Bloch functions, asymptotic variance, and geometric zero packing

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The Bergman projection

We let $\mathbb{D}$ be the open unit disk.

Let $\mathbf{P}$ denote the Bergman projection

$$\mathbf{P}f(z) := \int_{\mathbb{D}} \frac{f(w)}{(1 - z\bar{w})^2}\,\mathrm{d}A(w), \qquad z \in \mathbb{D},$$

which is well-defined if $f \in L^1(\mathbb{D})$ ($\mathrm{d}A$ is normalized area measure). It is well-known that $\mathbf{P}$ maps $L^p(\mathbb{D}) \to L^p(\mathbb{D})$ for $1 < p < +\infty$, and that

$$\mathbf{P}:\ L^2(\mathbb{D}) \to L^2(\mathbb{D})$$

is a norm contraction. We write $\langle \cdot, \cdot \rangle_{\mathbb{D}}$ for the sesquilinear form

$$\langle f, g \rangle_{\mathbb{D}} := \int_{\mathbb{D}} f(z)\bar{g}(z)\mathrm{d}A(z),$$

which is well-defined if $f\bar{g} \in L^1(\mathbb{D})$. We shall be concerned with the space $\mathbf{P}L^\infty(\mathbb{D})$, supplied with the canonical norm

$$\|f\|_{\mathbf{P}L^\infty(\mathbb{D})} := \inf\left\{\|\mu\|_{L^\infty(\mathbb{D})} :\ \mu \in L^\infty(\mathbb{D}) \quad \text{and} \quad f = \mathbf{P}\mu\right\}.$$

The Bloch space

It is well-known that as a space, $\mathbf{P}L^\infty(\mathbb{D}) = \mathcal{B}(\mathbb{D})$, the *Bloch space*. This seems to have been observed first in a 1976 paper by Coifman, Rochberg, Weiss [CRW]. We recall that the Bloch space consists of all holomorphic $f : \mathbb{D} \to \mathbb{C}$ subject to the seminorm boundedness condition

$$\|f\|_{\mathcal{B}(\mathbb{D})} := \sup_{z\in\mathbb{D}} (1-|z|^2)|f'(z)| < +\infty.$$

Indeed, if $f = \mathbf{P}\mu$, where $\|\mu\|_{L^\infty(\mathbb{D})} \le 1$, then

$$(1-|z|^2)|(\mathbf{P}\mu)'(z)| = 2(1-|z|^2)\left|\int_{\mathbb{D}} \frac{\bar{w}\mu(w)}{(1-z\bar{w})^3}\mathrm{d}A(w)\right|$$
$$\le 2(1-|z|^2)\int_{\mathbb{D}} \frac{|w|}{|1-z\bar{w}|^3}\mathrm{d}A(w) = 2(1-|z|^2)\sum_{j=0}^{+\infty} \frac{[(\frac{3}{2})_j]^2}{(j!)^2(j+\frac{3}{2})}|z|^{2j} \le \frac{8}{\pi}, \tag{1}$$

where the main loss of information is in the application of the triangle inequality.

The Bloch space, cont

On the other hand, if $f \in \mathcal{B}(\mathbb{D})$ with $f(0) = f'(0) = 0$, and we put

$$\mu_f(w) := (1 - |w|^2)\frac{f'(w)}{\bar{w}}, \qquad w \in \mathbb{D},$$

then $\mu_f \in L^\infty(\mathbb{D})$, and

$$|\mu_f(w)| = (1 - |w|^2)\left|\frac{f'(w)}{w}\right| \le (1 + \mathrm{o}(1))\|f\|_{\mathcal{B}(\mathbb{D})} \quad \text{as} \quad |w| \to 1,$$

while

$$\mathbf{P}\mu_f(z) = \int_{\mathbb{D}} \frac{(1 - |w|^2)f'(w)}{\bar{w}(1 - z\bar{w})^2}\mathrm{d}A(w) = f(z) - f(0) = f(z), \qquad z \in \mathbb{D}.$$

so in terms of boundary effects there would appear to be a gap of the size $8/\pi$ between the two norms. Perälä [Per] has shown that the bound $8/\pi$ in (1) is best possible.

