A GLOBAL ASYMPTOTIC EXPANSION OF THE POLYNOMIAL BERGMAN DENSITY

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Abstract. We find a global asymptotic formula for the polynomial Bergman densities with respect to a wide class of exponentially varying weights in the plane. Under appropriate conditions on the potential Q and the associated spectral droplets $\mathcal{S}_\tau$, the n-th polynomial Bergman density $\rho_{m,n}(z)$ in $L^2(\mathbb{C}, \mathrm{e}^{-2mQ}\mathrm{dA})$ satisfies an asymptotic expansion of the form

$$\rho_{m,n} \sim \rho_m \operatorname{erf}\left(-2\sqrt{m}V_\tau\right) + m^{-\frac{1}{2}}\mathcal{B}_{m,\tau}\,\mathrm{e}^{-2mV_\tau^2},$$

as $n = \tau m \to +\infty$. Here, $\rho_m \sim 2\Delta Q + \dots$ denotes the Bergman density of states for the space of all entire functions in $L^2(\mathbb{C}, \mathrm{e}^{-2mQ}\mathrm{dA})$, while V_τ is a certain smooth function vanishing along $\partial\mathcal{S}_\tau$ which is positive outside and negative inside $\mathcal{S}_\tau$. The expression $\mathcal{B}_{m,\tau}$ stands for a smooth asymptotic expansion in powers of m^{-1} and V_τ. The main novelty is a new calculus which bypasses the summation over the degree of the orthogonal polynomials in the calculation of $\rho_{m,n}$, which is needed due to the lack of Christoffel-Darboux-type formulae in the planar context. The result has direct applications to strong planar Szegő limit theorems for Gram determinants, the details of which will appear elsewhere.

1 INTRODUCTION

1.1 Polynomial Bergman densities

Fix a real-valued function Q ("the potential") on the complex plane $\mathbb{C}$, subject to the growth condition

$$\liminf_{|z|\to+\infty} \frac{Q(z)}{\log|z|} = \tau_\infty > 1. \tag{1.1}$$

For a positive integer $n \in \mathbb{Z}_{\geq 0}$, we denote by $\mathrm{Pol}_n(\mathbb{C})$ the space of polynomials of degree at most $n-1$ and by $\mathrm{dA}(z) = \frac{1}{\pi}\mathrm{d}x\mathrm{d}y$ the usual area element normalized by $\frac{1}{\pi}$. For n and Q as above and $m \in \mathbb{R}_{>0}$, we denote by $K_{m,n}(z,w)$ the polynomial Bergman kernel associated to the weight e^{-2mQ}. That is, $K_{m,n}(z,w)$ is the unique Hermitian kernel function which satisfies the reproducing property

$$p(z) = \int_{\mathbb{C}} p(w)K_{m,n}(z,w)\,\mathrm{e}^{-2mQ(w)}\mathrm{dA}(w)$$

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for any polynomial p with $\deg(p) \le n-1$, such for any fixed w the function $K_{m,n}(\cdot, w)$ belongs to $\mathrm{Pol}_n(\mathbb{C})$. If we denote by $\{P_{m,k}(z)\}_{k\le n-1}$ the standard sequence of normalized orthogonal polynomials in $L^2(\mathbb{C}, \mathrm{e}^{-2mQ}\mathrm{dA})$, the polynomial Bergman kernel may be expressed more concretely as the sum

$$K_{m,n}(z,w) = \sum_{k=0}^{n-1} P_{m,k}(z)\overline{P_{m,k}(w)}. \tag{1.2}$$

We also let $K_m(z,w)$ be the reproducing kernel for the full Bergman space of all entire functions in $L^2(\mathbb{C}, \mathrm{e}^{-2mQ}\mathrm{dA})$. Under mild assumptions on the potential, we may think of K_m as the large n-limit of $K_{m,n}$, for fixed m. Our aim in this paper is to derive a new calculus for the polynomial Bergman density

$$\rho_{m,n}(z) = m^{-1}K_{m,n}(z,z)\,\mathrm{e}^{-2mQ(z)}$$

in the regime when m and n tend to infinity in a proportional fashion. In particular, we will obtain a transparent asymptotic formula which expresses $\rho_{m,n}(z)$ in terms of the full Bergman density $\rho_m(z) \stackrel{\text{def}}{=} m^{-1}K_m(z,z)\mathrm{e}^{-2mQ(z)}$ (see Theorem 1.2 below). We note in passing that

$$\int_{\mathbb{C}} \rho_{m,n}(z)\mathrm{dA}(z) = \frac{n}{m} \stackrel{\text{def}}{=} \tau(m,n).$$

The large m-asymptotics of ρ_m has been studied extensively. We will discuss the relevant literature in Section 1.5 below, but let us mention already here that for smooth and strictly subharmonic potentials, the Bergman density satisfies a full asymptotic expansion, the first few terms of which are given by

$$\rho_m = 2\Delta Q + \frac{1}{2m}\Delta \log \Delta Q + \mathrm{O}\big(m^{-2}\big). \tag{1.3}$$

In contrast, the asymptotics of the polynomial density $\rho_{m,n}$ is much more recent and still not fully understood. Roughly speaking, if the parameters n and m are related by $n = \tau m$ with $\tau > 0$, there is a compact region $\mathcal{S}_\tau$ in the interior of which $\rho_{m,n}$ has "classical" behavior, i.e., the asymptotic expansion matches that in (1.3), see, e.g., [4,5,9]. On the other hand, outside $\Gamma_\tau \stackrel{\text{def}}{=} \mathcal{S}_\tau$ the density instead decays rapidly, with the transition taking place in a band around $\partial\mathcal{S}_\tau$ at scale $m^{-\frac{1}{2}}$. In particular, we have the convergence in the sense of distribution theory

$$\lim_{n=\tau m\to+\infty} \rho_{m,n} = 2\Delta Q\, 1_{\mathcal{S}_\tau}, \tag{1.4}$$

see, for instance, [15]. The sharp cut-off along the boundary interface $\Gamma_\tau = \partial\mathcal{S}_\tau$ ought to be understood better. A natural thing is blow up to the the microscopic characteristic scale $m^{-\frac{1}{2}}$, which amounts to the typical distance between the particles in the standard Coulomb gas models. Under suitable conditions on Q and $\mathcal{S}_\tau$, the macroscopic sharp cut-off exhibited in (1.4) gets smooth at the microscopic level, and instead governed by the Gaussian error function

$$\operatorname{erf}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-t^2/2} \mathrm{d}t. \tag{1.5}$$

This universal boundary behavior, which may be understood as the boundary universality of Coulomb gas ensembles with inverse temperature $\beta = 2$, was obtained recently in [18]. This may also be understood as universality of eigenvalue statistics of random normal matrix ensembles along smooth spectral interfaces.

1.2 Potential theoretic assumptions

We require require that the potential Q meets the logarithmic growth bound (1.1), so that any polynomial of degree n belongs to $L^2(\mathbb{C}, \mathrm{e}^{-2mQ}\mathrm{dA})$ provided that m is sufficiently large. For a positive parameter $\tau < \tau_\infty$, we let $\mu_\tau = \mu_{\tau,Q}$ be the unique minimizer of the weighted logarithmic energy functional

$$I_Q(\mu) = \int_{\mathbb{C}\times\mathbb{C}} \log \frac{1}{|z-w|} \mathrm{d}\mu(z)\mathrm{d}\mu(w) + 2\int_{\mathbb{C}} Q(z)\mathrm{d}\mu(z)$$

among all positive compactly supported Borel measures on $\mathbb{C}$ with total mass τ and finite logarithmic energy. The measure μ_τ is known as the weighted equilibrium measure, and we denote by $\mathcal{S}_\tau$ its support (the "droplet"). The logarithmic potential of a measure μ is given by $U^\mu(z) = \int \log|z-w|\,\mathrm{d}\mu(w)$. If Q is assumed to be C^2–smooth and subharmonic, there is a constant C_τ such that

$$\check{Q}_\tau(z) \stackrel{\text{def}}{=} U^{\mu_\tau}(z) + C_\tau = Q(z) \quad \text{ for } z \in \mathcal{S}_\tau.$$

On the exterior domain $\mathbb{C}\setminus\mathcal{S}_\tau$ we have the bound $Q(z) \geq \check{Q}_\tau(z)$.

Definition 1.1 (Admissible potentials)**.** Fix a potential Q subject to (1.1). We say that Q is τ-admissible if

(1) Q is C^ω–smooth and strictly subharmonic on a neighborhood of $\mathcal{S}_\tau$,
(2) the spectral boundary $\partial\mathcal{S}_\tau$ is a smooth Jordan curve,
(3) We have $Q(z) > \check{Q}_\tau(z)$ on the exterior domain $\mathbb{C}\setminus\mathcal{S}_\tau$.

The assumptions (2)–(3) correspond to the standard "off-critical one-cut regime" in 1D random matrix theory, and they are known to hold in many concrete situations as well as under certain convexity assumptions on Q.

If Q is τ_0-admissible, it is not difficult to see that it is also τ-admissible for each $\tau \in I_0 \stackrel{\text{def}}{=} \{\tau : |\tau - \tau_0| \leq \epsilon\}$, provided that $\epsilon > 0$ is chosen sufficiently small. We will assume that this is the case, and in addition that ϵ is small enough for the exterior conformal maps $\phi_\tau : \mathbb{C}\setminus\mathcal{S}_\tau \to \mathbb{D}_\mathrm{e} \stackrel{\text{def}}{=} \{z : |z| > 1\}$, $\tau \in I_0$, to extend conformally to a common domain $\mathbb{C}\setminus\mathcal{K}$. Here, $\mathcal{K} = \mathcal{K}_\epsilon$ is a compact subset with $\mathcal{S}_{\tau_0-2\epsilon} \subset \mathcal{K} \subset \mathcal{S}^\circ_{\tau_0-\epsilon}$, and the conformal maps are normalized by the conditions that $\phi_\tau(\infty) = \infty$ and $\phi_\tau'(\infty) > 0$. Below we will need to shrink ϵ further, but it will always remain strictly positive and independent of the main parameter m. The precise restriction placed upon ϵ depends upon a quantitative regularity measure of Q and the boundaries $\partial\mathcal{S}_\tau$, and is explained in Section 3.1 below.

For $f \in C(\partial\mathcal{S}_\tau)$, we let $\mathbf{P}_\tau f$ denote the Poisson extension of f to the exterior domain $\mathbb{C}\setminus\mathcal{S}_\tau$. If we define the function

$$\breve{Q}_\tau = \tau\log|\phi_\tau| + \mathbf{P}_\tau Q,$$

the above regularity assumptions guarantee that $\breve{Q}_\tau$ extends harmonically to some neighborhood of the exterior domain $\overline{\mathbb{C}\setminus\mathcal{S}_\tau}$. Possibly after decreasing ϵ, we may assume that this neighborhood coincides with $\mathbb{C}\setminus\mathcal{K}$. On $\mathbb{C}\setminus\mathcal{S}_\tau$, we have $\breve{Q}_\tau = \check{Q}_\tau$. It is well-known that $\check{Q}_\tau$ is $C^{1,1}$-smooth and that its gradient coincides with that of Q along $\partial\mathcal{S}_\tau$ (see [15]), from which we conclude that the difference $Q - \breve{Q}_\tau$ is non-negative in a neighborhood of $\overline{\mathbb{C}\setminus\mathcal{S}_\tau}$ and vanishes quadratically along $\partial\mathcal{S}_\tau$. We let V_τ denotes the unique real-analytic square root defined by

$$V_\tau^2 = Q - \breve{Q}_\tau,$$

where the sign is chosen such that $V_\tau < 0$ on $\mathcal{S}_\tau\setminus\mathcal{K}$ while $V_\tau > 0$ on $\mathbb{C}\setminus\mathcal{S}_\tau$. We tacitly extend V_τ to a smooth function on $\mathbb{C}$, such that V_τ is strictly negative on the interior of $\mathcal{S}_\tau$.

1.3 The main result

Our main result is an asymptotic formula for $\rho_{m,n}$, which in particular explains how it relates to the full density ρ_m, and highlights how the error function transition across $\partial\mathcal{S}_\tau$ comes about.

Theorem 1.2. *Suppose that Q is τ_0-admissible. Then there exists a positive number $\epsilon = \epsilon(Q,\tau_0)$ as well as smooth bounded functions $a_{j,\tau}$ and $b_{j,\ell,\tau}$, such that for any $\kappa \in \mathbb{Z}_{>0}$, we have the asymptotics*

$$\rho_{m,n} = \rho_m \operatorname{erf}\big(-2\sqrt{m}\,V_\tau\big) + \frac{1}{\sqrt{m}}\mathcal{B}_{m,\tau}\,\mathrm{e}^{-2mV_\tau^2} + \mathrm{O}\big(m^{-\kappa-\frac12}\mathrm{e}^{-2m(Q-\check{Q}_\tau)}\big),$$

as $n = \tau m \to \infty$ uniformly on a neighborhood $\mathscr{V}_\tau$ of $\mathcal{S}_\tau$, where the boundary correction $\mathcal{B}_{m,\tau} = \mathcal{B}^\kappa_{m,\tau}$ is given by

$$\mathcal{B}_{m,\tau} = \sum_{j=0}^{\kappa} m^{-j} a_{j,\tau} + \sum_{j=1}^{\kappa}\sum_{\ell=0}^{2j-1} m^{-2j+1} b_{j,\ell,\tau}(2mV_\tau)^\ell.$$

Moreover, the implicit constant is uniformly bounded throughout the complex plane provided that $|\tau-\tau_0| \le \epsilon$.

Remark 1.3. (a) Theorem 1.2 gives the asymptotics of $\rho_{m,n}$ on a fixed neighborhood $\mathscr{V}_\tau$ of $\mathcal{S}_\tau$. However, the globally valid a priori bound

$$\rho_{m,n} \le Cm\mathrm{e}^{-2m(Q-\check{Q}_\tau)}$$

combines with Assumption (3) from Definition 1.1 to ensure that the density is exponentially decaying in m on $\mathbb{C} \setminus \mathscr{V}_\tau$.

(b) All of the coefficient functions $a_{j,\tau}$ and $b_{j,\ell,\tau}$ can in principle be computed in an iterative fashion. Below, we will give explicit formulas for the first few terms.

(c) We note that the asymptotic expansion of $\mathcal{B}_{m,\tau}$ in Theorem 1.2 is somewhat non-standard. If we attempt to write a standard expansion of the form

$$\mathcal{B}_{m,\tau}(z) = \sum_{j=0}^{\kappa} m^{-j} \alpha_{j,\tau}(z)$$

for some smooth coefficients $\alpha_{j,\tau}$, these would have to be adjusted as the precision parameter κ varies. For instance, we would get

$$\alpha_{0,\tau} = a_{0,\tau} + \sum_{j=1}^{\kappa} b_{j,2j-1,\tau}(2V_\tau)^{2j-1}.$$

This means that the added terms produce a correction away from the droplet boundary loop $\Gamma_\tau = \partial\mathcal{S}_\tau$, which however gets suppressed by the later multiplication by the "Gaussian ridge" $\mathrm{e}^{-2mV_\tau^2}$. For instance, the uniform norm of $V_\tau^k\,\mathrm{e}^{-2mV_\tau^2}$ is of order $\mathrm{O}(m^{-k/2})$.

(c) The precise condition on the size of the parameter ϵ is outlined in Section 3.1 below. Roughly speaking, ϵ should be small enough for Q, ΔQ, ϕ_τ and a couple of additional functions of potential theoretical nature to remain real-analytic throughout $\bigcup_{|\tau-\tau_0|\le\epsilon} \partial\mathcal{S}_\tau$ in a quantified way. The regularity is measured in terms of certain uniform norms of the Hermitian-analytic polarizations of these functions.

Curiously, the proof of Theorem 5.1 does not rely on an a priori understanding of ρ_m in any essential way. Indeed, it is enough to know that the expansions of ρ_m and $\rho_{m,n}$ coincide deep inside the droplet. The calculus for $\rho_{m,n}$ thus gives a genuinely new algorithm for the deriving the classical bulk-asymptotics, provided that the region around a given point can be foliated by admissible droplet boundaries. In contrast to previous approaches which are entirely local, this method emphasizes the connection with the Hele-Shaw flow via the expansion of the domains $\mathcal{S}_\tau$.

It is natural to ask whether the asymptotics obtained in Theorem 1.2 can be polarized to give off-diagonal asymptotic information about the polynomial Bergman kernels. Certainly this is possible in a shrinking neighborhood of the diagonal, but it is less clear what happens when z and w are far away. Such descriptions are known to hold in several different regimes, such as when one of the points is fixed in $\mathbb{C} \setminus \mathcal{S}_\tau$ and the other varies in $\mathbb{C} \setminus \mathcal{S}_\tau$ [17] ("off-spectral" asymptotics), or when both points are on $\partial\mathcal{S}_\tau$ but remain separated from each other [3]. In particular, it would be interesting to interpolate this latter result of Ameur and Cronvall with the present formula.

