properties of the dynamics of Gauss-type maps on intervals

2.1 Notation for intervals. For a positive real γ, let $I_\gamma :=]-\gamma, \gamma[$ denote the corresponding symmetric open interval, and let $I_\gamma^+ :=]0, \gamma[$ be the positive side of the interval I_γ. At times, we will need the half-open intervals $\tilde{I}_\gamma :=]-\gamma, \gamma]$ and $\tilde{I}_\gamma^+ := [0, \gamma[$, as well as the closed intervals $\bar{I}_\gamma := [-\gamma, \gamma]$ and $\bar{I}_\gamma^+ := [0, \gamma]$.

2.2 Dual action notation. For a Lebesgue measurable subset E of the real line $\mathbb{R}$, we write

$$\langle f, g\rangle_E := \int_E f(t)g(t)\mathrm{d}t,$$

whenever$fg \in L^1(E)$. This will be of interest mainly when E is an open interval, and in this case, we use the same notation to describe the dual action of a distribution on a test function. For a set $E \subset \mathbb{R}$, 1_E stands for the characteristic function of E, which equals 1 on E and vanishes elsewhere. So, in particular, we see that

$$\langle f, g\rangle_E = \langle 1_E f, g\rangle_{\mathbb{R}} = \langle 1_E f, 1_E g\rangle_{\mathbb{R}}.$$

2.3 Gauss-type maps on intervals. For background material in Ergodic Theory, we refer to the book [5].

For $N = 2, 3, 4, \ldots$, the N**-step wandering subset** is given by

$$\mathcal{E}_{\beta,N} := \{x \in \bar{I}_\beta : \tau_\beta^n(x) \in \bar{I}_\beta \text{ for } n = 1, \ldots, N-1\}, \tag{2.3.1}$$

where $\tau_\beta^n := \tau_\beta \circ \cdots \circ \tau_\beta$ (n-fold composition). We also agree that $\mathcal{E}_{\beta,1} := \bar{I}_\beta$. The sets $\mathcal{E}_{\beta,N}$ get smaller as N increases, and we form their intersections

$$\mathcal{E}_{\beta,\infty} := \bigcap_{N=1}^{+\infty} \mathcal{E}_{\beta,N}. \tag{2.3.2}$$

On a given interval, the **cone of positive functions** consists of all integrable functions f with $f \geq 0$ a.e. on the respective interval. Similarly, we say that f is positive if $f \geq 0$ a.e. on the given interval, allowing for nonintegrable functions.

Proposition 2.3.1. *Fix* $0 < \beta \leq 1$. *Then we have the following assertions:*

(i) *The operators* $\mathbf{T}_\beta : L^1(I_1) \to L^1(I_1)$ *and* $\mathfrak{T}_\beta : L^1(I_1) \to L^1(I_1)$ *are both norm contractions, which preserve the respective cones of positive functions.*

(ii) *On the positive functions,* $\mathfrak{T}_\beta$ *acts isometrically with respect to the* $L^1(I_1)$ *norm.*

(iii) *If* $\mathcal{E}_{\beta,N}$ *denotes the* N*-step wandering subset given by* (2.3.1) *above, then* $\mathbf{T}_\beta^N f = \mathfrak{T}_\beta^N(1_{\mathcal{E}_{\beta,N}} f)$ *for* $f \in L^1(I_1)$ *and* $N = 1, 2, 3, \ldots$.

(iv) *For* $0 < \beta < 1$, *and* $f \in L^1(I_1)$, *we have that* $\|\mathbf{T}_\beta^N f\|_{L^1(I_1)} \to 0$ *as* $N \to +\infty$. *In particular,* $|\mathcal{E}_{\beta,N}| \to 0$ *as* $N \to +\infty$.

(v) *For* $\beta = 1$ *and* $f \in L^1(I_1)$ *with mean* $\langle f, 1\rangle_{I_1} = 0$, *we have that* $\|\mathbf{T}_1^N f\|_{L^1(I_1)} \to 0$ *as* $N \to +\infty$.

(vi) *For* $\beta = 1$ *and* $f \in L^1(I_1)$, *we have that* $\|1_{I_\eta} \mathbf{T}_1^N f\|_{L^1(I_1)} \to 0$ *as* $N \to +\infty$ *for each real* η *with* $0 < \eta < 1$.

This is a conglomerate of ingredients from Propositions 3.4.1, 3.10.1, 3.11.3, 3.13.1, 3.13.2, and 3.13.3 in [14].

2.4 An elementary observation extending the domain of definition for $\mathbf{T}_\beta$. We begin with the following elementary observation.

Observation. The subtransfer and transfer operators $\mathbf{T}_\beta$ and $\mathfrak{T}_\beta$, initially defined on L^1 functions, make sense for wider classes of functions. Indeed, if $f \geq 0$, then the formulae (1.8.3) and (1.8.5) make sense pointwise, with values in the extended nonnegative reals $[0, +\infty]$. More generally, if f is complex-valued, we may use the triangle inequality to dominate the convergence of $\mathbf{T}_\beta f$ by that of $\mathbf{T}_\beta |f|$. This entails that $\mathbf{T}_\beta f$ is well-defined a.e. if $\mathbf{T}_\beta |f| < +\infty$ holds a.e. The same goes for $\mathfrak{T}_\beta$ of course.

This means that $\mathbf{T}_\beta f$ will be well-defined for many functions f, not necessarily in $L^1(I_1)$.

2.5 Symmetry preservation of the subtransfer operator $\mathbf{T}_\beta$. The property that $\mathbf{T}_\beta$ preserves symmetry on $L^1(I_1)$ holds much more generally.

Proposition 2.5.1. *Fix $0 < \beta \le 1$. To the extent that $\mathbf{T}_\beta f$ is well-defined pointwise, we have the following:*
(i) *If f is odd, then $\mathbf{T}_\beta f$ is odd as well.*
(ii) *If f is even, then $\mathbf{T}_\beta f$ is even as well.*

This follows from Proposition 3.6.1 in [14].

Along with the symmetry, we can add constraints like monotonicity and convexity. Under such constraints on f, the pointwise values of $\mathbf{T}_\beta f$ are guaranteed to exist, and the constraint is preserved under $\mathbf{T}_\beta$.

Proposition 2.5.2. *Fix $0 < \beta \le 1$. We have the following:*
(i) *If $f : I_1 \to \mathbb{R}$ is odd and (strictly) increasing, then so is $\mathbf{T}_\beta f$.*
(ii) *If $f : I_1 \to \mathbb{R}$ is even and convex, and if $f \ge 0$, then so is $\mathbf{T}_\beta f$.*

This follows from Propositions 3.7.1 and 3.7.2 in [14].

2.6 Preservation of point values of continuous functions under $\mathbf{T}_\beta$. For γ with $0 < \gamma < +\infty$, let $C(\bar{I}_\gamma)$ denote the space of continuous functions on the compact symmetric interval $\bar{I}_\gamma = [-\gamma, \gamma]$.

Proposition 2.6.1. *Fix $0 < \beta \le 1$. If $f \in C(\bar{I}_\beta)$, then $\mathbf{T}_\beta f \in C(\bar{I}_1)$. Moreover, if in addition, f is odd, then $\mathbf{T}_\beta f(1) = \beta f(\beta)$.*

This result combines Propositions 3.8.1 and 3.8.2 in [14].

2.7 Subinvariance of certain key functions. Next, we consider the $\mathbf{T}_\beta$-iterates of the function

$$\kappa_\alpha(x) := \frac{\alpha}{\alpha^2 - x^2}, \quad x \in I_1, \tag{2.7.1}$$

where α is assumed confined to the interval $0 < \alpha \le 1$. This function is not in $L^1(I_1)$, although it is in $L^{1,\infty}(I_1)$. However, by the observation made in Subsection 2.4, we may still calculate the expression $\mathbf{T}_\beta \kappa_\alpha$ pointwise wherever $\mathbf{T}_\beta|\kappa_\alpha|(x) < +\infty$. Note that $\kappa_1(x)\mathrm{d}x$ is the invariant measure for the transformation $\tau_1(x) = \{-1/x\}_2$, which in terms of the transfer operator $\mathbf{T}_1$ means that $\mathbf{T}_1\kappa_1 = \kappa_1$.

Proposition 2.7.1. *Fix* $0 < \beta \le 1$. *For the function* $\kappa_\beta(x) = \beta/(\beta^2 - x^2)$, *we have that*

$$\mathbf{T}_\beta\kappa_\beta(x) = \mathbf{T}_\beta|\kappa_\beta|(x) = \kappa_1(x) = \frac{1}{1-x^2}, \quad a.e.\ x \in I_1.$$

As for the function $\kappa_1(x) = (1-x^2)^{-1}$, *we have the estimate*

$$0 \le \mathbf{T}_\beta^n\kappa_1(x) \le \beta^n\,\kappa_1(x) = \frac{\beta^n}{1-x^2}, \quad x \in I_1,\ n = 1, 2, 3, \ldots,$$

which for $0 < \beta < 1$ *may be replaced by the uniform estimate*

$$\mathbf{T}_\beta^n\kappa_1(x) \le \frac{2\beta^n}{1-\beta}, \quad x \in I_1,\ n = 1, 2, 3, \ldots.$$

Remark 2.7.2. As noted earlier, for $\beta = 1$, we have the equality $\mathbf{T}_1\kappa_1 = \kappa_1$.

3 Background material: The Hilbert transform on the line and related spaces.

3.1 The Szegő projections and the Hardy H^1-space. For a reference on the basic facts of Hardy spaces and BMO (bounded mean oscillation), we refer to, e.g., the monographs of Duren and Garnett [6], [9], as well as those of Stein [26], [27], and Stein and Weiss [28].

Let $H^1_+(\mathbb{R})$ and $H^1_-(\mathbb{R})$ be the subspaces of $L^1(\mathbb{R})$ consisting of those functions whose Poisson extensions to the upper half plane

$$\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$$

are holomorphic and conjugate-holomorphic, respectively. Here, we use the term conjugate-holomorphic (or anti-holomorphic) to mean that the complex conjugate of the function in question is holomorphic.

It is well-known that any function $f \in H^1_+(\mathbb{R})$ has vanishing integral,

$$\langle f, 1\rangle_{\mathbb{R}} = \int_{\mathbb{R}} f(t)\mathrm{d}t = 0, \quad f \in H^1_+(\mathbb{R}). \tag{3.1.1}$$

In other words, $H^1_+(\mathbb{R}) \subset L^1_0(\mathbb{R})$, where

$$L^1_0(\mathbb{R}) := \{f \in L^1(\mathbb{R}) : \langle f, 1\rangle_{\mathbb{R}} = 0\}. \tag{3.1.2}$$

By a version of Liouville's theorem,

$$H^1_+(\mathbb{R}) \cap H^1_-(\mathbb{R}) = \{0\},$$

which allows us to think of the space

$$H^1_{\circledast}(\mathbb{R}) := H^1_+(\mathbb{R}) \oplus H^1_-(\mathbb{R})$$

as a linear subspace of $L^1_0(\mathbb{R})$. We will call $H^1_{\circledast}(\mathbb{R})$ the **real H^1-space** of the line $\mathbb{R}$, although it is $\mathbb{C}$-linear and the elements are generally complex-valued. It is not difficult to show that $H^1_{\circledast}(\mathbb{R})$ is norm dense as a subspace of $L^1_0(\mathbb{R})$. The elements of $f \in H^1_{\circledast}(\mathbb{R})$ are just the functions $f \in L^1_0(\mathbb{R})$ which may be written in the form

$$f = f_1 + f_2, \quad \text{where } f_1 \in H^1_+(\mathbb{R}),\ f_2 \in H^1_-(\mathbb{R}). \tag{3.1.3}$$

As already mentioned, the decomposition (3.1.3) is unique. As for notation, we let $\mathbf{P}_+$ and $\mathbf{P}_-$ denote the projections $\mathbf{P}_+ f := f_1$ and $\mathbf{P}_- f := f_2$ in the decomposition (3.1.3). These **Szegő projections $\mathbf{P}_+$, $\mathbf{P}_-$** can of course be extended beyond this $H^1_{\circledast}(\mathbb{R})$ setting; more about this in the following subsection.

3.2 The Hilbert and the modified Hilbert transform. With respect to the dual action

$$\langle f, g\rangle_{\mathbb{R}} = \int_{\mathbb{R}} f(t)g(t)\mathrm{d}t,$$

we may identify the dual space of $H^1_{\circledast}(\mathbb{R})$ with $\mathrm{BMO}(\mathbb{R})/\mathbb{C}$. Here, $\mathrm{BMO}(\mathbb{R})$ is the space of functions of **bounded mean oscillation**; this is the celebrated Fefferman duality theorem [7], [8]. As for notation, we write "$\cdot/\mathbb{C}$" to express that we mod out with respect to the constant functions. One of the main results in the theory is the theorem of Fefferman and Stein [8] which tells us that

$$\mathrm{BMO}(\mathbb{R}) = L^\infty(\mathbb{R}) + \tilde{\mathbf{H}} L^\infty(\mathbb{R}), \tag{3.2.1}$$

or, in words, a function g is in $\mathrm{BMO}(\mathbb{R})$ if and only if it may be written in the form $g = g_1 + \tilde{\mathbf{H}} g_2$, where $g_1, g_2 \in L^\infty(\mathbb{R})$. Here, $\tilde{\mathbf{H}}$ denotes the **modified Hilbert transform**, defined for $f \in L^\infty(\mathbb{R})$ by the formula

$$\begin{aligned} \tilde{\mathbf{H}} f(x) &:= \frac{1}{\pi}\mathrm{pv}\int_{\mathbb{R}} f(t)\Big\{\frac{1}{x-t} + \frac{t}{1+t^2}\Big\}\mathrm{d}t \\ &= \lim_{\epsilon\to 0^+}\int_{\mathbb{R}\setminus[x-\epsilon, x+\epsilon]} f(t)\Big\{\frac{1}{x-t} + \frac{t}{1+t^2}\Big\}\mathrm{d}t. \end{aligned} \tag{3.2.2}$$

The decomposition (3.2.1) is clearly not unique. The non-uniqueness of the decomposition is equal to the intersection space

$$H^\infty_{\circledast}(\mathbb{R}) := L^\infty(\mathbb{R}) \cap \tilde{\mathbf{H}} L^\infty(\mathbb{R}), \tag{3.2.3}$$

which we refer to as the **real H^∞-space**.

We should compare the modified Hilbert transform $\tilde{\mathbf{H}}$ with the standard Hilbert transform $\mathbf{H}$, which acts boundedly on $L^p(\mathbb{R})$ for $1 < p < +\infty$, and maps $L^1(\mathbb{R})$ into $L^{1,\infty}(\mathbb{R})$ for $p = 1$. Here, $L^{1,\infty}(\mathbb{R})$ denotes the weak-L^1 space; see, e.g., (1.8.9). The Hilbert transform of a function f, assumed integrable on the line $\mathbb{R}$ with respect to the measure $(1+t^2)^{-1/2}\mathrm{d}t$, is defined as the principal value integral

$$\mathbf{H}f(x) := \frac{1}{\pi}\mathrm{pv}\int_{\mathbb{R}} f(t)\frac{\mathrm{d}t}{x-t} = \lim_{\epsilon\to 0^+}\frac{1}{\pi}\int_{\mathbb{R}\setminus[x-\epsilon,x+\epsilon]} f(t)\frac{\mathrm{d}t}{x-t}. \tag{3.2.4}$$

If $f \in L^p(\mathbb{R})$, where $1 \le p < +\infty$, then both $\mathbf{H}f$ and $\tilde{\mathbf{H}}f$ are well-defined a.e., and it is easy to see that the difference $\tilde{\mathbf{H}}f - \mathbf{H}f$ is equal to a constant. It is often useful to think of the natural harmonic extensions of the Hilbert transforms $\mathbf{H}f$ and $\tilde{\mathbf{H}}f$ to the upper half-plane $\mathbb{C}_+$ given by

$$\begin{aligned}\mathbf{H}f(z) &:= \frac{1}{\pi}\int_{\mathbb{R}} \frac{\operatorname{Re} z - t}{|z-t|^2} f(t)\mathrm{d}t,\\ \tilde{\mathbf{H}}f(z) &:= \frac{1}{\pi}\int_{\mathbb{R}} \left\{\frac{\operatorname{Re} z - t}{|z-t|^2} + \frac{t}{t^2+1}\right\} f(t)\mathrm{d}t.\end{aligned} \tag{3.2.5}$$

So, as a matter of normalization, we have that $\tilde{\mathbf{H}}f(\mathrm{i}) = 0$. This tells us the value of the constant mentioned above: $\tilde{\mathbf{H}}f - \mathbf{H}f = -\mathbf{H}f(\mathrm{i})$.

Returning to the real H^1-space, we note the following characterization of the space in terms of the Hilbert transform: for $f \in L^1(\mathbb{R})$,

$$f \in H^1_{\circledast}(\mathbb{R}) \iff f \in L^1_0(\mathbb{R}) \quad \text{and} \quad \mathbf{H}f \in L^1_0(\mathbb{R}). \tag{3.2.6}$$

The Szegő projections $\mathbf{P}_+$ and $\mathbf{P}_-$ which were mentioned in Subsection 3.1 are more generally defined in terms of the Hilbert transform:

$$\mathbf{P}_+ f := \frac{1}{2}(f + \mathrm{i}\mathbf{H}f), \quad \mathbf{P}_- f := \frac{1}{2}(f - \mathrm{i}\mathbf{H}f). \tag{3.2.7}$$

In a similar manner, for $f \in L^\infty(\mathbb{R})$, based on the modified Hilbert transform $\tilde{\mathbf{H}}$ we may define the corresponding projections (which are actually projections modulo the constant functions)

$$\tilde{\mathbf{P}}_+ f := \frac{1}{2}(f + \mathrm{i}\tilde{\mathbf{H}}f), \quad \tilde{\mathbf{P}}_- f := \frac{1}{2}(f - \mathrm{i}\tilde{\mathbf{H}}f), \tag{3.2.8}$$

so that, by definition, $f = \tilde{\mathbf{P}}_+ f + \tilde{\mathbf{P}}_- f$.

4 Operators on a space of distributions on the line.

4.1 The Hilbert transform on L^1. For background material on the Hilbert transform and related topics; see, e.g., the monographs [6], [9], [26], [27] and [28].

It is well-known that the Hilbert transform as given by (3.2.4) maps $\mathbf{H} : L^1(\mathbb{R}) \to L^{1,\infty}(\mathbb{R})$. Since functions in $L^{1,\infty}(\mathbb{R})$ have no obvious interpretation as distributions, it is better to define $\mathbf{H}f$ right away as a distribution for $f \in L^1(\mathbb{R})$.

The distributional interpretation is as follows:

$$\langle \varphi, \mathbf{H}f \rangle_{\mathbb{R}} := -\langle \mathbf{H}\varphi, f \rangle_{\mathbb{R}}, \tag{4.1.1}$$

where φ is a test function with compact support, and $f \in L^1(\mathbb{R})$. Note that $\mathbf{H}\varphi$, the Hilbert transform of the test function, may be defined without the need of the principal value integral:

$$\mathbf{H}\varphi(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\varphi(x-t) - \varphi(x+t)}{t} \mathrm{d}t;$$

it is a C^∞ function on $\mathbb{R}$ with decay $\mathbf{H}\varphi(x) = \mathrm{O}(|x|^{-1})$ as $|x| \to +\infty$. As a consequence, it is clear from (4.1.1) how to extend the notion $\mathbf{H}f$ to functions f with $x \mapsto (1+x^2)^{-1/2} f(x)$ in $L^1(\mathbb{R})$. Note that as a result of the work of Kolmogorov, the equivalence (3.2.6) holds equally well when $\mathbf{H}f$ is interpreted as a distribution and as a weak-L^1 function.

4.2 The real H^∞ space. The real H^∞ space, denoted by $H^\infty_\circledast(\mathbb{R})$, was introduced in (3.2.3). An alternative definition is to say that it consists of all functions $f \in L^\infty(\mathbb{R})$ of the form

$$f = f_1 + f_2, \quad f_1 \in H^\infty_+(\mathbb{R}),\ f_2 \in H^\infty_-(\mathbb{R}). \tag{4.2.1}$$

Here, $H^\infty_+(\mathbb{R})$ consists of all functions in $L^\infty(\mathbb{R})$ whose Poisson extension to the upper half-plane is holomorphic, while $H^\infty_-(\mathbb{R})$ consists of all functions in $L^\infty(\mathbb{R})$ whose Poisson extension to the upper half-plane is conjugate-holomorphic (alternatively, the Poisson extension to the lower half-plane is holomorphic). The decomposition (4.2.1) is unique up to additive constants.

4.3 The predual of the real H^∞ space. We shall be concerned with the following space of distributions on the line $\mathbb{R}$:

$$\mathfrak{L}(\mathbb{R}) := L^1(\mathbb{R}) + \mathbf{H}L^1_0(\mathbb{R}),$$

which we supply with the appropriate norm (1.8.8), that is,

$$\|u\|_{\mathfrak{L}(\mathbb{R})} := \inf\{ \|f\|_{L^1(\mathbb{R})} + \|g\|_{L^1(\mathbb{R})} : u = f + \mathbf{H}g,\ f \in L^1(\mathbb{R}),\ g \in L^1_0(\mathbb{R})\},$$

which makes $\mathfrak{L}(\mathbb{R})$ a Banach space.

We recall that $L^1_0(\mathbb{R})$ is the codimension-one subspace of $L^1(\mathbb{R})$ which consists of the functions whose integral over $\mathbb{R}$ vanishes. Given $f \in L^1(\mathbb{R})$ and $g \in L^1_0(\mathbb{R})$, the action of $u := f + \mathbf{H}g$ on a test function φ is (compare with (4.1.1))

$$\langle \varphi, f + \mathbf{H}g\rangle_{\mathbb{R}} = \langle \varphi, f\rangle_{\mathbb{R}} - \langle \mathbf{H}\varphi, g\rangle_{\mathbb{R}} = \langle \varphi, f\rangle_{\mathbb{R}} - \langle \tilde{\mathbf{H}}\varphi, g\rangle_{\mathbb{R}}; \tag{4.3.1}$$

we observe that the last identity uses that $\langle 1, g\rangle_{\mathbb{R}} = 0$ and the fact that the functions $\tilde{\mathbf{H}}\varphi$ and $\mathbf{H}\varphi$ differ by a constant.

It remains to identify the dual space of $\mathfrak{L}(\mathbb{R})$ with $H^\infty_\circledast(\mathbb{R})$.

Proposition 4.3.1. *Each continuous linear functional $\mathfrak{L}(\mathbb{R}) \to \mathbb{C}$ corresponds to a function $\varphi \in H^\infty_\circledast(\mathbb{R})$ in accordance with* (4.3.1). *In short, the dual space of $\mathfrak{L}(\mathbb{R})$ equals $H^\infty_\circledast(\mathbb{R})$.*

This is Proposition 7.3.1 in [14]. We will refer to $\mathfrak{L}(\mathbb{R})$ as the predual of the real H^∞ space, although there are alternative preduals.

Remark 4.3.2. Since an L^1-function f gives rise to an absolutely continuous measure $f(t)\mathrm{d}t$, it is natural to think of $\mathfrak{L}(\mathbb{R})$ as embedded into the space $\mathfrak{M}(\mathbb{R}) := M(\mathbb{R}) + \mathbf{H}M_0(\mathbb{R})$, where $M(\mathbb{R})$ denotes the space of complex-valued finite Borel measures on $\mathbb{R}$, and $M_0(\mathbb{R})$ is the subspace of measures $\mu \in M(\mathbb{R})$ with $\mu(\mathbb{R}) = 0$. The Hilbert transforms of singular measures noticeably differ from those of absolutely continuous measures; see [23].