The Bloch space as a dual space

It is known that with respect to $\langle\cdot,\cdot\rangle_{\mathbb{D}}$, $\mathcal{B}(\mathbb{D})$ can be identified with the dual space of the Bergman space $A^1(\mathbb{D})$ in much the same way that $\mathrm{BMOA}(\mathbb{D})$ is the dual space of $H^1(\mathbb{D})$ with respect to the dual action on the circle $\mathbb{T}$:

$$\langle f, g\rangle_{\mathbb{T}} := \int_{\mathbb{T}} f(z)\bar{g}(z)\mathrm{d}s(z),$$

where $\mathrm{d}s(z) := |\mathrm{d}z|/(2\pi)$ is normalized arc length measure. Indeed, the norm induced on $\mathcal{B}(\mathbb{D})$ by $A^1(\mathbb{D})$ is that of $\mathbf{P}L^\infty(\mathbb{D})$:

PROPOSITION 1

For $f \in \mathcal{B}(\mathbb{D})$, we have that

$$\|f\|_{\mathbf{P}L^\infty(\mathbb{D})} = \sup\left\{|\langle f, g\rangle_{\mathbb{D}}| :\ g \in A^2(\mathbb{D}),\ \ \|g\|_{A^1(\mathbb{D})} \le 1\right\}.$$

Remarks on Proposition 1

The assertion is a consequence of the Hahn-Banach theorem.

Indeed, a bounded linear functional on $A^1(\mathbb{D})$ is lifted to a bounded linear functional on $L^1(\mathbb{D})$ with the same norm. Such functionals then correspond to elements of $L^\infty(\mathbb{D})$.

This is used to see that with respect to $\langle \cdot, \cdot \rangle_{\mathbb{D}}$, the dual space to $A^1(\mathbb{D})$ is $\mathbf{P}L^\infty(\mathbb{D})$, isometrically and isomorphically.

Duality and dilations

For a function f, we let $f_r(z) := f(rz)$ denote its dilate. We shall need the following identities.

PROPOSITION 2

Suppose $g, h \in L^1(\mathbb{D})$ are both harmonic. Then, for $0 < r < 1$, we have

$$\langle g_r, h \rangle_{\mathbb{D}} = \langle g, h_r \rangle_{\mathbb{D}}$$

In other words, the dilation $\mathbf{D}_r f(z) = f(rz)$ is self-adjoint with respect to the duality $\langle \cdot, \cdot \rangle_{\mathbb{D}}$.

PROPOSITION 3

Suppose $g = \mathbf{P}\mu$, where $\mu \in L^\infty(\mathbb{D})$, and that $h \in L^\infty(\mathbb{D})$ is harmonic. Then

$$\langle z g_r, h \rangle_{\mathbb{T}} = \langle g_r, \partial h \rangle_{\mathbb{D}} = \langle g, (\partial h)_r \rangle_{\mathbb{D}} = \langle \mathbf{P}\mu, (\partial h)_r \rangle_{\mathbb{D}} = \langle \mu, (\partial h)_r \rangle_{\mathbb{D}}.$$

Duality and dilations, cont.

Remarks on Proposition 3

The first equality follows from Green's formula. The second step uses that the dilation is self-adjoint, which is easy to check using Taylor series expansions. The third equality expresses that $g = \mathbf{P}\mu$, while the fourth uses that $\mathbf{P}$ is self-adjoint and preserves the holomorphic functions.

A basic estimate

Corollary 4

Suppose $g = \mathbf{P}\mu$, where $\mu \in L^\infty(\mathbb{D})$, and that $h \in H^\infty(\mathbb{D})$. Then

$$|\langle zg_r, h\rangle_{\mathbb{T}}| \leq \|\mu\|_{L^\infty(\mathbb{D})}\|(h')_r\|_{A^1(\mathbb{D})} = \frac{\|\mu\|_{L^\infty(\mathbb{D})}}{r^2}\int_{\mathbb{D}(0,r)}|h'|\mathrm{d}A.$$

We now resort to the Cauchy-Schwarz inequality:

In the setting of Corollary 4, we have that

$$\begin{aligned}
&|\langle zg_r, h\rangle_{\mathbb{T}}| \leq \frac{\|\mu\|_{L^\infty(\mathbb{D})}}{r^2}\int_{\mathbb{D}(0,r)}|h'|\mathrm{d}A \\
&\leq \frac{\|\mu\|_{L^\infty(\mathbb{D})}}{r^2}\left(\int_{\mathbb{D}(0,r)}|h'(z)|^2(1-|z|^2)\mathrm{d}A(z)\right)^{1/2}\left(\int_{\mathbb{D}(0,r)}\frac{\mathrm{d}A(z)}{1-|z|^2}\right)^{1/2} \\
&\qquad \leq \frac{\|\mu\|_{L^\infty(\mathbb{D})}}{r^2}\left(\int_{\mathbb{D}}|h'(z)|^2(1-|z|^2)\mathrm{d}A(z)\right)^{1/2}\left(\log\frac{1}{1-r^2}\right)^{1/2}. \quad (2)
\end{aligned}$$