The proof of Theorem 1.2 is based on an approximate potential theoretic characterization of the polynomial Bergman density (see Problem 2.4 below). This shares some resemblance with the Fokas-Its-Kitaev characterization for orthogonal polynomials on the real line in terms of a Riemann-Hilbert problem [13] as well as the corresponding characterization of planar orthogonal polynomials in terms of a $\bar\partial$-problem (or soft Riemann-Hilbert problem) from [14, 20]. However, instead of the $\bar\partial$-operator the characterization involves the third order operator $\partial_\tau\Delta$, where τ is a continuous parameter playing the role of the quotient n/m. This characterization appears to be quite novel, given that the density function lacks any typical reproducing properties or defining orthogonality conditions. Such conditions are usually needed in the Riemann-Hilbert framework.

While it will be discussed in detail in Section 2, we should give the flavor of the potential theoretic problem already here. It involves a family P_τ of holomorphic functions of fractional growth

$$|P_\tau(z)| = |z|^{\tau m}(1+\mathrm{O}(|z|^{-1})) \quad \text{as } |z|\to+\infty,$$

with the necessary branch cut suitably chosen for $\tau m \notin \mathbb{Z}_{\geq 0}$, as well as a potential U_τ such that

$$\partial_\tau \mathit{\Delta} U_\tau = |P_\tau|^2 \mathrm{e}^{-2mQ}\left(1+\mathrm{O}\big(m^{-\kappa-1}\big)\right) \tag{1.6}$$

If U_τ satisfies the appropriate asymptotics at infinity along with a condition of rapid decay interiorly to $\partial\mathcal{S}_\tau$, then (1.6) encodes the approximate orthogonality of P_τ to lower order holomorphic functions of fractional growth and matching branch cut structure. In particular, for $n\in\mathbb{Z}_{\geq 0}$, P_τ will approximate the orthogonal polynomial $P_{m,n}$. Since integration in τ is closely related to summation over the right-hand sides of (1.6) over $n=\tau m\in\mathbb{Z}_{\geq 0}$, it turns out that $\mathit{\Delta} U_\tau$ approximates $\rho_{m,n}$ modulo (explicit) boundary corrections. The approximate potential U_τ will take the form

$$U_\tau = \Lambda_\tau \operatorname{erf}\left(2\sqrt{m}V_\tau\right) + \Omega \operatorname{erf}\left(-2\sqrt{m}V_\tau\right) + \frac{\Sigma_\tau}{\sqrt{2\pi m}}\,\mathrm{e}^{-2mV_\tau^2}, \tag{1.7}$$

where Λ_τ are harmonic in the exterior and of logarithmic growth, while Ω and Σ_τ are smooth, and we will algorithmically compute the coefficients. The first order approximation is given by $\Lambda_\tau \approx 2\check{Q}_\tau$ while $\Omega\approx 2Q$. Comparing with (1.4), we see that U_τ can be thought of as a "quantization" of the limiting potential

$$\lim_{n=\tau m\to+\infty}\int_{\mathbb{C}} \log|z-w|^2 \rho_{m,n}(w)\mathrm{dA}(w) = 2\check{Q}_\tau(z).$$

1.4 Consequences for the asymptotics of Gram determinants

For a potential Q as above, we consider the Gram matrices of complex moments

$$G_{m,n}(Q) = \left(\int_{\mathbb{C}} z^j\,\bar z^k \mathrm{e}^{-2mQ(z)}\mathrm{dA}\right)_{0\leq j,k\leq n-1}$$

and the large m-asymptotics of the Gram determinants $\det G_{m,n}(Q)$. As a direct consequence of Theorem 1.2 and the asymptotics of the potential (1.7) (cf. Lemma 5.1 and its proof), we will obtain the following theorem. It gives precise asymptotics of $\det G_{m,n}(Q)$ for a wide class of so-called Hele-Shaw potentials, i.e. those of the form

$$Q(z) = \frac{1}{2}|z|^2 - U^\mu(z)$$

where μ is a finite (signed) Borel measure. We put $Q^\lambda = Q(z) = \frac{1}{2}|z|^2 - \lambda U^\mu(z)$, $0 \le \lambda \le 1$. We denote by $\mathrm{cap}(S)$ the logarithmic capacity of a compact set S, and by $\mathbf{P}_\tau = \mathbf{P}_{\mathbb{C}\setminus\mathcal{S}_\tau}$ the Poisson extension operator for the exterior domain $\mathbb{C}\setminus\mathcal{S}_\tau$. We put

$$\Upsilon_\tau^\lambda = \Lambda_\tau^\lambda + \frac{1}{2m}\partial_\tau\Lambda_\tau^\lambda + \sum_{l=1}^{\lceil\frac{\kappa}{2}\rceil} m^{-2\ell}\frac{B_{2\ell}}{(2\ell)!}\partial_\tau^{2\ell}\Lambda_\tau^\lambda - \mathcal{C}_\tau^\lambda, \tag{1.8}$$

where $B_{2\ell}$ are the standard Bernoulli numbers and where $\mathcal{C}_\tau^\lambda$ are the uniquely determined constants such that $\Upsilon_\tau(z) = \tau \log|z|^2 + \mathrm{O}(|z|^{-1})$ at infinity. It is not difficult to see that $\mathcal{C}_\tau^\lambda$ admits a full asymptotic expansion in negative powers of m.

Theorem 1.4. *Let μ and $Q = Q_\mu$ be as above, and assume that for each λ, the interior of the droplet $\mathcal{S}_\tau^\lambda$ is simply connected with regular boundary and that* $\mathrm{supp}(\mu) \subset (\mathcal{S}_\tau^\lambda)^c$. *Then we have the asymptotics*

$$\log\det G_{m,n}(Q) = \log\det G_{m,n}(\tfrac{1}{2}|z|^2) + 2m^2\int_0^1\int_{\mathbb{C}}\Upsilon_\tau^\lambda(z)\mathrm{d}\mu(z)\mathrm{d}\lambda + \mathrm{O}(m^{-\kappa-1})$$

as $n = \tau m \to +\infty$. Moreover, Υ_τ^λ admits a full asymptotic expansion in powers of m^{-1}, and depends smoothly on λ.

The above expansion takes on a particularly simple form when μ is a finite sum of point masses.

Theorem 1.4 can be thought of as a (structural) strong Szegő limit theorem in the planar context. Equivalently, it amounts to having a full asymptotic expansion of the free energy in the determinantal 2D Coulomb gas model for Hele-Shaw potentials. We discuss this connection and related literature in more detail in Section 6.

1.5 Related work on Bergman kernel asymptotics

As mentioned above, there is a large literature on Bergman kernel expansions. We mention specifically the celebrated Tian-Catlin-Zelditch expansion [11, 26, 28] and the more recent developments [8, 12, 25] (this list is by no means complete). These works pertain to higher dimensional compact settings of relevance in complex geometry, but the methods can be adapted to our context and yield full asymptotic expansions of ρ_m, as well as bulk-asymptotics (that is, asymptotics inside $\mathcal{S}_\tau^\circ$) of $\rho_{m,n}$ [4, 5, 9].

The boundary universality result of [18] was derived as a corollary of a full asymptotic expansion of the planar orthogonal polynomials on shrinking neighborhoods of $\mathbb{C} \setminus \mathcal{S}_\tau$ followed by direct summation of the right-hand side of (1.2). We stress that in the planar context there is no Christoffel-Darboux formula, and thus polynomials of all orders need to be taken into account. Due to the complexity of the resulting calculations, this approach has inherent limitations. As a result, we were restricted to studying the leading order behavior of $\rho_{m,n}$, and more importantly, much remains to be said about the interpolation between different regimes.

We mention in passing that in the setting of polarized Kähler manifolds, the boundary behavior of the related *partial Bergman kernels* of holomorphic sections constrained to vanish to order ϵm along a divisor is of considerable interest. Ross-Singer [24] and Zelditch-Zhou [31] determined the near-boundary asymptotics under S^1-invariance assumptions, and just like in the planar context the boundary scaling limit is expressed in terms of the Gaussian error function. Moreover, the error function has recently been shown to govern interface transitions also for the class of *spectral partial Bergman kernels* [30], but the general universality question remains open. The interested reader will enjoy the survey [29].

1.6 Overview of the article

The paper is organized as follows. The potential theory problem and an Ansatz for its solution are described and motivated in Section 2. There we also give a detailed but non-technical outline of the proof of the main result. Section 3 is devoted to the asymptotic analysis of a non-linear semiclassical PDE which arises in the construction of the potential. We recall a convenient measure of real-analytic smoothness, and develop basic existence and regularity theory for an initial value problem (a "moving front Neumann jump problem"). This is then used to set up an approximate solution scheme for the semiclassical PDE. In Section 4 we perform a sequence of reductions, which show that the potential problem can be cast as a semiclassical PDE of the form treated in Section 3. The proof of the main result is supplied in Section 5. We conclude the paper in Section 6 by describing an application of Theorem 1.2 to strong Szegő limit theorems for planar Gram determinants. We keep the discussion brief and informal, with the intent of pursuing the finer details in future works.

1.7 Notation and conventions

We denote by $\operatorname{erf}(x)$ the Gaussian error function with the normalization

$$\operatorname{erf}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-t^2/2} \mathrm{d}t.$$

With this normalization, $\operatorname{erf}(x)$ satisfies $\lim_{x\to+\infty} \operatorname{erf}(x) = 1$ as well as the identity $\operatorname{erf}(x) + \operatorname{erf}(-x) = 1$. The derivative is given by $\operatorname{erf}'(x) = (2\pi)^{-\frac{1}{2}} \exp(-t^2/2)$. We also use the (non-standard) normalization

$$\varDelta = \partial\bar\partial = \frac{1}{4}\big(\partial_x^2 + \partial_y^2\big)$$

for the Laplacian, where $\partial = \partial_z$ and $\bar\partial = \bar\partial_z$ are the standard Wirtinger (or Cauchy-Riemann) operators.

For $f, g \in L^2(\mathbb{C}, \mathrm{e}^{-2mQ}\mathrm{dA})$, we denote by $\|f\|_{2mQ}$ and $\langle f, g\rangle_{2mQ}$ the usual norm and sesquilinear inner product in the space $L^2(\mathbb{C}, \mathrm{e}^{-2mQ}\mathrm{dA})$, respectively.

To keep the notation light and since no confusion should occur, we will mostly suppress the subscripts m and n and the truncation parameter κ. However, it will crucial to keep track of whether or not different function depend on the parameter τ, so we will keep this index at all times.

2 THE POTENTIAL OF THE BERGMAN DENSITY

2.1 The single wave potential

In the recent work [19], we found that the n-th normalized orthogonal polynomial $P_{m,n}$ in $L^2(\mathbb{C}, \mathrm{e}^{-2mQ}\mathrm{dA})$ is uniquely characterized by the following problem.

Problem 2.1. Determine a pair $(\mathscr{U}, P)$ of smooth functions on $\mathbb{C}$ such that $\mathscr{U}$ is real-valued, and such that

$$\begin{cases} \varDelta\mathscr{U} = |P|^2\mathrm{e}^{-2mQ}, \\ \bar\partial P \equiv 0, \\ P^{-1}\partial\,\mathscr{U} \in C^2(\mathbb{C}), \end{cases}$$

with asymptotic behavior

$$\mathscr{U}(z) = \log|z|^2 + \mathrm{O}(|z|^{-1}) \qquad \text{and} \qquad P(z) = \kappa^{-1}\, z^n\big(1 + \mathrm{O}(|z|^{-1})\big)$$

as $|z| \to +\infty$, for some positive constant $\kappa = \kappa_{m,n}$.

If $(\mathscr{U}, P)$ is a solution to Problem 2.1, then P coincides with the orthogonal polynomial $P_{m,n}$ and $\mathscr{U}$ is twice the logarithmic potential of $|P_{m,n}|^2\mathrm{e}^{-2mQ}$

$$\mathscr{U}(z) = \int_{\mathbb{C}} \log|z-w|^2\, |P_{m,n}(z)|^2\mathrm{e}^{-2mQ(z)}\mathrm{dA}(z)$$

(see Proposition 2.1 in [19]). Roughly speaking, the reason for this is the following computation. Assume for simplicity that the zeros of P are simple[1]. For any polynomial q of degree at most $n-1$, the quotient $q(z)/P(z)$ decays like z^{-1} at infinity,

[1]The present argument can be adapted to the situation with zeros with higher multiplicity. as well, but in that situation the proof in [19] is preferable.

and the same holds for $\partial\mathscr{U}$. Along with the assumed simplicity of the zeros of P, this allows us to apply Green's formula in the sense of distributions, to obtain

$$\begin{aligned}\int_{\mathbb{C}} q(z)\overline{P(z)}\mathrm{e}^{-2mQ(z)}\mathrm{dA}(z) &= \int_{\mathbb{C}} \frac{q(z)}{P(z)}\Delta\mathscr{U}(z)\mathrm{dA}(z)\\ &= -\int_{\mathbb{C}} \bar\partial\frac{q(z)}{P(z)}\partial\mathscr{U}(z)\mathrm{dA}(z)\\ &= -\sum_{\{\zeta:P(\zeta)=0\}}\frac{q(\zeta)}{P'(\zeta)}\partial\mathscr{U}(\zeta).\end{aligned}$$

Since the quotient $\partial\mathscr{U}/P$ is smooth we must have $\partial\mathscr{U}(\zeta)=0$ for each zero ζ of P. Hence the right-hand side vanishes, and since q was an arbitrary element of $\mathrm{Pol}_{n-1}(\mathbb{C})$ the orthogonality of P follows.

Remark 2.2. (a) Notice that the above computation also gives a converse statement. If $P=P_{m,n}$ is the orthogonal polynomial and $\mathscr{U}$ its potential, then if P has simple zeros, we have for any $q\in\mathrm{Pol}_n(\mathbb{C})$ that

$$\int_{\mathbb{C}} q(z)\overline{P(z)}\mathrm{e}^{-2mQ(z)}\mathrm{dA}(z) = \sum_{\{\zeta:P(\zeta)=0\}}\frac{q(\zeta)}{P'(\zeta)}\partial\mathscr{U}(\zeta). \tag{2.1}$$

For a given zero ζ of P, there exists interpolating polynomials $q_\zeta\in\mathrm{Pol}_{n-1}(\mathbb{C})$ such that $q_\zeta(\zeta)=1$ while $q_\zeta(\zeta')=0$ for any zero $\zeta'\neq\zeta$ of P. From the orthogonality of P to q_ζ, it follows that $\partial\mathscr{U}(\zeta)=0$. Hence, the division condition on $\partial\mathscr{U}/P$ is also necessary, at least when the zeros of P are simple.

(b) Looking again at (2.1), one realizes that if P is close to the orthogonal polynomial $P_{m,n}$, then $\partial\mathscr{U}$ needs to be small on the zero set of P (we do not make precise what "close" or "small" mean here). The zero set of P should in turn be close to that of $P_{m,n}$, which is located inside $\mathcal{S}_\tau$. When looking for the potential $\mathscr{U}$ of an approximation P of $P_{m,n}$, it is hence natural to ask that that $|\partial\mathscr{U}(z)|$ decays rapidly in the interior direction to $\mathcal{S}_\tau$ at the boundary $\partial\mathcal{S}_\tau$. For us it will be natural to impose that $|\partial\mathscr{U}(z)|=\mathrm{O}(\mathrm{e}^{-2mV_\tau^2})$ for $z\in\mathcal{S}_\tau\setminus\mathcal{K}$.

There is a natural generalization of Problem 2.1, which can be thought of as a characterization of orthogonal polynomials of fractional growth. To formulate it, fix a point $z_0\in\mathcal{K}$ and denote by $(z-z_0)^s$ the principal branch of the s-th power. We denote by $\lceil\alpha\rceil$ the smallest integer greater than or equal to α, and consider the space of "fractional polynomials"

$$P(z)=(z-z_0)^{\alpha-\lceil\alpha\rceil}p(z),\qquad \deg(p)\le\lceil\alpha\rceil.$$

We also let $\{\alpha\}$ denote the fractional part of α. Note that while a fractional polynomial $P(z)$ has a branch cut along a slit emanating from z_0, the modulus $|P(z)|$ is well-defined throughout.

Problem 2.3. For given reals $\alpha > -1$ and $m > 0$, find a pair $(\mathscr{U}, P)$ of smooth functions on $\mathbb{C}$ such that $\mathscr{U}$ is real-valued, and such that

$$\begin{cases} \Delta\mathscr{U} = |P|^2 \mathrm{e}^{-2mQ}, \\ P = (z - z_0)^{\tau m - \lceil \tau m \rceil} p, \quad \deg(p) \leq \lceil \tau m \rceil \\ p^{-1}\partial\mathscr{U} \in C^2(\mathbb{C}), \end{cases}$$

with prescribed asymptotic behavior

$$\begin{cases} \mathscr{U}(z) = \log|z|^2 + \mathrm{O}(|z|^{-1}), \\ p(z) = \kappa^{-1} z^{\lceil \tau m \rceil}\big(1 + \mathrm{O}(|z|^{-1})\big) \end{cases}$$

as $|z| \to +\infty$, for some positive constant $\kappa = \kappa_{m,\tau}$.