4.4 The "valeur au point" function associated with an element of $\mathfrak{L}(\mathbb{R})$. We recall that $\mathfrak{L}(\mathbb{R})$ consists of distributions on the real line. However, the definition

$$\mathfrak{L}(\mathbb{R}) = L^1(\mathbb{R}) + \mathbf{H}L^1_0(\mathbb{R})$$

would allow us to also think of this space as a subspace of $L^{1,\infty}(\mathbb{R})$, the weak L^1-space. It is a natural question to ask for the relationship between the distribution and the $L^{1,\infty}$ function. We stick with the distribution theory definition of $\mathfrak{L}(\mathbb{R})$, and associate with a given $u \in \mathfrak{L}(\mathbb{R})$ the "valeur au point" function $\mathrm{vap}[u]$ at almost all points of the line. The precise definition of $\mathrm{vap}[u]$ is as follows.

Definition 4.4.1. For a fixed $x \in \mathbb{R}$, let $\chi = \chi_x$ be a compactly supported C^∞-smooth function on $\mathbb{R}$ with $\chi(t) = 1$ for all t in an open neighborhood of the point x. Also, let

$$P_{x+\mathrm{i}\epsilon}(t) := \pi^{-1}\frac{\epsilon}{\epsilon^2 + (x-t)^2}$$

be the Poisson kernel. The **valeur au point function** associated with the distribution u on $\mathbb{R}$ is the function $\mathrm{vap}[u] = \mathrm{vap}[u\chi]$ given by

$$\mathrm{vap}[u](x) := \lim_{\epsilon \to 0^+} \langle \chi P_{x+\mathrm{i}\epsilon}, u\rangle_{\mathbb{R}}, \quad x \in \mathbb{R}, \tag{4.4.1}$$

wherever the limit exists.

In principle, $\mathrm{vap}[u](x)$ might depend on the choice of the cut-off function χ. Lemma 7.4.2 in [14] guarantees that this is not the case, and that almost everywhere, it gives the same result as the weak-L^1 interpretation of the Hilbert transform on $L^1(\mathbb{R})$. A basic result is the following.

Proposition 4.4.2 (Kolmogorov)**.** *The mapping*

$$\mathrm{vap}:\mathfrak{L}(\mathbb{R})\to L^{1,\infty}(\mathbb{R}),\quad u\mapsto \mathrm{vap}[u],$$

is injective and continuous.

This is a combination of Propositions 7.4.3 and 7.4.4 in [14].

4.5 The restriction of $\mathfrak{L}(\mathbb{R})$ to an interval. If u is a distribution on an open interval J, then the restriction of u to an open subinterval I, denoted $u|_I$, is the distribution defined by

$$\langle\varphi, u|_I\rangle_I := \langle\varphi, u\rangle_J,$$

where φ is a C^∞-smooth test function whose support is compact and contained in I.

Definition 4.5.1. Let I be an open interval of the real line. Then $u\in\mathfrak{L}(I)$ means by definition that u is a distribution on I such that there exists a distribution $v\in\mathfrak{L}(\mathbb{R})$ with $u = v\,|_I$.

Kolmogorov's theorem (Proposition 4.4.2) has a local version as well.

Proposition 4.5.2. (Kolmogorov) *Let I be a nonempty open interval of the line $\mathbb{R}$. Then the "valeur au point" mapping is injective and continuous*

$$\mathrm{vap}:\mathfrak{L}(I)\to L^{1,\infty}(I).$$

This is a combination of Corollaries 7.6.3 and 7.6.6 in [14].

5 Background material: Function spaces on the circle

5.1 The Hardy space H^1 on the circle. Let $L^1(\mathbb{R}/2\mathbb{Z})$ denote the space of (equivalence classes of) 2-periodic Borel measurable functions $f:\mathbb{R}\to\mathbb{C}$ subject to the integrability condition

$$\|f\|_{L^1(\mathbb{R}/2\mathbb{Z})} := \int_{I_1}|f(t)|\mathrm{d}t < +\infty,$$

where $I_1 =]-1,1[$ as before. Via the exponential mapping $t\mapsto \mathrm{e}^{\mathrm{i}\pi t}$, which is 2-periodic and maps the real line $\mathbb{R}$ onto the unit circle $\mathbb{T}$, we may identify the space $L^1(\mathbb{R}/2\mathbb{Z})$ with the standard Lebesgue space $L^1(\mathbb{T})$ of the unit circle. This will allow us to develop the elements of Hardy space theory in the setting of 2-periodic functions. We shall need the subspace $L^1_0(\mathbb{R}/2\mathbb{Z})$ consisting of all $f \in L^1(\mathbb{R}/2\mathbb{Z})$ with

$$\langle f, 1\rangle_{I_1} = \int_{I_1} f(t)\mathrm{d}t = 0;$$

it has codimension 1 in $L^1(\mathbb{R}/2\mathbb{Z})$. The Hardy space $H^1_+(\mathbb{R}/2\mathbb{Z})$ is defined as the subspace of $L^1(\mathbb{R}/2\mathbb{Z})$ consisting of functions $g \in L^1(\mathbb{R}/2\mathbb{Z})$ whose Poisson extension to the unit disk $\mathbb{D}$ is holomorphic and vanishes at the origin, and analogously, $H^1_-(\mathbb{R}/2\mathbb{Z})$ consists of the functions g in $L^1(\mathbb{R}/2Z)$ whose complex conjugate $\bar{g}$ is in $H^1_+(\mathbb{R}/2\mathbb{Z})$. In terms of the Poisson extensions to the upper half-plane instead, $f \in H^1_+(\mathbb{R}/2\mathbb{Z})$ if the extension is holomorphic and vanishes at $+\mathrm{i}\infty$, whereas $f \in H^1_-(\mathbb{R}/2\mathbb{Z})$ if the extension is conjugate-holomorphic and vanishes at $+\mathrm{i}\infty$. We then introduce the **real H^1-space**

$$H^1_\circledast(\mathbb{R}/2\mathbb{Z}) := H^1_+(\mathbb{R}/2\mathbb{Z}) \oplus H^1_-(\mathbb{R}/2\mathbb{Z}),$$

where we think of the elements of the sum space as 2-periodic functions on $\mathbb{R}$. As before the symbol $\oplus$ means direct sum, which is possible since

$$H^1_+(\mathbb{R}/2\mathbb{Z}) \cap H^1_-(\mathbb{R}/2\mathbb{Z}) = \{0\}.$$

We note that, for instance, $H^1_\circledast(\mathbb{R}/2\mathbb{Z}) \subset L^1_0(\mathbb{R}/2\mathbb{Z})$.

5.2 The Hilbert transform on 2-periodic functions and distributions. For $f \in L^1(\mathbb{R}/2\mathbb{Z})$, we define $\mathbf{H}_2$ be the convolution operator by

$$\mathbf{H}_2 f(x) := \frac{1}{2}\,\mathrm{pv}\int_{I_1} f(t)\cot\frac{\pi(x-t)}{2}\mathrm{d}t, \tag{5.2.1}$$

where again pv stands for principal value, which means we take the limit as $\epsilon \to 0^+$ of the integral where the set

$$\{x\} + 2\mathbb{Z} + [-\epsilon, \epsilon]$$

is removed from the interval $I_1 =]-1, 1[$. It is obvious from the periodicity of the cotangent function that $\mathbf{H}_2 f$, if it exists as a limit, is 2-periodic. Alternatively, by a change of variables, we have that

$$\mathbf{H}_2 f(x) = \frac{1}{2}\lim_{\epsilon\to 0^+}\int_{I_1\setminus I_\epsilon} f(x-t)\cot\frac{\pi t}{2}\mathrm{d}t, \tag{5.2.2}$$

where $I_\epsilon =]-\epsilon, \epsilon[$. It is well-known that the operator $\mathbf{H}_2$ is just the natural extension of the Hilbert transform $\mathbf{H}$ to the 2-periodic functions. We observe the peculiarity that $\mathbf{H}_2 1 = 0$, which follows from the fact that the cotangent function is odd. Like the situation for the real line $\mathbb{R}$, the periodic Hilbert transform $\mathbf{H}_2$ maps $L^1(\mathbb{R}/2\mathbb{Z})$ into the weak L^1-space $L^{1,\infty}(\mathbb{R}/2\mathbb{Z})$. However, to work within the framework of distribution theory, we proceed as follows.

Let $C^\infty(\mathbb{R}/2\mathbb{Z})$ denote the space of C^∞-smooth 2-periodic functions on $\mathbb{R}$. It is easy to see that

$$\varphi \in C^\infty(\mathbb{R}/2\mathbb{Z}) \implies \mathbf{H}_2\varphi \in C^\infty(\mathbb{R}/2\mathbb{Z}).$$

To emphasize the importance of the circle $\mathbb{T} \cong \mathbb{R}/2\mathbb{Z}$, we write

$$\langle f, g\rangle_{\mathbb{R}/2\mathbb{Z}} := \int_{-1}^{1} f(t)g(t)\mathrm{d}t, \tag{5.2.3}$$

for the dual action when f and g are 2-periodic.

Definition 5.2.1. For a test function $\varphi \in C^\infty(\mathbb{R}/2\mathbb{Z})$ and a distribution u on the circle $\mathbb{R}/2\mathbb{Z}$, we set

$$\langle \varphi, \mathbf{H}_2 u\rangle_{\mathbb{R}/2\mathbb{Z}} := -\langle \mathbf{H}_2\varphi, u\rangle_{\mathbb{R}/2\mathbb{Z}}.$$

This defines the Hilbert transform $\mathbf{H}_2 u$ for any distribution u on the circle $\mathbb{R}/2\mathbb{Z}$.

5.3 The real H^∞-space of the circle. The **real H^∞-space** on the circle $\mathbb{R}/2\mathbb{Z}$ is denoted by $H^\infty_\circledast(\mathbb{R}/2\mathbb{Z})$, and consists of all the functions in $H^\infty_\circledast(\mathbb{R})$ that are 2-periodic. It has the characterization

$$f \in H^\infty_\circledast(\mathbb{R}/2\mathbb{Z}) \iff f, \mathbf{H}_2 f \in L^\infty(\mathbb{R}/2\mathbb{Z}). \tag{5.3.1}$$

5.4 A predual of 2-periodic real H^∞. We set

$$\mathfrak{L}(\mathbb{R}/2\mathbb{Z}) := L^1(\mathbb{R}/2\mathbb{Z}) + \mathbf{H}_2 L^1_0(\mathbb{R}/2\mathbb{Z}),$$

which should be understood as a space of 2-periodic distributions on the line $\mathbb{R}$. More precisely, if $u = f + \mathbf{H}_2 g$, where $f \in L^1(\mathbb{R}/2\mathbb{Z})$ and $g \in L^1_0(\mathbb{R}/2\mathbb{Z})$, then the action on a test function $\varphi \in C^\infty(\mathbb{R}/2\mathbb{Z})$ is given by

$$\langle \varphi, u\rangle_{\mathbb{R}/2\mathbb{Z}} := \langle \varphi, f\rangle_{\mathbb{R}/2\mathbb{Z}} - \langle \mathbf{H}_2\varphi, g\rangle_{\mathbb{R}/2\mathbb{Z}}. \tag{5.4.1}$$

But a 2-periodic distribution should be possible to think of as a distribution on the line, which means that we need to understand the action on standard test functions in $C^\infty_c(\mathbb{R})$. If $\psi \in C^\infty_c(\mathbb{R})$, we simply put

$$\langle \psi, u\rangle_{\mathbb{R}/2\mathbb{Z}} := \langle \Pi_2\psi, u\rangle_{\mathbb{R}/2\mathbb{Z}}, \tag{5.4.2}$$

where $\Pi_2\psi \in C^\infty(\mathbb{R}/2\mathbb{Z})$ is given by

$$\Pi_2\psi(x) := \sum_{j\in\mathbb{Z}} \psi(x+2j). \tag{5.4.3}$$

We will refer to Π_2 as the **periodization operator**.

As in the case of the line $\mathbb{R}$, we may identify $\mathfrak{L}(\mathbb{R}/2\mathbb{Z})$ with the predual of the real H^∞-space $H^\infty_\circledast(\mathbb{R}/2\mathbb{Z})$:

$$\mathfrak{L}(\mathbb{R}/2\mathbb{Z})^* = H^\infty_\circledast(\mathbb{R}/2\mathbb{Z})$$

with respect to the standard dual action $\langle\cdot,\cdot\rangle_{\mathbb{R}/2\mathbb{Z}}$.

The definition of the "valeur au point function" $\mathrm{vap}[u]$ makes sense for $u \in \mathfrak{L}(\mathbb{R}/2\mathbb{Z})$ and as in the case of the line, it does not depend on the choice of the particular cut-off function. The following assertion is the analogue of Proposition 4.4.2; the proof is omitted.

Proposition 5.4.1 (Kolmogorov). *The "valeur au point" mapping*

$$\mathrm{vap} : \mathfrak{L}(\mathbb{R}/2\mathbb{Z}) \to L^{1,\infty}(\mathbb{R}/2\mathbb{Z}), \quad u \mapsto \mathrm{vap}[u],$$

is injective and continuous.

6 A sum of two preduals and its localization to intervals

6.1 The sum space $\mathfrak{L}(\mathbb{R}) \oplus \mathfrak{L}(\mathbb{R}/2\mathbb{Z})$. Suppose u is a distribution on the line $\mathbb{R}$ of the form

$$u = v + w, \quad \text{where } v \in \mathfrak{L}(\mathbb{R}),\ w \in \mathfrak{L}(\mathbb{R}/2\mathbb{Z}). \tag{6.1.1}$$

The natural question appears as to whether the distributions v, w on the right-hand side are unique for a given u. This is indeed so (Proposition 9.1.1 in [14]):

$$\mathfrak{L}(\mathbb{R}) \cap \mathfrak{L}(\mathbb{R}/2\mathbb{Z}) = \{0\}. \tag{6.1.2}$$

In view of (6.1.2), it makes sense to write $\mathfrak{L}(\mathbb{R})\oplus\mathfrak{L}(\mathbb{R}/2\mathbb{Z})$ for the space of tempered distributions u of the form (6.1.1). We endow $\mathfrak{L}(\mathbb{R}) \oplus \mathfrak{L}(\mathbb{R}/2\mathbb{Z})$ with the induced Banach space norm

$$\|u\|_{\mathfrak{L}(\mathbb{R})\oplus\mathfrak{L}(\mathbb{R}/2\mathbb{Z})} := \|v\|_{\mathfrak{L}(\mathbb{R})} + \|w\|_{\mathfrak{L}(\mathbb{R}/2\mathbb{Z})},$$

provided u, v, w are related via (6.1.1).

6.2 The localization of $\mathfrak{L}(\mathbb{R})\oplus\mathfrak{L}(\mathbb{R}/2\mathbb{Z})$ to a bounded open interval. In the sense of Subsection 4.5, we may restrict a given distribution $u\in\mathfrak{L}(\mathbb{R})\oplus\mathfrak{L}(\mathbb{R}/2\mathbb{Z})$ to a given open interval I. It is natural to wonder what the space of such restrictions looks like.

Proposition 6.2.1. *The restriction of the space $\mathfrak{L}(\mathbb{R})\oplus\mathfrak{L}(\mathbb{R}/2\mathbb{Z})$ to a bounded open interval I equals the space $\mathfrak{L}(I)$.*

This is Proposition 9.2.1 in [14].

7 An involution, its adjoint, and the periodization operator.

7.1 An involution. For each positive real number β, let $\mathbf{J}_\beta$ denote the involution given by

$$\mathbf{J}_\beta f(x) := \frac{\beta}{x^2} f(-\beta/x), \quad x \in \mathbb{R}^\times.$$

We use the standard notation $\mathbb{R}^\times := \mathbb{R}\setminus\{0\}$. If $f \in L^1(\mathbb{R})$ and $\varphi \in L^\infty(\mathbb{R})$, the change-of-variables formula yields that

$$\langle\varphi, \mathbf{J}_\beta f\rangle_{\mathbb{R}} = \int_{\mathbb{R}} \varphi(t) f(-\beta/t)\,\frac{\beta \mathrm{d}t}{t^2} = \int_{\mathbb{R}} \varphi(-\beta/t) f(t)\,\mathrm{d}t = \langle \mathbf{J}_\beta^*\varphi, f\rangle_{\mathbb{R}}, \tag{7.1.1}$$

where $\mathbf{J}_\beta^*$ is the involution

$$\mathbf{J}_\beta^*\varphi(t) := \varphi(-\beta/t), \quad t \in \mathbb{R}^\times.$$

It is a consequence of the change-of-variables formula that $\mathbf{J}_\beta$ is an isometric isomorphism $L^1(\mathbb{R}) \to L^1(\mathbb{R})$.

Next, we extend $\mathbf{J}_\beta$ to a bounded operator $\mathfrak{L}(\mathbb{R}) \to \mathfrak{L}(\mathbb{R})$. The arguments in Subsection 10.1 of [14] show that the correct extension of $\mathbf{J}_\beta$ to an operator $\mathfrak{L}(\mathbb{R}) \to \mathfrak{L}(\mathbb{R})$ reads as follows.

Definition 7.1.1. For $u \in \mathfrak{L}(\mathbb{R})$ of the form $u = f + \mathbf{H}g \in \mathfrak{L}(\mathbb{R})$, where $f \in L^1(\mathbb{R})$ and $g \in L^1_0(\mathbb{R})$, we define $\mathbf{J}_\beta u$ to be the distribution on $\mathbb{R}$ given by the formula

$$\langle\varphi, \mathbf{J}_\beta u\rangle_{\mathbb{R}} = \langle\varphi, \mathbf{J}_\beta(f+\mathbf{H}g)\rangle_{\mathbb{R}} := \langle\varphi, \mathbf{J}_\beta f\rangle_{\mathbb{R}} + \langle\varphi, \mathbf{H}\mathbf{J}_\beta g\rangle_{\mathbb{R}} = \langle\varphi, \mathbf{J}_\beta f\rangle_{\mathbb{R}} - \langle\tilde{\mathbf{H}}\varphi, \mathbf{J}_\beta g\rangle_{\mathbb{R}},$$

for test functions $\varphi \in H^\infty_{\circledast}(\mathbb{R})$.

The involutive properties of $\mathbf{J}_\beta$ and its adjoint are then naturally preserved (Proposition 10.1.4 in [14]).

7.2 The periodization operator. We recall the definition of the **periodization operator** Π_2:

$$\Pi_2 f(x) := \sum_{j\in\mathbb{Z}} f(x+2j).$$

In (5.4.3), we defined the Π_2 on test functions. It is however clear that it remains well-defined with much less smoothness required of f. The terminology comes from the property that whenever it is well-defined, the function $\Pi_2 f$ is 2-periodic automatically. It is obvious from the definition that Π_2 acts contractively $L^1(\mathbb{R}) \to L^1(\mathbb{R}/2\mathbb{Z})$.

The basic property of the periodization operator is the following, for $f \in L^1(\mathbb{R})$ and $F \in L^\infty(\mathbb{R}/2\mathbb{Z})$, see, e.g., (10.2.2) in [14]:

$$\langle F, \Pi_2 f\rangle_{\mathbb{R}/2\mathbb{Z}} = \langle F, f\rangle_{\mathbb{R}}, \quad n \in \mathbb{Z}. \tag{7.2.1}$$

We need to extend Π_2 in a natural fashion to the space $\mathfrak{L}(\mathbb{R})$. If $\varphi \in C^\infty(\mathbb{R}/2\mathbb{Z})$ is a test function on the circle, we glance at (7.2.1), and for $u \in \mathfrak{L}(\mathbb{R})$ with $u = f + \mathbf{H}g$, where $f \in L^1(\mathbb{R})$ and $g \in L^1_0(\mathbb{R})$, we set

$$\langle \varphi, \Pi_2 u\rangle_{\mathbb{R}/2\mathbb{Z}} := \langle \varphi, u\rangle_{\mathbb{R}} = \langle \varphi, f\rangle_{\mathbb{R}} - \langle \tilde{\mathbf{H}}\varphi, g\rangle_{\mathbb{R}}. \tag{7.2.2}$$

This defines $\Pi_2 u$ as a distribution on the circle (compare with (4.3.1)).

Proposition 7.2.1. *For $u \in \mathfrak{L}(\mathbb{R})$ of the form $u = f + \mathbf{H}g$, where $f \in L^1(\mathbb{R})$ and $g \in L^1_0(\mathbb{R})$, we have that $\Pi_2 u = \Pi_2 f + \mathbf{H}_2 \Pi_2 g$. In particular, Π_2 maps $\mathfrak{L}(\mathbb{R}) \to \mathfrak{L}(\mathbb{R}/2\mathbb{Z})$ continuously.*

This is Proposition 10.2.2 in [14].

8 Reformulation of the spanning problem of Theorem 1.9.2

8.1 An equivalence. Let us write $\mathbb{Z}_{+,0} := \{0, 1, 2, \ldots\}$.

Lemma 8.1.1. *Fix $0 < \beta < +\infty$. Then the following statements are equivalent:*

(a) *The linear span of the functions*

$$e_n(t) := \mathrm{e}^{\mathrm{i}\pi n t}, \; e_m^{\langle\beta\rangle}(t) := \mathrm{e}^{-\mathrm{i}\pi\beta m/t}, \quad m, n \in \mathbb{Z}_{+,0},$$

is weak-star dense in $H^\infty_+(\mathbb{R})$.

(b) *For $f \in L^1_0(\mathbb{R})$, the following implication holds:*

$$\Pi_2 f, \; \Pi_2 \mathbf{J}_\beta f \in H^1_+(\mathbb{R}/2\mathbb{Z}) \Longrightarrow f \in H^1_+(\mathbb{R}).$$

Observe that the functions $e^{i\pi nt}$ and $e^{-i\pi\beta m/t}$ for $m, n \in \mathbb{Z}_{+,0}$ belong to $H^\infty_+(\mathbb{R})$, since they have bounded holomorphic extensions to $\mathbb{C}_+$. Thus that part (a) makes sense.

Proof of Lemma 8.1.1. With respect to the dual action $\langle\cdot,\cdot\rangle_{\mathbb{R}}$ on the line, the predual of $H^\infty_+(\mathbb{R})$ is the quotient space $L^1(\mathbb{R})/H^1_+(\mathbb{R})$. With this in mind, the assertion of part (a) is seen to be equivalent to the following: For any $f \in L^1(\mathbb{R})$, the implication

$$\{\forall m, n \in \mathbb{Z}_{+,0} : \langle e_n, f\rangle_{\mathbb{R}} = \langle e^{\langle\beta\rangle}_m, f\rangle_{\mathbb{R}} = 0\} \Longrightarrow f \in H^1_+(\mathbb{R}) \tag{8.1.1}$$

holds. By testing with, e.g., $n = 0$, we note that we might as well assume that $f \in L^1_0(\mathbb{R})$ in (8.1.1). By the basic property (7.2.1) of the periodization operator Π_2, we have that

$$\langle e_n, f\rangle_{\mathbb{R}} = \langle e_n, \Pi_2 f\rangle_{\mathbb{R}/2\mathbb{Z}}, \tag{8.1.2}$$

from which we conclude that

$$\{\forall n \in \mathbb{Z}_{+,0} : \langle e_n, f\rangle_{\mathbb{R}} = 0\} \Longleftrightarrow \Pi_2 f \in H^1_+(\mathbb{R}/2\mathbb{Z}).$$

Since $\mathbf{J}^*_\beta e_m = e^{\langle\beta\rangle}_m$, where $\mathbf{J}^*_\beta$ is the involution studied in Subsection 7.1, a repetition of the above gives that for $f \in L^1_0(\mathbb{R})$, we have the equivalence

$$\{\forall m \in \mathbb{Z}_{+,0} : \langle e^{\langle\beta\rangle}_m, f\rangle_{\mathbb{R}} = 0\} \Longleftrightarrow \Pi_2 \mathbf{J}_\beta f \in H^1_+(\mathbb{R}/2\mathbb{Z}).$$

By splitting the annihilation conditions in (8.1.1), we see that they are equivalent to having both $\Pi_2 f$ and $\Pi_2\mathbf{J}_\beta f$ in $H^1_+(\mathbb{R}/2\mathbb{Z})$. In other words, the statement in (a) and (b) are equivalent. □

Remark 8.1.2. By the argument involving point separation in $\mathbb{C}_+$ from [13], it is necessary that $\beta \le 1$ for part (a) of Lemma 8.1.1 to hold. Actually, as mentioned in the introduction, the methods of [4] supply infinitely many linearly independent counterexamples for $\beta > 1$.