Application of the Littlewood-Paley identity

If $h(0) = 0$, the Littlweood-Paley identity asserts that

$$\|h\|^2_{L^2(\mathbb{T})} = \int_{\mathbb{D}} |h'(z)|^2 \log \frac{1}{|z|^2} \mathrm{d}A(z) \geq \int_{\mathbb{D}} |h'(z)|^2 (1 - |z|^2) \mathrm{d}A(z)$$

by the elementary inequality

$$1 - |z|^2 \leq \log \frac{1}{|z|^2}.$$

It now follows from (2) that

$$|\langle zg_r, h\rangle_{\mathbb{T}}| \leq \frac{\|\mu\|_{L^\infty(\mathbb{D})}}{r^2} \|h\|_{L^2(\mathbb{T})} \left(\log \frac{1}{1 - r^2} \right)^{1/2},$$

and with $h = zg_r$ we obtain

$$\|g_r\|_{L^2(\mathbb{T})} \leq \frac{\|\mu\|_{L^\infty(\mathbb{D})}}{r^2} \left(\log \frac{1}{1 - r^2} \right)^{1/2}. \tag{3}$$

The asymptotic variance of a Bloch function

Definition

For a given function g in the Bloch space, its **asymptotic variance** is the quantity

$$\sigma(g)^2 := \limsup_{r\to 1^-} \frac{\|g_r\|^2_{L^2(\mathbb{T})}}{\log \frac{1}{1-r^2}}.$$

The estimate (3) shows that for $g = \mathbf{P}\mu$,

$$\sigma(g)^2 \le \|\mu\|^2_{L^\infty(\mathbb{D})}.$$

Definition

The **universal asymptotic variance** for $\mathbf{P}L^\infty(\mathbb{D})$ is the quantity

$$\Sigma^2 := \sup_{\|\mu\|_{L^\infty(\mathbb{D})}\le 1} \sigma(\mathbf{P}\mu)^2.$$

From (3) it is clear that $\Sigma^2 \le 1$. This estimate was obtained by other means in [AIPP].

Why is Σ^2 of interest?

Σ^2 is related to quasiconformal mappings of with small $k \sim 0$:

$$|\bar{\partial}\varphi| \leq k|\partial\varphi|.$$

Let $\mathcal{S}_k$ denote the class of φ which are conformal on $\mathbb{D}$ and have a global k-quasiconformal extension. For $t \in \mathbb{C}$, we define

$$\beta_\varphi(t) := \limsup_{r \to 1^-} \frac{\log \int_{\mathbb{T}} |(\varphi_r')^t| \mathrm{d}s}{\log \log \frac{1}{1-r}}$$

and put (quasiconformal integral means spectrum)

$$B(k,t) := \sup_{\varphi \in \mathcal{S}_k} \beta_\varphi(t).$$

THEOREM (Ivrii)

For $k, t \sim 0$, we have that

$$B(k,t) \sim \frac{\Sigma^2}{4} k^2 |t|^2.$$

See [Ivr].

Dimesion of quasicircles

Let $D(k)$ denote the maximal Hausdorff (or Minkowski) dimension of a k-quasicircle (the image of the unit circle under a k-quasiconformal mapping). It turns out that it is also related to Σ^2:

$$D(k) = 1 + \Sigma^2 k^2 + O(k^{8/3}) \quad \text{as} \quad k \to 0.$$

See [Ivr]. It is an open question whether the error bound can be improved to $O(k^3)$.

The bound $\Sigma^2 < 1$

Let

$$\rho_{\alpha,\beta}(\mathbb{H}) := \liminf_{r\to 1^-} \inf_f \frac{\int_{\mathbb{D}(0,r)} ((1-|z|^2)^\alpha |f(z)|^\beta - 1)^2 \frac{\mathrm{d}A(z)}{1-|z|^2}}{\int_{\mathbb{D}(0,r)} \frac{\mathrm{d}A(z)}{1-|z|^2}}.$$

where the infimum is over all polynomials f.

THEOREM

We have that $\Sigma^2 = 1 - \rho_{1,1}(\mathbb{H})$. Moreover, $\rho_{1,1}(\mathbb{H}) \geq 2.3 \times 10^{-8}$ so that in particular $\Sigma^2 < 1$.

A Hilbert space lemma

The following lemma is very helpful.