If $(\mathscr{U}, P)$ is a solution to Problem 2.3, then P is orthogonal to any fractional polynomial of the form $(z - z_0)^{\alpha - \lceil \alpha \rceil} q(z)$ with $\deg(q) \leq \lceil \tau m \rceil - 1$. Equivalently, the polynomial $p(z) = (z - z_0)^{\lceil \tau m \rceil - \tau m} P(z)$ is the normalized orthogonal polynomial of degree $\lceil \tau m \rceil$ with respect to the planar measure $|z - z_0|^{2(\tau m - \lceil \tau m \rceil)} \mathrm{e}^{-2mQ} \mathrm{dA}$. We will not use this problem outright, and therefore leave the verification of these properties to the interested reader. However, below it will be useful to keep in mind this type of embedding of the orthogonal polynomial sequence for a given potential Q into a continuously indexed family exhibiting fractional growth.

2.2 The integrated potential

In view of the identity $|P_{m,n}|^2 \mathrm{e}^{-2mQ} = \rho_{m,n} - \rho_{m,n-1}$, it is clear that if $(\mathscr{U}_k, P_{k-1})_{k=1}^n$ is a sequence of pairs such that for each k, $(\mathscr{U}_{k+1} - \mathscr{U}_k, P_k)$ is a solution of Problem 2.1 with degree parameter $n = k$, then we recover the polynomial Bergman density as

$$\rho_{m,n}(z) = m^{-1} \Delta\mathscr{U}_n(z).$$

Our main new idea is that there is an approximate characterization of $\rho_{m,n}$ in terms of a similar problem, but for a continuously indexed pair (U_τ, h_τ). Here, the function h_τ plays the role of $\log|P_{m\tau}|^2 - 2m\check{Q}_\tau$ (cf. Problem 2.3), and the discrete difference $\mathscr{U}_k - \mathscr{U}_{k-1}$ is replaced with differentiation in the "continuous degree parameter" τ. As motivated in Remark 2.1 (b), the divisibility condition

$$P_k^{-1} \partial(\mathscr{U}_k - \mathscr{U}_{k-1}) \in C^2(\mathbb{C})$$

is replaced by an interior decay condition. The problem reads as follows.

For the formulation, recall the compact subset $\mathcal{K}$ of $\mathcal{S}^{\circ}_{\tau_0 - \epsilon}$ from Section 1.2. We moreover fix an open set $\mathcal{V} = \mathcal{V}_\epsilon$ with $\mathcal{S}_{\tau_0 + \epsilon} \subset \mathcal{V} \subset \mathcal{S}_{\tau_0 + 2\epsilon}$.

Problem 2.4. Fix an accuracy parameter $\kappa \in \mathbb{Z}_{>0}$. Find a pair $U_\tau = U_{\tau,m}^{\langle\kappa\rangle}$ and $h_\tau = h_{\tau,m}^{\langle\kappa\rangle}$ of functions $\mathbb{C} \setminus \mathcal{K} \times I_0 \to \mathbb{R}$ such that

$$\begin{cases} \partial_\tau \Delta U_\tau = \sqrt{\frac{m}{2\pi}}\, \mathrm{e}^{h_\tau - 2mV_\tau^2}\Big(1 + \mathrm{O}\big(m^{-\kappa-1}\big)\Big) \;\text{ on }\; \mathcal{V} \setminus \mathcal{K}, , \\ \Delta h_\tau = 0 \;\text{ on }\; \mathbb{C} \setminus \mathcal{K}, \\ |\partial \partial_\tau U_\tau(z)| = \mathrm{O}(\mathrm{e}^{-2m(Q-\check{Q})(z)}) \;\text{ for }\; z \in \mathcal{S}_\tau \setminus \mathcal{K}, \end{cases} \tag{2.2}$$

with prescribed asymptotic behavior

$$U_\tau(z) = \tau \log |z|^2 + \mathrm{O}(1) \quad \text{ and } \quad h_\tau(z) = \mathrm{O}(1) \quad \text{ as } |z| \to +\infty.$$

If (U_τ, h_τ) is a solution to Problem 2.4, we expect ΔU_τ to be a good approximation of the Bergman density $\rho_{m,n}$. In the following section, we proceed with a non-technical description of why this is.

2.3 Outline of the proof of Theorem 1.2

By introducing suitable cut-off functions, it is possible to modify U_τ to a globally defined approximate potential $\mathcal{U}_\tau$, such that the approximate potential equation

$$\partial_\tau \Delta \mathcal{U}_\tau = \chi \sqrt{\frac{m}{2\pi}} \mathrm{e}^{h_\tau - 2mV_\tau^2}\Big(1 + \mathrm{O}\big(m^{-\kappa-1}\big)\Big) \tag{2.3}$$

holds throughout the plane. Here, χ is a cut-off function which vanishes on $\mathcal{K}$ and equals one on a neighborhood of $\mathbb{C} \setminus \mathcal{S}_\tau$. In addition, the requisite decay and asymptotic properties from Problem 2.4 are retained in the truncation procedure.

Whenever τ is an integer fraction n/m of m, the exponential on the right-hand side of (2.3) can be written as a weighted squared modulus

$$\sqrt{\frac{m}{2\pi}} \mathrm{e}^{h_\tau - 2mV_\tau^2} = |F_n(z)|^2 \mathrm{e}^{-2mQ(z)},$$

where F_n is a holomorphic function on $\mathbb{C} \setminus \mathcal{K}$ of polynomial growth, i.e. $F_n(z) = z^n + \mathrm{O}(|z|^{n-1})$ as $|z| \to +\infty$ (that is, F_n is a "quasipolynomial" in the terminology of [18]). Moreover, $|F_n|^2 \mathrm{e}^{-2mQ}$ decays rapidly interiorly to $\partial \mathcal{S}_\tau$. Using a scheme involving Hörmander's L^2-estimate for the $\bar{\partial}$-operator, we may moreover find a polynomial P_τ of degree n, such that $|P_\tau|^2 \mathrm{e}^{-2mQ}$ approximates $\chi |F_n|^2 \mathrm{e}^{-2mQ}$ to exponentially decaying accuracy in m. This is by now standard, and is explained in detail e.g. in Section 3.6 of [18]. The fact that (U_τ, h_τ) is a solution to Problem 2.4 translates to saying that $(\partial_\tau U_\tau, P_\tau)$ is an approximate solution to Problem 2.1, and hence P_τ must be close to the true orthogonal polynomial $P_{m,n}$. More specifically, the squared modulus of the n-th orthonormal polynomial is asymptotically given by

$$|P_{m,n}|^2 \mathrm{e}^{-2mQ(z)} = \chi(z) \sqrt{\frac{m}{2\pi}} \mathrm{e}^{h_{\tau(n,m)} - 2mV_{\tau(n,m)}^2} + \mathrm{O}\big(m^{-\kappa - \frac{1}{2}} \mathrm{e}^{-2m(Q - \check{Q}_\tau)}\big), \tag{2.4}$$

where $\tau(n,m) = n/m \in I_0$ (cf. Theorem 5.1 in [19]).

Using the approximation (2.4) together with the Euler-MacLaurin summation formula, the difference $\rho_{m,n} - \rho_{m,k}$ may then be expressed as

$$\begin{aligned} \rho_{m,n} - \rho_{m,k} &= \sum_{j=k}^{n-1} \chi \sqrt{\frac{1}{2\pi m}} \mathrm{e}^{h_{\tau(j,m)} - 2mV_{\tau(j,m)}^2} + \mathrm{O}\big(m^{-\kappa-\frac{1}{2}} \mathrm{e}^{-2m(Q-\check{Q}_\tau)}\big) \\ &= \int_{\tau'}^{\tau} \chi \sqrt{\frac{m}{2\pi}} \mathrm{e}^{h_t - 2mV_t^2} \mathrm{d}t + \mathcal{B}_\kappa + \mathrm{O}\big(m^{-\kappa-\frac{1}{2}} \mathrm{e}^{-2m(Q-\check{Q}_\tau)}\big) \\ &= \int_{\tau'}^{\tau} \partial_t \Delta \mathcal{U}_t \, \mathrm{d}t + B_{m,\tau}^{\kappa} + \mathrm{O}\big(m^{-\kappa-\frac{1}{2}} \mathrm{e}^{-2m(Q-\check{Q}_\tau)}\big), \end{aligned} \tag{2.5}$$

where $k = \tau' m$ and $n = \tau m$. Here, $B_{m,\tau}^{\kappa}$ is a sum of boundary terms that arise from the application of Euler-MacLaurin's formula. The first term on the right-hand side can be further simplified using the fundamental theorem of calculus, which gives

$$\int_{\tau'}^{\tau} \partial_t \Delta \mathcal{U}_t \, \mathrm{d}t = \Delta\big(\mathcal{U}_\tau - \mathcal{U}_{\tau'}\big).$$

Combining this with the equation (2.5) and a well-known Bernstein-Walsh type exterior decay estimate for $\rho_{m,k}(z)$, we obtain

$$\rho_{m,n}(z) = \Delta \mathcal{U}_\tau(z) + B_{m,\tau}^{\kappa}(z) + \mathrm{O}\big(m^{-\kappa-1} \mathrm{O}(m^{-\kappa-\frac{1}{2}} \mathrm{e}^{-2m(Q-\check{Q}_\tau)})\big), \tag{2.6}$$

where the error term is initially only controlled for z in some neighbourhood $\mathscr{V}_\tau$ and at a fixed positive distance from $\mathcal{S}_{\tau'}$. In our construction of $\mathcal{U}_\tau$ we will find that the Laplacian of $\mathcal{U}_\tau$ takes the form

$$\Delta \mathcal{U}_\tau = \rho_m \operatorname{erf}\big(-2\sqrt{m} V_\tau\big) + A_{m,\tau} \, \mathrm{e}^{-2mV_\tau^2} + \mathrm{O}\big(m^{-\kappa-1} \mathrm{O}(m^{-\kappa-\frac{1}{2}} \mathrm{e}^{-2m(Q-\check{Q}_\tau)})\big), \tag{2.7}$$

where the boundary correction $A_{m,\tau}$ admits an asymptotic expansion in negative powers of m with smooth coefficients. Hence, if we put $\mathcal{B}_{m,\tau}^{\kappa} = A_{m,\tau}^{\kappa} + B_{m,\tau}^{\kappa}$, the asymptotic formula of Theorem 1.2 follows by combining (2.7) and (2.6).

Since (2.6) only holds for $z \in \mathscr{V}_\tau$ away from $\backslash \mathcal{S}_{\tau'}$, (where $\Delta \mathcal{U}_{\tau'}$ has negligible influence), the asymptotic formula for $\rho_{m,n}$ initially holds only on that same set. However, deep inside $\mathcal{S}_\tau$ the two densities $\rho_{m,n}$ and ρ_m have matching asymptotic expansions to any order, and there we also have

$$\operatorname{erf}(-2\sqrt{m} V_\tau) = 1 + \mathrm{O}(\mathrm{e}^{-\delta_0 m}), \qquad \mathrm{e}^{-2mV_\tau^2} = \mathrm{O}(\mathrm{e}^{-\delta_0 m})$$

for some $\delta_0 > 0$. As a consequence, the asymptotic formula is readily extended to all of $\mathscr{V}_\tau$.

The remainder of the paper is devoted to carrying out the scheme outlined in this section in detail. The main step is to supply an algorithm that produces the approximate potential U_τ satisfying (2.2). This will be done in Lemma 5.1 below, but we proceed to describe some of the main ingredients already now, beginning with a structural Ansatz for U_τ.

2.4 The Ansatz for U_τ

We start from an Ansatz which guarantees that U_τ and $\partial\partial_\tau U_\tau$ have the correct asymptotics at infinity and appropriate interior decay at $\partial\mathcal{S}_\tau$, respectively. To accomplish this, we look for a potential $U_\tau = U_{\tau,m}^{\langle\kappa\rangle}$ of the form

$$U_\tau = \Lambda_\tau \operatorname{erf}\big(2\sqrt{m}V_\tau\big) + \Omega \operatorname{erf}\big(-2\sqrt{m}V_\tau\big) + \frac{\Sigma_\tau}{\sqrt{2\pi m}}\,\mathrm{e}^{-2mV_\tau^2} \tag{2.8}$$

for some functions Λ_τ, Ω and Σ_τ. Recall that the error function erf is given by (1.5), and that V_τ is the unique real-analytic function with $V_\tau^2 = Q - \check{Q}_\tau$ such that V_τ is negative on the interior of the droplet $\mathcal{S}_\tau$ and positive on $\mathbb{C}\setminus\mathcal{S}_\tau$. This means that Λ_τ dominates the asymptotics of U_τ in the exterior of $\mathcal{S}_\tau$ while Ω dominates in the interior. The two regimes are smoothly interpolated in a band around $\partial\mathcal{S}_\tau$ at the scale $m^{-\frac{1}{2}}$.

The functions Ω and Σ_τ are required to be real-analytically smooth functions on $\mathcal{V}\setminus\mathcal{K}$, while we ask that Λ_τ is harmonic throughout the exterior domain $\mathbb{C}\setminus\mathcal{K}$. Up to an additive constant, U_τ should approximate twice the logarithmic potential of a measure of mass τ whose density with respect to dA decays rapidly at infinity, so we should ask that Λ_τ meets the growth condition

$$\Lambda_\tau(z) = \tau\log|z|^2 + \mathrm{O}(1) \qquad \text{as } \ |z|\to+\infty. \tag{2.9}$$

The functions Λ_τ and Σ_τ will depend on τ (in addition to m and κ). However, we will require that Ω is independent of τ. Heuristically, this can be motivated as follows. For integer values of $k = \tau m$, the measure $|P_{m,k}|^2\mathrm{e}^{-2mQ}\mathrm{dA}$ supplies a good approximation to harmonic measure for $\mathbb{C}\setminus\mathcal{S}_\tau$. Thus, adding additional wave functions should at most cause the potential to change by a constant inside $\mathcal{S}_\tau$ (plus an error term which is $\mathrm{O}(m^{-K})$ for any $K>0$). Since varying τ essentially has this effect, we expect that $\partial_\tau\Omega$ should be a constant. By Eq. (2.9), the potential U_τ is only defined up to an additive constant, so we may just as well ask that Ω is independent of τ altogether.

We finally ask that the functions Λ_τ, Ω and Σ_τ are given by truncated asymptotic expansions

$$\Lambda_\tau = \sum_{j=0}^{\kappa+2} m^{-j}\Lambda_{j,\tau}, \qquad \Omega = \sum_{j=0}^{\kappa+2} m^{-j}\Omega_j, \qquad \Sigma_\tau = \sum_{j=0}^{\kappa+2} m^{-j}\Sigma_{j,\tau}, \tag{2.10}$$

for some coefficient functions $\Lambda_{j,\tau}$, Ω_j and $\Sigma_{j,\tau}$. We finally put

$$\Xi_\tau = \Lambda_\tau - \Omega - 2V_\tau\Sigma_\tau \tag{2.11}$$

and use the notation $\Xi_{j,\tau} = \Lambda_{j,\tau} - \Omega_j - 2V_\tau\Sigma_{j,\tau}$ for its coefficient functions. To find U_τ, our task is to understand what the structural requirement of (2.3) entails for the coefficient functions in (2.10). That is, the coefficients will get uniquely determined up to the given accuracy from the condition that $\partial_\tau \mathit{\Delta} U_\tau$ is essentially the exponential of a bounded harmonic function times $\sqrt{\frac{m}{2\pi}}\mathrm{e}^{-2mV_\tau^2}$.