Remark 8.1.3. If we think of $\Pi_2 f$ and $\Pi_2\mathbf{J}_\beta f$ as 2-periodic “shadows” of f and $\mathbf{J}_\beta f$, the issue at hand in part (b) of Lemma 8.1.1 is whether knowing that the two “shadows” are in the right space we may conclude the function comes from the space $H^1_+(\mathbb{R})$. We note here that the main result of [13] may be understood as the assertion that f is determined uniquely by the two “shadows” $\Pi_2 f$ and $\Pi_2\mathbf{J}_\beta f$ if and only if $\beta \le 1$.

8.2 An alternative statement in terms of the space $\mathfrak{L}(\mathbb{R})$. Let $\mathfrak{L}_0(\mathbb{R})$ denote the space

$$\mathfrak{L}_0(\mathbb{R}) := L^1_0(\mathbb{R}) + \mathbf{H}L^1_0(\mathbb{R}) \subset \mathfrak{L}(\mathbb{R}),$$

which has codimension 1 in $\mathfrak{L}(\mathbb{R})$.

Lemma 8.2.1. *Fix $0 < \beta \leq 1$. Then (a) $\Longrightarrow$ (b), where (a) and (b) are the following assertions:*

(a) *For $u \in \mathfrak{L}_0(\mathbb{R})$, the following implication holds:*

$$\Pi_2 u = \Pi_2 \mathbf{J}_\beta u = 0 \Longrightarrow u = 0.$$

(b) *For $f \in L^1_0(\mathbb{R})$, the following implication holds:*

$$\Pi_2 f,\ \Pi_2 \mathbf{J}_\beta f \in H^1_+(\mathbb{R}/2\mathbb{Z}) \Longrightarrow f \in H^1_+(\mathbb{R}).$$

Proof. We connect $u \in \mathfrak{L}_0(\mathbb{R})$ with $f \in L^1_0(\mathbb{R})$ via the conjugate-holomorphic Szegő projection $u := \mathbf{P}_- f = \frac{1}{2}(f - \mathrm{i}\mathbf{H}f)$. If $\Pi_2 f \in H^1_+(\mathbb{R}/2\mathbb{Z})$, then by a Liouville-type argument, $\Pi_2 u = 0$ holds. Analogously, if $\Pi_2 \mathbf{J}_\beta f \in H^1_+(\mathbb{R}/2\mathbb{Z})$, then we obtain that $\Pi_2 \mathbf{J}_\beta u = 0$. So, from the implication of part (a), we obtain from the assumptions in (b) that $u = 0$, that is, that $f \in H_+(\mathbb{R})$. This means that the implication of (a) implies that of (b), as claimed. □

Remark 8.2.2. Condition (b) of Lemma 8.2.1 has acquired the same general appearance as in the analysis of the $L^\infty(\mathbb{R})$ problem, but at the cost of considering the larger space $\mathfrak{L}_0(\mathbb{R})$ in place of $L^1_0(\mathbb{R})$. This is unavoidable, as the weak-star topology of the real Hardy space $H^\infty_\circledast(\mathbb{R})$ is finer than that of $L^\infty(\mathbb{R})$. Our proof of Theorem 1.9.2 passes through Lemmas 8.1.1 and 8.2.1, and we ultimately show that the implication (a) of Lemma 8.2.1 is valid for $0 < \beta \leq 1$. It then follows from Lemmas 8.1.1 and 8.2.1 that assertion (a) of Lemma 8.1.1 is valid in the range $0 < \beta \leq 1$. In its turn, the proof that the implication (a) of Lemma 8.2.1 holds for $0 < \beta \leq 1$ is based on an extension of ergodic theory for Gauss-type maps, developed in Sections 9–14.

9 A subtransfer operator on a space of distributions.

9.1 Restrictions of $\mathfrak{L}(\mathbb{R})$ to a symmetric interval and to its complement. In Subsection 4.5, we dealt with the restriction of $\mathfrak{L}(\mathbb{R})$ to an open interval. Here we deal with the restriction to the complement of a closed interval as well. For a positive real parameter γ, we consider the symmetric interval I_γ and its closure $\bar{I}_\gamma$ as in Subsection 2.1,

$$I_\gamma =]-\gamma, \gamma[, \quad \bar{I}_\gamma = [-\gamma, \gamma].$$

We recall that by Definition 4.5.1, the space $\mathfrak{L}(I_\gamma)$ is defined as

$$\mathfrak{L}(I_\gamma) := \{u \in \mathcal{D}'(I_\gamma) : \exists U \in \mathfrak{L}(\mathbb{R}) \text{ with } U|_{I_\gamma} = u\}$$

and analogously we may define $\mathfrak{L}(\mathbb{R} \setminus \bar{I}_\gamma)$ for the complementary interval $\mathbb{R} \setminus \bar{I}_\gamma$:

$$\mathfrak{L}(\mathbb{R} \setminus \bar{I}_\gamma) := \{u \in \mathcal{D}'(\mathbb{R} \setminus \bar{I}_\gamma) : \exists U \in \mathfrak{L}(\mathbb{R}) \text{ with } U|_{\mathbb{R}\setminus\bar{I}_\gamma} = u\}.$$

Here, $\mathcal{D}'$ has the standard interpretation of the space of Schwartzian distributions on the given interval. Of course, in the sense of distribution theory, taking the restriction to an open subset has the interpretation of considering the linear functional restricted to test functions supported on that given open subset. The norm on each of the spaces $\mathfrak{L}(I_\gamma)$ and $\mathfrak{L}(\mathbb{R} \setminus \bar{I}_\gamma)$ is the associated quotient norm, where we mod out with respect to the distributions in $\mathfrak{L}(\mathbb{R})$ whose support is contained in the complementary closed set; cf. Subsection 4.5.

We will need to work with restrictions to I_γ and $\mathbb{R} \setminus \bar{I}_\gamma$ repeatedly, so it is a good idea to introduce appropriate notation.

Definition 9.1.1. We let $\mathbf{R}_\gamma$ denote the operation of restricting a distribution to the interval I_γ. Analogously, we let $\mathbf{R}_\gamma^\dagger$ denote the operation of restricting a distribution to the open set $\mathbb{R} \setminus \bar{I}_\gamma$.

9.2 The involution on the local spaces. We need to understand the action of the involution $\mathbf{J}_\beta$ defined in Subsection 7.1 on the local spaces $\mathfrak{L}(I_\gamma)$ and $\mathfrak{L}(\mathbb{R} \setminus \bar{I}_\gamma)$.

Proposition 9.2.1. *Fix $0 < \beta, \gamma < +\infty$. The involution $\mathbf{J}_\beta$ defines continuous maps*

$$\mathbf{J}_\beta : \mathfrak{L}(I_\gamma) \to \mathfrak{L}(\mathbb{R} \setminus \bar{I}_{\beta/\gamma}) \quad \textit{and} \quad \mathbf{J}_\beta : \mathfrak{L}(\mathbb{R} \setminus \bar{I}_\gamma) \to \mathfrak{L}(I_{\beta/\gamma}).$$

Proof. The assertion is rather immediate from the mapping properties of $\mathbf{J}_\beta$, see Subsection 7.1, and the localization procedure. □

9.3 Splitting of the periodization operator. We split the periodization operator Π_2 in two parts: $\Pi_2 = \mathbf{I} + \Sigma_2$, where $\mathbf{I}$ is the identity and Σ_2 is the operator defined by

$$\Sigma_2 u(x) := \sum_{j \in \mathbb{Z}^\times} u(x + 2j),$$

whenever the right-hand side is meaningful in the sense of distributions. Here, we use the notation $\mathbb{Z}^\times := \mathbb{Z} \setminus \{0\}$. By using Proposition 7.2.1, the proof of the following proposition is immediate.

Proposition 9.3.1. *The operator* Σ_2 *maps* $\mathfrak{L}(\mathbb{R}) \to \mathfrak{L}(\mathbb{R}) \oplus \mathfrak{L}(\mathbb{R}/2\mathbb{Z})$ *continuously.*

Definition 9.3.2. Let

$$\Sigma_2^{①} : \mathfrak{L}(\mathbb{R} \setminus \bar{I}_1) \to \mathfrak{L}(I_1)$$

be defined as follows. Given a distribution $u \in \mathfrak{L}(\mathbb{R} \setminus \bar{I}_1)$, we find a $U \in \mathfrak{L}(\mathbb{R})$ whose restriction is $\mathbf{R}_1 U = u$. Then, using Proposition 6.2.1, we set

$$\Sigma_2^{①} u := \mathbf{R}_1 \Sigma_2 U \in \mathbf{R}_1(\mathfrak{L}(\mathbb{R}) \oplus \mathfrak{L}(\mathbb{R}/2\mathbb{Z})) = \mathfrak{L}(I_1).$$

We will call $\Sigma_2^{①}$ the **compression** of Σ_2. However, we still need to verify that this definition is consistent, that is, that the right-hand side $\mathbf{R}_1 \Sigma_2 U$ is independent of the choice of the extension U.

Proposition 9.3.3. *The operator* $\Sigma_2^{①} : \mathfrak{L}(\mathbb{R} \setminus \bar{I}_1) \to \mathfrak{L}(I_1)$ *is well-defined and bounded. Moreover, we have that* $\mathbf{R}_1 \Sigma_2 U = \Sigma_2^{①} \mathbf{R}_1^{\dagger} U$ *holds for* $U \in \mathfrak{L}(\mathbb{R})$.

Proof. To see that $\Sigma_2^{①}$ is well-defined, we need to check that if $U \in \mathfrak{L}(\mathbb{R})$ and its restriction to $\mathbb{R} \setminus \bar{I}_1$ vanishes (this means that $\operatorname{supp} U \subset \bar{I}_1$), then $\mathbf{R}_1 \Sigma_2 U = 0$. From the definition of the operator Σ_2, we know that

$$\operatorname{supp} \Sigma_2 U \subset \operatorname{supp} U + 2\mathbb{Z}^{\times} \subset \bar{I}_1 + 2\mathbb{Z}^{\times} = \mathbb{R} \setminus I_1.$$

In particular, the restriction to I_1 of $\Sigma_2 U$ vanishes, as required. Similarly, we argue that $\Sigma_2^{①}$ is bounded, based on Proposition 6.2.1 and Definition 9.3.2. Finally, the asserted identity

$$\mathbf{R}_1 \Sigma_2 U = \Sigma_2^{①} \mathbf{R}_1^{\dagger} U$$

just expresses how the operator $\Sigma_2^{①}$ is defined. □

9.4 Further analysis of the uniqueness problem. The complementary restriction operators have the following properties:

$$\mathbf{R}_1^{\dagger} \mathbf{J}_{\beta} u = \mathbf{J}_{\beta} \mathbf{R}_{\beta} u, \quad u \in \mathfrak{L}(\mathbb{R}), \tag{9.4.1}$$

and, for $0 < \beta \le \gamma < +\infty$,

$$\mathbf{R}_1^{\dagger} \mathbf{J}_{\beta} u = \mathbf{J}_{\beta} \mathbf{R}_{\beta} u, \quad u \in \mathfrak{L}(I_{\gamma}). \tag{9.4.2}$$

They will help us analyze further the tentative implication (a) of Lemma 8.2.1.

Proposition 9.4.1. *Fix $0 < \beta \le 1$. Suppose that for $u \in \mathfrak{L}(\mathbb{R})$ we have $\Pi_2 u = 0$ and $\Pi_2 \mathbf{J}_\beta u = 0$. Then the restrictions $u_0 := \mathbf{R}_1 u \in \mathfrak{L}(I_1)$ and $u_1 := \mathbf{R}_1^\dagger u \in \mathfrak{L}(\mathbb{R} \setminus \bar{I}_1)$ each solve the equations*

$$u_0 = \Sigma_2^{\textcircled{1}} \mathbf{J}_\beta \mathbf{R}_\beta \Sigma_2^{\textcircled{1}} \mathbf{J}_\beta \mathbf{R}_\beta u_0, \quad u_1 = \mathbf{R}_1^\dagger \mathbf{J}_\beta \Sigma_2^{\textcircled{1}} \mathbf{R}_1^\dagger \mathbf{J}_\beta \Sigma_2^{\textcircled{1}} u_1,$$

and are given in terms of each other by

$$u_0 = -\Sigma_2^{\textcircled{1}} u_1, \quad u_1 = -\mathbf{R}_1^\dagger \mathbf{J}_\beta \Sigma_2^{\textcircled{1}} \mathbf{J}_\beta \mathbf{R}_\beta u_0.$$

Proof. To begin with, we write the given conditions $\Pi_2 u = 0$ and $\Pi_2 \mathbf{J}_\beta u = 0$ in the form

$$u = -\Sigma_2 u, \quad \mathbf{J}_\beta u = -\Sigma_2 \mathbf{J}_\beta u;$$

then, we restrict to the interval I_1:

$$\mathbf{R}_1 u = -\Sigma_2^{\textcircled{1}} \mathbf{R}_1^\dagger u, \quad \mathbf{R}_1 \mathbf{J}_\beta u = \mathbf{J}_\beta \mathbf{R}_\beta^\dagger u = -\Sigma_2^{\textcircled{1}} \mathbf{R}_1^\dagger \mathbf{J}_\beta u = -\Sigma_2^{\textcircled{1}} \mathbf{J}_\beta \mathbf{R}_\beta u.$$

Now we simplify the second condition a little by applying $\mathbf{J}_\beta$ to both sides:

$$\mathbf{R}_1 u = -\Sigma_2^{\textcircled{1}} \mathbf{R}_1^\dagger u, \quad \mathbf{R}_\beta^\dagger u = -\mathbf{J}_\beta \Sigma_2^{\textcircled{1}} \mathbf{J}_\beta \mathbf{R}_\beta u. \tag{9.4.3}$$

By combining these two identities in two separate ways, we find that

$$\mathbf{R}_1 u = \Sigma_2^{\textcircled{1}} \mathbf{J}_\beta \mathbf{R}_\beta \Sigma_2^{\textcircled{1}} \mathbf{J}_\beta \mathbf{R}_\beta u, \quad \mathbf{R}_\beta^\dagger u = \mathbf{J}_\beta \Sigma_2^{\textcircled{1}} \mathbf{R}_1^\dagger \mathbf{J}_\beta \Sigma_2^{\textcircled{1}} \mathbf{R}_1^\dagger u. \tag{9.4.4}$$

The assertions now follow, if we use (9.4.1) and (9.4.2). □

9.5 Two subtransfer operators on spaces of distributions. As usual, we assume that $0 < \beta \le 1$, and consider the operators

$$\mathbf{T}_\beta := \Sigma_2^{\textcircled{1}} \mathbf{J}_\beta \mathbf{R}_\beta : \mathfrak{L}(I_1) \to \mathfrak{L}(I_1), \tag{9.5.1}$$

and

$$\mathbf{W}_\beta := \mathbf{R}_1^\dagger \mathbf{J}_\beta \Sigma_2^{\textcircled{1}} : \mathfrak{L}(\mathbb{R} \setminus \bar{I}_1) \to \mathfrak{L}(\mathbb{R} \setminus \bar{I}_1). \tag{9.5.2}$$

These operators are extensions to the respective space of distributions of standard subtransfer operators. We already encountered $\mathbf{T}_\beta$ back in Subsection 1.8. Indeed, if $u \in L^1(I_1)$ and $v \in L^1(\mathbb{R} \setminus \bar{I}_1)$, then

$$\mathbf{T}_\beta u(x) = \sum_{j \in \mathbb{Z}^\times} \frac{\beta}{(x+2j)^2} u\Big(-\frac{\beta}{x+2j}\Big), \quad x \in I_1, \tag{9.5.3}$$

and

$$\mathbf{W}_\beta v\ (x) = \frac{\beta}{x^2}\sum_{j\in\mathbb{Z}^\times} v\ \Big(-\frac{\beta}{x}+2j\Big), \quad x\in\mathbb{R}\setminus\bar{I}_1. \tag{9.5.4}$$

In terms of these two subtransfer operators, the formulation of Proposition 9.4.1 simplifies considerably.

Proposition 9.5.1. *Fix* $0<\beta\le 1$*. Suppose that for* $u\in\mathfrak{L}(\mathbb{R})$ *we have* $\Pi_2 u=0$ *and* $\Pi_2\mathbf{J}_\beta u=0$*. Then the restrictions* $u_0:=\mathbf{R}_1 u\in\mathfrak{L}(I_1)$ *and* $u_1:=\mathbf{R}_1^\dagger u\in\mathfrak{L}(\mathbb{R}\setminus\bar{I}_1)$ *satisfy*

$$u_0=\mathbf{T}_\beta^2 u_0,\quad u_1=\mathbf{W}_\beta^2 u_1,\quad u_0=-\Sigma_2^{①} u_1,\quad u_1=-\mathbf{R}_1^\dagger\mathbf{J}_\beta\mathbf{T}_\beta u_0.$$

Proof. The proof is immediate from the definitions of $\mathbf{T}_\beta$ and $\mathbf{W}_\beta$. □

Proposition 9.5.2. *Suppose that for* $u\in\mathfrak{L}(\mathbb{R})$ *we have that the two restrictions vanish, i.e.,* $\mathbf{R}_1 u=0$ *and* $\mathbf{R}_1^\dagger u=0$ *as elements of* $\mathfrak{L}(I_1)$ *and* $\mathfrak{L}(\mathbb{R}\setminus\bar{I}_1)$*, respectively. Then* $u=0$*.*

Proof. The assumption implies that the valeur au point function vap$[u]$ vanishes on $\mathbb{R}^\times=\mathbb{R}\setminus\{0\}$. But then vap$[u]$ vanishes a.e., so that by Kolmogorov's Proposition 4.4.2, the claim $u=0$ follows. □

Remark 9.5.3. Fix $0<\beta\le 1$. Suppose that we are given a distribution $u_0\in\mathfrak{L}(I_1)$ which is a fixed point for the subtransfer operator: $\mathbf{T}_\beta^2 u_0=u_0$. Then the formula

$$u_1:=-\mathbf{R}_1^\dagger\mathbf{J}_\beta\mathbf{T}_\beta u_0$$

defines a distribution $u_1\in\mathfrak{L}(\mathbb{R}\setminus\bar{I}_1)$. We can quickly show that $u_1=\mathbf{W}_\beta^2 u_1$ and $u_0=-\Sigma_2^{①} u_1$, so that all the conditions of Proposition 9.5.1 are indeed accounted for. This means that all the solutions pairs (u_0,u_1) can be parametrized by the distribution u_0 alone, which is one of the key points in what follows.

9.6 The subtransfer operator $\mathbf{T}_\beta$ acting on valeur au point functions.

The subtransfer operators $\mathbf{T}_\beta$ and $\mathbf{W}_\beta$ are defined on distributions, but the formulas (9.5.3) and (9.5.4) often make sense pointwise in the almost everywhere sense for functions which are not summable on the respective interval. In the sequel, we focus on $\mathbf{T}_\beta$; the case of $\mathbf{W}_\beta$ is analogous. The question appears whether for a given distribution $u\in\mathfrak{L}(I_1)$, with valeur au point function vap$[u]\in L^{1,\infty}(I_1)$, the action of $\mathbf{T}_\beta$ on vap$[u]$ by formula (9.5.3) when it converges a.e. has the same result as taking vap$[\mathbf{T}_\beta u]$. To analyze this, we need the finite sum operators ($N = 2, 3, 4, \ldots$)

$$\mathbf{T}_\beta^{[N]} u(x) = \beta \sum_{j\in\mathbb{Z}^\times : |j|\le N} \frac{1-\frac{|j|}{N}}{(x+2j)^2}\, u\Big(-\frac{\beta}{x+2j}\Big), \quad x \in I_1. \tag{9.6.1}$$

This finite sum operator naturally acts both on the distribution u and on its valeur au point function vap$[u]$. As for the distributional interpretation, it is more properly understood as

$$\mathbf{T}_\beta^{[N]} := \Sigma_{2,N}^{①} \mathbf{J}_\beta \mathbf{R}_\beta, \tag{9.6.2}$$

where

$$\Sigma_{2,N}^{①} : \mathfrak{L}(\mathbb{R}\setminus \bar{I}_1) \to \mathfrak{L}(I_1)$$

is defined in the same fashion as $\Sigma_2^{①}$ based on the operator

$$\Sigma_{2,N} V(x) := \sum_{j\in\mathbb{Z}^\times : |j|\le N} \Big(1 - \frac{|j|}{N}\Big)\, V(x+2j),$$

which maps $\mathfrak{L}(\mathbb{R}) \to \mathfrak{L}(\mathbb{R})$. Whether we apply the operator valeur au point before or after $\mathbf{T}_\beta^{[N]}$ does not affect the result:

Proposition 9.6.1. *For $u \in \mathfrak{L}(I_1)$, we have that*

$$\mathrm{vap}[\mathbf{T}_\beta^{[N]} u](x) = \mathbf{T}_\beta^{[N]} \mathrm{vap}[u](x)$$

almost everywhere on the interval I_1.

Proof. Since the sum defining $\mathbf{T}_\beta^{[N]} u$ is finite, it suffices to handle a single term. This amounts to showing that

$$\mathrm{vap}\Big[\frac{\beta}{(x+2j)^2}\, u\Big(-\frac{\beta}{x+2j}\Big)\Big] = \frac{\beta}{(x+2j)^2}\, \mathrm{vap}[u]\Big(-\frac{\beta}{x+2j}\Big)$$

holds almost everywhere on I_1, which is elementary. □

We can now show that $\mathbf{T}_\beta^{[N]} u$ approximates $\mathbf{T}_\beta u$ as $N \to +\infty$ in terms of the valeur au point.

Proposition 9.6.2. *For $u \in \mathfrak{L}(I_1)$, we have that $\mathbf{T}_\beta^{[N]}(\mathrm{vap}[u]) \to \mathrm{vap}[\mathbf{T}_\beta u]$ as $N \to +\infty$ in the quasinorm of $L^{1,\infty}(I_1)$.*

Proof. We use the factorization (9.6.2), which says that $\mathbf{T}_\beta^{[N]} = \Sigma_{2,N}^{①} \mathbf{J}_\beta \mathbf{R}_\beta$. For $v \in \mathfrak{L}(\mathbb{R}\setminus \bar{I}_1)$, we have the convergence $\Sigma_{2,N}^{①} v \to \Sigma_2^{①} v$ in $\mathfrak{L}(I_1)$ as $N \to +\infty$ (cf. the proof of Proposition 6.2.1), which leads to $\mathbf{T}_\beta^{[N]} u \to \mathbf{T}_\beta u$ in $\mathfrak{L}(I_1)$ as $N \to +\infty$, for fixed $u \in \mathfrak{L}(I_1)$. The asserted convergence now follows from a combination of Proposition 9.6.2 with the weak-type estimate (Proposition 4.4.2). □

The Hilbert transform $\mathbf{H}$ maps $L^1_0(\mathbb{R}) \to \mathbf{H}L^1_0(\mathbb{R}) \subset \mathfrak{L}(\mathbb{R})$, and the restriction $\mathbf{R}_1$ maps $\mathfrak{L}(\mathbb{R}) \to \mathfrak{L}(I_1)$, so that $\mathbf{R}_1\mathbf{H}$ maps $L^1_0(\mathbb{R}) \to \mathfrak{L}(I_1)$. By considering also the function $P_{\mathrm{i}}(t) = \pi^{-1}(1+t^2)^{-1}$, which is in $L^1(\mathbb{R})$ but not in $L^1_0(\mathbb{R})$, we realize that $\mathbf{R}_1\mathbf{H}$ maps $L^1(\mathbb{R})$ into $\mathfrak{L}(I_1)$. We formalize this as a lemma.

Lemma 9.6.3. *The operator* $\mathbf{R}_1\mathbf{H}$ *maps* $L^1(\mathbb{R})$ *into* $\mathfrak{L}(I_1)$.