LEMMA

Let $\mathcal{H}$ be a real-linear Hilbert space, and suppose $u, v \in \mathcal{H}$ and $0 \le \theta \le 1$. TFAE:

(a) $\forall c \in \mathbb{R}$: $\|u - cv\|_{\mathcal{H}} \ge \theta \|u\|_{\mathcal{H}}$,

(b) $\forall c \in \mathbb{R}$: $\|u - cv\|_{\mathcal{H}} \ge |c| \theta \|v\|_{\mathcal{H}}$,

(c) $|\langle u, v \rangle_{\mathcal{H}} \le (1 - \theta^2)^{1/2} \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}}$.

Alternative definition of Σ^2

It is natural to also consider the possibility of changing the order of the operations in the definition of Σ^2:

$$\tilde{\Sigma}^2 := \limsup_{r\to 1^-} \sup_{\|\mu\|_{L^\infty(\mathbb{D})}\le 1} \frac{\|g_r\|^2_{L^2(\mathbb{T})}}{\log\frac{1}{1-r^2}}, \qquad g = \mathbf{P}\mu.$$

Then clearly $\tilde{\Sigma}^2 \ge \Sigma^2$, and one might guess that $\tilde{\Sigma}^2 = \Sigma^2$. In terms of the dilation operator $\mathbf{D}_r f(z) = f(rz)$, it is a matter of definition that

$$\tilde{\Sigma}^2 := \limsup_{r\to 1^-} \frac{\|\mathbf{D}_r\|^2_{\mathbf{P}L^\infty(\mathbb{D})\to L^2(\mathbb{T})}}{\log\frac{1}{1-r^2}}.$$

Since $\mathbf{D}_r$ is self-adjoint by Proposition 2, and the $L^2(\mathbb{T})$ norm is the same as the norm on $H^2(\mathbb{D})$ norm on holomorphic functions, we must also have

$$\tilde{\Sigma}^2 = \limsup_{r\to 1^-} \frac{\|\mathbf{D}_r\|^2_{(H^2(\mathbb{D}))^*\to A^1(\mathbb{D})}}{\log\frac{1}{1-r^2}},$$

where $(H^2(\mathbb{D}))^*$ is the dual with respect to $\langle\cdot,\cdot\rangle_{\mathbb{D}}$.

The identity $\Sigma^2 = 1 - \rho_{1,1}(\mathbb{H})$

Up to inessential contributions, the space $(H^2(\mathbb{D}))^*$ can be identified with the Hilbert space $A^2_1(\mathbb{D})$ of holomorphic functions f with finite norm

$$\|f\|^2_{A^2_1(\mathbb{D})} := \int_{\mathbb{D}} |f(z)|^2(1-|z|^2)\mathrm{d}A(z) < +\infty.$$

It follows that

$$\tilde{\Sigma}^2 = \limsup_{r\to 1^-} \sup_f \frac{\|f_r\|^2_{A^1(\mathbb{D})}}{\|f\|^2_{A^2_1(\mathbb{D})} \log \frac{1}{1-r^2}},$$

and Lemma ?? now shows that $\tilde{\Sigma}^2 = 1 - \rho_{1,1}(\mathbb{H})$. Actually, a recent theorem of Wennman is needed as well. Finally, an argument involving the construction of a single quasioptimizing μ out of a sequence of optimizers shows that $\tilde{\Sigma}^2 = \Sigma^2$.

The theorem $\rho_{1,1}(\mathbb{H}) \geq 2.3 \times 10^{-8}$

Basically, the argument begins with a local estimate which obtains a universal lower bound for

$$\int_{\mathbb{D}(0,\frac{1}{2})} ((1-|z|^2)|f(z)|-1)^2 \frac{\mathrm{d}A(z)}{1-|z|^2}.$$

This estimate is the moved around by a Möbius mapping to other disks and after suitable integration the desired result is obtained.

Analogous planar densities

We consider the asymptotic density

$$\rho_\beta(\mathbb{C}) := \liminf_{R\to+\infty} R^{-2} \inf_f \int_{\mathbb{D}(0,R)} (e^{-|z|^2}|f(z)|^\beta - 1)^2 \mathrm{d}A(z),$$

where again f runs over all polynomials. Here, $\beta > 0$ is assumed. Again, one shows that $\rho_\beta(\mathbb{C}) > 0$, so the density is nontrivial. This is a (local) limit case of the hyperbolic density, since

$$\left(1 - \frac{|z|^2}{\alpha}\right)^\alpha \to e^{-|z|^2}$$

as $\alpha \to +\infty$.