2.5 The approximate potential equation for the coefficients

Taking first the $\bar\partial$-derivative of the Ansatz (2.8) for U_τ, we find that

$$\bar\partial U_\tau = \bar\partial\Lambda_\tau \operatorname{erf}\left(2\sqrt{m}V_\tau\right) + \bar\partial\Omega \operatorname{erf}\left(-2\sqrt{m}V_\tau\right) + \left(\frac{2\sqrt{m}}{\sqrt{2\pi}}\Xi_\tau\bar\partial V_\tau + \frac{\bar\partial\Sigma_\tau}{\sqrt{2\pi m}}\right)\mathrm{e}^{-2mV_\tau^2}.$$

Proceeding to take the ∂-derivative of the right-hand side, we obtain

$$\begin{aligned}\mathit{\Delta} U_\tau = \mathit{\Delta}\Omega \operatorname{erf}\left(-2\sqrt{m}V_\tau\right) - 8\frac{m^{\frac{3}{2}}}{\sqrt{2\pi}}\Xi_\tau V_\tau|\bar\partial V_\tau|^2\,\mathrm{e}^{-2mV_\tau^2} \\ + \frac{m^{\frac12}}{\sqrt{2\pi}}\Big(2\partial V_\tau\big(\bar\partial\Lambda_\tau - \bar\partial\Omega - 2V_\tau\bar\partial\Sigma_\tau\big) + 2\partial\Xi_\tau\bar\partial V_\tau + 2\Xi_\tau\mathit{\Delta} V_\tau\Big)\,\mathrm{e}^{-2mV_\tau^2} \\ + \frac{m^{-\frac12}}{\sqrt{2\pi}}\mathit{\Delta}\Sigma_\tau\,\mathrm{e}^{-2mV_\tau^2}.\end{aligned}$$

Cleaning up the formula using the identity $V_\tau\bar\partial\Sigma_\tau = \bar\partial(V_\tau\Sigma_\tau) - \Sigma_\tau\bar\partial V_\tau$, we obtain

$$\mathit{\Delta} U_\tau = \mathit{\Delta}\Omega \operatorname{erf}\left(-2\sqrt{m}V_\tau\right) + \sqrt{\frac{m}{2\pi}}\left(mX_{0,\tau} + X_{1,\tau} + m^{-1}X_{2,\tau}\right)\mathrm{e}^{-2mV_\tau^2}, \tag{2.12}$$

where the functions $X_{j,\tau}$ are given by

$$\begin{cases} X_{0,\tau} = & -8\Xi_\tau V_\tau|\partial V_\tau|^2 \\ X_{1,\tau} = & 4\Sigma_\tau|\bar\partial V_\tau|^2 + 4\mathrm{Re}\left\{\partial\Xi_\tau\bar\partial V_\tau\right\} + 2\Xi_\tau\mathit{\Delta} V_\tau \\ X_{2,\tau} = & \mathit{\Delta}\Sigma_\tau. \end{cases} \tag{2.13}$$

Taking the ∂_τ derivative of the right-hand side of (2.12), we obtain a formula for $\partial_\tau\mathit{\Delta} U_\tau$, which reads

$$\partial_\tau\mathit{\Delta} U_\tau = \sqrt{\frac{m}{2\pi}}\Big(m^2Y_{0,\tau} + mY_{1,\tau} + Y_{2,\tau} + m^{-1}Y_{3,\tau}\Big)\,\mathrm{e}^{-2mV_\tau^2} \tag{2.14}$$

where the functions $Y_{j,\tau}$ are given in terms of the functions $X_{j,\tau}$ and the potential theoretic data by

$$\begin{cases} Y_{0,\tau} = -4V_\tau\partial_\tau V_\tau X_{0,\tau} \\ Y_{1,\tau} = -4V_\tau\partial_\tau V_\tau X_{1,\tau} + \partial_\tau X_{0,\tau} \\ Y_{2,\tau} = -2\partial_\tau V_\tau\mathit{\Delta}\Omega - 4V_\tau\partial_\tau V_\tau X_{2,\tau} + \partial_\tau X_{1,\tau} \\ Y_{3,\tau} = \partial_\tau X_{2,\tau}. \end{cases} \tag{2.15}$$

In order to solve Problem 2.4, we need to find coefficient functions Λ_τ, Ω, Σ_τ and h_τ, all with asymptotic expansions in powers of m^{-1} and subject to the appropriate smoothness and harmonicity conditions, such that the potential U_τ given by (2.8) satisfies

$$\partial_\tau \mathit{\Delta} U_\tau = \sqrt{\frac{2m}{\pi}}\mathrm{e}^{h_\tau}\mathrm{e}^{-2mV_\tau^2}\Big(1+\mathrm{O}\big(m^{-\kappa-1}\big)\Big)$$

in $\mathcal{V}\setminus\mathcal{K}$. In terms of the functions $Y_{j,\tau}$, this amounts to asking that

$$m^2 Y_{0,\tau} + mY_{1,\tau} + Y_{2,\tau} + m^{-1}Y_{3,\tau} = \mathrm{e}^{h_\tau} + \mathrm{O}(m^{-\kappa-1}). \tag{2.16}$$

In the form (2.16), the equation is not very illuminating. We will return to it in Section 4, where the following result will be obtained. For the formulation, we put

$$\begin{cases} \Xi_\tau &= -\frac{1}{2}m^{-1} + m^{-2}\Xi_\tau^{(2)}, \\ \Omega &= 2Q + m^{-1}\Omega^{(1)}. \end{cases}$$

Proposition (see Proposition 4.2)**.** *The approximate potential equation* (2.16) *is equivalent to the reduced problem*

$$V_\tau^2 A_\tau \Xi_\tau^{(2)} + V_\tau B_\tau \tfrac{(\mathbf{I}-\mathbf{P}_\tau)\Omega^{(1)}}{V_\tau} + f_\tau + m^{-1}\mathbf{S}_\tau = \mathrm{e}^{h_\tau} + \mathrm{O}(m^{-\kappa-1}) \tag{2.17}$$

for the functions $\Xi_\tau^{(2)}$, $\Omega^{(1)}$ and h_τ. Here, A_τ, B_τ and f_τ are non-vanishing real-analytic functions with f_τ strictly positive, and $\mathbf{S}_\tau$ is a bounded affine differential expression in the coefficients $\Xi_\tau^{(2)}$ and $\Omega_\tau^{(1)}$.

For the explicit representations of f_τ and $\mathbf{S}_\tau$, see the formulation of Proposition 4.1 below. We also note that the coefficients Λ_τ and Σ_τ are determined by Ξ_τ and Ω (cf. (4.10)).

Before performing the reductions which transform (2.16) to (2.17), we discuss how approximate semiclassical equations of the form (2.17) can be solved. This is the focus of the upcoming section.

3 A SEMICLASSICAL EQUATION ALONG MOVING FRONTS

3.1 Spaces of quantitatively real-analytic functions

Throughout the proof, we need to keep track of the regularity of various functions defined by iterative procedures. A prototypical example is that we are given a family of functions $f_\tau : \partial\mathcal{S}_\tau \to \mathbb{R}$, $\tau \in I_0$, and want to ensure that all the functions $\mathbf{P}_\tau f_\tau$ with $\tau \in I_0$ extend harmonically to a common domain $\mathbb{C}\setminus\mathcal{K}$. It turns out to be convenient to express regularity in terms of a certain chain of Banach spaces. The use such measures of regularity in the context of non-linear PDE can be traced back to work by Nishida [23] and Nirenberg [22], and it was recently used successfully in [14, 19] in a similar context as this one.

We let $\rho, \sigma \in (0,1)$ and $\delta > 0$ denote three positive parameters. We write $\mathbb{A}(\rho)$ for the annulus $\{z : \rho \le |z| \le \rho^{-1}\}$ and $\widehat{\mathbb{A}}(\sigma)$ for the fattened diagonal annulus

$$\widehat{\mathbb{A}}(\sigma) = \big\{(z,w) \in \mathbb{A}^2(\rho(\sigma)) : |z-w| \le 2\sigma\big\},$$

where the number $\rho(\sigma)$ is given by[2]

$$\rho(\sigma) = \frac{1}{\sigma + \sqrt{1+\sigma^2}}.$$

For a real-analytic function $f(z)$, we denote by $f^{\diamond}(z,w)$ its Hermitian-analytic polarization. We denote by $\mathscr{H}^{\infty}_{\sigma}$ the space of real-analytic functions $f(z)$ on $\phi_{\tau_0}^{-1}\big(\mathbb{A}(\rho(\sigma))\big)$, such that the Hermitian-analytic polarization of $f \circ \phi_{\tau_0}$ is bounded on $\widehat{\mathbb{A}}(\sigma)$. Denote moreover by $\mathscr{H}^{\infty}_{\sigma,\delta}$ the class of functions $f_\tau(z)$ which are holomorphic in $\tau \in \mathbb{D}(\tau_0,\delta)$, such that for fixed $\tau \in \mathbb{D}(\tau_0,\delta)$ we have that $f_\tau(z) \in \mathscr{H}^{\infty}_{\sigma}$.

It is immediate that these spaces grow as the parameters σ and τ decrease. Endowed with the uniform norms

$$\|f\|_{\sigma} = \big\|(f\circ\phi_{\tau_0})^{\diamond}(z,w)\big\|_{\widehat{A}(\sigma)}$$

and

$$\|f_\tau\|_{\sigma,\delta} = \big\|(f_\tau\circ\varphi)^{\diamond}(z,w)\big\|_{\widehat{A}(\sigma)\times\mathbb{D}(\tau_0,\delta)},$$

respectively, the spaces $\mathscr{H}^{\infty}_{\sigma}$ and $\mathscr{H}^{\infty}_{\sigma,\delta}$ become commutative Banach algebras. We refer to Section 5.2 of [14] or Section 9.1 of [19] for more details on these spaces. Here we will only need a few basic results, which are summarized in the following extensive remark. Most of the necessary estimates can be found verbatim in Section 5.2 of [14], but we leave some routine extensions to the interested reader.

Remark 3.1. (a) For any element f_τ of $\mathscr{H}^{\infty}_{\sigma,\delta}$, the Poisson extension $\mathbf{P}_\tau f_\tau$ belongs to $\mathscr{H}^{\infty}_{\sigma,\delta}$ as well, and provided that σ remains bounded away from zero the norm $\|\mathbf{P}_\tau f_\tau\|_{\sigma,\delta}$ comparable to $\|f_\tau\|_{\sigma,\delta}$. If f_τ vanishes along $\partial\mathcal{S}_\tau$, then the quotient $f_\tau(z)/V_\tau(z)$ belongs to $\mathscr{H}^{\infty}_{\sigma',\delta}$ for any parameter σ' with $\sigma' < \sigma$, and the norm grows at most by a factor proportional to $(\sigma-\sigma')^{-1}$. The same type of bound holds for complex gradients and derivatives in τ.

(b) It is clear that if f is a real-analytic and non-vanishing function in a neighborhood of $\partial\mathcal{S}_{\tau_0}$, then both f and its reciprocal belong to the space $\mathscr{H}^{\infty}_{\sigma}$ for some $\sigma \in (0,1)$. Similarly, for any function $f_\tau(z)$ which is real-analytic and non-vanishing for z on a neighborhood of $\partial\mathcal{S}_{\tau_0}$ and τ near τ_0, we have $f_\tau, \frac{1}{f_\tau} \in \mathscr{H}^{\infty}_{\sigma,\delta}$ for some $\sigma \in (0,1)$, $\delta > 0$. In particular, there exist parameters $\sigma \in (0,1)$, $\delta > 0$ such that all the functions

$$\partial_\tau V_\tau, \quad \bar\partial V_\tau, \quad \Delta V_\tau, \quad \partial_\tau V_\tau,$$

as well as their reciprocals belong to $\mathscr{H}^{\infty}_{\sigma,\delta}$. We may moreover arrange thing so that $\Delta Q, 1/\Delta Q \in \mathscr{H}^{\infty}_{\sigma}$ and that $V_\tau(z) \in \mathscr{H}^{\infty}_{\sigma,\delta}$.

[2]Why this particular choice is convenient is explained in Section 6 of [18].

(c) The domains $(\mathcal{S}_\tau)_\tau$ undergo a weighted Hele-Shaw-type flow as τ varies [15]. More precisely, the exterior domains $\mathbb{C}\setminus\mathcal{S}_\tau$ evolve according to the weighted Laplacian growth flow with weight $2\Delta Q$, and a sink of unit strength at infinity. Informally, this means that the boundary $\partial\mathcal{S}_\tau$ moves at speed

$$\nu(z) = (4\Delta Q(z))^{-1}|\phi'(z)|$$

in the outward normal direction to $\mathcal{S}_\tau$. This scalar velocity field will play a role below, and we note that both ν and its reciprocal belong to $\mathscr{H}^\infty_{\sigma,\delta}$ for some $\sigma,\delta>0$. The same holds for $|\phi_\tau'|$ and $1/|\phi_\tau'|$ (see, e.g., the main result of [16]).

(d) For τ in some neighborhood I of τ_0, $(\partial\mathcal{S}_\tau)_{\tau\in I}$ is a flow of simple smooth loops which foliates a ring domain $\mathcal{D}$. For $z\in\mathcal{D}$ we denote by $\tau(z)$ the "radial Laplacian growth coordinate", that is, the unique parameter τ for which $z\in\partial\mathcal{S}_\tau$. For sufficiently small σ, we have that $\tau(z)\in\mathscr{H}^\infty_\sigma$ (this also follows from [16]). We restrict attention to σ for which this holds.

We fix the parameters $\sigma\in(0,1)$, $\delta>0$ and $\epsilon>0$ as follows. First we require that σ and δ are small enough to ensure that all the functions from Remark 3.1 (b)–(d) belong to $\mathscr{H}^\infty_{\sigma,\delta}$. After possibly shrinking σ, we may also assume that the polarized Hele-Shaw coordinate satisfies

$$\tau^\diamond(z,w)\in\mathbb{D}\big(\tau_0,\tfrac12\delta\big)\qquad\text{whenever}\ \ (\phi_{\tau_0}(z),\phi_{\tau_0}(w))\in\widehat{\mathbb{A}}(\sigma). \tag{3.1}$$

We then let $\epsilon\in(0,\frac12\delta)$, be small enough for the conformal mapping ϕ_{τ_0} to map the ring domain $\mathcal{D}=\bigcup_{|\tau-\tau_0|\le\epsilon}\partial\mathcal{S}_\tau$ into the annulus $\mathbb{A}(\rho(\frac12\sigma))$. With ϵ given, we fix once and for all $I_0=\{\tau:|\tau-\tau_0|<\epsilon\}$.

Below we will need to shrink the parameters σ and δ finitely many times. We will at all times ensure that the final (smallest) parameters σ' and δ' satisfy $\sigma'\ge\frac12\sigma$ and $\delta'\ge\frac12\delta$, respectively. This is significant because, due to the above parameter choices, we have $\phi_{\tau_0}\big(\mathcal{V}\setminus\mathcal{K}\big)\subset\mathbb{A}\big(\rho(\frac12\sigma)\big)$ and $I_0\subset\mathbb{D}(0,\frac12\delta)$. As a consequence, any element f_τ of $\mathscr{H}_{\sigma',\delta'}$ is defined and real-analytic for $z\in\mathcal{V}\setminus\mathcal{K}$ and $\tau\in I_0$. In addition, all the functions $\mathbf{P}_\tau f_\tau$ with $\tau\in I_0$ extend to bounded harmonic functions on $\mathbb{C}\setminus\mathcal{K}$.

3.2 Moving front Neumann jump problems

In the process of obtaining approximate solutions to the equation (2.16) and thereby to Problem 2.4, we will encounter the following type of initial value problem repeatedly. We are given a real-analytic function $F_\tau:\partial\mathcal{S}_\tau\to\mathbb{R}$, $\tau\in I_0$, and we want to find a solution ω to the problem

$$\partial_{\mathrm{n}}(\mathbf{I}-\mathbf{P}_\tau)\omega\big|_{\partial\mathcal{S}_\tau}=F_\tau,\qquad\qquad \tau\in I_0$$

where $\mathbf{P}_\tau$ denotes the Poisson extension operator to the exterior domain $\mathbb{C}\setminus\mathcal{S}_\tau$, and where n denotes the outward unit normal to $\mathcal{S}_\tau$. Here, the solution ω should be a τ-independent, real-valued and real-analytic function on $\mathcal{V}\setminus\mathcal{K}$.

Lemma 3.2. *Suppose that F_τ, $\tau \in I_0$, are real-valued real-analytic functions on a fixed neighborhood of $\mathcal{V} \setminus \mathcal{K}$. Suppose moreover that we are given an initial datum f_0 on $\partial\mathcal{S}_{\tau_0-\epsilon}$, whose Poisson extension is harmonic on $\mathbb{C} \setminus \mathcal{K}$. Then there exists a unique real-analytically smooth solution ω on $\mathcal{V} \setminus \mathcal{K}$ to the equation*

$$\partial_{\mathrm{n}}(\mathbf{I} - \mathbf{P}_\tau)\,\omega\big|_{\mathcal{S}_\tau} = F_\tau\big|_{\partial\mathcal{S}_\tau}, \qquad |\tau - \tau_0| \le \epsilon \tag{3.2}$$

with initial value $\omega\big|_{\partial\mathcal{S}_{\tau_0-\epsilon}} = f_0$. If moreover $F_\tau \in \mathscr{H}_{\sigma',\delta'}$ and $f_0 \in \mathscr{H}^\infty_{\sigma'}$, then $\omega \in \mathscr{H}^\infty_{\sigma'}$ as well.

Remark 3.3. Below we will fix $f_0 = 0$. We will denote by $\omega = \mathbf{N}[F_\tau]$ the unique solution to (3.2) corresponding to this initial value.