9.7 Norm expansiveness of the transfer operator on $\mathfrak{L}(I_1)$. We now supply the proof of Theorem 1.8.1.

Proof of Theorem 1.8.1. Since $\mathbf{T}_\beta = \Sigma_2^{①}\mathbf{J}_\beta\mathbf{R}_\beta$, and $\mathbf{R}_\beta$ maps $\mathfrak{L}(I_1)$ into $\mathfrak{L}(I_\beta)$ boundedly, it follows from Propositions 9.2.1 and 9.3.3 that $\mathbf{T}_\beta$ is also bounded.

We turn to the assertion that the norm of $\mathbf{T}_\beta$ exceeds 1 as an operator on $\mathfrak{L}(I_1)$. We recall that the norm on the space $\mathfrak{L}(I_1)$ is induced as a quotient norm based on (1.8.8). It is straightforward to identify the dual space of $\mathfrak{L}(\mathbb{R})$ with $H^\infty_{\circledast}(\mathbb{R})$, where the norm on $H^\infty_{\circledast}(\mathbb{R})$ that is dual to (1.8.8) is given by

$$\|g\|_{\circledast} := \max\left(\|g\|_{L^\infty(\mathbb{R})}, \inf_{c\in\mathbb{C}} \|\tilde{\mathbf{H}}g + c\|_{L^\infty(\mathbb{R})}\right).$$

In the same fashion, the dual space of $\mathfrak{L}(I_1)$ is identified with

$$H^\infty_{\circledast}(I_1) = \{g \in H^\infty_{\circledast}(\mathbb{R}) : \operatorname{supp} g \subset \bar{I}_1\},$$

and the corresponding norm on $H^\infty_{\circledast}(I_1)$ is $\|\cdot\|_{\circledast}$. Now, we know that

$$\|\mathbf{T}_1\| = \|\mathbf{T}_1^*\|,$$

where the adjoint

$$\mathbf{T}_1^* = \mathbf{R}_\beta^*\mathbf{J}_\beta^*(\Sigma_2^{①})^* : H^\infty_{\circledast}(I_1) \to H^\infty_{\circledast}(I_1)$$

and the space $H^\infty_{\circledast}(I_1)$ is endowed with the norm $\|\cdot\|_{\circledast}$. It is clear that involution $\mathbf{J}_\beta^*$ takes $H^\infty_{\circledast}(\mathbb{R}\setminus I_1)$ onto $H^\infty_{\circledast}(I_\beta)$ isometrically. In addition, $\mathbf{R}_\beta^*$ is just the canonical injection $H^\infty_{\circledast}(I_\beta) \to H^\infty_{\circledast}(I_1)$, which is isometric as well. In conclusion, we see that $\|\mathbf{T}_1^*\| = \|(\Sigma_2^{①})^*\|$, where $(\Sigma_2^{①})^*$ maps $H^\infty_{\circledast}(I_1) \to H^\infty_{\circledast}(\mathbb{R}\setminus \bar{I}_1)$ and both spaces are endowed with the norm $\|\cdot\|_{\circledast}$. Here, $H^\infty_{\circledast}(\mathbb{R}\setminus \bar{I}_1)$ denotes the subspace

$$H^\infty_{\circledast}(\mathbb{R}\setminus I_1) = \{g \in H^\infty_{\circledast}(\mathbb{R}) : \operatorname{supp} g \subset \mathbb{R}\setminus I_1\}.$$

We proceed to show that $\|(\Sigma_2^{①})^*\| > 1$. It should be mentioned that the space $H^\infty_{\circledast}(I_1)$ may be identified with $H^\infty_0(\mathbb{C}\setminus \bar{I}_1)$, the space of bounded holomorphic functions in the slit plane $\mathbb{C}\setminus I_1$ which also vanish at infinity. The identification is via the Cauchy transform, it is an isomorphism but it is not isometric; actually, arguably, the supremum norm on $\mathbb{C}\setminus I_1$ might be more natural than the norm on $H^\infty_{\circledast}(I_1)$

coming from the chosen norm (1.8.8) on $\mathfrak{L}(I_1)$. For $0 < \gamma \le 1$, let us consider the function

$$G_\gamma(z) = -z^2 + z(z+\gamma)\sqrt{\frac{z-\gamma}{z+\gamma}} + \frac{\gamma^2}{2},$$

where the square root is given by the principal branch of the argument in $\mathbb{C} \setminus \bar{\mathbb{R}}_-$. Then $G_\gamma \in H^\infty_0(\mathbb{C} \setminus \bar{I}_1)$, and the corresponding element of $H^\infty_{\circledast}(I_1)$ is

$$g_\gamma(x) := x\sqrt{\gamma^2 - x^2}\,1_{I_\gamma}(x),$$

which is odd, with Hilbert transform

$$\mathbf{H}g_\gamma(x) = x^2 - \frac{\gamma^2}{2} - 1_{\mathbb{R}\setminus I_\gamma}(x)|x|\sqrt{x^2-\gamma^2},$$

which is even. Both g_γ and $\mathbf{H}g_\gamma$ are Hölder continuous, with $\|g_\gamma\|_{L^\infty(\mathbb{R})} = \frac{1}{2}\gamma^2$ and

$$\inf_{c\in\mathbb{C}} \|\tilde{\mathbf{H}}g_\gamma + c\|_{L^\infty(\mathbb{R})} = \inf_{c_0\in\mathbb{C}} \|\mathbf{H}g_\gamma + c_0\|_{L^\infty(\mathbb{R})} = \|\mathbf{H}g_\gamma\|_{L^\infty(\mathbb{R})} = \frac{\gamma^2}{2},$$

which we see from a calculation of the range of the function $\mathbf{H}g_\gamma$, which equals the interval $[-\frac{1}{2}\gamma^2, \frac{1}{2}\gamma^2]$. This gives that $\|g_\gamma\|_{\circledast} = \frac{1}{2}\gamma^2$. We proceed to estimate the norm $\|(\Sigma_2^{①})^* g_\gamma\|_{\circledast}$ from below. From the definition of the operator $\Sigma_2^{①}$, we see that

$$(\Sigma_2^{①})^* g_\gamma(x) = \sum_{j\in\mathbb{Z}^\times} g_\gamma(x+2j), \qquad x \in \mathbb{R} \setminus \bar{I}_1,$$

and the corresponding Hilbert transform is

$$\mathbf{H}(\Sigma_2^{①})^* g_\gamma(x) = \sum_{j\in\mathbb{Z}^\times} \mathbf{H}g_\gamma(x+2j) = \sum_{j=1}^{+\infty}(\mathbf{H}g_\gamma(x+2j) + \mathbf{H}g_\gamma(x-2j)), \qquad x \in \mathbb{R}.$$

In the sums in the last display, it is important to consider symmetric partial sums. As the sum defining $(\Sigma_2^{①})^* g_\gamma(x)$ has at most one nonzero term for each given $x \in \mathbb{R}$, we see that $\|(\Sigma_2^{①})^* g_\gamma\|_{L^\infty(\mathbb{R}\setminus I_1)} = \frac{1}{2}\gamma^2$. In order to obtain the norm $\|(\Sigma_2^{①})^* g_\gamma\|_{\circledast}$, we proceed to evaluate

$$\inf_{c_0\in\mathbb{C}} \|\mathbf{H}(\Sigma_2^{①})^* g_\gamma(x) - c_0\|_{L^\infty(\mathbb{R})}.$$

Since the functions involved are Hölder continuous and real-valued, we realize that if we may find two points $x_1, x_2 \in \mathbb{R}$ with

$$\mathbf{H}(\Sigma_2^{①})^* g_\gamma(x_1) - \mathbf{H}(\Sigma_2^{①})^* g_\gamma(x_2) > \gamma^2, \tag{9.7.1}$$

then it would follow that

$$\inf_{c\in\mathbb{C}} \|\mathbf{H}(\Sigma_2^{①})^* g_\gamma(x) - c\|_{L^\infty(\mathbb{R})} > \frac{\gamma^2}{2},$$

and as a consequence, $\|\Sigma_2^{①}\| = \|(\Sigma_2^{①})^*\| > 1$, as claimed. We will restrict our attention to values of γ that are close to 0. Taylor's formula applied to the square root function shows that

$$\mathbf{H}g_\gamma(x) = \frac{\gamma^4}{8x^2} + \mathrm{O}\Big(\frac{\gamma^6}{x^4}\Big)$$

uniformly for $|x| > 1$. Since $\mathbf{H}g$ is even, the value at the point $x_2 := 2$ of the function $\mathbf{H}(\Sigma_2^{①})^* g_\gamma$ then equals

$$\begin{aligned}\mathbf{H}(\Sigma_2^{①})^* g_\gamma(x_2) = \mathbf{H}(\Sigma_2^{①})^* g_\gamma(2) &= \mathbf{H}g_\gamma(0) + \mathbf{H}g(2) + 2\sum_{j=2}^{+\infty}\mathbf{H}g_\gamma(2j)\\ &= -\frac{\gamma^2}{2} + \frac{\pi^2-3}{96}\gamma^4 + \mathrm{O}(\gamma^6),\end{aligned}$$

while the value at $x_1 := \gamma + 2N$ tends to the following value as $N \to +\infty$ through the integers:

$$\begin{aligned}\lim_{N\to+\infty}\mathbf{H}(\Sigma_2^{①})^* g_\gamma(\gamma+2N) &= \mathbf{H}g_\gamma(\gamma) + \sum_{j=1}^{+\infty}(\mathbf{H}g_\gamma(\gamma+2j) + \mathbf{H}g_\gamma(2j-\gamma))\\ &= \frac{\gamma^2}{2} + 2\sum_{j=1}^{+\infty}\mathbf{H}g_\gamma(2j) + \mathrm{O}(\gamma^6) = \frac{\gamma^2}{2} + \frac{\pi^2\gamma^4}{96} + \mathrm{O}(\gamma^6).\end{aligned}$$

Finally, since

$$\lim_{N\to+\infty}\mathbf{H}(\Sigma_2^{①})^* g_\gamma(\gamma+2N) - \mathbf{H}(\Sigma_2^{①})^* g_\gamma(2) = \gamma^2 + \frac{\gamma^4}{32} + \mathrm{O}(\gamma^6) > \gamma^2$$

for small values of γ, we obtain (9.7.1) for $x_1 = \gamma + 2N$ and $x_2 = 2$, provided γ is small and the positive integer N is large. □

9.8 An operator identity of commutator type. We recall that by Lemma 9.6.3, the operator $\mathbf{R}_1\mathbf{H}$ maps $L^1(\mathbb{R}) \to \mathfrak{L}(I_1)$.

Proposition 9.8.1. *Fix $0 < \beta \le 1$. For $f \in L^1(I_1)$, extended to vanish off I_1, we have the identity*

$$\mathbf{T}_\beta\mathbf{R}_1\mathbf{H}f = \mathbf{R}_1\mathbf{H}\mathfrak{T}_\beta f + \mathbf{T}_\beta\mathbf{R}_1\mathbf{H}\mathbf{J}_\beta\mathfrak{T}_\beta f - \mathbf{R}_1\mathbf{H}\mathbf{J}_\beta f,$$

as elements of the space $\mathfrak{L}(I_1)$.

Proof. In line with the presentation in the introduction, in particular, (1.8.2), we show that the claimed equality holds for $f = \delta_\xi$, i.e.,

$$\mathbf{T}_\beta\mathbf{R}_1\mathbf{H}(\delta_\xi - \mathbf{J}_\beta\mathfrak{T}_\beta\delta_\xi) = \mathbf{R}_1\mathbf{H}(\mathfrak{T}_\beta\delta_\xi - \mathbf{J}_\beta\delta_\xi) \tag{9.8.1}$$

holds, for almost every $\xi \in I_1$. The equality then holds for all $f \in L^1(I_1)$ by "averaging", as in (1.8.2). The canonical extension of the involution $\mathbf{J}_\beta$ and the transfer operator $\mathfrak{T}_\beta$ to such point masses δ_ξ reads:

$$\mathbf{J}_\beta\delta_\xi = \delta_{-\beta/\xi}, \quad \mathfrak{T}_\beta\delta_\xi = \delta_{\{-\beta/\xi\}_2}, \tag{9.8.2}$$

where, as in Subsection 2.1, the expression $\{t\}_2$ stands for the real number in the interval $]-1, 1]$ with the property that $t - \{t\}_2 \in 2\mathbb{Z}$. It follows that

$$\mathfrak{T}_\beta\delta_\xi - \mathbf{J}_\beta\delta_\xi = \delta_{\{-\beta/\xi\}_2} - \delta_{-\beta/\xi}, \quad \mathbf{J}_\beta\mathfrak{T}_\beta\delta_\xi = \delta_{-\beta/\{-\beta/\xi\}_2}, \tag{9.8.3}$$

so that for $\xi \in I_1 \setminus \bar{I}_\beta$,

$$\delta_\xi - \mathbf{J}_\beta\mathfrak{T}_\beta\delta_\xi = 0 \quad \text{and} \quad \mathfrak{T}_\beta\delta_\xi - \mathbf{J}_\beta\delta_\xi = 0.$$

It follows that for $\xi \in I_1 \setminus \bar{I}_\beta$, both the left-hand and the right-hand sides of the claimed equality (9.8.1) vanish, and the equality is trivially true. It remains to consider $\xi \in \bar{I}_\beta$. For $\eta \in \mathbb{R}$, the canonical extension of the Hilbert transform to a Dirac point mass at η is

$$\mathbf{H}\delta_\eta = \frac{1}{\pi}\,\mathrm{pv}\,\frac{1}{x-\eta},$$

and we calculate that for two points $\eta, \eta' \in \mathbb{R}^\times$,

$$\mathbf{T}_\beta\mathbf{R}_1\mathbf{H}(\delta_\eta - \delta_{\eta'}) = \frac{1}{\pi}\,\mathrm{pv}\sum_{j\in\mathbb{Z}^\times}\Big(\frac{1}{x+2j+\frac{\beta}{\eta}} - \frac{1}{x+2j+\frac{\beta}{\eta'}}\Big) \quad \text{on } I_1;$$

here, we may observe that the principal value interpretation is only needed with respect to at most two terms of the series. A particular instance is when

$$\frac{\beta}{\eta'} = \frac{\beta}{\eta} - 2k, \quad \text{for some } k \in \mathbb{Z},$$

in which case we get telescopic cancellation:

$$\begin{aligned}\mathbf{T}_\beta\mathbf{R}_1\mathbf{H}[\delta_\eta - \delta_{\eta'}] &= \frac{1}{\pi}\,\mathrm{pv}\sum_{j\in\mathbb{Z}^\times}\Big(\frac{1}{x+2j+\frac{\beta}{\eta}} - \frac{1}{x+2(j-k)+\frac{\beta}{\eta}}\Big)\\ &= \frac{1}{\pi}\,\mathrm{pv}\Big\{\frac{1}{x-2k+\frac{\beta}{\eta}} - \frac{1}{x+\frac{\beta}{\eta}}\Big\}\end{aligned}$$

on the interval I_1. We apply this to the case $\eta := \xi \in I_1$ and $\eta' := -\beta/\{-\beta/\xi\}_2$, in which case $k \in \mathbb{Z}$ is given by

$$2k = \frac{\beta}{\xi} + \{-\beta/\xi\}_2,$$

and obtain that

$$\mathbf{T}_\beta \mathbf{R}_1 \mathbf{H}(\delta_\xi - \delta_{-\beta/\{-\beta/\xi\}_2}) = \frac{1}{\pi}\,\mathrm{pv}\left\{\frac{1}{x-\{-\beta/\xi\}_2} - \frac{1}{x+\frac{\beta}{\xi}}\right\} \quad \text{on } I_1. \tag{9.8.4}$$

The natural requirements that $\xi \neq 0$ and that $\{-\beta/\xi\}_2 \neq 0$ exclude a countable collection of $\xi \in I_1$, which has Lebesgue measure 0. By (9.8.3), this is the left-hand side expression of (9.8.1), and another application (9.8.3) gives that the right-hand side expression of (9.8.1) equals

$$\mathbf{R}_1 \mathbf{H}(\delta_{\{-\beta/\xi\}_2} - \delta_{-\beta/\xi}) = \frac{1}{\pi}\,\mathrm{pv}\left\{\frac{1}{x-\{-\beta/\xi\}_2} - \frac{1}{x+\frac{\beta}{\xi}}\right\} \quad \text{on } I_1. \tag{9.8.5}$$

From equations (9.8.4) and (9.8.5), together with (9.8.3), we find that the claimed identity (9.8.1) is correct for almost every $\xi \in I_1$. □

Proposition 9.8.2. *Fix $0 < \beta \le 1$. For $f \in L^1(I_1)$, extended to vanish off I_1, we have the identity*

$$\mathbf{T}_\beta^n \mathbf{R}_1 \mathbf{H} f = \mathbf{R}_1 \mathbf{H} \mathfrak{T}_\beta^n f + \sum_{j=0}^{n-1}\{\mathbf{T}_\beta^{n-j}\mathbf{R}_1\mathbf{H}\mathbf{J}_\beta \mathfrak{T}_\beta^{j+1} f - \mathbf{T}_\beta^{n-j-1}\mathbf{R}_1\mathbf{H}\mathbf{J}_\beta\mathfrak{T}_\beta^j f\},$$

as elements of the space $\mathfrak{L}(I_1)$, for $n = 2, 3, 4, \ldots$.

Proof. We argue by induction. First, the identity actually holds for $n = 1$, by Proposition 9.8.1; here, the sum from $j = 0$ to $j = -1$ should be understood as 0.

Next, we assume that the identity is true for a positive integer n, thus we have

$$\begin{aligned}\mathbf{T}_\beta^{n+1}\mathbf{R}_1\mathbf{H} f &= \mathbf{T}_\beta \mathbf{T}_\beta^n \mathbf{R}_1 \mathbf{H} f \\ &= \mathbf{T}_\beta \mathbf{R}_1 \mathbf{H}\mathfrak{T}_\beta^n f + \sum_{j=0}^{n-1}\{\mathbf{T}_\beta^{n-j+1}\mathbf{R}_1\mathbf{H}\mathbf{J}_\beta\mathfrak{T}_\beta^{j+1} f - \mathbf{T}_\beta^{n-j}\mathbf{R}_1\mathbf{H}\mathbf{J}_\beta\mathfrak{T}_\beta^j f\}.\end{aligned} \tag{9.8.6}$$

By Proposition 9.8.1 again, we have

$$\mathbf{T}_\beta\mathbf{R}_1\mathbf{H}\mathfrak{T}_\beta^n f = \mathbf{R}_1\mathbf{H}\mathfrak{T}_\beta^{n+1} f + \mathbf{T}_\beta\mathbf{R}_1\mathbf{H}\mathbf{J}_\beta\mathfrak{T}_\beta^{n+1} f - \mathbf{R}_1\mathbf{H}\mathbf{J}_\beta\mathfrak{T}_\beta^n f,$$

and applied to (9.8.6), we obtain that

$$\begin{aligned}\mathbf{T}_\beta^{n+1}\mathbf{R}_1\mathbf{H} f &= \mathbf{R}_1\mathbf{H}\mathfrak{T}_\beta^{n+1} f + \mathbf{T}_\beta\mathbf{R}_1\mathbf{H}\mathbf{J}_\beta\mathfrak{T}_\beta^{n+1} f - \mathbf{R}_1\mathbf{H}\mathbf{J}_\beta\mathfrak{T}_\beta^n f \\ &\qquad + \sum_{j=0}^{n-1}\{\mathbf{T}_\beta^{n-j+1}\mathbf{R}_1\mathbf{H}\mathbf{J}_\beta\mathfrak{T}_\beta^{j+1} f - \mathbf{T}_\beta^{n-j}\mathbf{R}_1\mathbf{H}\mathbf{J}_\beta\mathfrak{T}_\beta^j f\} \\ &= \mathbf{R}_1\mathbf{H}\mathfrak{T}_\beta^{n+1} f + \sum_{j=0}^{n}\{\mathbf{T}_\beta^{n-j+1}\mathbf{R}_1\mathbf{H}\mathbf{J}_\beta\mathfrak{T}_\beta^{j+1} f - \mathbf{T}_\beta^{n-j}\mathbf{R}_1\mathbf{H}\mathbf{J}_\beta\mathfrak{T}_\beta^j f\}.\end{aligned}$$

The induction is complete. □

10 $\mathbf{T}_\beta$-iterates of Hilbert transforms

10.1 Smooth Hilbert transforms. We fix $0 < \beta \le 1$. Recall that for a function $g \in L^1(\mathbb{R})$, its Hilbert transform is

$$\mathbf{H}g(x) = \frac{1}{\pi}\,\mathrm{pv}\int_{\mathbb{R}} \frac{g(t)}{x-t}\mathrm{d}t, \qquad x \in \mathbb{R}. \tag{10.1.1}$$

In here, we are interested in the specific case when the function g vanishes on the interval I_β. Then the Hilbert transform $\mathbf{H}g$ is smooth on I_β, and there is no need to consider principal values when we restrict our attention to I_β. In terms of the involution

$$\mathbf{J}_\beta g(x) = \frac{\beta}{x^2}\, g\Big(-\frac{\beta}{x}\Big), \tag{10.1.2}$$

we see that $\mathbf{J}_\beta g \in L^1(I_1)$ and that

$$\mathbf{H}g(x) = \frac{1}{\pi}\int_{\mathbb{R}\setminus I_\beta} \frac{g(t)}{x-t}\mathrm{d}t = \frac{1}{\pi}\int_{I_1} \frac{t}{\beta+tx}\mathbf{J}_\beta g(t)\mathrm{d}t, \qquad x \in I_\beta; \tag{10.1.3}$$

the advantage is that we now integrate over the symmetric unit interval I_1. In terms of the kernel

$$Q_\beta(t,x) := \frac{t}{\beta+tx}$$

and the associated integral operator

$$\mathbf{Q}_\beta f(x) := \frac{1}{\pi}\int_{I_1} Q_\beta(t,x) f(t)\mathrm{d}t = \frac{1}{\pi}\int_{I_1}\frac{t}{\beta+tx} f(t)\mathrm{d}t, \qquad x \in I_\beta, \tag{10.1.4}$$

equation (10.1.3) simply asserts that

$$\mathbf{Q}_\beta f(x) = \mathbf{H}\mathbf{J}_\beta f(x), \qquad x \in I_\beta, \tag{10.1.5}$$

for $f \in L^1(I_1)$, extended to vanish off I_1. It is elementary to estimate that

$$|Q_\beta(t,x)| = \frac{|t|}{\beta+tx} \le \frac{2\beta}{\beta^2-x^2} = 2\kappa_\beta(x), \qquad x\in I_\beta,\ t\in \bar{I}_1, \tag{10.1.6}$$

where κ_β is as in (2.7.1), which yields that

$$|\mathbf{Q}_\beta f(x)| \le \frac{1}{\pi}\int_{I_1} |Q_\beta(t,x) f(t)|\mathrm{d}t \le \frac{2}{\pi}\,\|f\|_{L^1(I_1)}\kappa_\beta(x), \qquad x\in I_\beta. \tag{10.1.7}$$

In general, $\mathbf{Q}_\beta f$ is not in $L^1(I_\beta)$. But at least (10.1.7) guarantees that $\mathbf{Q}_\beta f$ is well-defined pointwise with an effective bound. We will want to consider the $\mathbf{T}_\beta$-iterates of the function $\mathbf{Q}_\beta f$. Since, as a matter of fact, the subtransfer operator $\mathbf{T}_\beta$ only cares about the values of the function in question on the interval I_β, we may use the above estimate (10.1.7) together with the observation made in Subsection 2.4 to see that the $\mathbf{T}_\beta$-iterates of $\mathbf{Q}_\beta f$ are well-defined pointwise. We can also provide an effective estimate of those iterates, which we first do for $0 < \beta < 1$.

Proposition 10.1.1. *Fix* $0 < \beta < 1$. *Suppose* $f \in L^1(I_1)$. *Then we have the estimate*

$$|\mathbf{T}_\beta^n \mathbf{Q}_\beta f(x)| \le \frac{4\beta^{n-1}}{\pi(1-\beta)} \|f\|_{L^1(I_1)}, \quad x \in I_1,\ n = 2, 3, 4, \ldots,$$

so that $\mathbf{T}_\beta^n \mathbf{Q}_\beta f \to 0$ *geometrically as* $n \to +\infty$, *uniformly on the interval* I_1.