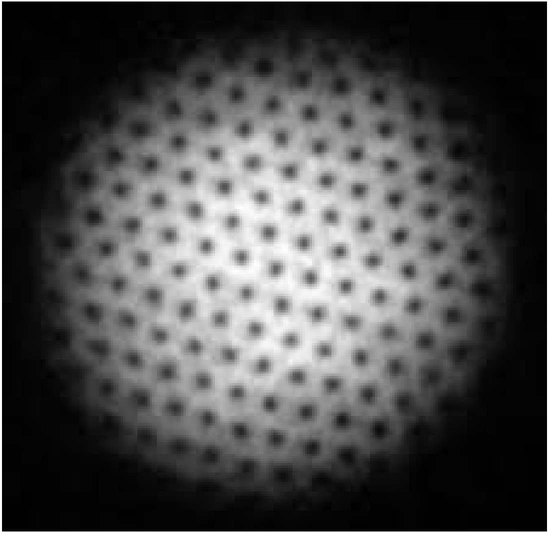
Abrikosov's work on superconductivity à la Nier

In view of Lemma ??,

$$\frac{1}{1-\rho_\beta(\mathbb{C})} = \liminf_{R\to+\infty} \frac{R^2 \int_{\mathbb{D}(0,R)} |f(z)|^{2\beta} e^{-2|z|^2} \mathrm{d}A(z)}{\left(\int_{\mathbb{D}(0,R)} |f(z)|^{\beta} e^{-|z|^2} \mathrm{d}A(z)\right)^2}.$$

This density appears in the work of Aftalion, Blanc, and Nier [ABN] for $\beta = 2$ in the context of Abrikosov's analysis ([Abr], 1957) of type II superconductors.

The equilateral triangular lattice (or honeycomb) appears naturally in the physical context ($\beta = 2$). The associated zeros are located inside the grayish dots.



What about compact surfaces?

Let $\mathcal{S}$ be a compact Riemann surface. How to formulate a zero packing problem? Let $U(z,w) = U_{\mathcal{S}}(z,w)$ be a real-valued function with

$$\Delta^{\mathcal{S}} U(\cdot, w))\mathrm{d}A_{\mathcal{S}} = \frac{1}{2}\delta_w - \frac{1}{2a(\mathcal{S}}\mathrm{d}A_{\mathcal{S}},$$

where $\mathrm{d}A_{\mathcal{S}}$ is the area measure on $\mathcal{S}$ (with constant Gaussian curvature!) and $\Delta^{\mathcal{S}}$ the Laplace-Beltrami operator, both normalized. This means that for $z \sim w$, using chart coordinates,

$$U(\cdot, w)) = \log|z-w| + O(1).$$

We call $U(\cdot, w)$ a *logarithmic monopole* (but they are known as Green functions in the literature). These functions are a substitute for polynomials!

Zero packing density for a compact surface

DEFINITION

We put

$$\rho_{n,\beta}(\mathcal{S}) := \inf_{b,z_1,\ldots,z_n} \frac{1}{a(\mathcal{S})} \int_{\mathcal{S}} \left(b e^{\beta U(\cdot,z_1)+\cdots+U(\cdot,z_n)} - 1 \right)^2 \mathrm{d}A_{\mathcal{S}},$$

where the infimum is over all positive reals b and all points $z_1, \ldots, z_n$ on the surface $\mathcal{S}$.

PROPOSITION

The function $\beta \mapsto \rho_{n,\beta}(\mathcal{S})$ is monotonically increasing.

Relationship with zero packing on the universal cover

Suppose the compact surface $\mathcal{S}$ has genus ≥ 2. Then the universal covering surface is the hyperbolic plane, which can be modelled by the disk $\mathbb{D}$. Then $\mathcal{S} \cong \mathbb{D}/\Gamma$, where Γ is a Fuchsian group.

THEOREM

Suppose $f : \mathbb{D} \to \mathbb{C}$ is holomorphic, and that the function $z \mapsto (1-|z|^2)^\alpha |f(z)|^\beta$ is Γ-periodic, where $2\alpha a(\mathbb{D}/\Gamma) = n\beta$, where $n \geq 1$ is an integer. Then

$$\lim_{r\to 1^-} \frac{1}{\log \frac{1}{1-r^2}} \int_{\mathbb{D}(0,r)} \left((1-|z|^2)^\alpha |f(z)|^\beta - 1\right)^2 \frac{\mathrm{d}A(z)}{1-|z|^2} = \frac{1}{a(\mathbb{D}/\Gamma)} \int_{\mathbb{D}/\Gamma} \left((1-|z|^2)^\alpha |f(z)|^\beta - 1\right)^2 \mathrm{d}A_{\mathbb{H}}(z).$$

The proof uses the ergodicity of geodesic flow for negatively curved surfaces (Hopf).

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