Proof. We look for a solution $\omega(z)$ of the form

$$\omega(z) = \mathbf{P}_{\tau_0-\epsilon} f_0(z) + \int_{\tau_0-\epsilon}^{\tau(z)} \mathbf{P}_s g_s(z)\,\mathrm{d}s, \qquad z \in \mathcal{V} \setminus \mathcal{K} \tag{3.3}$$

where g_s, $s \in I_0$, are real-analytic functions subject to the appropriate regularity conditions. For a point $z \in \mathcal{S}_\tau$ sufficiently close to the boundary $\partial\mathcal{S}_\tau$, we claim that

$$(\mathbf{I} - \mathbf{P}_\tau)\,\omega(z) = \int_{\tau}^{\tau(z)} \mathbf{P}_s g_s(z)\,\mathrm{d}s. \tag{3.4}$$

Indeed, for $z \in \partial\mathcal{S}_\tau$ we have $\tau(z) = \tau$, and hence the restriction of ω to $\partial\mathcal{S}_\tau$ satisfies

$$\omega(z) = \mathbf{P}_{\tau_0-\epsilon} f_0(z) + \int_{\tau_0-\epsilon}^{\tau} \mathbf{P}_s g_s(z)\,\mathrm{d}s.$$

Moreover, considered as a function on $\mathbb{C} \setminus \mathcal{S}_\tau$, the right-hand side defines a bounded harmonic function, and hence we have

$$\mathbf{P}_\tau\,\omega(z) = \mathbf{P}_{\tau_0-\epsilon} f_0(z) + \int_{\tau_0-\epsilon}^{\tau} \mathbf{P}_s g_s(z)\,\mathrm{d}s, \qquad z \in \mathbb{C} \setminus \mathcal{K}$$

from which (3.4) is immediate. We next compute the normal derivative at $\partial\mathcal{S}_\tau$ from the inside of $\mathcal{S}_\tau$. To that end, fix $\tau \in I_0$ and let $z = z_0 - \mathrm{n}y$, where $y > 0$ and $z_0 \in \partial\mathcal{S}_\tau$. It then holds that

$$\begin{aligned}(\mathbf{I} - \mathbf{P}_\tau)\,\omega(z) &= -\int_{\tau - y/\nu(z_0) + \mathrm{O}(y^2)}^{\tau} \mathbf{P}_s g_s(z_0 - \mathrm{n}y)\,\mathrm{d}s \\ &= -y\frac{g_\tau(z_0)}{\nu(z_0)} + \mathrm{O}(y^2),\end{aligned}$$

where $\nu(z)$ denotes the normal velocity of the Hele-Shaw flow $(\partial\mathcal{S}_t)_t$. Hence

$$\partial_{\mathrm{n}}(\mathbf{I} - \mathbf{P}_\tau)\,\omega(z) = -\frac{g_\tau(z)}{\nu(z)}, \qquad z \in \partial\mathcal{S}_\tau$$

and we may solve for g_τ in (3.2) to obtain

$$g_\tau(z) = -\nu(z)F_\tau(z), \qquad z \in \partial\mathcal{S}_\tau.$$

With this choice, (3.3) supplies a real-analytic solution to the equation (3.2).

In order to see that the solution is unique, we simply need to show that if

$$\partial_{\mathrm{n}}(\mathbf{I}-\mathbf{P}_\tau)u = 0 \qquad \text{for} \quad \tau\in I_0 \tag{3.5}$$

with $u=0$ on one given spectral boundary (say $\partial\mathcal{S}_{\tau_0-\epsilon}$), then $u\equiv 0$. But this follows from the following argument. The function $\partial_\tau\mathbf{P}_\tau u(z)$ is bounded and harmonic on $\mathbb{C}\setminus\mathcal{S}_\tau$. Moreover, if we denote by z_η the point on $\partial\mathcal{S}_{\tau+\eta}$ which is closest to $z\in\partial\mathcal{S}_\tau$, we have

$$z_\eta = z+\eta\nu(z)\mathrm{n}+\mathrm{O}(\eta^2) = z+\eta\nu(z)\mathrm{n}_\eta+\mathrm{O}(\eta^2),$$

where n and n_η denote the outward unit normals to $\partial\mathcal{S}_\tau$ and $\partial\mathcal{S}_{\tau+\eta}$ at z and z_η, respectively. But by (3.5), the function $(\mathbf{I}-\mathbf{P}_\tau)u$ vanishes quadratically around $\partial\mathcal{S}_{\tau+\eta}$, so an application of Taylor's formula in normal coordinates around $\partial\mathcal{S}_{\tau+\epsilon}$ shows that

$$(\mathbf{I}-\mathbf{P}_{\tau+\eta})u(z) = (\mathbf{I}-\mathbf{P}_\tau)u\big(z_\eta-\eta\nu(z)\big)+\mathrm{O}(\eta^3) = 2\Delta u(z_\eta)(\nu(z)\eta)^2+\mathrm{O}(\eta^3).$$

But this then shows that

$$\begin{aligned}\partial_\tau\mathbf{P}_\tau u(z) = -\partial_\tau(\mathbf{I}-\mathbf{P}_\tau)u(z) &= -\lim_{\eta\to 0}\eta^{-1}(\mathbf{I}-\mathbf{P}_{\tau+\eta})u(z)\\ &= -\lim_{\eta\to 0} 2\Delta u(z_\eta)\nu^2(z_\eta)\eta+\mathrm{O}(\eta^2) = 0,\end{aligned}$$

from which we deduce that $\partial_\tau\mathbf{P}_\tau u$ vanishes along the boundary $\partial\mathcal{S}_\tau$. But $\partial_\tau\mathbf{P}_\tau u$ us bounded and harmonic, so $\partial_\tau\mathbf{P}_\tau u$ must vanishes identically. As a consequence, $\mathbf{P}_\tau u$ is constant in τ, so since $\mathbf{P}_{\tau_0}u\equiv 0$ holds by the initial condition this gives that u vanishes identically.

Turning to the fine regularity of ω, we note that since $F_\tau,\nu\in\mathscr{H}^\infty_{\sigma,\delta}$, we have $g_\tau\in\mathscr{H}^\infty_{\sigma,\delta}$ as well. The polarization of $\omega_1(z)\overset{\text{def}}{=}\int_{\tau_0-\epsilon}^{\tau(z)}\mathbf{P}_s g_s(z)\,\mathrm{d}s$ is given by

$$\omega_1^\diamond(z,w) = \int_{\tau_0-\epsilon}^{\tau^\diamond(z,w)}\big(\mathbf{P}_s g_s\big)^\diamond(z,w)\mathrm{d}s,$$

where, by (3.1), the integration contour can be taken to be an arc in $\mathbb{D}(\tau_0,\frac{1}{2}\delta)$. From this it follows by inspection that $\omega_1\in\mathscr{H}^\infty_{\sigma',\delta'}$. Since $\mathbf{P}_{\tau_0}[f_0]\in\mathscr{H}^\infty_{\sigma',\delta'}$ by assumption, the regularity assertion follows and the proof is complete. □

3.3 A classical limiting equation

In Proposition 4.2 below, we will show that the potential equation from Problem 2.4 induces a non-linear problem of a semiclassical character. Our goal for the remainder of this section is to develop tools for solving such equations approximately to high precision. We begin by looking closely at a related equation, which corresponds to collapsing the semiclassical problem to a classical one by taking the limit $m\to+\infty$. The limiting equation can be solved by hands, and the solution scheme forms the basis for an iterative approach to the full semiclassical problem.

The limiting classical problem reads as follows. We are given strictly positive real-analytic functions f_τ defined on the neighborhood $\mathcal{V}\setminus\mathcal{K}$ of $\partial\mathcal{S}_{\tau_0}$, and consider an equation for three unknown functions $\mathcal{E}_\tau$, $\mathcal{F}_\tau$ and $\mathcal{G}_\tau$ of the form

$$V_\tau^2\,\mathcal{E}_\tau + V_\tau\,\mathcal{F}_\tau + \mathcal{G}_\tau \equiv f_\tau \qquad \text{on } \mathcal{V}\setminus\mathcal{K}$$

where $\mathcal{F}_\tau$ is subject to the structural requirement that

$$\mathcal{F}_\tau = H_\tau \frac{\big(\mathbf{I}-\mathbf{P}_\tau\big)\,\mathfrak{f}}{V_\tau} \tag{3.6}$$

for some (τ-independent) function $\mathfrak{f}$ and a given non-vanishing real-analytic function H_τ, while $\mathcal{G}_\tau$ takes the form

$$\mathcal{G}_\tau = \mathrm{e}^{\mathfrak{g}_\tau} \tag{3.7}$$

for some bounded harmonic function $\mathfrak{g}_\tau$ on a neighborhood of $\mathbb{C}\setminus\mathcal{K}$. How would we go about solving such an equation?

First restriction. Notice that the function $\mathcal{G}_\tau$ is uniquely determined from f_τ by restricting to $\partial\mathcal{S}_\tau$. Indeed, since V_τ vanishes there, we find that $\mathcal{G}_\tau = f_\tau$ along the boundary, so that

$$\mathfrak{g}_\tau = \mathbf{P}_\tau[\log f_\tau]. \tag{3.8}$$

Second restriction. The difference $f_\tau - \mathcal{G}_\tau$ vanishes along $\partial\mathcal{S}_\tau$, hence $g_\tau \overset{\text{def}}{=} \frac{f_\tau-\mathcal{G}_\tau}{V_\tau}$ is a real-analytic function. After adding $\mathcal{G}_\tau$ on both sides followed by dividing by V_τ, our equation reads

$$V_\tau\,\mathcal{E}_\tau + \mathcal{F}_\tau \equiv g_\tau,$$

where only $\mathcal{E}_\tau$ and $\mathcal{F}_\tau$ remain to be determined. Restricting anew to $\partial\mathcal{S}_\tau$, we see that

$$\mathcal{F}_\tau = g_\tau \qquad \text{along } \partial\mathcal{S}_\tau.$$

Incorporating the structural assumption (3.6) and expanding both numerator and denominator of (3.6) using Taylor's formula in normal coordinates, we see that

$$\mathcal{F}_\tau\big|_{\partial\mathcal{S}_\tau} = H_\tau\frac{(\mathbf{I}-\mathbf{P}_\tau)\,\mathfrak{f}}{V_\tau}\Big|_{\partial\mathcal{S}_\tau} = H_\tau(\partial_\mathrm{n}V_\tau)^{-1}\partial_\mathrm{n}(\mathbf{I}-\mathbf{P}_\tau)\,\mathfrak{f}.$$

As a consequence of this formula, the restriction condition for $\mathcal{F}_\tau$ turns into the "moving front Neumann jump problem"

$$\partial_\mathrm{n}(\mathbf{I}-\mathbf{P}_\tau)\,\mathfrak{f} = \frac{g_\tau\partial_\mathrm{n}V_\tau}{H_\tau} =: F_\tau \qquad \text{along } \partial\mathcal{S}_\tau. \tag{3.9}$$

for $\mathfrak{f}$ (cf. Section 3.2). Up to the addition of an inessential bounded harmonic function, the jump problem (3.9) determines $\mathfrak{f}$ uniquely as a linear function $\mathfrak{f} = \mathbf{N}[F_\tau]$ of F_τ (see Lemma 3.2).

The full solution. By construction, the solution $\mathfrak{f} = \mathbf{N}[F_\tau]$ to the above moving front Neumann problem satisfies

$$g_\tau - H_\tau \frac{(\mathbf{I} - \mathbf{P}_\tau)\mathfrak{f}}{V_\tau} = 0 \qquad \text{along } \partial\mathcal{S}_\tau,$$

so the quotient $\frac{g_\tau - \mathcal{F}_\tau}{V_\tau}$ is real analytic on $\mathcal{V}\backslash\mathcal{K}$. Our original equation thus transforms into

$$\mathcal{E}_\tau \equiv \frac{g_\tau - \mathcal{F}_\tau}{V_\tau},$$

where the right-hand side is a real-analytic function, uniquely determined by the datum f_τ. Taking this as our definition of $\mathcal{E}_\tau$, the full equation has been solved.

In summary, the equation (3.8) admits a solution

$$\begin{cases} \mathcal{G}_\tau = \mathrm{e}^{\mathfrak{g}_\tau}, & \text{where } \; \mathfrak{g}_\tau = \mathbf{P}_\tau[\log f_\tau], \\ \mathcal{F}_\tau = H_\tau \frac{(\mathbf{I}-\mathbf{P}_\tau)\mathfrak{f}}{V_\tau}, & \text{where } \; \mathfrak{f} = \mathbf{N}\big[\frac{\partial_{\mathrm{n}} V_\tau}{H_\tau} \frac{f_\tau - \mathcal{G}_\tau}{V_\tau}\big], \\ \mathcal{E}_\tau = \frac{g_\tau - \mathcal{F}_\tau}{V_\tau}. \end{cases} \tag{3.10}$$

Moreover, the solution is unique up to the freedom of adding a bounded harmonic function to $\mathfrak{f}$, which does not affect $\mathcal{E}_\tau$, $\mathcal{F}_\tau$ or $\mathcal{G}_\tau$. If f_τ, $\log f_\tau$, H_τ and $1/H_\tau$ all belong to to the space $\mathscr{H}^\infty_{\sigma',\delta'}$, the regularity assertion of Lemma 3.2 and Remark 3.1 (a) imply that $\mathcal{E}_\tau$, $\mathcal{F}_\tau$ and $\mathcal{G}_\tau$ belong to $\mathscr{H}^\infty_{\sigma'',\delta'}$ whenever $0 < \sigma'' < \sigma'$.

3.4 Approximate solutions to the full semiclassical problem

The problem we have at our hands is of a similar form, but with one essential complication. Namely, the right-hand side f_τ appearing in the toy problem was given to us in advance, while in the actual problem it depends on the unknowns. However, the right-hand side is of the "semiclassical" form

$$f_\tau + m^{-1}\mathbf{T}_\tau(\mathcal{E}_\tau, \mathcal{F}_\tau),$$

where f_τ is a given strictly positive real-analytic function, and where $\mathbf{T}_\tau = \mathbf{T}_{\tau,m}$ is a certain well-behaved affine operator acting on tuples of real-analytic functions. By "well-behaved" we mean that they admit a Lipschitz-type estimates in the Banach scales introduced in Section 3.1. More specifically, we ask that for some positive constant $C = C(\sigma' - \sigma'', \delta' - \delta'')$ the bound

$$\big\|\mathbf{T}_\tau\big(\mathcal{E}_\tau, \mathcal{F}_\tau\big) - \mathbf{T}_\tau\big(\mathcal{E}'_\tau, \mathcal{F}'_\tau\big)\big\|_{\sigma'',\delta''} \le C \max\Big\{\big\|\mathcal{E}_\tau - \mathcal{E}'_\tau\big\|_{\sigma',\delta'}, \big\|\mathcal{F}_\tau - \mathcal{F}'_\tau\big\|_{\sigma',\delta'}\Big\}, \tag{3.11}$$

holds for any pairs $(\mathcal{E}_\tau, \mathcal{F}_\tau)$ and $(\mathcal{E}'_\tau, \mathcal{F}'_\tau)$ of functions in $\mathscr{H}^\infty_{\sigma',\delta'}$.

While the added perturbation prevents us from solving the equation exactly, the semiclassical nature ensures that we can still construct approximate solutions by iterating the above exact solution algorithm. More precisely, we have the following lemma.

Lemma 3.4. *Let f_τ, H_τ and $\mathbf{T}_\tau$ be be as above, and fix $\kappa \in \mathbb{Z}_{>0}$. Then there exist triples $(\mathcal{E}_\tau, \mathcal{F}_\tau, \mathcal{G}_\tau)$ of real-analytic functions in $\mathcal{H}^\infty_{\frac12\sigma,\frac12\delta}$ subject to the structural constraints* (3.6) *and* (3.7) *and with asymptotic expansions in powers of m^{-1}, such that*

$$\big\|V_\tau^2\,\mathcal{E}_\tau + V_\tau\,\mathcal{F}_\tau + \mathcal{G}_\tau - f_\tau - m^{-1}\mathbf{T}_\tau(\mathcal{E}_\tau,\mathcal{F}_\tau)\big\|_{\frac12\sigma,\frac12\delta} = \mathrm{O}\big(m^{-\kappa-1}\big). \tag{3.12}$$

Proof. Fix thee sequences of shrinking positive parameters $\{\sigma_k\}_{k=0}^\kappa$, $\{\sigma'_k\}_{k=0}^\kappa$ and $\{\sigma''_k\}_{k=0}^\kappa$ with the interlacing property

$$\tfrac12\sigma < \sigma_\kappa < \sigma'_\kappa < \sigma''_\kappa < \ldots < \sigma_0 < \sigma'_0 < \sigma''_0 < \sigma.$$

We similarly let $\{\delta_k\}_{k=0}^\kappa$ and $\{\delta'_k\}_{k=0}^\kappa$ be interlacing increasing sequences bounded between $\frac12\delta$ and δ. We formally put $\mathcal{E}_{-1,\tau}\equiv\mathcal{F}_{-1,\tau}\equiv 0$.