Proof. As observed above, by definition, $\mathbf{T}_\beta g$ is only concerned with the behavior of g on the interval I_β. It follows from the positivity of the operator $\mathbf{T}_\beta$ that

$$|\mathbf{T}_\beta \mathbf{Q}_\beta f(x)| \le \frac{2}{\pi} \|f\|_{L^1(I_1)} \mathbf{T}_\beta \kappa_\beta(x) = \frac{2}{\pi} \|f\|_{L^1(I_1)} \kappa_1(x), \quad x \in I_1, \tag{10.1.8}$$

where in the last step, we used Proposition 2.7.1. Now, the same kind of argument of Proposition 2.7.1 yields

$$|\mathbf{T}_\beta^n \mathbf{Q}_\beta f(x)| \le \frac{2}{\pi} \|f\|_{L^1(I_1)} \mathbf{T}_\beta^{n-1} \kappa_1(x) \le \frac{4\beta^{n-1}}{\pi(1-\beta)} \|f\|_{L^1(I_1)}, \quad x \in I_1,$$

as claimed. □

For $\beta = 1$, the situation is much more involved.

Proposition 10.1.2. *Fix* $\beta = 1$. *Suppose* $f \in L^1(I_1)$. *We then have the estimate*

$$|\mathbf{T}_1^n \mathbf{Q}_1 f(x)| \le \frac{2}{\pi}(1 - x^2)^{-1} \|f\|_{L^1(I_1)}, \quad x \in I_1,\ n = 1, 2, 3, \ldots,$$

and in addition, $\mathbf{T}_1^n \mathbf{Q}_1 f(x) \to 0$ *as* $n \to +\infty$, *uniformly on compact subsets of* I_1.

Proof. The derivation of (10.1.8) applies also in the case $\beta = 1$, so that

$$|\mathbf{T}_1 \mathbf{Q}_1 f(x)| \le \frac{2}{\pi} \|f\|_{L^1(I_1)} \mathbf{T} \kappa_1(x) = \frac{2}{\pi} \|f\|_{L^1(I_1)} \kappa_1(x), \quad x \in I_1, \tag{10.1.9}$$

which is the claimed estimate for $n = 1$. For $n > 1$, we use the positivity of $\mathbf{T}_1$ again, to obtain from (10.1.9) that

$$|\mathbf{T}_1^n \mathbf{Q}_1 f(x)| \le \frac{2}{\pi} \|f\|_{L^1(I_1)} \mathbf{T}^{n-1} \kappa_1(x) = \frac{2}{\pi} \|f\|_{L^1(I_1)} \kappa_1(x), \quad x \in I_1, \tag{10.1.10}$$

which establishes the claimed estimate.

We proceed to obtain the uniform convergence to 0 locally on compact subsets of I_1. To this end, we use the representation (10.1.4) to see that

$$\mathbf{T}_1^n \mathbf{Q}_1 f(x) = \frac{1}{\pi} \int_{I_1} \mathbf{T}_1^n Q_1(t, \cdot)(x) f(t) \mathrm{d}t. \tag{10.1.11}$$

We verify that for $0 < a < 1$,

$$|Q_1(t,x)| \le Q_1(a,x) = \frac{a}{1+ax}, \quad t \in [0,a],\ x \in I_1,$$

and that

$$|Q_1(t,x)| \le -Q_1(-a,x) = \frac{a}{1-ax}, \quad t \in [-a,0],\ x \in I_1.$$

As a consequence, using the positivity of $\mathbf{T}_1$, we may derive that

$$|\mathbf{T}_1^n Q_1(t,\cdot)(x)| \le \mathbf{T}_1^n Q_1(a,\cdot)(x) \le 2\kappa_1(x), \quad t \in [0,a],\ x \in I_1,$$

and that

$$|\mathbf{T}_1^n Q_1(t,\cdot)(x)| \le \mathbf{T}_1^n(-Q_1(-a,\cdot))(x) \le 2\kappa_1(x), \quad t \in [-a,0],\ x \in I_1.$$

Next, we apply the triangle inequality to the integral (10.1.11):

$$\begin{aligned} |\mathbf{T}_1^n \mathbf{Q}_1 f(x)| &\le \frac{1}{\pi}\int_{I_a} |\mathbf{T}_1^n Q_1(t,\cdot)(x) f(t)|\mathrm{d}t + \frac{1}{\pi}\int_{I_1\setminus I_a} |\mathbf{T}_1^n Q_1(t,\cdot)(x) f(t)|\mathrm{d}t \\ &\le \frac{1}{\pi}\mathbf{T}_1^n Q_1(a,\cdot)(x)\int_{[0,a]} |f(t)|\mathrm{d}t \\ &\quad + \frac{1}{\pi}\mathbf{T}_1^n(-Q_1(-a,\cdot))(x)\int_{[-a,0]} |f(t)|\mathrm{d}t \\ &\quad + \frac{2}{\pi}\kappa_1(x)\int_{I_1\setminus I_a} |f(t)|\mathrm{d}t. \end{aligned} \tag{10.1.12}$$

Note that in the last term, we used the estimate (10.1.6) with $\beta = 1$. By Proposition 2.3.1(vi), $\mathbf{T}_1^n Q_1(a,\cdot) \to 0$ and $\mathbf{T}_1^n Q_1(-a,\cdot) \to 0$ as $n \to +\infty$ in the L^1 sense on compact subintervals of I_1. It is a consequence of the regularity of the functions $Q_1(a,\cdot)$ and $Q_1(-a,\cdot)$ that the convergence is actually uniform on compact subintervals. By fixing a so close to 1 that the rightmost integral of (10.1.12) is as small as we like, we see that $\mathbf{T}_1^n \mathbf{Q}_1 f \to 0$ as $n \to +\infty$, uniformly on compact subsets of I_1. This completes the proof. □

11 Asymptotic decay of the $\mathbf{T}_\beta$-orbit of a distribution in $\mathfrak{L}(I_1)$ for $0 < \beta < 1$.

11.1 An application of asymptotic decay for $0 < \beta < 1$. We now supply the argument which shows how, in the subcritical parameter regime $\alpha\beta < 1$, Theorem 1.9.2 follows from the asymptotic decay result Theorem 1.8.2, which is of extended ergodicity type.

Proof of Theorem 1.9.2 for $\alpha\beta < 1$. As observed right after the formulation of Theorem 1.9.2, a scaling argument allows us to reduce the redundancy and fix $\alpha = 1$, in which case the condition $0 < \alpha\beta < 1$ reads $0 < \beta < 1$. In view of Subsections 8.1 and 8.2, it will be sufficient to show that for $u \in \mathfrak{L}(\mathbb{R})$,

$$\Pi_2 u = \Pi_2 \mathbf{J}_\beta u = 0 \Longrightarrow u = 0. \tag{11.1.1}$$

So, we assume that $u \in \mathfrak{L}(\mathbb{R})$ has $\Pi_2 u = \Pi_2 \mathbf{J}_\beta u = 0$. Let $u_0 := \mathbf{R}_1 u \in \mathfrak{L}(I_1)$ and $u_1 := \mathbf{R}_1^\dagger u \in \mathfrak{L}(\mathbb{R} \setminus \bar{I}_1)$ denote the restrictions of the distribution u to the symmetric interval I_1 and to the complement $\mathbb{R} \setminus \bar{I}_1$, respectively. We will be done once we are able to show that $u_0 = 0$, because then u_1 vanishes as well, as a result of Proposition 9.5.1:

$$u_1 = -\mathbf{R}_1^\dagger \mathbf{J}_\beta \mathbf{T}_\beta u_0 = 0.$$

Indeed, we have Proposition 9.5.2, which tells us that

$$u_0 = \mathbf{R}_1 u = 0 \quad \text{and} \quad u_1 = \mathbf{R}_1^\dagger u = 0$$

together imply that $u = 0$.

Finally, to obtain that $u_0 = 0$, we observe that in addition, Proposition 9.5.1 says that u_0 has the important property $u_0 = \mathbf{T}_\beta^2 u_0$. By iterating we find that $u_0 = \mathbf{T}_\beta^{2n} u_0$ for $n = 1, 2, 3, \ldots$, and by letting $n \to +\infty$, Theorem 1.8.2 tells us that $u_0 = 0$ is the only solution in $\mathfrak{L}(I_1)$, which completes the proof. □

11.2 The proof of the asymptotic decay result for $0 < \beta < 1$. We now proceed with the proof of Theorem 1.8.2. Note that we have to be particularly careful because the operator $\mathbf{T}_\beta : \mathfrak{L}(I_1) \to \mathfrak{L}(I_1)$ has norm > 1, by Theorem 1.8.1. However, it clear that it acts contractively on the subspace $L^1(I_1)$.

Proof of Theorem 1.8.2. We decompose $u_0 = f + \mathbf{R}_1 \mathbf{H} g$, where $f \in L^1(I_1)$ and $g \in L^1_0(\mathbb{R})$, and observe that by Proposition 2.3.1(iv),

$$\|\mathbf{T}_\beta^N f\|_{L^1(I_1)} \to 0 \quad \text{as } N \to +\infty. \tag{11.2.1}$$

So the iterates $\mathbf{T}_\beta^N f$ tend to 0 in $L^1(I_1)$ and hence in $L^{1,\infty}(I_1)$ as well. We turn to the $\mathbf{T}_\beta^2$-iterates of $\mathbf{R}_1 \mathbf{H} g$. First, we split

$$g = g_1 + g_2, \quad \text{where } g_1 \in L^1(I_\beta),\ g_2 \in L^1(\mathbb{R} \setminus I_\beta);$$

here, it is tacitly assumed that the functions g_1, g_2 are extended to vanish on the rest of the real line $\mathbb{R}$. As the operator $\mathbf{J}_\beta$ maps $L^1(\mathbb{R} \setminus I_\beta) \to L^1(I_1)$ isometrically, and $\mathbf{H} g_2 = \mathbf{Q}_\beta \mathbf{J}_\beta g_2$ holds on I_1 by (10.1.5), Proposition 10.1.1 gives us the pointwise estimate (we write "vap" although it is not needed at all)

$$|\mathrm{vap}[\mathbf{T}_\beta^N \mathbf{R}_1 \mathbf{H} g_2](x)| \le \frac{4\beta^{N-1}}{(1-\beta)\pi} \|g_2\|_{L^1(\mathbb{R})}, \quad x \in I_1. \tag{11.2.2}$$

In particular, the $\mathbf{T}_\beta$-iterates of $\mathbf{R}_1\mathbf{H}g_2$ tend to 0 geometrically in $L^\infty(I_1)$. We still need to analyze the $\mathbf{T}_\beta^2$-iterates of $\mathbf{R}_1\mathbf{H}g_1$. We apply $\mathbf{T}_\beta^k$ to the two sides of the identity of Proposition 9.8.2, with g_1 in place of f and with $k = 2, 3, 4, \ldots$, to obtain that

$$\begin{aligned}\mathbf{T}_\beta^{n+k}\mathbf{R}_1\mathbf{H}g_1 &= \mathbf{T}_\beta^k\mathbf{R}_1\mathbf{H}\mathfrak{T}_\beta^n g_1\\ &\quad+\sum_{j=0}^{n-1}\{\mathbf{T}_\beta^{n+k-j}\mathbf{R}_1\mathbf{H}\mathbf{J}_\beta\mathfrak{T}_\beta^{j+1}g_1-\mathbf{T}_\beta^{n+k-j-1}\mathbf{R}_1\mathbf{H}\mathbf{J}_\beta\mathfrak{T}_\beta^j g_1\}.\end{aligned}\tag{11.2.3}$$

For $l = 0, 1, 2, \ldots$, the function $\mathfrak{T}_\beta^l g_1$ is in $L^1(I_1)$, so that again by Proposition 10.1.1, since $\mathbf{R}_1\mathbf{H}\mathbf{J}_\beta = \mathbf{Q}_\beta$, we have

$$|\mathrm{vap}[\mathbf{T}_\beta^r\mathbf{R}_1\mathbf{H}\mathbf{J}_\beta\mathfrak{T}_\beta^l g_1](x)| \le \frac{4\beta^{r-1}}{\pi(1-\beta)}\|g_1\|_{L^1(I_\beta)},\qquad x\in I_1,\ r=2,3,4,\ldots,\tag{11.2.4}$$

where we use that the transfer operator $\mathfrak{T}_\beta$ acts contractively on $L^1(I_1)$, by Proposition 2.3.1(i). An application of the valeur au point estimate (11.2.4) to each term of the sum on the right-hand side of the identity (11.2.3) gives that

$$|\mathrm{vap}[\mathbf{T}_\beta^{n+k}\mathbf{R}_1\mathbf{H}g_1-\mathbf{T}_\beta^k\mathbf{R}_1\mathbf{H}\mathfrak{T}_\beta^n g_1](x)| \le \frac{8\beta^{k-1}}{\pi(1-\beta)^2}\|g_1\|_{L^1(I_\beta)},\qquad \text{a.e. } x\in I_1.\tag{11.2.5}$$

Next, we split the function g_1 as follows:

$$g_1 = h_{0,n} + h_{1,n},\quad h_{0,n}\in L^1(\mathcal{E}_{\beta,n+1}),\ h_{1,n}\in L^1(I_\beta\setminus\mathcal{E}_{\beta,n+1}),$$

where the set $\mathcal{E}_{\beta,n+1}$ is as in (2.3.1), and with the understanding that $h_{0,n}, h_{1,n}$ both vanish elsewhere on the real line. Next, we observe that $\mathfrak{T}_\beta^n h_{1,n}\in L^1(I_1\setminus\bar{I}_\beta)$. This can be seen from the defining property of the set $\mathcal{E}_{\beta,n+1}$ and the relation between the map τ_β and the corresponding transfer operator $\mathfrak{T}_\beta$; see (1.8.2). We then apply Proposition 10.1.1 to arrive at

$$\begin{aligned}|\mathrm{vap}[\mathbf{T}_\beta^k\mathbf{R}_1\mathbf{H}\mathfrak{T}_\beta^n h_{1,n}](x)| &\le \frac{4\beta^{k-1}}{\pi(1-\beta)}\|\mathfrak{T}_\beta^n h_{1,n}\|_{L^1(I_1)}\\ &\le \frac{4\beta^{k-1}}{\pi(1-\beta)}\|h_{1,n}\|_{L^1(I_\beta)}\\ &\le \frac{4\beta^{k-1}}{\pi(1-\beta)}\|g_1\|_{L^1(I_\beta)},\qquad x\in I_1.\end{aligned}\tag{11.2.6}$$

By combining (11.2.5) with the estimate (11.2.6), we obtain that

$$|\mathrm{vap}[\mathbf{T}_\beta^{n+k}\mathbf{R}_1\mathbf{H}g_1 - \mathbf{T}_\beta^{k}\mathbf{R}_1\mathbf{H}\mathfrak{T}_\beta^{n}h_{0,n}](x)| \le \frac{12\beta^{k-1}}{\pi(1-\beta)^2}\|g_1\|_{L^1(I_\beta)}, \quad \text{a.e. } x\in I_1. \tag{11.2.7}$$

The norm of $h_{0,n}\in L^1(\mathcal{E}_{\beta,n+1})$ is equal to

$$\int_{\mathcal{E}_{\beta,n+1}}|h_{0,n}(t)|\mathrm{d}t = \int_{\mathcal{E}_{\beta,n+1}}|g_1(t)|\mathrm{d}t = \|1_{\mathcal{E}_{\beta,n+1}}g_1\|_{L^1(I_1)},$$

and it approaches 0 as $n\to+\infty$, by Proposition 2.3.1(iv). Since the transfer operator $\mathfrak{T}_\beta$ is a norm contraction on $L^1(I_1)$, we know that

$$\|\mathfrak{T}_\beta^n h_{0,n}\|_{L^1(I_1)} \le \|h_{0,n}\|_{L^1(I_1)} = \|1_{\mathcal{E}_{\beta,n+1}}g_1\|_{L^1(I_1)},$$

and, consequently, for fixed k we have that

$$\mathbf{T}_\beta^{k}\mathbf{R}_1\mathbf{H}\mathfrak{T}_\beta^{n}h_{0,n}\to 0 \quad \text{in } \mathfrak{L}(I_1),\ \text{as } n\to+\infty.$$

As convergence in $\mathfrak{L}(I_1)$ entails convergence in $L^{1,\infty}(I_1)$ for the corresponding "valueur au point" function, we obtain from (11.2.7), by application of the $L^{1,\infty}(I_1)$ quasinorm triangle inequality, that

$$\limsup_{n\to+\infty}\|\mathrm{vap}[\mathbf{T}_\beta^{n+k}\mathbf{R}_1\mathbf{H}g_1]\|_{L^{1,\infty}(I_1)} \le \frac{24\beta^{k-1}}{\pi(1-\beta)^2}\|g_1\|_{L^1(I_\beta)}, \quad \text{a.e. } x\in I_1. \tag{11.2.8}$$

Note that the limit on the left-hand side does not depend on the parameter k. This allows us to let $k\to+\infty$ in a second step, and we obtain that

$$\lim_{N\to+\infty}\|\mathrm{vap}[\mathbf{T}_\beta^{N}\mathbf{R}_1\mathbf{H}g_1]\|_{L^{1,\infty}(I_1)} = 0. \tag{11.2.9}$$

Finally, gathering the terms, we obtain from (11.2.1), (11.2.2) and (11.2.9) that

$$\begin{aligned}\mathrm{vap}[\mathbf{T}_\beta^{N}u_0] &= \mathrm{vap}[\mathbf{T}_\beta^{N}(f+\mathbf{R}_1\mathbf{H}g)]\\ &= \mathrm{vap}[\mathbf{T}_\beta^{N}f] + \mathrm{vap}[\mathbf{T}_\beta^{N}\mathbf{R}_1\mathbf{H}g_1] + \mathrm{vap}[\mathbf{T}_\beta^{N}\mathbf{R}_1\mathbf{H}g_2]\to 0\\ &\qquad\qquad \text{as } N\to+\infty,\end{aligned} \tag{11.2.10}$$

in the quasinorm of $L^{1,\infty}(I_1)$, as claimed. □

Remark 11.2.1. One may wonder if Theorem 1.8.2 (and hence Corollary 1.8.3) would remain true if the space $\mathfrak{L}(\mathbb{R})$ were to be replaced by the larger space $L^{1,\infty}(I_1)$. To look into this issue, we keep $0<\beta<1$, and consider the function

$$f(x) := \frac{1}{x-x_1} - \frac{1}{x-x_2}, \quad \text{where } x_1 := 1+\sqrt{1-\beta},\ x_2 := -1+\sqrt{1-\beta}.$$

Then $x_1x_2 = -\beta$, so that $\frac{\beta}{x_1} = -x_2$ and $\frac{\beta}{x_2} = -x_1$, and, in addition,

$$\frac{\beta}{x_1} - \frac{\beta}{x_2} = x_1 - x_2 = 2,$$

which leads to

$$\begin{aligned}
\mathbf{T}_\beta f(x) &= \sum_{j\in\mathbb{Z}^\times} \frac{\beta}{(2j+x)^2} f\Big(-\frac{\beta}{2j+x}\Big) \\
&= \sum_{j\in\mathbb{Z}^\times} \frac{\beta}{(2j+x)^2}\Big(\frac{1}{-\frac{\beta}{2j+x}-x_1} - \frac{1}{-\frac{\beta}{2j+x}-x_2}\Big) \\
&= \sum_{j\in\mathbb{Z}^\times} \Big(\frac{\beta}{(2j+x)(\beta+(2j+x)x_2)} - \frac{\beta}{(2j+x)(\beta+(2j+x)x_1)}\Big) \\
&= \sum_{j\in\mathbb{Z}^\times} \Big(\frac{x_1}{\beta+(2j+x)x_1} - \frac{x_2}{\beta+(2j+x)x_2}\Big) \\
&= \sum_{j\in\mathbb{Z}^\times} \Big(\frac{1}{2j+x+\frac{\beta}{x_1}} - \frac{1}{2j+x+\frac{\beta}{x_2}}\Big) \\
&= \frac{1}{x+\frac{\beta}{x_2}} - \frac{1}{x+\frac{\beta}{x_1}} = \frac{1}{x-x_1} - \frac{1}{x-x_2} = f(x),
\end{aligned}$$

by telescoping sums. The function f is a nontrivial element of $L^{1,\infty}(I_1)$ and it is $\mathbf{T}_\beta$-invariant: $\mathbf{T}_\beta f = f$. Many other choices of the points x_1, x_2 would work as well. For $\beta = 1$, the indicated points x_1, x_2 coincide, so that $f = 0$, but it is enough to choose instead $x_1 := 2 + \sqrt{3}$ and $x_2 = -2 + \sqrt{3}$ to obtain a nontrivial function f in $L^{1,\infty}(I_1)$ which is $\mathbf{T}_1$-invariant. This illustrates how Theorem 1.8.2 and Corollary 1.8.3 would utterly fail to hold if the space $\mathfrak{L}(\mathbb{R})$ were to be replaced by $L^{1,\infty}(I_1)$.

12 The Hilbert kernel and its dynamical decomposition

12.1 Odd and even parts of the Hilbert kernel. As in Subsection 10.1, we write

$$Q_1(t,x) := \frac{t}{1+tx},$$

which is a variant of the Hilbert kernel. Indeed, it arises in connection with the Hilbert transform; see, e.g., (10.1.5). We split the function Q_1 according to odd and even parts:

$$Q_1(t,x) = Q_1^I(t,x) - Q_1^{II}(t,x), \quad Q_1^I(t,x) := \frac{t}{1-x^2t^2}, \quad Q_1^{II}(t,x) := \frac{t^2x}{1-t^2x^2}.$$

For fixed $t \in I_1 =]-1, 1[$, we may calculate the action of the transfer operator $\mathbf{T}_1$ on the function $Q_1(t, \cdot)$ using standard trigonometric identities:

$$\begin{aligned}
\mathbf{T}_1 Q_1(t, \cdot)(x) &= \sum_{j \in \mathbb{Z}^\times} \frac{1}{(2j+x)^2} \frac{t}{1 + t(-\frac{1}{2j+x})} \\
&= \sum_{j \in \mathbb{Z}^\times} \left\{ \frac{1}{2j + x - t} - \frac{1}{2j+x} \right\} \\
&= \frac{\pi}{2} \cot\left(\frac{\pi}{2}(x-t)\right) - \frac{\pi}{2} \cot\left(\frac{\pi x}{2}\right) - \frac{t}{x(x-t)}.
\end{aligned} \tag{12.1.1}$$

12.2 The dynamically reduced Hilbert kernel. Next, let q_1 be the associated function

$$q_1(t, x) := (\mathbf{I} - \mathbf{T}_1) Q_1(t, \cdot)(x),$$

so that by (12.1.1),

$$q_1(t, x) = \frac{t}{x(x-t)} + \frac{t}{1+tx} + \frac{\pi}{2} \cot\left(\frac{\pi x}{2}\right) - \frac{\pi}{2} \cot\left(\frac{\pi}{2}(x-t)\right). \tag{12.2.1}$$

The function $x \mapsto q_1(t, x)$ has removable singularities at $x = 0$ and $x = t$, and poles at $x = -2 + t$ and $x = 2 + t$. Therefore, the function $x \mapsto q_1(t, x)$ has a Taylor series representation at the origin with radius of convergence equal to $2 - |t|$, for $t \in I_1$. For fixed $t \in I_1$, we know that $q_1(t, \cdot) \in L^1_0(I_1)$, where

$$L^1_0(I_1) := \{f \in L^1(I_1) : \langle 1, f \rangle_{I_1} = 0\},$$

the reason being that

$$\begin{aligned}
\langle 1, q_1(t, \cdot) \rangle_{I_1} &= \langle 1, Q_1(t, \cdot) \rangle_{I_1} - \langle 1, \mathbf{T}_1 Q_1(t, \cdot) \rangle_{I_1} \\
&= \langle 1, Q_1(t, \cdot) \rangle_{I_1} - \langle \mathbf{L}_1 1, Q_1(t, \cdot) \rangle_{I_1} = 0,
\end{aligned}$$

since $\mathbf{L}_1 1 = 1$. We will refer to $q_1(t, x)$ as the **dynamically reduced Hilbert kernel**.

For the endpoint parameter value $t = 1$, the expression for the kernel q_1 is

$$q_1(1, x) = -\frac{1}{x} - \frac{2x}{1-x^2} + \frac{\pi}{2} \cot\left(\frac{\pi x}{2}\right) + \frac{\pi}{2} \tan\left(\frac{\pi}{2} x\right). \tag{12.2.2}$$

The function $x \mapsto q_1(1, x)$ has removable singularities at $x = 0$, $x = 1$, and $x = -1$, and the radius of convergence for its Taylor series at the origin equals 2. So, in particular, the function $x \mapsto q_1(1, x)$ extends to a smooth function on the closed interval $\bar{I}_1 = [-1, 1]$.

12.3 Odd and even parts of the dynamically reduced Hilbert kernel. We split the dynamically reduced Hilbert kernel $q_1(t,x)$ according to odd and even parts with respect to x:

$$q_1(t,x) = q_1^I(t,x) - q_1^{II}(t,x),$$

where

$$q_1^I(t,x) := \frac{1}{2}(q_1(t,x) + q_1(t,-x)), \quad q_1^{II}(t,x) := \frac{1}{2}(q_1(t,-x) - q_1(t,x)). \tag{12.3.1}$$

Obviously, $Q_1^I(t,x)$ and $q_1^I(t,x)$ are even functions of x, while $Q_1^{II}(t,x)$ and $q_1^{II}(t,x)$ are odd. By Proposition 2.5.1, the operator $\mathbf{T}_1$ preserves even and odd symmetry, and it follows that

$$q_1^I(t,x) = (\mathbf{I} - \mathbf{T}_1)Q_1^I(t,\cdot)(x), \quad q_1^{II}(t,x) = (\mathbf{I} - \mathbf{T}_1)Q_1^{II}(t,\cdot)(x). \tag{12.3.2}$$

By inspection, the function $q_1(1,\cdot)$ is odd, so that $q_1^I(1,\cdot) = 0$ and $q_1^{II}(1,\cdot) = -q_1(1,\cdot)$. This of course corresponds to the observation that $Q_1^I(1,x) = (1-x^2)^{-1}$ is the density of the invariant measure. Based on (12.3.2), the standard Neumann series cancellation shows the following: for fixed $t \in I_1$, we have the decompositions

$$\begin{cases} \sum_{j=0}^{n-1} \mathbf{T}_1^j q_1^I(t,\cdot)(x) = Q_1^I(t,x) - \mathbf{T}_1^n Q_1^I(t,\cdot)(x), \\ \sum_{j=0}^{n-1} \mathbf{T}_1^j q_1^{II}(t,\cdot)(x) = Q_1^{II}(t,x) - \mathbf{T}_1^n Q_1^{II}(t,\cdot)(x). \end{cases} \tag{12.3.3}$$

Note that on the right-hand side of (12.3.3), the first term will tend to dominate as $n \to +\infty$, by Proposition 2.3.1(vi). We proceed to analyze the odd part.

Proposition 12.3.1 (Dynamic decomposition). *For fixed $t \in I_1$, we have the decomposition*

$$Q_1^{II}(t,x) = \sum_{j=0}^{+\infty} \mathbf{T}_1^j q_1^{II}(t,\cdot)(x), \quad x \in I_1,$$

with uniform convergence on compact subsets of I_1, as well as norm convergence in $L^1(I_1)$. (See Figure 12.3.1)

Proof. For fixed $t \in I_1$, we have that $Q_1^{II}(1,\cdot) \in L_0^1(I_1)$, so an application of Proposition 2.3.1(v) shows that $\mathbf{T}_1^n Q_1^{II}(t,\cdot) \to 0$ in norm in $L^1(I_1)$ as $n \to +\infty$. Combined with (12.3.3), this shows that the Neumann series converges in the $L^1(I_1)$ norm, as required. Next, the uniform convergence on compact subsets follows from the $L^1(I_1)$ convergence, combined with a comparison with the invariant measure density and a normal families argument. We omit the easy details. □

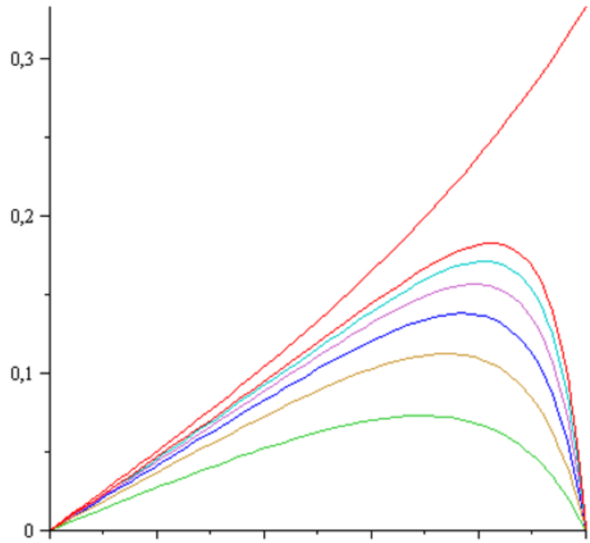

Figure 12.3.1. Illustration of the dynamic decomposition (Proposition 12.3.1). The top curve is $Q_1^{II}(t,\cdot)$, while the curves below are the partial sums $\sum_{j=0}^{N}\mathbf{T}_1^j q_1^{II}(t,\cdot)$, with $N = 0, 1, 2, 3, 4, 5$. We use the parameter value $t = 0.5$.

12.4 The fundamental estimate of the odd part of the dynamically reduced Hilbert kernel. We will focus our attention on the odd part, which involves Q_1^{II} and q_1^{II}. Note that from the work in the previous subsection, especially formula (12.1.1), the function q_1^{II} may be expanded in the series

$$q_1^{II}(t,x) = \frac{t^2x}{1-t^2x^2} + \sum_{j=1}^{+\infty}\left\{\frac{2x}{4j^2-x^2} - \frac{x}{(2j-t)^2-x^2} - \frac{x}{(2j+t)^2-x^2}\right\}. \tag{12.4.1}$$

We need effective control from above and below of the summands in Proposition 12.3.1. Our result reads as follows.

Theorem 12.4.1 (Uniform control of summands)**.** *We have the following estimate, for fixed $t \in I_1$ and $j = 0, 1, 2, 3, \ldots$*

$$0 < \mathbf{T}_1^j q_1^{II}(t,\cdot)(x) < \mathbf{T}_1^j q_1^{II}(1,\cdot)(x), \quad x \in I_1^+.$$

The proof is delivered in pieces. The first instalment is the estimate from above, which is obtained in Proposition 12.5.3 below. The estimate from below is considerably more involved, and is presented in Subsection 13.3, based on estimates of the Hurwitz zeta function and notions from the theory of totally positive matrices. We should mention that by the odd symmetry with respect to x, there is a corresponding estimate which holds on the left-side interval $I_1^- :=]-1,0[$ as well.

12.5 Estimate from above of the odd part of the dynamically reduced Hilbert kernel. The estimate from above in Theorem 12.4.1 will be obtained as a consequence of the following property.

Proposition 12.5.1. *For fixed $t \in I_1$, the function $x \mapsto q_1^{II}(1,x) - q_1^{II}(t,x)$ is odd and strictly increasing on I_1.*

Remark 12.5.2. In view of Proposition 2.6.1, $q_1^{II}(t,1) = q_1^{II}(t,-1) = 0$ holds for each fixed $t \in I_1$. The function $x \mapsto q_1^{II}(t,x)$ is odd, and it will be shown later that it is increasing on some interval I_η with $0 < \eta < 1$, and decreasing on the remainder set $I_1 \setminus I_\eta$ (where the parameter $\eta = \eta(t)$ depends on t). However, the issue is more delicate for $t = 1$. In particular, the endpoint value at $x = \pm 1$ is different, as $q_1^{II}(1,1) = \frac{1}{2}$ and $q_1^{II}(1,-1) = -\frac{1}{2}$.

Proof of Proposition 12.5.1. It is obvious that the function

$$x \mapsto q_1^{II}(1,x) - q_1^{II}(t,x)$$

is odd. In view of the identity (12.4.1),

$$\begin{aligned} q_1(1,x) - q_1(t,x) &= (\mathbf{I} - \mathbf{T}_1)[Q_1(1,\cdot) - Q_1(t,\cdot)](x) \\ &= \frac{1}{1+x} - \frac{t}{1+tx} + \sum_{j\in\mathbb{Z}^\times} \left\{ \frac{1}{2j+x-t} - \frac{1}{2j+x-1} \right\}, \end{aligned}$$

and by forming the odd part with respect to the variable x, we obtain that

$$\begin{aligned} & q_1^{II}(1,x) - q_1^{II}(t,x) \\ &\quad = (\mathbf{I} - \mathbf{T}_1)[Q_1^{II}(1,\cdot) - Q_1^{II}(t,\cdot)](x) = \frac{x}{1-x^2} - \frac{t^2x}{1-t^2x^2} \\ &\qquad + \frac{1}{2}\sum_{j\in\mathbb{Z}^\times} \left\{ \frac{1}{2j-x-t} - \frac{1}{2j-x-1} - \frac{1}{2j+x-t} + \frac{1}{2j+x-1} \right\} \\ &\quad = \frac{x}{1-x^2} - \frac{t^2x}{1-t^2x^2} \\ &\qquad + \sum_{j=1}^{+\infty} \left\{ \frac{x}{(2j-t)^2-x^2} + \frac{x}{(2j+t)^2-x^2} - \frac{x}{(2j-1)^2-x^2} - \frac{x}{(2j+1)^2-x^2} \right\}. \end{aligned}$$

In terms of the function

$$F(t,x) := \partial_x Q_1^{II}(t,x) = t^2 \frac{1+t^2x^2}{(1-t^2x^2)^2},$$

we compute

$$\begin{aligned} &\partial_x(q_1^{II}(1,x) - q_1^{II}(t,x)) \\ &\qquad = F(1,x) - F(t,x) \\ &\qquad\quad + \sum_{j=1}^{+\infty} (F(r_j(t),x) + F(r_j(-t),x) - F(r_j(1),x) - F(r_j(-1),x)), \end{aligned} \tag{12.5.1}$$

where we use the notation $r_j(t) := 1/(2j-t)$ (then $t \mapsto r_j(t)$ is a positive and increasing function for $j = 1, 2, 3, \ldots$). Since the right-hand side of (12.5.1) expresses an even function of t, we may restrict our attention to $t \in I_1^+$. The derivative with respect to t of the function $F(t,x)$ is

$$G(t,x) := \partial_t F(t,x) = \partial_t \partial_x Q_1^{II}(t,x) = 2t \frac{1+3t^2x^2}{(1-t^2x^2)^4},$$

and by representing differences as definite integrals of the derivative, we obtain that

$$\begin{aligned} &\partial_x(q_1^{II}(1,x) - q_1^{II}(t,x)) \\ &\qquad = \int_t^1 G(\tau,x)\mathrm{d}\tau + \sum_{j=1}^{+\infty} \left(\int_{r_j(-1)}^{r_j(-t)} G(\tau,x)\mathrm{d}\tau - \int_{r_j(t)}^{r_j(1)} G(\tau,x)\mathrm{d}\tau \right). \end{aligned} \tag{12.5.2}$$

Here, we used the trivial observation that $r_{j+1}(1) = r_j(-1)$. Moreover, as the function $t \mapsto G(t,x)$ is monotonically strictly increasing, we have that

$$(r_j(-t) - r_j(-1))\, G(r_j(-1),x) < \int_{r_j(-1)}^{r_j(-t)} G(\tau,x)\mathrm{d}\tau,$$

$$\int_{r_{j+1}(t)}^{r_j(-1)} G(\tau,x)\mathrm{d}\tau < (r_j(-1) - r_{j+1}(t))\, G(r_j(-1),x),$$

and since a trivial calculation shows that

$$r_j(-t) - 2r_j(-1) + r_{j+1}(t) = \frac{2(t-1)^2}{(2j+t)(2j+2-t)(2j+1)} > 0$$

for $j = 1, 2, 3, \ldots$ and $t \in I_1^+$, we obtain that

$$\int_{r_j(-1)}^{r_j(-t)} G(\tau,x)\mathrm{d}\tau - \int_{r_{j+1}(t)}^{r_j(-1)} G(\tau,x)\mathrm{d}\tau > 0.$$

Then, by (12.5.2), and the observation that

$$\int_t^{1/(2-t)} G(\tau, x)\mathrm{d}\tau > 0, \quad t \in I_1^+,\ x \in I_1,$$

it follows that the function $x \mapsto q_1^{II}(1, x) - q_1^{II}(t, x)$ is strictly increasing, as claimed.□

We may now derive the upper bound in Theorem 12.4.1.

Proposition 12.5.3. *For $j = 0, 1, 2, \ldots$ and for fixed $t \in I_1$, the function $\mathbf{T}_1^j[q_1^{II}(1, \cdot) - q_1^{II}(t, \cdot)]$ is odd and increasing. In particular, we have that for $j = 0, 1, 2, \ldots$ and $t \in I_1$,*

$$\mathbf{T}_1^j q_1^{II}(t, \cdot)(x) < \mathbf{T}_1^j q_1^{II}(1, \cdot)(x), \quad x \in I_1^+.$$

Proof. This follows from a combination of Proposition 12.5.1 and Proposition 2.5.2(i). □

Remark 12.5.4. In general, the positivity of all the powers $q_1^{II}(t,\cdot)$ on $I_1^+ =]0, 1[$ cannot be deduced from the simple observation that $Q_1^{II}(t, \cdot)$, $0 < t \le 1$, is odd, increasing, and positive on I_1^+. For instance, the function $f(x) = x^3$ is odd, increasing, and positive on I_1^+. However, it can be seen that the function $(\mathbf{I} - \mathbf{T}_1)f = f - \mathbf{T}_1 f$ changes signs on I_1^+.

13 Power series, Hurwitz zeta function, and total positivity

13.1 A class of power series with at most one positive zero. The lower bound in Theorem 12.4.1 requires a more sophisticated analysis. To this end, we introduce a class of Taylor series.

Let $\mathfrak{P}(\gamma)$ denote the class of convergent Taylor series

$$f(x) = \sum_{j=0}^{+\infty} \hat{f}(j)x^j, \quad x \in I_\gamma =]-\gamma, \gamma[;$$

in short, we write $f \in \mathfrak{P}(\gamma)$. Moreover, we write $f \in \mathfrak{P}_{\mathbb{R}}(\gamma)$ to express that the Taylor coefficients are real, that is, $\hat{f}(j) \in \mathbb{R}$ holds for all $j = 0, 1, 2, \ldots$.

Definition 13.1.1. Fix $0 < \gamma < +\infty$. If $f \in \mathfrak{P}_{\mathbb{R}}(\gamma)$, we write $f \in \mathfrak{P}_{\mathbb{R}}^{\downarrow}(\gamma)$ to express that either

(a) $\hat{f}(j) \ge 0$ for all $j = 0, 1, 2, \ldots$, or
(b) $\hat{f}(j) \le 0$ for all $j = 0, 1, 2, \ldots$, or
(c) there exists an index $j_0 = j_0(f) \in \mathbb{Z}_{+,0}$ such that $\hat{f}(j) \ge 0$ for $j \le j_0$ while $\hat{f}(j) \le 0$ for $j > j_0$.

Under (c), assuming we have excluded the cases (a) and (b), we let $j_0(f)$ be the maximal index with the property (among all the possibilities). Then $j_0(f) \in \mathbb{Z}_{+,0}$.

Proposition 13.1.2. *Suppose $f \in \mathfrak{P}_{\mathbb{R}}^{\downarrow}(\gamma)$, where $\gamma > 0$, is not the null function. Then there exists at most one zero of f on the interval $]0, \gamma[$. If such a point $x_0 \in]0, \gamma[$ with $f(x_0) = 0$ exists, then $f(x) > 0$ holds for $0 < x < x_0$, while $f(x) < 0$ for $x_0 < x < \gamma$.*

Proof. Note that in cases (a) or (b) of Definition 13.1.1, i.e., when $\forall j : \hat{f}(j) \geq 0$ or $\forall j : \hat{f}(j) \leq 0$, then, on the interval $[0, \gamma[$, f is either strictly increasing with $f(0) = \hat{f}(0) \geq 0$, or strictly decreasing with $f(0) \leq 0$, and in each case f has no zero in the interval $]0, \gamma[$.

It remains to deal with the case (c) of Definition 13.1.1, assuming that neither (a) nor (b) is fulfilled.

We proceed by induction on the index $j_0(f)$. First assume that $j_0(f) = 0$. In this case, $f(x)$ is decreasing on $[0, \gamma[$, and it is strictly decreasing unless it is constant. If $f(x)$ is constant, then the constant cannot be 0, and then $f(x)$ would have no zeros at all. If $f(x)$ is strictly decreasing instead, then it obviously can have at most one zero in the given interval, and if such a zero exists, then the values of $f(x)$ are positive to the left of x_0 and negative to the right.

Assume now that $j_0(f) = r \geq 1$ and that the assertion of the statement has been already proved for all $f \in \mathfrak{S}_{\mathbb{R}}^{\downarrow}(\gamma)$ with $j_0(f) = r - 1$. Then the derivative $f'(x)$ is also in the class $\mathfrak{P}_{\mathbb{R}}^{\downarrow}(\gamma)$, and $j_0(f') = r - 1$. By the induction hypothesis, there are two possibilities:

Case (i) There is no point x_0 on $]0, \gamma[$ such that $f'(x_0) = 0$, and

Case (ii) There is x_1 such that $f'(x_1) = 0$.

In case (i), f' must have constant sign on the interval $]0, \gamma[$. If the sign is positive, then $f(x)$ is increasing there, and since $f(0) \geq 0$ the function $f(x)$ cannot have any zeros in $]0, \gamma[$. If instead the sign of f' is negative, then $f(x)$ is decreasing. If $f(0) = 0$ the function $f(x)$ has no zeros at all on $]0, \gamma[$. If $f(0) > 0$, then either $f(x)$ is positive on $]0, \gamma[$, or it has precisely one zero $x_0 \in]0, \gamma[$, in which case $f(x)$ is positive to the left and negative to the right of x_0.

In case (ii), then by the induction hypothesis, we also know that $f'(x) > 0$ for $0 < x < x_1$ and that $f'(x) < 0$ for $x_1 < x < \gamma$. This means that $f(x)$ is strictly increasing on $]0, x_1]$ and as $f(0) \geq 0$, it follows that $f(x) > 0$ for $0 < x \leq x_1$. In the remaining interval $x_1 < x < \gamma$, $f(x)$ is decreasing, and it is either positive throughout or has exactly one zero $x_0 \in]x_1, \gamma[$, in which case $f(x)$ is positive to the left on x_0 and negative to the right. The proof is complete. □

13.2 The Taylor coefficients of the odd part of the dynamically reduced Hilbert kernel. We now analyze the symmetrized dynamically reduced Hilbert kernel $q_1^{II}(t,x)$.

Proposition 13.2.1. *For a fixed $0<t<1$, the function $x\mapsto q_1^{II}(t,x)$ is odd and belongs to the class $\mathfrak{S}_{\mathbb{R}}^{\downarrow}(\gamma)$ with $\gamma=2-t$. Indeed, $q_1^{II}(t,\cdot)$ meets condition (c) of Definition 13.1.1.*

Before we supply the full proof of the proposition, we need to do some preparatory work. The kernel $q_1^{II}(t,x)$ is given by (12.4.1). It is an odd function of x, and has the Taylor expansion

$$q_1^{II}(t,x)=\sum_{j=0}^{+\infty}\varkappa_j(t)x^{2j+1},\qquad x\in I_1, \tag{13.2.1}$$

with radius of convergence $2-t$, where the coefficients can be readily calculated:

$$\varkappa_j(t):=\frac{2^{-2j-2}}{(2j+1)!}\Big\{2\psi^{(2j+1)}(1)-\psi^{(2j+1)}\Big(1-\frac{t}{2}\Big)-\psi^{(2j+1)}\Big(1+\frac{t}{2}\Big)\Big\}+t^{2j+2}.$$

Here,

$$\psi(x):=\frac{\Gamma'(x)}{\Gamma(x)},\qquad \psi^{(m)}(x)=\frac{\mathrm{d}^m}{\mathrm{d}x^m}\frac{\Gamma'(x)}{\Gamma(x)} \tag{13.2.2}$$

is the poly-Gamma function, and $\Gamma(x)$ is the standard Gamma function. A more convenient expression is obtained by direct Taylor expansion of the terms in (12.4.1):

$$\varkappa_j(t)=t^{2j+2}+\sum_{k=1}^{+\infty}\Big\{\frac{2}{(2k)^{2j+2}}-\frac{1}{(2k-t)^{2j+2}}-\frac{1}{(2k+t)^{2j+2}}\Big\}.$$

In terms of the Hurwitz zeta function

$$\zeta(s,x):=\sum_{k=0}^{+\infty}(x+k)^{-s},$$

the expression for $\varkappa_j(x)$ equals

$$\varkappa_j(t)=t^{2j+2}+2^{-2j-2}\Big\{2\zeta(2j+2,1)-\zeta\Big(2j+2,1-\frac{t}{2}\Big)-\zeta\Big(2j+2,1+\frac{t}{2}\Big)\Big\}.$$

Moreover, since $\zeta(s,x)=x^{-s}+\zeta(s,1+x)$, we may rewrite this as

$$\begin{aligned}\varkappa_j(t)&\\ &=t^{2j+2}-(2-t)^{-2j-2}\\ &\quad+2^{-2j-2}\Big\{2\zeta(2j+2,1)-\zeta\Big(2j+2,2-\frac{t}{2}\Big)-\zeta\Big(2j+2,1+\frac{t}{2}\Big)\Big\}.\end{aligned} \tag{13.2.3}$$

We need the following result.

Lemma 13.2.2. *For fixed τ with $0 < \tau \le \frac{1}{2}$, the function*

$$\Lambda_\tau(s) := (1-\tau)^s\{2\zeta(s,1) - \zeta(s,2-\tau) - \zeta(s,1+\tau)\}$$

is positive and strictly decreasing on the interval $[3, +\infty[$, *with limit*

$$\lim_{s\to+\infty} \Lambda_\tau(s) = 0.$$

Proof. We will keep the variables τ and s confined to the indicated intervals $0 < \tau \le \frac{1}{2}$ and $3 \le s < +\infty$.

By comparing term by term in the Hurwitz zeta sum, we see that the function $\Lambda_\tau(s)$ is positive. Moreover, as $s \to +\infty$, the first term x^{-s} becomes dominant in the Hurwitz zeta series $\zeta(s,x)$, and we obtain that $\Lambda_\tau(s)$ has the indicated limit.

Next, we split the function in the following way:

$$\Lambda_\tau(s) = \lambda_\tau(s) + R_\tau(s), \quad \lambda_\tau(s) := 2(1-\tau)^s - \Big(\frac{1-\tau}{1+\tau}\Big)^s \tag{13.2.4}$$

where $R_\tau(s)$ is given by

$$R_\tau(s) := (1-\tau)^s\{2\zeta(s,2) - \zeta(s,2-\tau) - \zeta(s,2+\tau)\}.$$

In the identity (13.2.4), we should think of $\lambda_\tau(s)$ as the main term and of $R_\tau(s)$ as the remainder. Clearly, we see that $\lambda_\tau(s) > 0$, and that $\lambda_\tau(s)$ is decreasing in s:

$$\begin{aligned} \lambda'_\tau(s) &= 2(1-\tau)^s \log(1-\tau) - \Big(\frac{1-\tau}{1+\tau}\Big)^s \log\frac{1-\tau}{1+\tau} \\ &= (1-\tau)^s\Big\{\log(1-\tau^2) - [1-(1+\tau)^{-s}]\log\frac{1+\tau}{1-\tau}\Big\} < 0. \end{aligned} \tag{13.2.5}$$

Moreover, by direct calculation

$$\partial_s \frac{\lambda'_\tau(s)}{(1-\tau)^s} = -(1+\tau)^{-s}\{\log(1+\tau)\}\log\frac{1+\tau}{1-\tau} < 0,$$

so that by (13.2.5), we have that

$$\begin{aligned} \lambda'_\tau(s) &\le (1-\tau)^s \frac{\lambda'_\tau(3)}{(1-\tau)^3} \\ &= (1-\tau)^s\Big\{\log(1-\tau^2) - [1-(1+\tau)^{-3}]\log\frac{1+\tau}{1-\tau}\Big\} \\ &= (1-\tau)^s\Big\{\log(1-\tau^2) - \frac{\tau(3+3\tau+\tau^2)}{(1+\tau)^3}\log\frac{1+\tau}{1-\tau}\Big\} \\ &\le -\tau^2(1-\tau)^s\Big\{1 + \frac{6+6\tau+2\tau^2}{(1+\tau)^4}\Big\} \le -\frac{233}{81}\tau^2(1-\tau)^s, \end{aligned} \tag{13.2.6}$$

by an elementary estimate of the logarithm function, and by using our constraints on s and τ.