The initial guess. We start with a first guess $(\mathcal{E}_{0,\tau},\mathcal{F}_{0,\tau},\mathcal{G}_{0,\tau})$, obtained by solving the equation from Section 3.3 for the unperturbed (classical) right-hand $f_{0,\tau} = f_\tau$. As a consequence, we obtain

$$V_\tau^2\mathcal{E}_{0,\tau} + V_\tau\mathcal{F}_{0,\tau} + \mathcal{G}_{0,\tau} - f_\tau - m^{-1}\mathbf{T}_\tau(\mathcal{E}_{0,\tau},\mathcal{F}_{0,\tau}) = -m^{-1}\mathbf{T}_\tau(\mathcal{E}_{0,\tau},\mathcal{F}_{0,\tau}), \tag{3.13}$$

The three elements of the triple $(\mathcal{E}_{0,\tau},\mathcal{F}_{0,\tau},\mathcal{G}_{0,\tau})$ belong to the space $\mathscr{H}^\infty_{\sigma'_0,\delta}$, and hence by the assumption on $\mathbf{T}_\tau$ we have that

$$\mathbf{T}_\tau(\mathcal{E}_{0,\tau},\mathcal{F}_{0,\tau}) \in \mathscr{H}^\infty_{\sigma_0,\delta'_0}.$$

Combining this by (3.13), we find that

$$\big\|V_\tau^2\mathcal{E}_{0,\tau} + V_\tau\mathcal{F}_{0,\tau} + \mathcal{G}_{0,\tau} - f_\tau + m^{-1}\mathbf{T}_\tau(\mathcal{E}_{0,\tau},\mathcal{F}_{0,\tau})\big\|_{\sigma_0,\delta_0} = \mathrm{O}(m^{-1}),$$

where we used the fact that $\delta_0 < \delta'_0$ in the last step. Hence, this guess gives (3.12) with $\kappa = 0$.

The iteration step. We enter this step having obtained an exact solution triple $(\mathcal{E}_{k,\tau},\mathcal{F}_{k,\tau},\mathcal{G}_{k,\tau})$ to the equation

$$V_\tau^2\mathcal{E}_{k,\tau} + V_\tau\mathcal{F}_{k,\tau} + \mathcal{G}_{k,\tau} = f_\tau + m^{-1}\mathbf{T}_\tau(\mathcal{E}_{k-1,\tau},\mathcal{F}_{k-1,\tau}) =: f_{k-1,\tau},$$

such that

$$\begin{cases} \mathcal{E}_{k,\tau} = \mathcal{E}_{k-1,\tau} + \mathrm{O}(m^{-k}),\\ \mathcal{F}_{k,\tau} = \mathcal{F}_{k-1,\tau} + \mathrm{O}(m^{-k}),\\ \mathcal{G}_{k,\tau} = \mathcal{G}_{k-1,\tau} + \mathrm{O}(m^{-k}) \end{cases} \tag{3.14}$$

where the errors refer to the norm in $\mathscr{H}^\infty_{\sigma_k,\delta_k}$. Hence, the triple is an approximate solution to the equation (3.12) in the sense that

$$\begin{aligned} V_\tau^2\mathcal{E}_{k,\tau} &+ V_\tau\mathcal{F}_{k,\tau} + \mathcal{G}_{k,\tau} - f_\tau - m^{-1}\mathbf{T}_\tau(\mathcal{E}_{k,\tau},\mathcal{F}_{k,\tau}) \\ &= m^{-1}\Big(\mathbf{T}_\tau(\mathcal{E}_{k-1,\tau},\mathcal{F}_{k-1,\tau}) - \mathbf{T}_\tau(\mathcal{E}_{k,\tau},\mathcal{F}_{k,\tau})\Big) = \mathrm{O}(m^{-k-1}), \end{aligned}$$

the last line referring to the norm in the space $\mathscr{H}^{\infty}_{\sigma''_{k+1},\delta'_{k+1}}$.

We define an improved right-hand side by

$$f_{k,\tau} = f_\tau + m^{-1}\mathbf{T}_\tau(\mathcal{E}_{k,\tau}, \mathcal{F}_{k,\tau}),$$

which by the induction assumption belongs to $\mathscr{H}_{\sigma''_{k+1},\delta'_{k+1}}$. Using the Lipschitz bounds for $\mathbf{T}_\tau$ together with the approximation (3.14), it is not difficult to see that the two successive right-hand sides meet the bound

$$\|f_{k,\tau} - f_{k-1,\tau}\|_{\sigma''_{k+1},\delta'_{k+1}} = \mathrm{O}(m^{-k-1}),$$

uniformly for $\tau \in I_0$. Hence, denoting by $(\mathcal{E}_{k+1,\tau}, \mathcal{F}_{k+1,\tau}, \mathcal{G}_{k+1,\tau})$ the solution triple to the exact equation

$$V_\tau^2\mathcal{E}_{k+1,\tau} + V_\tau\mathcal{F}_{k+1,\tau} + \mathcal{G}_{k+1,\tau} = f_{k,\tau},$$

an inspection of the solution formula (3.10) together with the approximation (3.14) shows that

$$\begin{cases} \mathcal{E}_{k+1,\tau} = \mathcal{E}_{k,\tau} + \mathrm{O}(m^{-k-1}), \\ \mathcal{F}_{k+1,\tau} = \mathcal{F}_{k,\tau} + \mathrm{O}(m^{-k-1}), \\ \mathcal{G}_{k+1,\tau} = \mathcal{G}_{k,\tau} + \mathrm{O}(m^{-k-1}) \end{cases}$$

where the errors are taken in the norm of $\mathscr{H}^{\infty}_{\sigma'_{k+1},\delta'_{k+1}}$. As a consequence, it holds that

$$\big\|\mathbf{T}_\tau(\mathcal{E}_{k+1,\tau}, \mathcal{F}_{k+1,\tau}) - \mathbf{T}_\tau(\mathcal{E}_{k,\tau}, \mathcal{F}_{k,\tau})\big\|_{\sigma_{k+1},\delta_{k+1}} = \mathrm{O}(m^{-k-1}),$$

which implies that

$$\big\|f_{k,\tau} - f_\tau - m^{-1}\mathbf{T}_\tau(\mathcal{E}_{k+1,\tau}, \mathcal{F}_{k+1,\tau})\big\|_{\sigma_{k+1},\delta_{k+1}} = \mathrm{O}(m^{-k-2}). \tag{3.15}$$

But we have

$$\begin{aligned} V_\tau^2\mathcal{E}_{k+1,\tau} + V_\tau\mathcal{F}_{k+1,\tau} + \mathcal{G}_{k+1,\tau} - f_\tau - m^{-1}\mathbf{T}(\mathcal{E}_{k+1,\tau}, \mathcal{F}_{k+1,\tau}) \\ = f_{k,\tau} - f_\tau - m^{-1}\mathbf{T}_\tau(\mathcal{E}_{k+1,\tau}, \mathcal{F}_{k+1,\tau}), \end{aligned}$$

which when combined with (3.15) shows that the new triple solves the original equation up to error $\mathrm{O}(m^{-k-2})$ in the space $\mathscr{H}^{\infty}_{\sigma_{k+1},\delta_{k+1}}$.

The assertion of the lemma now follows by induction on the number k of iterations. □

Remark 3.5. By the argument of Lemma 10.2 in [14], we may write

$$\mathcal{E}_\tau = \sum_{j=0}^{\kappa} m^{-j}\mathcal{E}_{j,\tau} + m^{-\kappa-1}E,$$

for a sequence $\{\mathcal{E}_{j,\tau}\}_{j=0}^{\kappa}$ of m-independent coefficients and an function ("error") $E = E_{m,\kappa,\tau}$ belonging to the space $\mathscr{H}^{\infty}_{\frac12\sigma,\frac12\delta}$. Similar expansions hold for the remaining functions $\mathcal{F}_\tau, \mathcal{G}_\tau$, $\mathfrak{f}$ and $\mathfrak{g}_\tau$ as well.

4 A NON-LINEAR PDE FOR THE BERGMAN POTENTIAL

4.1 The potential equation as a semiclassical PDE

In the form (2.16), the approximate potential equation is not very illuminating. Our goal in this subsection is to bring it into the form of a semiclassically perturbed (or "quantized") non-linear equation for the unknowns Ξ_τ and Ω. This type of equation can be solved by iterating a non-linear operator, see Section 3 above.

Proposition 4.1. *We have the identity*

$$\partial_\tau \mathit{\Delta} U_\tau = \sqrt{\frac{m}{2\pi}}\Big(m^2 V_\tau^2 A_\tau \Xi_\tau + mV_\tau B_\tau \Sigma_\tau - 2\partial_\tau V_\tau \mathit{\Delta}\Omega + \mathbf{Q}_\tau\Big)\,\mathrm{e}^{-2mV_\tau^2},$$

where A_τ and B_τ are non-vanishing real-analytic functions on $\mathcal{V}\setminus\mathcal{K}$ given by

$$\begin{cases} A_\tau &= 32\partial_\tau V_\tau |\bar\partial V_\tau|^2 \\ B_\tau &= -16\partial_\tau V_\tau |\bar\partial V_\tau|^2, \end{cases} \tag{4.1}$$

and where $\mathbf{Q}_\tau = \mathbf{Q}_{\tau,m}(\Lambda_\tau,\Omega,\Sigma_\tau)$ is a linear operator acting on the coefficient functions, explicitly given by

$$\begin{aligned} \mathbf{Q}_\tau = -m\Big(&16V_\tau\partial_\tau V_\tau \mathrm{Re}\left\{\partial\Xi_\tau\bar\partial V_\tau\right\} + 8V_\tau\partial_\tau V_\tau \Xi_\tau \mathit{\Delta} V_\tau + 8\partial_\tau\Big(\Xi_\tau V_\tau|\partial V_\tau|^2\Big)\Big) \\ &- 4V_\tau\partial_\tau V_\tau \mathit{\Delta}\Sigma_\tau + 2\partial_\tau\Big(2\Sigma_\tau|\bar\partial V_\tau|^2 + 2\mathrm{Re}\left\{\partial\Xi_\tau\bar\partial V_\tau\right\} + \Xi_\tau\mathit{\Delta} V_\tau\Big) + m^{-1}\partial_\tau\mathit{\Delta}\Sigma_\tau. \end{aligned} \tag{4.2}$$

Proof. Recall the definitions of $X_{j,\tau}$ and $Y_{j,\tau}$ from (2.13) and (2.15). To isolate the important terms, we first decompose $X_{1,\tau}$ as

$$X_{1,\tau} = 4\Sigma_\tau|\bar\partial V_\tau|^2 + \widetilde{X}_{1,\tau}$$

where $\widetilde{X}_{1,\tau}$ is given by

$$\widetilde{X}_{1,\tau} = 4\mathrm{Re}\left\{\partial\Xi_\tau\bar\partial V_\tau\right\} + 2\Xi_\tau\mathit{\Delta} V_\tau.$$

Similarly, we introduce the decompositions

$$\begin{cases} Y_{1,\tau} &= -16V_\tau\partial_\tau V_\tau|\bar\partial V_\tau|^2\Sigma_\tau + \widetilde{Y}_{1,\tau}, \\ Y_{2,\tau} &= -2\partial_\tau V_\tau\mathit{\Delta}\Omega + \widetilde{Y}_{2,\tau} \end{cases}$$

where $\widetilde{Y}_{1,\tau}$ and $\widetilde{Y}_{2,\tau}$ are given by

$$\begin{cases} \widetilde{Y}_{1,\tau} &= -4V_\tau\partial_\tau V_\tau\widetilde{X}_{1,\tau} + \partial_\tau X_{0,\tau}. \\ \widetilde{Y}_{2,\tau} &= -4V_\tau\partial_\tau V_\tau X_{2,\tau} + \partial_\tau X_{1,\tau}. \end{cases} \tag{4.3}$$

Using these decompositions, we may rewrite the right-hand side of (2.14) as

$$\partial_\tau \mathit{\Delta} U_\tau = \sqrt{\frac{m}{2\pi}}\Big(m^2 A_\tau V_\tau^2 \Xi_\tau + mV_\tau B_\tau \Sigma_\tau - 2\partial_\tau V_\tau \mathit{\Delta}\Omega + \mathbf{Q}_\tau\Big)\,\mathrm{e}^{-2mV_\tau^2},$$

where A_τ and B_τ are readily checked to be given by the explicit formulas in the proposition, and where $\mathbf{Q}_\tau$ is given by

$$\mathbf{Q}_\tau = m\widetilde{Y}_{1,\tau} + \widetilde{Y}_{2,\tau} + m^{-1}Y_{3,\tau}.$$

Expressing the right-hand side in terms of the coefficient functions Ξ_τ, Ω and Σ_τ using the definitions (2.15) and (4.3), it is readily verified that $\mathbf{Q}_\tau$ is indeed given by (4.16). The claim follows. □

4.2 Reduction of the semiclassical PDE

Starting from Proposition 4.1, we proceed to obtain the first few terms in the asymptotic expansion of Ξ_τ, and the leading order coefficients $\Lambda_{0,\tau}$, Ω_0 and $\Sigma_{0,\tau}$. This will solve the equation (2.16) approximately to order $\mathrm{O}(1)$, and will bring it exactly into the form (3.12). The outcome is summarized in Proposition 4.2 below.

The coefficient $\Xi_{0,\tau}$. In view of Proposition 4.1, the approximate potential equation (2.16) is equivalent to

$$m^2V_\tau^2A_\tau\Xi_\tau + mV_\tau B_\tau\Sigma_\tau - 2\partial_\tau V_\tau \Delta\Omega + \mathbf{Q}_\tau = \mathrm{e}^{h_\tau} + \mathrm{O}(m^{-\kappa-2}), \tag{4.4}$$

where A_τ and B_τ are non-vanishing real-analytic functions defined in (4.1), and where $\mathbf{Q}_\tau$ is a linear expression in the coefficient functions, explicitly given by (4.16). It is immediate from (4.16) that $\mathbf{Q}_\tau = \mathrm{O}(m)$, from which it follows that the left-hand side of (4.4) equals

$$m^2V_\tau^2A_\tau\Xi_{0,\tau} + \mathrm{O}(m)$$

while the right-hand side is bounded. Since A_τ is non-vanishing on $\mathcal{V}\setminus\mathcal{K}$, we see that the coefficient $\Xi_{\tau,0}$ has to vanish identically, and we write

$$\Xi_\tau = m^{-1}\Xi_\tau^{(1)}. \tag{4.5}$$

With this choice, (4.4) is equivalent to the approximate equation

$$mV_\tau^2A_\tau\Xi_\tau^{(1)} + mV_\tau B_\tau\Sigma_\tau - 2\partial_\tau V_\tau \Delta\Omega + \mathbf{Q}_\tau = \mathrm{e}^{h_\tau} + \mathrm{O}(m^{-\kappa-2}). \tag{4.6}$$

By the definition (2.11) of Ξ_τ, we must moreover have

$$\begin{cases} \Lambda_{0,\tau} = \tau\log|\phi_\tau|^2 + \mathbf{P}_\tau[\Omega_0] \\ \Sigma_{0,\tau} = \frac{\Lambda_{0,\tau}-\Omega_0}{2V_\tau}. \end{cases} \tag{4.7}$$

Note that these relations do not determine $\Lambda_{0,\tau}$, Ω_0 and $\Sigma_{0,\tau}$ uniquely. Rather, they express $\Lambda_{0,\tau}$ and $\Sigma_{0,\tau}$ as functions of the unknown Ω_0, which will be determined in the next step. Observe however that $\Lambda_{0,\tau} - \Omega_0$ vanishes along the boundary $\partial\mathcal{S}_\tau$, so the division by V_τ does not create any singularity. Hence these relations are compatible with the properties we ask of the coefficient functions, no matter the choice of Ω_0.