We proceed to estimate the remainder term. Since for positive τ and a C^2-smooth function f, we have that

$$f(\tau)+f(-\tau)-2f(0)=\int_{-\tau}^{\tau}(\tau-|\theta|)f''(\theta)\mathrm{d}\theta,$$

and, in particular,

$$\begin{aligned}\zeta(s,2-\tau)+\zeta(s,2+\tau)-2\zeta(s,2)&=\int_{-\tau}^{\tau}(\tau-|\theta|)\partial_\theta^2\zeta(s,2+\theta)\mathrm{d}\theta\\&=s(s+1)\int_{-\tau}^{\tau}(\tau-|\theta|)\zeta(s+2,2+\theta)\mathrm{d}\theta,\end{aligned}$$

and consequently

$$R_\tau(s)=-s(s+1)(1-\tau)^s\int_{-\tau}^{\tau}(\tau-|\theta|)\zeta(s+2,2+\theta)\mathrm{d}\theta.$$

By direct inspection, then, we see that $R_\tau(s)<0$. Moreover, by differentiating the above formula with respect to s, we obtain

$$\begin{aligned}R_\tau'(s)&=-(2s+1)(1-\tau)^s\int_{-\tau}^{\tau}(\tau-|\theta|)\zeta(s+2,2+\theta)\mathrm{d}\theta\\&\quad-s(s+1)\int_{-\tau}^{\tau}(\tau-|\theta|)\partial_s[(1-\tau)^s\zeta(s+2,2+\theta)]\mathrm{d}\theta\\&=\frac{2s+1}{s(s+1)}R_\tau(s)\\&\quad-s(s+1)\int_{-\tau}^{\tau}(\tau-|\theta|)\partial_s[(1-\tau)^s\zeta(s+2,2+\theta)]\mathrm{d}\theta\\&\le-s(s+1)\int_{-\tau}^{\tau}(\tau-|\theta|)\partial_s[(1-\tau)^s\zeta(s+2,2+\theta)]\mathrm{d}\theta.\end{aligned}\tag{13.2.7}$$

We proceed to compute the derivative which appears in (13.2.7):

$$-\partial_s[(1-\tau)^s\zeta(s+2,2+\theta)]=(1-\tau)^{-2}\sum_{k=0}^{+\infty}\left(\frac{2+\theta+k}{1-\tau}\right)^{-s-2}\log\frac{2+\theta+k}{1-\tau}.\tag{13.2.8}$$

To analyze this derivative, we need the following computation

$$\begin{aligned}\partial_T\{T^{-s-2}\log T\}&=-T^{-s-3}((s+2)\log T-1)\\&\le-T^{-s-3}(5\log T-1)<0,\qquad\frac{3}{2}\le T<+\infty.\end{aligned}\tag{13.2.9}$$

In other words, $T\mapsto T^{-s-2}\log T$ is decreasing on the interval $[\frac{3}{2},+\infty[$. As a first application of the property (13.2.9), we apply it to the identity (13.2.8) and obtain that

$$0 \le -\partial_s[(1-\tau)^s\zeta(s+2,2+\theta)] \le -\partial_s[(1-\tau)^s\zeta(s+2,2-\tau)], \quad -\tau \le \theta \le \tau,$$

which we may implement into (13.2.7):

$$\begin{aligned} R'_\tau(s) &\le -s(s+1)\int_{-\tau}^{\tau}(\tau-|\theta|)\partial_s[(1-\tau)^s\zeta(s+2,2+\theta)]\mathrm{d}\theta \\ &\le -s(s+1)\int_{-\tau}^{\tau}(\tau-|\theta|)\partial_s[(1-\tau)^s\zeta(s+2,2-\tau)]\mathrm{d}\theta \\ &= -s(s+1)\tau^2\partial_s[(1-\tau)^s\zeta(s+2,2-\tau)]. \end{aligned} \tag{13.2.10}$$

Next, we implement the property (13.2.9) again, in the context of (13.2.8) with $\theta=-\tau$:

$$\begin{aligned} -\partial_s&[(1-\tau)^s\zeta(s+2,2-\tau)] \\ &= (1-\tau)^{-2}\sum_{k=0}^{+\infty}\Big(\frac{2-\tau+k}{1-\tau}\Big)^{-s-2}\log\frac{2-\tau+k}{1-\tau} \\ &\le (1-\tau)^s(2-\tau)^{-s-2}\log\frac{2-\tau}{1-\tau} \\ &\quad + (1-\tau)^{-2}\int_0^{+\infty}\Big(\frac{2-\tau+x}{1-\tau}\Big)^{-s-2}\log\frac{2-\tau+x}{1-\tau}\mathrm{d}x. \end{aligned} \tag{13.2.11}$$

Here, we kept the first term, with $k=0$, and replaced each later term indexed by k by a corresponding integral over the adjacent interval $[k-1,k]$. The integral expression in (13.2.11) may be computed explicitly

$$\begin{aligned} \int_0^{+\infty}&\Big(\frac{2-\tau+x}{1-\tau}\Big)^{-s-2}\log\frac{2-\tau+x}{1-\tau}\mathrm{d}x \\ &= (s+1)^{-2}(1-\tau)^{s+2}(2-\tau)^{-s-1}\Big[1+(s+1)\log\frac{2-\tau}{1-\tau}\Big]. \end{aligned} \tag{13.2.12}$$

Finally, putting (13.2.10), (13.2.11), and (13.2.12) together, and obtain that

$$\begin{aligned} R'_\tau(s) &\le -s(s+1)\tau^2\partial_s[(1-\tau)^s\zeta(s+2,2-\tau)] \\ &\le \tau^2(1-\tau)^s\Big\{s(s+1)(2-\tau)^{-s-2}\log\frac{2-\tau}{1-\tau} \\ &\qquad + \frac{s}{s+1}(2-\tau)^{-s-1}\Big[1+(s+1)\log\frac{2-\tau}{1-\tau}\Big]\Big\}. \end{aligned} \tag{13.2.13}$$

The expression within brackets is optimized at the right endpoint $\tau=\frac{1}{2}$:

$$\begin{aligned} s(s+1)&(2-\tau)^{-s-2}\log\frac{2-\tau}{1-\tau}+\frac{s}{s+1}(2-\tau)^{-s-1}\Big[1+(s+1)\log\frac{2-\tau}{1-\tau}\Big] \\ &\le s(s+1)\Big(\frac{3}{2}\Big)^{-s-2}\log 3+\frac{s}{s+1}\Big(\frac{3}{2}\Big)^{-s-1}[1+(s+1)\log 3] \le \frac{27}{10}, \end{aligned} \tag{13.2.14}$$

where the rightmost inequality is elementary given that $3 \le s < +\infty$. It follows from (13.2.13) and (13.2.14) that

$$R'_\tau(s) \le \frac{27}{10}\tau^2(1-\tau)^s. \tag{13.2.15}$$

Finally, a combination of (13.2.6) and (13.2.15) gives the desired result

$$\begin{aligned}\Lambda'_\tau(s) = \lambda'_\tau(s) + R'_\tau(s) &\le \Big(\frac{27}{10} - \frac{233}{81}\Big)\tau^2(1-\tau)^s \\ &= -\frac{143}{810}\tau^2(1-\tau)^s < 0.\end{aligned}$$

The proof is complete. □

Proposition 13.2.3. *For fixed t, $0 < t < 1$, the function*

$$j \mapsto (2-t)^{2j+2}\varkappa_j(t)$$

is strictly decreasing on $\mathbb{Z}_+ = \{1, 2, \ldots\}$, with limit

$$\lim_{j\to+\infty}(2-t)^{2j+2}\varkappa_j(t) = -1.$$

Proof. In view of (13.2.3), we know that

$$(2-t)^{2j+2}\varkappa_j(t) = [t(2-t)]^{2j+2} - 1 + \Lambda_{t/2}(2j+2).$$

Since $0 < t(2-t) < 1$ holds for $t \in I_1^+$, the function $j \mapsto [t(2-t)]^{2j+2}$ is decreasing, and the statement of the proposition immediately follows from Lemma 13.2.2. □

Proof of Proposition 13.2.1. It is clear from the known radius of convergence for $q_1^{II}(t,\cdot)$ that $q_1^{II}(t,\cdot) \in \mathfrak{P}(2-t)$. Moreover, $q_1^{II}(t,\cdot)$ is odd, and all the Taylor coefficients (see (13.2.1)) are clearly real-valued, while the coefficient of the linear term is explicit and positive:

$$\varkappa_0(t) := \frac{\pi^2}{12} + \frac{1}{t^2} - \frac{\pi^2/4}{\sin^2(\frac{\pi}{2}t)} + t^2 > 0.$$

Now, the proof of the proposition is an immediate consequence of Proposition 13.2.3. □

13.3 Positivity of the odd part of the dynamically reduced Hilbert kernel and totally positive matrices. The transfer operator $\mathbf{T}_1$ can be applied to polynomials, or, more generally, convergent power series. For $j = 0, 1, 2, \ldots,$

let u_j denote the monomial $u_j(x) := x^{2j+1}$. The action of $\mathbf{T}_1$ on odd power series can be analyzed in terms of the infinite matrix $\mathbf{B} = \{b_{j,k}\}_{j,k=0}^{+\infty}$ with entries $b_{j,k}$ given by

$$\mathbf{T}_1 u_j(x) = \sum_{k=0}^{+\infty} b_{j,k} u_k, \qquad x \in I_1, \tag{13.3.1}$$

since the transfer operator $\mathbf{T}_1$ preserves oddness.

We recall the notion of a totally positive matrix [21]. An infinite matrix $\mathbf{A} = \{a_{j,k}\}_{j,k=0}^{+\infty}$ is said to be **totally positive** if all its minors are nonnegative, and **strictly totally positive** if all its minors are strictly positive. Here, a minor is the determinant of a square submatrix $\{a_{j_s,k_t}\}_{s,t=1}^{r}$ where $j_1 < \cdots < j_r$ and $k_1 < \cdots < k_r$. This is a much stronger property than the usual positive definiteness of a matrix, which would correspond to considering only symmetric squares.

Proposition 13.3.1. *The matrix* $\mathbf{B} = \{b_{j,k}\}_{j,k=0}^{+\infty}$ *with coefficients given by* (13.3.1) *is strictly totally positive.*

Proof. We read off from the definition of $\mathbf{T}_1$ that

$$\mathbf{T}_1 u_j(x) = -\sum_{n \in \mathbb{Z}^\times} (x + 2n)^{-2j-3},$$

and observe that the right-hand side may be written in the form

$$\mathbf{T}_1 u_j(x) = \frac{1}{2^{2j+2}(2j+2)!} \left\{ \psi^{(2j+2)}\left(1 + \frac{x}{2}\right) - \psi^{(2j+2)}\left(1 - \frac{x}{2}\right) \right\},$$

where $\psi^{(m)}(x)$ is the poly-Gamma function (see (13.2.2)). From this, we immediately obtain

$$b_{j,k} = \frac{\psi^{(2j+2k+3)}(1)}{2^{2j+2k+3}(2j+2)!(2k+1)!}.$$

Since strict total positivity remains unchanged as we multiply a column or a row by a positive number, the strict total positivity of the matrix $\mathbf{B}$ is equivalent to the strict total positivity of the infinite matrix with entries $\{c_{j+k}\}_{j,l=0}^{+\infty}$ where $c_j := \psi^{(2j+3)}(1)$. This is a Hankel matrix, and in view of Theorem 4.4 [21], its total positivity is equivalent to the strict positive definiteness of all the finite square matrices $\{c_{j+l}\}_{j,l=0}^{N}$ and $\{c_{j+l+1}\}_{j,l=0}^{N-1}$, for every $N = 1, 2, 3, \ldots$. Following the digression in Section 4.6 of [21], we know that this is equivalent to having the c_j be the moments of a positive measure (the Stieltjes moment problem). However, it is known that

$$c_j = \psi^{(2j+3)}(1) = \int_0^{+\infty} t^{2j+3} \frac{\mathrm{e}^{-t}}{1 - \mathrm{e}^{-t}} \mathrm{d}t = \int_0^{+\infty} t^j \frac{t\,\mathrm{e}^{-\sqrt{t}}}{1 - \mathrm{e}^{-\sqrt{t}}} \mathrm{d}t,$$

which means that the c_j are indeed the moments of a positive measure. This completes the proof. □

We need to have a precise definition of the notion of counting sign changes; see [21].

Definition 13.3.2. Let $\mathbf{a} = \{a_j\}_j$, $j = 0, \ldots, N$, be a finite sequence of real numbers.

(a) The number $S^-(\mathbf{a})$ counts the number of sign changes in the sequence with zero terms discarded. This is the number of **strong sign changes**.

(b) The number $S^+(\mathbf{a})$ counts the maximal number of sign changes in the sequence, where zero terms are arbitrarily replaced by $+1$ or -1. This is the number of **weak sign changes**.

Obviously, the number of weak sign changes exceeds the number of strong sign changes, i.e., $S^-(\mathbf{a}) \le S^+(\mathbf{a})$.

Corollary 13.3.3. *Fix $1 \le \gamma < +\infty$. If $f \in \mathfrak{P}^{\downarrow}_{\mathbb{R}}(\gamma)$ is odd, then $\mathbf{T}_1 f$ is odd as well, and $\mathbf{T}_1 f \in \mathfrak{P}^{\downarrow}_{\mathbb{R}}(2 - \frac{1}{\gamma})$.*

Proof. Based on the explicit expression (1.8.5) for $\mathbf{T}_1 f$, it is a straightforward exercise in the analysis of power series to check that if $f \in \mathfrak{S}(\gamma)$, then $\mathbf{T}_1 f \in \mathfrak{P}(2 - \frac{1}{\gamma})$. Moreover, it is clear that the property of having real Taylor coefficients is preserved under $\mathbf{T}_1$. To finish the proof, we pick an odd $f \in \mathfrak{P}^{\downarrow}_{\mathbb{R}}(\gamma)$, and expand it in a Taylor series:

$$f(x) = \sum_{j=0}^{+\infty} \hat{f}(2j+1) u_j(x), \quad -\gamma < x < \gamma,$$

where as before $u_j(x) = x^{2j+1}$. Then, in view of (13.3.1),

$$\begin{aligned} \mathbf{T}_1 f(x) &= \sum_{j=0}^{+\infty} \hat{f}(2j+1) \mathbf{T}_1 u_j(x) \\ &= \sum_{k=0}^{+\infty} \Bigg\{ \sum_{j}^{+\infty} b_{j,k} \hat{f}(2j+1) \Bigg\} u_k(x), \quad -1 < x < 1, \end{aligned} \tag{13.3.2}$$

and, as noted before, the right-hand side Taylor series converges in the interval $]-2+\frac{1}{\gamma}, 2-\frac{1}{\gamma}[$. The assertion of the corollary is trivial if $f(x) \equiv 0$, so we may assume that f does not vanish identically. From the definition of the class $\mathfrak{P}^{\downarrow}_{\mathbb{R}}(\gamma)$, we read off that for $N = 1, 2, 3, \ldots$, the finite sequence $\{\hat{f}(2j+1)\}_{j=0}^N$ has at most one strong sign change. Next, by Proposition 13.3.1, we may apply the Variation Diminishing Theorem for strictly totally positive matrices (see Theorem 3.3 in [21]), which asserts that the sequence $\{F_{k,N}\}_{k=0}^N$, where

$$F_{k,N} := \sum_{j=0}^{N} b_{j,k} \hat{f}(2j+1), \tag{13.3.3}$$

has at most one weak sign change in the index interval $\{0, \dots, N\}$. Moreover, if there is a weak sign change in the sequence $\{F_{k,N}\}_{k=0}^{N}$, then it is from ≥ 0 on the left to ≤ 0 on the right. More precisely, we have the following three possibilities:

(i) $F_{k,N} \geq 0$ for all $k = 0, \dots, N$, or
(ii) $F_{k,N} \leq 0$ for all $k = 0, \dots, N$, or
(iii) there exists an index $k_0 \in \{0, \dots, N-1\}$ such that $F_{k,N} \geq 0$ for $k = 0, \dots, k_0$ while $F_{k,N} \leq 0$ for $k = k_0 + 1, \dots, N$.

As we let $N \to +\infty$, the coefficients $F_{k,N}$ converge to

$$F_k := \sum_{j=0}^{+\infty} b_{j,k} \hat{f}(2j+1),$$

where the right-hand side is absolutely convergent because all the coefficients, except possibly a finite number of them, have the same sign. From the properties (i)–(iii), we see that the sequence $\{F_k\}_k$ has one of the following three properties:

(i') $F_k \geq 0$ for all $k = 0, 1, 2, \dots$, or
(ii') $F_k \leq 0$ for all $k = 0, 1, 2, \dots$, or
(iii') there exists an index $k_0 \in \mathbb{Z}_{+,0}$ such that $F_k \geq 0$ for $k = 0, \dots, k_0$ while $F_{k,N} \leq 0$ for $k = k_0 + 1, k_0 + 2, \dots$.

We remark that while it is clear that property (i) converges to (i'), and that (ii) converges to (ii'), the case (iii) is less stable and might degenerate into (i') or (ii'), as $N \to +\infty$. No matter which of these cases (i')–(iii') we are in, the corresponding Taylor series

$$\mathbf{T}_1 f(x) = \sum_{k=0}^{+\infty} F_k x^{2k+1}, \quad -1 < x < 1,$$

is odd and belongs to $\mathfrak{P}^{\downarrow}_{\mathbb{R}}(2 - \frac{1}{\gamma})$. The proof is complete. □

We now turn to the proof of Theorem 12.4.1.

Proof of Theorem 12.4.1. As the required estimate from above was obtained back in Proposition 12.5.3, we may concentrate on the estimate from below.

The function $t \mapsto q_1^{II}(t, x)$ is clearly even, and then the iterates $\mathbf{T}_1^j q_1^{II}(t, \cdot)$ are also even with respect to the parameter t. By symmetry, it will be enough to treat the case $0 < t < 1$. So, we assume $0 < t < 1$, and observe that Proposition 13.2.1 asserts that $q_1^{II}(t, \cdot)$ is odd and belongs to the class $\mathfrak{P}^{\downarrow}_{\mathbb{R}}(2 - t)$. Next, by applying Corollary 13.3.3 once, we have that $\mathbf{T}_1 q_1^{II}(t, \cdot)$ is odd as well and belongs to $\mathfrak{P}^{\downarrow}_{\mathbb{R}}(2 - \frac{1}{2-t})$. We note that $1 < 2 - t < 2$ while $1 < 2 - \frac{1}{2-t} < \frac{3}{2}$. By applying Corollary 13.3.3 iteratively, we find more generally that $\mathbf{T}_1^j q_1^{II}(t, \cdot) \in \mathfrak{P}^{\downarrow}_{\mathbb{R}}(\gamma_j(t))$, for some $\gamma_j(t)$ with $\gamma_j(t) > 1$. Moreover, since the function $q_1^{II}(t, \cdot)$ is odd, we may apply repeatedly Proposition 2.6.1, to see that

$$\mathbf{T}_1^j q_1^{II}(t,\cdot)(1) = \mathbf{T}_1^{j-1} q_1^{II}(t,\cdot)(1) = \cdots = \mathbf{T}_1 q_1^{II}(t,\cdot)(1) = q_1^{II}(t,\cdot)(1) = 0. \tag{13.3.4}$$

Next, by (13.3.4) and Proposition 13.1.2, we find that

$$\mathbf{T}_1^j q_1^{II}(t,\cdot)(x) > 0, \quad 0 < x < 1,\ j = 1,2,3,\ldots,$$

unless the function $\mathbf{T}_1^j q_1^{II}(t,\cdot) \in \mathfrak{P}_{\mathbb{R}}^{\downarrow}(\gamma_j(t))$ vanishes identically. To rule out the latter possibility, we argue as follows. If $\mathbf{T}_1^j q_1^{II}(t,\cdot) = 0$, then we would have that

$$0 = \mathbf{T}_1^j q_1^{II}(t,\cdot) = \mathbf{T}_1^j(\mathbf{I} - \mathbf{T}_1) Q_1^{II}(t,\cdot) = (\mathbf{I} - \mathbf{T}_1)\mathbf{T}_1^j Q_1^{II}(t,\cdot),$$

that is, $\mathbf{T}_1^j Q_1^{II}(t,\cdot) \in L^1(I_1)$ would be an eigenfunction for the operator $\mathbf{T}_1$ corresponding to the eigenvalue 1, which by Proposition 10.1.2 is only possible when $\mathbf{T}_1^j Q_1^{II}(t,\cdot) = 0$. This is absurd, as $\mathbf{T}_1$ preserves the class of odd strictly increasing functions; see Proposition 2.5.2(i). The proof is complete. □

14 Asymptotic decay of the $\mathbf{T}_1$-orbit of an odd distribution in $\mathfrak{L}(I_1)$

14.1 An application of asymptotic decay for $\beta = 1$. We now show how to obtain, in the critical parameter regime $\alpha\beta = 1$, Theorem 1.9.2 as a consequence of Theorem 1.8.4. The argument would be considerably simpler if we were to appeal to Theorem 1.8.6, but then our presentation would not be self-contained.

Proof of Theorem 1.9.2 for $\alpha\beta = 1$. As observed right after the formulation of Theorem 1.9.2, a scaling argument allows us to reduce the redundancy and fix $\alpha = 1$, in which case the condition $0 < \alpha\beta = 1$ reads $\beta = 1$. From the work in Subsections 8.1 and 8.2, it will be sufficient to show that for $u \in \mathfrak{L}_0(\mathbb{R})$,

$$\Pi_2 u = \Pi_2 \mathbf{J}_1 u = 0 \implies u = 0. \tag{14.1.1}$$

Here, we recall the notation

$$\mathfrak{L}_0(\mathbb{R}) := L^1_0(\mathbb{R}) + \mathbf{H}L^1_0(\mathbb{R}) \subset \mathfrak{L}(\mathbb{R}).$$

So, we assume that $u \in \mathfrak{L}_0(\mathbb{R})$ has $\Pi_2 u = \Pi_2 \mathbf{J}_1 u = 0$. The distribution u has a decomposition $u = f + \mathbf{H}g$, where $f, g \in L^1_0(\mathbb{R})$. We write

$$f^I(t) = \frac{1}{2}(f(t) + f(-t)), \quad f^{II}(t) = \frac{1}{2}(f(-t) - f(t)), \tag{14.1.2}$$

and

$$g^I(t) = \frac{1}{2}(g(t) + g(-t)), \quad g^{II}(t) = \frac{1}{2}(g(-t) - g(t)), \tag{14.1.3}$$

so that the functions $f^I, g^I \in L^1_0(\mathbb{R})$ are even while $f^{II}, g^{II} \in L^1_0(\mathbb{R})$ are odd. We then put

$$u^I = f^I - \mathbf{H}g^{II}, \quad u^{II} = f^{II} - \mathbf{H}g^I,$$

so that $u^I \in \mathfrak{L}_0(\mathbb{R})$ is an even distribution, while u^{II} is odd. This is so because the Hilbert transform is symmetry reversing, odd is mapped to even, and even to odd.

Step I. We first prove that the implication (14.1.1) holds for odd u, that is, when $u = -u^{II}$:

$$\Pi_2 u^{II} = \Pi_2 \mathbf{J}_1 u^{II} = 0 \Longleftrightarrow u^{II} = 0. \tag{14.1.4}$$

The added arrow to the left is of course a trivial implication. Let

$$u^{II}_0 := \mathbf{R}_1 u^{II} \in \mathfrak{L}(I_1) \quad \text{and} \quad u^{II}_1 := \mathbf{R}^\dagger_1 u^{II} \in \mathfrak{L}(\mathbb{R} \setminus \bar{I}_1)$$

denote the restrictions of the distribution u^{II} to the symmetric interval I_1 and to the complement $\mathbb{R} \setminus \bar{I}_1$, respectively. Clearly, u^{II}_0 and u^{II}_1 are odd, because u^{II} is. We will be done with this step once we are able to show that $u^{II}_0 = 0$, because then u^{II}_1 vanishes as well, as a result of Proposition 9.5.1:

$$u^{II}_1 = -\mathbf{R}^\dagger_1 \mathbf{J}_1 \mathbf{T}_1 u^{II}_0 = 0.$$

Indeed, we have Proposition 9.5.2, which tells us that

$$u^{II}_0 = \mathbf{R}_1 u^{II} = 0 \quad \text{and} \quad u^{II}_1 = \mathbf{R}^\dagger_1 u^{II} = 0$$

together imply that $u^{II} = 0$. Finally, to obtain that $u^{II}_0 = 0$, we observe that in addition, Proposition 9.5.1 says that the odd distribution $u^{II}_0 \in \mathfrak{L}(I_1)$ has the important property $u^{II}_0 = \mathbf{T}^2_1 u^{II}_0$. By iteration, then, we have $u^{II}_0 = \mathbf{T}^{2n}_1 u^{II}_0$ for $n = 1, 2, 3, \ldots$, and by letting $n \to +\infty$, we realize from Theorem 1.8.4 that $u_0 = 0$ is the only possible solution in $\mathfrak{L}(I_1)$.

Step II. We now prove, based on Step I, that the implication (14.1.1) holds for an arbitrary distribution $u \in \mathfrak{L}_0(\mathbb{R})$, regardless of symmetry. So, we take a distribution $u \in \mathfrak{L}_0(\mathbb{R})$ for which $\Pi_2 u = 0$ and $\Pi_2 \mathbf{J}_1 u = 0$. We split $u = u^I - u^{II}$ as above, and note that since the operators Π_2 and $\mathbf{J}_1$ both respect odd-even symmetry,

$$0 = \Pi_2 u = \Pi_2 u^I - \Pi_2 u^{II} \quad \text{and} \quad 0 = \Pi_2 \mathbf{J}_1 u = \Pi_2 \mathbf{J}_1 u^I - \Pi_2 \mathbf{J}_1 u^{II}$$

correspond to the splitting of the 0 distribution into odd-even parts inside the space

$$\mathfrak{L}_0(\mathbb{R}/2\mathbb{Z}) := L^1_0(\mathbb{R}/2\mathbb{Z}) + \mathbf{H}_2 L^1_0(\mathbb{R}/2\mathbb{Z}) \subset \mathfrak{L}(\mathbb{R}/2\mathbb{Z}).$$

This means that each part must vanish separately, that is,

$$\Pi_2 u^I = 0, \quad \Pi_2 \mathbf{J}_1 u^I = 0, \quad \Pi_2 u^{II} = 0, \quad \Pi_2 \mathbf{J}_1 u^{II} = 0. \tag{14.1.5}$$

By Step I, we know that the implication (14.1.5) holds for the odd distribution u^{II}, so it is an immediate consequence of (14.1.5) that $u^{II} = 0$. We need to understand the result obtained in Step I better, and write the equivalence (14.1.4) in terms of the functions f^{II} and g^{I}:

$$f^{II} = \mathbf{H}g^{I} \iff \begin{cases} \Pi_2 f^{II} = \Pi_2 \mathbf{H} g^{I}, \\ \Pi_2 \mathbf{J}_1 f^{II} = \Pi_2 \mathbf{H} \mathbf{J}_1 g^{I}. \end{cases} \tag{14.1.6}$$

Next, since we know that $\Pi_2 \mathbf{H} = \mathbf{H}_2 \Pi_2$ as operators on $L^1_0(\mathbb{R})$, we may rewrite (14.1.6) as

$$f^{II} = \mathbf{H}g^{I} \iff \begin{cases} \Pi_2 f^{II} = \mathbf{H}_2 \Pi_2 g^{I}, \\ \Pi_2 \mathbf{J}_1 f^{II} = \mathbf{H}_2 \Pi_2 \mathbf{J}_1 g^{I}. \end{cases} \tag{14.1.7}$$

Since we already know that $u^{II} = 0$, it remains to show why $u^{I} = 0$ must hold as well. The relation (14.1.5) also contains the conditions $\Pi_2 u^{I} = \Pi_2 \mathbf{J}_1 u^{I} = 0$, which in terms of f^{I} and g^{II} amount to having

$$\begin{cases} \Pi_2 f^{I} = \mathbf{H}_2 \Pi_2 g^{II}, \\ \Pi_2 \mathbf{J}_1 f^{I} = \mathbf{H}_2 \Pi_2 \mathbf{J}_1 g^{II}. \end{cases}$$

Let us apply the periodic Hilbert transform $\mathbf{H}_2$ to the left-hand and right-hand sides, which is an invertible transformation on $\mathfrak{L}_0(\mathbb{R}/2\mathbb{Z})$ with $\mathbf{H}_2^2 = -\mathbf{I}$. The result is

$$\begin{cases} \mathbf{H}_2 \Pi_2 f^{I} = -\Pi_2 g^{II}, \\ \mathbf{H}_2 \Pi_2 \mathbf{J}_1 f^{I} = -\Pi_2 \mathbf{J}_1 g^{II}. \end{cases}$$

But this falls inside the setting of (14.1.7), only with $-g^{II}$ in place of f^{II}, and f^{I} in place of g^{I}. So we get from (14.1.7) that $-g^{II} = \mathbf{H}f^{I}$, which after application of $\mathbf{H}$ reads $f^{I} = \mathbf{H}g^{II}$. This means that $u^{I} = f^{I} - \mathbf{H}g^{II} = 0$, as desired. Finally, since both u^{I} and u^{II} vanish, we obtain $u = u^{I} - u^{II} = 0$. This proves that the implication (14.1.1) holds for every $u \in \mathfrak{L}_0(\mathbb{R})$, which completes the proof. □

14.2 The proof of asymptotic decay for $\beta = 1$. We now proceed with the proof of Theorem 1.8.4. As in the proof of Theorem 1.8.2, we have to be particularly careful because the operator $\mathbf{T}_1 : \mathfrak{L}(I_1) \to \mathfrak{L}(I_1)$ has norm > 1. However, it clearly acts contractively on $L^1(I_1)$.

Proof of Theorem 1.8.4. Since $u_0 \in \mathfrak{L}(I_1)$, we know that there exist functions $f \in L^1(\mathbb{R})$ and $g \in L^1_0(\mathbb{R})$ such that $u_0 = \mathbf{R}_1(f + \mathbf{H}g)$.

Step I. There exists an odd extension of u_0 to all of $\mathbb{R}$. Let the functions f^I, f^{II}, g^I, g^{II} be given by (14.1.2) and (14.1.3), and put

$$u^I = f^I - \mathbf{H}g^{II}, \quad u^{II} = f^{II} - \mathbf{H}g^I,$$

so that $u^I \in \mathfrak{L}(\mathbb{R})$ is an even distribution, while $u^{II} \in \mathfrak{L}(\mathbb{R})$ is odd. The way things are set up, we have that $u_0 = \mathbf{R}_1 u$, where $u := u^I - u^{II}$. Since it is given that u_0 is odd, we must have that $\mathbf{R}_1 u^I = 0$, and that $u_0 = -\mathbf{R}_1 u^{II}$. The distribution $-u^{II}$ is odd on all of $\mathbb{R}$, and provides an extension of u_0 beyond the interval I_1. We will focus our attention on the odd distribution $u^{II} = f^{II} - \mathbf{H}g^I$, which has $\mathbf{R}_1 u^{II} = -u_0$.

Step II. Without loss of generality, we may require of the even function $g^I \in L^1_0(\mathbb{R})$ that in addition

$$\langle 1, g^I\rangle_{I_1} = \langle 1, g^I\rangle_{\mathbb{R}\setminus I_1} = 0. \tag{14.2.1}$$

To this end, we consider the even function $h^I \in L^1_0(\mathbb{R})$ given by

$$h^I(x) := 1_{I_1}(x) - x^{-2} 1_{\mathbb{R}\setminus I_1}(x), \quad x \in \mathbb{R},$$

with Hilbert transform

$$\mathbf{H}h^I(x) = \frac{1}{\pi} \log\left|\frac{x+1}{x-1}\right| + \frac{1}{\pi x^2} \log\left|\frac{x+1}{x-1}\right| - \frac{2}{\pi x}, \quad x \in \mathbb{R},$$

and note that $\mathbf{H}h^I \in L^1_0(\mathbb{R})$ is odd. Now, if (14.2.1) is not fulfilled to begin with, then we consider instead the functions

$$F^{II} := f^{II} + \frac{1}{2}\langle g^I, 1\rangle_{I_1} \mathbf{H}h^I, \quad G^I := g^I - \frac{1}{2}\langle g^I, 1\rangle_{I_1} h^I.$$

Indeed, we see that $F^{II}, G^I \in L^1_0(\mathbb{R})$ where F^{II} is odd and G^I is even, that (14.2.1) holds with G^I in place of g^I, and that $u^{II} = f^{II} - g^I = F^{II} - \mathbf{H}G^I$. This allows us to assume that $f^{II}, g^I \in L^1_0(\mathbb{R})$ are chosen so that (14.2.1) holds, and completes the proof of Step II.

Step III. Splitting of the functions f^{II} and g^I according to intervals. We split $f = f_1 + f_2$ and $g = g_1 + g_2$, where

$$f^{II}_1 := f^{II} 1_{I_1} \in L^1_0(I_1), \quad f^{II}_2 := f^{II} 1_{\mathbb{R}\setminus I_1} \in L^1_0(\mathbb{R} \setminus I_1),$$

and

$$g_1 := g 1_{I_1} \in L^1_0(I_1), \quad g_2 := g 1_{\mathbb{R}\setminus I_1} \in L^1_0(\mathbb{R} \setminus I_1).$$

Here, we used in fact Step II. Note that the functions f^{II}_1, f^{II}_2 are odd, while g^I_1, g^I_2 are even. We write $u^{II}_0 := \mathbf{R}_1 u^{II}$, so that $u^{II}_0 = -u_0$. We note that

$$u^{II}_0 = \mathbf{R}_1(f^{II} + \mathbf{H}g^I) = \mathbf{R}_1(f^{II}_1 + f^{II}_2 + \mathbf{H}g^I_1 + \mathbf{H}g^I_2) = f^{II}_1 + \mathbf{R}_1\mathbf{H}g^I_1 + \mathbf{R}_1\mathbf{H}g^I_2. \tag{14.2.2}$$

By applying the operator $\mathbf{T}_1^N$ for $N = 2, 3, 4, \ldots$ to the leftmost and rightmost sides of (14.2.2), we obtain

$$\mathbf{T}_1^N u_0^{II} = \mathbf{T}_1^N f_1^{II} + \mathbf{T}_1^N \mathbf{R}_1 \mathbf{H} g_1^I + \mathbf{T}_1^N \mathbf{R}_1 \mathbf{H} g_2^I,$$

and after application of the valeur au point operation, this identity reads, for $N = 2, 3, 4, \ldots$,

$$\begin{aligned} &\mathrm{vap}[\mathbf{T}_1^N u_0^{II}](x) \\ &= \mathbf{T}_1^N f_1^{II}(x) + \mathrm{vap}[\mathbf{T}_1^N \mathbf{R}_1 \mathbf{H} g_1^I](x) + \mathrm{vap}[\mathbf{T}_1^N \mathbf{R}_1 \mathbf{H} g_2^I](x), \qquad \text{a.e. } x \in I_1. \end{aligned} \tag{14.2.3}$$

Next, by Propositions 2.3.1(v) and 10.1.2, we have for each fixed η, $0 < \eta < 1$, the convergence

$$\mathbf{T}_1^N f_1^{II} \to 0 \quad \text{and} \quad 1_{I_\eta} \mathrm{vap}[\mathbf{T}_1^N \mathbf{R}_1 \mathbf{H} g_2^I] \to 0, \tag{14.2.4}$$

the first one in the norm of $L^1(I_1)$ as $N \to +\infty$. That is, two terms on the right-hand side of (14.2.3) vanish in the limit on compact subintervals, and we are left to analyze the remaining middle term.

By rearranging the terms in the finite expansion of Proposition 9.8.2 with $n := N$, applied to the even function $g_1^I \in L_0^1(I_1)$ in place of f, we obtain that

$$\begin{aligned} \mathbf{T}_1^N \mathbf{R}_1 \mathbf{H} g_1^I &= \mathbf{R}_1 \mathbf{H} \mathfrak{T}_1^N g_1^I - \mathbf{T}_1^{N-1} \mathbf{R}_1 \mathbf{H} \mathbf{J}_1 g_1^I + \mathbf{T}_1 \mathbf{R}_1 \mathbf{H} \mathbf{J}_1 \mathfrak{T}_1^N g_1^I \\ &\quad + \sum_{j=1}^{N-1} (\mathbf{T}_1^2 - \mathbf{I}) \mathbf{T}_1^{N-j-1} \mathbf{R}_1 \mathbf{H} \mathbf{J}_1 \mathfrak{T}_1^j g_1^I. \end{aligned} \tag{14.2.5}$$

Here, of course, $\mathfrak{T}_1 = \mathbf{T}_1$ as operators, but we keep writing $\mathfrak{T}_1^m g_1^I$ to emphasize that the function is extended to vanish off the interval I_1; this is a relevant issue, since the Hilbert transform is nonlocal. Since we know that $g^I \in L_0^1(I_1)$, Proposition 2.3.1(v) implies that $\mathfrak{T}_1^N g_1^I = \mathbf{T}_1^N g_1^I \to 0$ in norm in $L^1(I_1)$ as $N \to +\infty$, so that

$$\mathrm{vap}[\mathbf{R}_1 \mathbf{H} \mathfrak{T}_1^N g_1^I] \to 0 \quad \text{and} \quad \mathrm{vap}[\mathbf{T}_1 \mathbf{R}_1 \mathbf{H} \mathbf{J}_1 \mathfrak{T}_1^N g_1^I] \to 0, \tag{14.2.6}$$

in $L^{1,\infty}(I_1)$ as $N \to +\infty$. Moreover, by Proposition 10.1.2, $\mathbf{T}_1^{N-1} \mathbf{R}_1 \mathbf{J}_1 g_1^I \to 0$ as $n \to +\infty$, uniformly on compact subsets of I_1, so that in particular,

$$1_{I_\eta} \mathrm{vap} \mathbf{T}_1^{N-1} \mathbf{R}_1 \mathbf{J}_1 g_1^I \to 0 \tag{14.2.7}$$

in $L^1(I_1)$, for each fixed η, $0 < \eta < 1$. We know, from (14.2.6) and (14.2.7), that the first three terms on the right-hand side (14.2.5) tend to zero as N tends to $+\infty$. Thus we only need to deal with the the summation term in (14.2.5).

Step IV. Application of kernel techniques. As in Subsection 10.1, we may write

$$\text{vap}\,\mathbf{H}\mathbf{J}_1\mathfrak{T}_1^j g_1^I(x) = \frac{1}{\pi}\int_{-1}^{1} Q_1(t,x)\,\mathfrak{T}_1^j g_1^I(t)\mathrm{d}t, \quad x \in I_1, \tag{14.2.8}$$

where $Q_1(t,x) = t/(1+tx)$. We recall the odd-even decomposition of $Q_1(t,x)$:

$$Q_1(t,x) = Q_1^I(t,x) - Q_1^{II}(t,x), \quad \text{where } Q_1^I(t,x) = \frac{t}{1-x^2t^2},\ Q_1^{II}(t,x) = \frac{t^2x}{1-t^2x^2}.$$

As, by inspection, the kernel $t \mapsto Q_1^I(t,x)$ is odd, and since the function $\mathfrak{T}_1^j g_1^I$ is even, we may rewrite (14.2.8) in the form

$$\mathbf{H}\mathbf{J}_1\mathfrak{T}_1^j g_1^I(x) = -\frac{2}{\pi}\int_0^1 Q_1^{II}(t,x)\,\mathfrak{T}_1^j g_1^I(t)\mathrm{d}t, \quad x \in I_1. \tag{14.2.9}$$

Using (14.2.9), we may decompose the sum in (14.2.5) in the following way:

$$\begin{aligned}
&\sum_{j=1}^{N-1}(\mathbf{T}_1^2-\mathbf{I})\mathbf{T}_1^{N-j-1}\mathbf{R}_1\mathbf{H}\mathbf{J}_1\mathfrak{T}_1^j g_1^I(x)\\
&\qquad = -\frac{2}{\pi}\int_0^1\sum_{j=1}^{N-1}\mathbf{T}_1^{N-j-1}(\mathbf{T}_1^2-\mathbf{I})Q_1^{II}(t,\cdot)(x)\,\mathfrak{T}_1^j g_1^I(t)\mathrm{d}t\\
&\qquad = \frac{2}{\pi}\sum_{j=1}^{N-1}\int_0^1(\mathbf{T}_1^{N-j-1}+\mathbf{T}_1^{N-j})q_1^{II}(t,\cdot)(x)\,\mathfrak{T}_1^j g_1^I(t)\mathrm{d}t.
\end{aligned} \tag{14.2.10}$$

All the sums appearing in (14.2.10) are odd functions of x, so we need only estimate them in the righthand interval $I_1^+ = [0,1[$. By appealing to the fundamental estimate of Theorem 12.4.1, we may obtain a pointwise estimate in (14.2.10), for $x \in I_1^+$, as follows:

$$\begin{aligned}
&\left|\text{vap}\sum_{j=1}^{N-1}(\mathbf{T}_1^2-\mathbf{I})\mathbf{T}_1^{N-j-1}\mathbf{R}_1\mathbf{H}\mathbf{J}_1\mathfrak{T}_1^j g_1^I(x)\right|\\
&\qquad \le \frac{2}{\pi}\sum_{j=1}^{N-1}\int_0^1|(\mathbf{T}_1^{N-j-1}+\mathbf{T}_1^{N-j})q_1^{II}(t,\cdot)(x)\,\mathfrak{T}_1^j g_1^I(t)|\mathrm{d}t\\
&\qquad \le \frac{1}{\pi}\sum_{j=1}^{N-1}(\mathbf{T}_1^{N-j-1}+\mathbf{T}_1^{N-j})q_1^{II}(1,\cdot)(x)\,\|\mathfrak{T}_1^j g_1^I\|_{L^1(I_1)}.
\end{aligned} \tag{14.2.11}$$

As observed previously, since $g_1^I \in L_0^1(I_1)$, Proposition 2.3.1(v) tells us that

$$\mathfrak{T}_1^j g_1^I = \mathbf{T}_1^j g_1^I \to 0$$

in norm in $L^1(I_1)$ as $j \to +\infty$. It follows that we may, for a given positive real ϵ, find a positive integer $n_0 = n_0(\epsilon)$ such that $\|\mathfrak{T}_1^j g_1^I\|_{L^1(I_1)} \le \epsilon$ for $j \ge n_0(\epsilon)$. We split the summation in (14.2.11) accordingly, for $N > n_0(\epsilon)$, and use that the transfer operator $\mathfrak{T}_1 = \mathbf{T}_1$ is a contraction on $L^1(I_1)$:

$$\begin{aligned}
&\left|\mathrm{vap}\sum_{j=1}^{N-1}(\mathbf{T}_1^2-\mathbf{I})\mathbf{T}_1^{N-j-1}\mathbf{R}_1\mathbf{H}\mathbf{J}_1\mathfrak{T}_1^j g_1^I(x)\right| \\
&\le \|g_1^I\|_{L^1(I_1)}\frac{1}{\pi}\sum_{j=1}^{n_0(\epsilon)-1}(\mathbf{T}_1^{N-j-1}+\mathbf{T}_1^{N-j})q_1^{II}(1,\cdot)(x) \\
&\quad+\frac{\epsilon}{\pi}\sum_{j=n_0(\epsilon)}^{N-1}(\mathbf{T}_1^{N-j-1}+\mathbf{T}_1^{N-j})q_1^{II}(1,\cdot)(x),
\end{aligned} \tag{14.2.12}$$

where again $x \in I_1^+$ is assumed. The odd part of the dynamically reduced Hilbert kernel $x \mapsto q_1^{II}(1,x)$ is odd and smooth on $\bar{I}_1$, so Proposition 2.3.1(v) tells us that for fixed ϵ,

$$\sum_{j=1}^{n_0(\epsilon)-1}\|(\mathbf{T}_1^{N-j-1}+\mathbf{T}_1^{N-j})q_1^{II}(1,\cdot)\|_{L^1(I_1)} \to 0 \quad \text{as } N \to +\infty.$$

As for the second sum on the right-hand side of (14.2.12), we use finite Neumann series summation (12.3.3) together with Lemma 2.5.2 to obtain that

$$\sum_{j=n_0(\epsilon)}^{N-1}(\mathbf{T}_1^{N-j-1}+\mathbf{T}_1^{N-j})q_1^{II}(1,\cdot)(x) \le (\mathbf{I}+\mathbf{T}_1)Q_1^{II}(1,\cdot)(x) \le \frac{2}{1-x^2}, \quad x \in I_1^+.$$

Note that in the last step, we compared $Q_1^{II}(1,x)$ with the invariant density $\kappa_1(x) = (1-x^2)^{-1}$. It now follows from the estimate (14.2.12) and symmetry that for fixed η with $0 < \eta < 1$,

$$\limsup_{N\to+\infty}\left\|1_{I_\eta}\sum_{j=1}^{N-1}(\mathbf{T}_1^2-\mathbf{I})\mathbf{T}_1^{N-j-1}\mathbf{R}_1\mathbf{H}\mathbf{J}_1\mathfrak{T}_1^j g_1^I(x)\right\|_{L^1(I_1)} \le \frac{4\epsilon}{\pi}\log\frac{1+\eta}{1-\eta}. \tag{14.2.13}$$

As this is true for any $\epsilon > 0$, it follows that for fixed η with $0 < \eta < 1$,

$$\limsup_{N\to+\infty}\left\|1_{I_\eta}\sum_{j=1}^{N-1}(\mathbf{T}_1^2-\mathbf{I})\mathbf{T}_1^{N-j-1}\mathbf{R}_1\mathbf{H}\mathbf{J}_1\mathfrak{T}_1^j g_1^I(x)\right\|_{L^1(I_1)} = 0. \tag{14.2.14}$$

This means that also the last term on the right-hand side of (14.2.5) tends to 0 in the mean on all compact subintervals. Putting things together in the context of the decomposition (14.2.3), we see from the convergences (14.2.4) and the further decomposition (14.2.5), together with the associated convergences (14.2.6), (14.2.7), and (14.2.13), that $1_{I_\eta}\mathrm{vap}[\mathbf{T}_1^N u_0^{II}] \to 0$ in $L^{1,\infty}(I_1)$, which is the claimed assertion, because $u_0 = -u_0^{II}$. The proof is complete. □

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Haakan Hedenmalm
DEPARTMENT OF MATHEMATICS
KTH ROYAL INSTITUTE OF TECHNOLOGY
S–10044 STOCKHOLM, SWEDEN
email: haakanh@kth.se

Alfonso Montes-Rodríguez
DEPARTMENT OF MATHEMATICAL ANALYSIS
UNIVERSITY OF SEVILLA
SEVILLA, SPAIN
email: amontes@us.es

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