The coefficients $\Xi_{1,\tau}$, $\Lambda_{0,\tau}$, Ω_0 and $\Sigma_{0,\tau}$. Incorporating the first-order coefficients of Ξ_τ, Ω and Σ_τ from (4.5) and (4.7) into the definition (4.16) of $\mathbf{Q}_\tau$, we find that

$$(4.8)\quad \mathbf{Q}_\tau = -16V_\tau\partial_\tau V_\tau \mathrm{Re}\left\{\partial\Xi_\tau^{(1)}\bar\partial V_\tau\right\} - 8V_\tau\partial_\tau V_\tau\Xi_\tau^{(1)}\mathit{\Delta}V_\tau - 8\partial_\tau\Big(\Xi_\tau^{(1)}V_\tau|\partial V_\tau|^2\Big)$$
$$- 4V_\tau\partial_\tau V_\tau\mathit{\Delta}\Sigma_\tau + 2\partial_\tau\Big(2\Sigma_\tau|\bar\partial V_\tau|^2 + 2\mathrm{Re}\left\{\partial\Xi_\tau\bar\partial V_\tau\right\} + \Xi_\tau\mathit{\Delta}V_\tau\Big) + m^{-1}\partial_\tau\mathit{\Delta}\Sigma_\tau.$$

Despite the initial appearance in (4.16), this shows that $\mathbf{Q}_\tau = \mathrm{O}(1)$. Expanding the left-hand side of the approximate potential equation (4.4) to order $\mathrm{O}(1)$, we see that

$$m^2V_\tau^2A_\tau\Xi_\tau + mV_\tau B_\tau\Sigma_\tau - 2\partial_\tau V_\tau\mathit{\Delta}\Omega + \mathbf{Q}_\tau = mV_\tau^2A_\tau\Xi_\tau^{(1)} + mV_\tau B_\tau\Sigma_{0,\tau} + \mathrm{O}(1),$$

which when inserted into (4.4) implies that

$$(4.9)\qquad V_\tau^2A_\tau\Xi_{1,\tau} + V_\tau B_\tau\Sigma_{0,\tau} \equiv 0.$$

As the function V_τ vanishes only along the boundary $\partial\mathcal{S}_\tau$ we also conclude that

$$V_\tau A_\tau\Xi_{1,\tau} + B_\tau\Sigma_{0,\tau} \equiv 0.$$

In order for this equation to hold, it must in particular hold along the boundary $\partial\mathcal{S}_\tau$, which since B_τ is non-vanishing implies that $\Sigma_{0,\tau}|_{\partial\mathcal{S}_\tau} = 0$. But in view of the relation (4.7) between $\Sigma_{0,\tau}$ and the other coefficients $\Lambda_{0,\tau}$ and Ω_0, this condition is equivalent to

$$\frac{\Omega_0 - \Lambda_{0,\tau}}{V_\tau}\Big|_{\partial\mathcal{S}_\tau} = 0,$$

which by an application of Taylor's formula at $\partial\mathcal{S}_\tau$ is seen to be equivalent to the normal derivative condition

$$\partial_{\mathrm{n}}\big(\Lambda_{\tau,0} - \Omega_0\big)\big|_{\partial\mathcal{S}_\tau} = 0.$$

This is an instance of the "moving front Neumann jump problem" which was explored above in Section 3.2. In particular, Lemma 3.2 shows that this problem always admits a solution, and moreover that the solution is unique up to the addition of a bounded harmonic function on $\mathbb{C}\setminus\mathcal{K}$. Due to cancellations between the terms Λ_τ and Ω, the apparent freedom that arises from this non-uniqueness is illusory, and we may choose the initial conditions at will. A quick inspection shows that one solution is supplied by $\Omega_0 = 2Q$ and $\Lambda_{0,\tau} = 2\breve{Q}_\tau$. Once $\Lambda_{0,\tau}$ and Ω_0 are determined we obtain $\Sigma_{0,\tau}$ from (4.7), giving the explicit formula

$$\Sigma_{0,\tau} = \frac{2\breve{Q}_\tau - 2Q}{2V_\tau} = -V_\tau.$$

Finally, solving for $\Xi_{1,\tau}$ in (4.9), we obtain

$$\Xi_{1,\tau} = \frac{\Sigma_{0,\tau}}{2V_\tau} = -\frac{1}{2}.$$

With the functions $\Xi_{1,\tau}$ and $\Lambda_{0,\tau}$, Ω_0 and $\Sigma_{\tau,0}$ determined, we write

$$
\begin{cases}
\Xi_\tau &= -\frac{1}{2}m^{-1} + m^{-2}\Xi_\tau^{(2)}, \\
\Lambda_\tau &= 2\breve{Q}_\tau + m^{-1}\Lambda_\tau^{(1)} \\
\Omega &= 2Q + m^{-1}\Omega^{(1)}, \\
\Sigma_\tau &= -V_\tau + m^{-1}\Sigma_\tau^{(1)},
\end{cases}
\tag{4.10}
$$

where all the functions $\Xi_\tau^{(2)}$, $\Lambda_\tau^{(1)}$ $\Omega^{(1)}$ and $\Sigma_\tau^{(1)}$ admits asymptotic expansions in non-negative powers of m^{-1}. The former admits an expansion up to order $\mathrm{O}(m^{-\kappa-2})$, and the latter three to order $\mathrm{O}(m^{-\kappa-1})$. The conclusion from (4.9) is that (4.6) is equivalent to having (4.10), where $\Xi_\tau^{(2)}$, $\Omega^{(1)}$ and $\Sigma_\tau^{(1)}$ satisfy

$$
V_\tau^2 A_\tau \Xi_\tau^{(2)} + V_\tau B_\tau \Sigma_\tau^{(1)} + \mathbf{Q}_\tau = \mathrm{e}^{h_\tau} + \mathrm{O}(m^{-\kappa-1}).
\tag{4.11}
$$

The operator $\mathbf{Q}_\tau$ and the final reduction. It now only remains to rewrite the expression (4.8) for $\mathbf{Q}_\tau$ in a more illuminating form. Since the right-hand side of (4.4) positive for any choice of h_τ, it will be important to show that the left-hand side is positive as well. To that end, we note that

$$
m^2 V_\tau^2 A_\tau \Xi_\tau + m V_\tau B_\tau \Sigma_\tau - 2\partial_\tau V_\tau \varDelta\Omega + \mathbf{Q}_\tau \Big|_{\partial\mathcal{S}_\tau} = -4\partial_\tau V_\tau \varDelta\Omega + \mathbf{Q}_\tau \Big|_{\partial\mathcal{S}_\tau}.
$$

Since $\Omega = 2Q + \mathrm{O}(m^{-1})$ and since $-\partial_\tau V_\tau \varDelta Q > 0$, it would suffice to show that $\mathbf{Q}_\tau$ vanishes to leading order along $\partial\mathcal{S}_\tau$. For the reader's convenience, we recall that

$$
\begin{aligned}
\mathbf{Q}_\tau = &-16 V_\tau \partial_\tau V_\tau \mathrm{Re}\left\{\partial\Xi_\tau^{(1)}\bar\partial V_\tau\right\} - 8V_\tau\partial_\tau V_\tau \Xi_\tau^{(1)}\varDelta V_\tau - 8\partial_\tau\Big(\Xi_\tau^{(1)} V_\tau |\partial V_\tau|^2\Big) \\
&- 4V_\tau\partial_\tau V_\tau \varDelta\Sigma_\tau + 2\partial_\tau\Big(2\Sigma_\tau|\bar\partial V_\tau|^2 + 2\mathrm{Re}\left\{\partial\Xi_\tau\bar\partial V_\tau\right\} + \Xi_\tau\varDelta V_\tau\Big) + m^{-1}\partial_\tau\varDelta\Sigma_\tau,
\end{aligned}
$$

Separating out the terms of leading order, we get the expression

$$
\begin{aligned}
\mathbf{Q}_\tau = &-16V_\tau\partial_\tau V_\tau \mathrm{Re}\left\{\partial\Xi_{1,\tau}\bar\partial V_\tau\right\} - 8V_\tau\partial_\tau V_\tau \Xi_{1,\tau}\varDelta V_\tau \\
&- 8\partial_\tau\Big(\Xi_{1,\tau}V_\tau|\partial V_\tau|^2\Big) - 4V_\tau\partial_\tau V_\tau\varDelta\Sigma_{0,\tau} + 4\partial_\tau\Sigma_{0,\tau}|\bar\partial V_\tau|^2 + m^{-1}\mathbf{R}_\tau,
\end{aligned}
\tag{4.12}
$$

where $\mathbf{R}_\tau$ is a (bounded) affine expression in the coefficient functions, explicitly given by

$$
\begin{aligned}
\mathbf{R}_\tau = &-4V_\tau\partial_\tau V_\tau\varDelta\Sigma_\tau^{(1)} - 8\partial_\tau\Big(\Xi_\tau^{(2)}V_\tau|\partial V_\tau|^2\Big) - 16V_\tau\partial_\tau V_\tau \mathrm{Re}\left\{\partial\Xi_\tau^{(2)}\bar\partial V_\tau\right\} \\
&+ 4\partial_\tau\Sigma_\tau|\bar\partial V_\tau|^2 + 2\partial_\tau\Big(2\mathrm{Re}\left\{\partial\Xi_\tau^{(1)}\bar\partial V_\tau\right\} + \Xi_\tau^{(1)}\varDelta V_\tau\Big) + \partial_\tau\varDelta\Sigma_\tau.
\end{aligned}
$$

Simplifying the main term of (4.12) using the known expressions for $\Xi_{1,\tau}$ and $\Sigma_{0,\tau}$ (see (4.10)), we arrive at

$$
\begin{aligned}
\mathbf{Q}_\tau = &\ 4V_\tau\partial_\tau|\partial V_\tau|^2 + 8V_\tau\partial_\tau V_\tau\varDelta V_\tau + m^{-1}\mathbf{R}_\tau \\
&= 4V_\tau\big(\partial_\tau|\partial V_\tau|^2 + 2\partial_\tau V_\tau\varDelta V_\tau\big) + m^{-1}\mathbf{R}_\tau
\end{aligned}
\tag{4.13}
$$

From the representation (4.13), it is clear that $\mathbf{Q}_\tau$ vanishes to leading order along the boundary $\partial\mathcal{S}_\tau$.

The expression for $\mathbf{R}_\tau$ is readily rewritten using (4.10), which yields an expression solely in terms of $\Xi_\tau^{(2)}$, $\Omega^{(1)}$, $\Sigma_\tau^{(1)}$ and known functions. More specifically, it gives that

$$\begin{aligned}\mathbf{R}_\tau = -4V_\tau\partial_\tau V_\tau\mathit{\Delta}\Sigma_\tau^{(1)} - 8\partial_\tau\Big(\Xi_\tau^{(2)}V_\tau|\partial V_\tau|^2\Big) - 16V_\tau\partial_\tau V_\tau \mathrm{Re}\left\{\partial\Xi_\tau^{(2)}\bar\partial V_\tau\right\}\\ + \frac{1}{m}\Big(4\partial_\tau\Sigma_\tau^{(1)}|\bar\partial V_\tau|^2 + 2\partial_\tau\Big(2\mathrm{Re}\{\partial\Xi_\tau^{(2)}\bar\partial V_\tau\} + \Xi_\tau^{(2)}\mathit{\Delta}V_\tau\Big) + \partial_\tau\mathit{\Delta}\Sigma_\tau^{(1)}\Big)\\ - 4\partial_\tau V_\tau|\bar\partial V_\tau|^2 - 2\partial_\tau\mathit{\Delta}V_\tau.\end{aligned} \tag{4.14}$$

Note also that despite the appearance of $\Sigma_\tau^{(1)}$ in its definition we can interpret $\mathbf{R}_\tau$ as an affine operator acting only on the pair $(\Xi_\tau^{(2)}, \Omega^{(1)})$. Indeed, it holds that that $\Xi_\tau^{(1)} = \Lambda_\tau^{(1)} - \Omega^{(1)}$ along the boundary, whence

$$\Lambda_\tau^{(1)} = \mathbf{P}_\tau\big[\Omega^{(1)} + \Xi_\tau^{(1)}\big] = \mathbf{P}_\tau\Omega^{(1)} + m^{-1}\mathbf{P}_\tau\Xi_\tau^{(2)}.$$

Solving for $\Sigma_\tau^{(1)}$, we find that

$$\Sigma_\tau^{(1)} = \frac{(\mathbf{I}-\mathbf{P}_\tau)\Omega^{(1)}}{V_\tau} + m^{-1}\frac{(\mathbf{I}-\mathbf{P}_\tau)\Xi_\tau^{(2)}}{V_\tau}, \tag{4.15}$$

which can be substituted into (4.14) to give the conclusion.

The following proposition summarizes our progress so far.

Proposition 4.2. *The approximate potential equation* (4.4) *is equivalent to the reduced problem*

$$V_\tau^2 A_\tau\Xi_\tau^{(2)} + V_\tau B_\tau\tfrac{(\mathbf{I}-\mathbf{P}_\tau)\Omega^{(1)}}{V_\tau} + f_\tau + m^{-1}\mathbf{S}_\tau = \mathrm{e}^{h_\tau} + \mathrm{O}(m^{-\kappa-1})$$

for the functions $\Xi_\tau^{(2)}$, $\Omega^{(1)}$ and h_τ along with the condition (4.10)*. Here, f_τ is a strictly positive real-analytic function given explicitly by*

$$\begin{aligned}f_\tau = -4\partial_\tau V_\tau\mathit{\Delta}Q - 4V_\tau\big(\partial_\tau|\partial V_\tau|^2 + 2\partial_\tau V_\tau\mathit{\Delta}V_\tau\big)\\ + m^{-1}\big(4\partial_\tau V_\tau|\bar\partial V_\tau|^2 + 2\partial_\tau\mathit{\Delta}V_\tau\big),\end{aligned}$$

and $\mathbf{S}_\tau$ is a bounded affine expression in the coefficients $\Xi_\tau^{(2)}$ and $\Omega_\tau^{(1)}$, given by

$$\mathbf{S}_\tau = \mathbf{R}_\tau + B_\tau(\mathbf{I}-\mathbf{P}_\tau)\Omega^{(1)} - 2\partial_\tau V_\tau\mathit{\Delta}\Omega^{(1)}. \tag{4.16}$$

Proof. The equation (4.11) asserts that the approximate potential equation (4.4) is equivalent to the problem

$$V_\tau^2 A_\tau\Xi_\tau^{(2)} + V_\tau B_\tau\Sigma_\tau^{(1)} - 2\partial_\tau V_\tau\mathit{\Delta}\Omega + \mathbf{Q}_\tau = \mathrm{e}^{h_\tau} + \mathrm{O}(m^{-\kappa-1}), \tag{4.17}$$

provided that we subscribe to (4.10). Rewriting the left-hand side using the formula (4.13), we arrive at the identity

$$V_\tau^2 A_\tau \Xi_\tau^{(2)} + V_\tau B_\tau \Sigma_\tau^{(1)} - 2\partial_\tau V_\tau \varDelta\Omega + \mathbf{Q}_\tau = V_\tau^2 A_\tau \Xi_\tau^{(2)} + V_\tau B_\tau \Sigma_\tau^{(1)} + m^{-1}\mathbf{R}_\tau \\ - 2\partial_\tau V_\tau \varDelta\Omega + 4V_\tau\big(\partial_\tau|\partial V_\tau|^2 + 2\partial_\tau V_\tau \varDelta V_\tau\big).$$

The desired formulas now follow by inserting the expressions (4.10) and (4.15) for Ω and $\Sigma_\tau^{(1)}$ into the right-hand side of (4.17).

It only remains to prove that the function f_τ defined above is strictly positive. Since we have obtained that

$$\begin{aligned} f_\tau &= -4\partial_\tau V_\tau \varDelta Q - 4V_\tau\big(\partial_\tau|\partial V_\tau|^2 + 2\partial_\tau V_\tau \varDelta V_\tau\big) + \mathrm{O}(m^{-1}) \\ &= -4\partial_\tau V_\tau \varDelta Q + \mathrm{O}(m^{-1} + |V_\tau|), \end{aligned}$$

it suffices to show that $-4\partial_\tau V_\tau \varDelta Q > 0$. To that end, we argue that $\partial_\tau V_\tau$ is strictly negative along $\partial\mathcal{S}_\tau$. Indeed, this follows e.g. by an application of the implicit function theorem along with the expansion of the sets $\{z \in \mathbb{C}\setminus\mathcal{K} : V_\tau(z) = 0\}$ with τ. Since the boundaries $\partial\mathcal{S}_\tau$ deform according to a weighted Laplacian growth flow with weight $2\varDelta Q$, these sets expand at a speed proportional to $(\varDelta Q)^{-1}|\phi_\tau'|$, where ϕ_τ are the conformal mappings from $\mathbb{C}\setminus\mathcal{S}_\tau$ onto the exterior unit disk (see, e.g., Lemma 2.5 in [18]). The claim then follows by combining the above observation with the strict subharmonicity of Q. □

5 PROOF OF THE MAIN THEOREM

5.1 The approximate potential

We are now ready to supply the approximate solution to Problem 2.4.

Lemma 5.1. *For any fixed $\kappa \in \mathbb{Z}_{>0}$, there exist bounded harmonic functions $\{h_{j,\tau}\}_{0\le j\le\kappa}$ on $\mathbb{C}\setminus\mathcal{K}$ as well as coefficient functions $\{\Omega_j\}_{0\le j\le\kappa+2}$, $\{\Sigma_{j,\tau}\}_{0\le j\le\kappa+2}$, $\{\Lambda_{j,\tau}\}_{0\le j\le\kappa+2}$ with the properties detailed in Section 2.4 such that if U_τ is given by* (2.8)*, we have that*

$$\partial_\tau \varDelta U_\tau = \sqrt{\frac{m}{2\pi}}\mathrm{e}^{h_\tau - 2mV_\tau^2}\Big(1 + \mathrm{O}\big(m^{-\kappa-1}\big)\Big)$$

where implicit constant is uniformly bounded in $\tau \in I_0$ on any compact subset of $\mathcal{V}\setminus\mathcal{K}$. Moreover, the coefficients all belong to $\mathscr{H}^\infty_{\frac12\sigma,\frac12\delta}$, with a norm depending only on κ, Q, τ_0, and the initial parameters σ,δ and ϵ.

Proof. By Proposition 4.2, the approximate potential equation is equivalent to the semiclassical problem

$$V_\tau^2 A_\tau \Xi_\tau^{(2)} + V_\tau B_\tau \tfrac{(\mathbf{I}-\mathbf{P}_\tau)\Omega^{(1)}}{V_\tau} + f_\tau + m^{-1}\mathbf{S}_\tau = \mathrm{e}^{h_\tau} + \mathrm{O}(m^{-\kappa-1}),$$

where A_τ, B_τ, f_τ and $\mathbf{S}_\tau$ are as in the proposition. Note that A_τ and B_τ are fixed and non-vanishing elements of $\mathscr{H}^\infty_{\sigma,\delta}$, and that f_τ is strictly positive for m large enough. By rearranging the terms and putting $\mathcal{E}_\tau = -A_\tau \Xi^{(2)}_\tau$, $\mathcal{F}_\tau = H_\tau \frac{(\mathbf{I}-\mathbf{P}_\tau)\mathfrak{f}}{V_\tau}$ where $\mathfrak{f} = \Omega^{(1)}$ and $H_\tau = -B_\tau$, and finally $\mathcal{G} = \mathrm{e}^{h_\tau}$, we bring the approximate potential equation to the form

$$V_\tau^2\mathcal{E}_\tau + V_\tau\mathcal{F}_\tau + \mathcal{G}_\tau = f_\tau + m^{-1}\mathbf{T}_\tau\big(\mathcal{E}_\tau, \mathcal{F}_\tau\big),$$

where $\mathbf{T}_\tau = -\mathbf{S}_\tau$. An inspection of the definition (4.16) of $\mathbf{S}_\tau$ and the auxiliary definition (4.14) of $\mathbf{R}_\tau$ shows that $\mathbf{T}_\tau$ satisfies the required Lipschitz-bound (3.11). Hence the claim follows by an application of Lemma 3.4. This completes the proof. □

5.2 The single wave potential

We let $\Lambda_{j,\tau}$, Ω_j, $\Sigma_{j,\tau}$ and $h_{j,\tau}$ be the coefficients produced by applying Lemma 5.1, and put as before

$$\Lambda_\tau = \sum_{j=0}^{\kappa+2} m^{-j}\Lambda_{j,\tau}, \qquad \Omega = \sum_{j=0}^{\kappa+2} m^{-j}\Omega_j,$$

$$\Sigma_\tau = \sum_{j=0}^{\kappa+2} m^{-j}\Sigma_{j,\tau}, \qquad h_\tau = \sum_{j=0}^{\kappa} m^{-j}h_{j,\tau}.$$

Recall that $\mathcal{K}$ and $\mathcal{V}$ denote compact and open sets, respectively, subject to the inclusions

$$\mathcal{S}_{\tau_0-2\epsilon} \subset \mathcal{K} \subset \mathcal{S}^\circ_{\tau_0-\epsilon} \subset \mathcal{S}_{\tau_0+\epsilon} \subset \mathcal{V} \subset \mathcal{S}_{\tau_0+2\epsilon}.$$

We denote by $\chi(z)$ and $\chi'(z)$ two cut-off functions with the following properties: We ask that χ vanishes identically on $\mathcal{K}$ while it equals 1 on $\mathcal{S}_{\tau_0-\epsilon}$. The function χ' instead vanishes on $\mathcal{V}^c$, while it equals 1 on $\mathcal{S}_{\tau_0+\epsilon}$.

Lemma 5.2. *The n-th orthogonal polynomial in $L^2(\mathbb{C}, \mathrm{e}^{-2mQ})$ satisfies*

$$|P_{m,n}(z)|^2\mathrm{e}^{-2mQ(z)} = \chi(z)\sqrt{\frac{m}{2\pi}}\mathrm{e}^{h_\tau(z)-2mV_\tau^2(z)} + \mathrm{O}\big(m^{-\kappa-1}\big)$$

as $n = \tau m \to \infty$ with $\tau \in I_0$.

Proof sketch. The proof is based on introducing the truncated potential

$$\mathcal{U}_\tau = \chi\Lambda_\tau \operatorname{erf}\big(2\sqrt{m}V_\tau\big) + \chi\chi'\Omega\operatorname{erf}\big(-2\sqrt{m}V_\tau\big) + \chi\chi'\frac{\Sigma_\tau}{\sqrt{2\pi m}}\,\mathrm{e}^{-2mV_\tau^2}\,. \tag{5.1}$$

Thanks to the truncation it is readily checked that the potential satisfies

$$\partial_\tau \Delta\mathcal{U}_\tau(z) = \chi(z)\sqrt{\frac{m}{2\pi}}\mathrm{e}^{h_\tau(z)-2mV_\tau^2(z)}\big(1+\mathrm{O}(m^{-\kappa-1})\big) \tag{5.2}$$

throughout the plane. This means that $\partial_\tau\mathcal{U}_\tau$ is an approximate potential for a single wave as treated in [19]. Specifically, the claim follows by repeating verbatim the proof of Theorem 8.2 in [19], but with a less precise error term. We skip the necessary details. $\square$

5.3 The Euler MacLaurin formula

We need means to convert integration in τ into summation over the orthogonal polynomials. At hand we have the Euler-MacLaurin formula, which we need in the following form. For the formulation, we use the notation $[f(t)]_A^B = f(B) - f(A)$ and denote by $(B_k)_{k\geq 0}$ the Bernoulli numbers.

Proposition 5.3. *Assume that $F(z,\tau) \in \mathscr{H}^\infty_{\sigma',\delta'}$ for some $\frac{1}{2}\sigma \leq \sigma' \leq \sigma$ and $\frac{1}{2}\sigma \leq \delta' \leq \delta$. Then, for any fixed $\kappa \in \mathbb{Z}_{>0}$, we have*

$$\sum_{j=n_0}^{n_1-1} F(z,j/m)\mathrm{e}^{-2mV^2_{j/m}(z)}$$
$$= m\int_{n_0/m}^{n_1/m} F(z,\tau)\mathrm{e}^{-2mV_\tau^2(z)}\mathrm{d}\tau - \frac{1}{2}\big[F(z,\tau)\mathrm{e}^{-2mV_\tau^2(z)}\big]_{\tau=n_0/m}^{n_1/m}$$
$$+\sum_{j=1}^{\kappa} m^{-2j+1}\frac{B_{2j}}{(2j)!}\Big[\partial_\tau^{2j-1}\Big(F(z,\tau)\mathrm{e}^{-2mV_\tau^2(z)}\Big)\Big]_{\tau=n_0/m}^{n_1/m} + \mathrm{O}(m^{-\kappa+\frac{1}{2}}).$$

The implicit constant is uniform as $m \to +\infty$ provided that n_1/m and m_2/m are confined to compact subsets of I_0 and $z \in \phi_{\tau_0}^{-1}(\mathbb{A}(\frac{1}{2}\sigma))$. Moreover, the expansion on the right-hand side belongs to $\mathscr{H}^\infty_{\sigma',\delta''}$ for any $\delta'' < \delta'$.

Proof. The classical Euler-MacLaurin formula asserts that

$$\sum_{n=n_0}^{n_1-1} f(n) = \int_{n_0}^{n_1} f(t)\mathrm{d}t - \frac{1}{2}[f(t)]_{n_0}^{n_1} + \sum_{k=1}^{\kappa}\frac{B_{2k}}{(2k)!}\big[f^{(2k-1)}(t)\big]_{n_0}^{n_1} + R_{\kappa,n_1,n_2}(f),$$

where the error enjoys the bound

$$|R_{\kappa,n_1,n_2}(f)| \leq C(n_2-n_1)\|\partial^{2\kappa+1}f\|_\infty$$

for some constant C depending only on κ. Applying it to our situation, we obtain

$$\sum_{j=n_0}^{n_1-1} F(j/m,z)\mathrm{e}^{-2mV^2_{j/m}(z)}$$
$$= \int_{n_0}^{n_1} F(s/m,z)\mathrm{e}^{-2mV^2_{s/m}(z)}\mathrm{d}s - \frac{1}{2}\big[F\big(\tfrac{s}{m},z\big)\mathrm{e}^{-2mV^2_{s/m}(z)}\big]_{n_0}^{n_1}$$
$$+\sum_{j=1}^{\kappa}\frac{B_{2j}}{(2j)!}\Big[\partial_s^{2j-1}\Big(F(z,\tfrac{s}{m})\mathrm{e}^{-2mV^2_{s/m}(z)}\Big)\Big]_{n_0}^{n_1}$$
$$+\mathrm{O}\Big(n\Big\|\partial_s^{2\kappa+1}\Big(F(z,\tfrac{s}{m})\mathrm{e}^{-2mV^2_{s/m}(z)}\Big)\Big\|_{L^\infty(\mathbb{C}\times I_0)}\Big),$$

where we again recall the notation $[f(s)]_A^B = f(B)-f(A)$. By the change of variables $\tau = s/m$ we have that

$$\int_{n_0}^{n_1} F(s/m,z)\mathrm{e}^{-2mV_{s/m}^2(z)}\mathrm{d}s = m\int_{n_0/m}^{n_1/m} F(\tau,z)\mathrm{e}^{-2mV_\tau^2(z)}\mathrm{d}\tau,$$

and likewise that

$$\sum_{j=1}^{\kappa}\frac{B_{2j}}{(2j)!}\Big[\partial_s^{2j-1}\Big(F(z,\tfrac{s}{m})\mathrm{e}^{-2mV_{s/m}^2(z)}\Big)\Big]_{n_0}^{n_1} = \sum_{j=1}^{\kappa} m^{-2j+1}\frac{B_{2j}}{(2j)!}\Big[\partial_t^{2j-1}\Big(F(z,t)\mathrm{e}^{-2mV_t^2(z)}\Big)\Big]_{t=n_0/m}^{n_1/m}.$$

Hence, it only remains to prove that the O-term is in fact of the order claimed in the lemma, which amounts to

$$\Big|\partial_s^{2\kappa+1}\Big(F(z,s/m)\mathrm{e}^{-2mV_{s/m}^2(z)}\Big)\Big| = \mathrm{O}(m^{-\kappa-\frac{1}{2}}).$$

Suppressing some (for now) unimportant details, we realize that, for some sequence $(C_{\kappa,j})_{j=0}^{2\kappa+1}$ of smooth bounded coefficient functions, it holds that

$$\partial_s^{2\kappa+1}\Big(F(z,s/m)\mathrm{e}^{-2mV_{s/m}^2(z)}\Big) = \sum_{j=1}^{2\kappa-1} C_{\kappa,j}(z)m^{-2\kappa+j}\big(V_\tau(z)\big)^j\mathrm{e}^{-2mV_\tau^2(z)}\Big|_{\tau=s/m}.$$

Moreover, it is a standard calculus exercise to show that

$$\sup_{x\in\mathbb{R}}|x|^j\mathrm{e}^{-2mx^2} = \frac{1}{2}j^{\frac{j}{2}}\mathrm{e}^{-2j}m^{-\frac{j}{2}} = \mathrm{O}(m^{-\frac{j}{2}}).$$

Applying this with $x = V_\tau(z)$ we obtain the estimate

$$\Big|\partial_s^{2\kappa+1}\Big(F(\cdot,s/m)\mathrm{e}^{-2mV_{s/m}^2}\Big)\Big| = \mathrm{O}\Big(\sum_{j=0}^{2\kappa-1} m^{-2\kappa+j}m^{-j/2}\Big) = \mathrm{O}(m^{-\kappa-\frac{1}{2}})$$

which completes the proof. □

5.4 Global asymptotics of the polynomial kernel

We recall the truncated potential $\mathcal{U}_\tau$ defined above in (5.1). Theorem 1.2 is now an almost immediate corollary of Lemma 5.2 and the potential equation (5.2) for $\mathcal{U}_\tau$ and h_τ. Before we proceed, we recall that by (2.12), we have

$$\varDelta U_\tau = \varDelta\Omega\,\mathrm{erf}\left(-2\sqrt{m}V_\tau\right) + \sqrt{\frac{m}{2\pi}}\Big(mX_{0,\tau} + X_{1,\tau} + m^{-1}X_{1,\tau}\Big)\,\mathrm{e}^{-2mV_\tau^2}\,.$$

where $X_{j,\tau}$ are given by (2.13). As the X_j-s still depend on m, they are *not* the terms in a bona fide asymptotic expansion. To obtain such an expansion, we rewrite the equation (2.12) as

$$\varDelta U_\tau = \varDelta\Omega\ \mathrm{erf}\left(2\sqrt{m}V_\tau\right) + \sqrt{\frac{m}{2\pi}}\sum_{j=0}^{\kappa} m^{-j}\mathcal{X}_{j,\tau}\,\mathrm{e}^{-2mV_\tau^2} \tag{5.3}$$

where the coefficients in the asymptotic expansion are given by

$$\begin{cases}\mathcal{X}_{0,\tau} = -8\Xi_{0,\tau}V_\tau|\bar\partial V_\tau|^2 \\ \mathcal{X}_{1,\tau} = -8\Xi_{1,\tau}V_\tau|\bar\partial V_\tau|^2 + 4\Sigma_{0,\tau}|\bar\partial V_\tau|^2 + 4\mathrm{Re}\,\{\partial\Xi_{0,\tau}\bar\partial V_\tau\} + 2\Xi_{0,\tau}\mathit{\Delta} V_\tau \\ \mathcal{X}_{j,\tau} = -8\Xi_{j,\tau}V_\tau|\bar\partial V_\tau|^2 + 4\Sigma_{j-1,\tau}|\bar\partial V_\tau|^2 + 4\mathrm{Re}\,\{\partial\Xi_{j-1,\tau}\bar\partial V_\tau\} + 2\Xi_{j-1,\tau}\mathit{\Delta} V_\tau + \mathit{\Delta}\Sigma_{j-2,\tau},\end{cases}$$

the last line referring to indices $j = 2, \ldots, \kappa$.

Proof of Theorem 1.2. We introduce the density

$$\rho_{m,n,k}(z) = \frac{1}{m}\sum_{j=k}^{n-1}|P_{m,j}(z)|^2\mathrm{e}^{-2mQ(z)}$$

where $k = \tau' m$ and $n = \tau m$ are such that

$$\tau_0 - \epsilon < \tau' < \tau < \tau_0 + \epsilon.$$

For integers $j = tm$, $t \in I_0$, we have by Lemma 5.2 that

$$|P_{m,j}(z)|^2\mathrm{e}^{-2mQ(z)} = \chi\sqrt{\frac{m}{2\pi}}\mathrm{e}^{h_t(z)-2mV_t^2(z)} + \mathrm{O}\big(m^{-\kappa-1}\big).$$

Inserting this asymptotic formula into the above definition of $\rho_{m,n,k}$, we obtain

$$\rho_{m,n,k}(z) = \frac{1}{m}\sum_{j=k}^{n-1}\chi\sqrt{\frac{m}{2\pi}}\mathrm{e}^{h_{j/m}(z)-2mV_{j/m}^2(z)} + \mathrm{O}\big(m^{-\kappa-\frac{1}{2}}\big).$$

The right-hand side is in a form suitable for the application of the Euler-MacLaurin summation formula. Specifically, by applying Proposition 5.3 we find that

$$\begin{aligned}\rho_{m,n,k}(z) = \int_{\tau'}^{\tau}\chi\sqrt{\frac{m}{2\pi}}\mathrm{e}^{h_t(z)-2mV_t^2(z)}\mathrm{d}t + \frac{\chi}{\sqrt{8\pi m}}\big[\mathrm{e}^{h_t(z)-2mV_t^2(z)}\big]_{t=\tau'}^{\tau} \\ + \frac{1}{\sqrt{2\pi m}}\sum_{j=1}^{\kappa} m^{-2j+1}\frac{B_{2j}}{(2j)!}\Big[\partial_t^{2j-1}\mathrm{e}^{h_t(z)-2mV_t^2(z)}\Big]_{t=\tau'}^{\tau} + \mathrm{O}\big(m^{-\kappa-\frac{1}{2}}\big),\end{aligned}$$

which when combined with the potential equation (5.2) gives the representation

$$\begin{aligned}\rho_{m,n,k}(z) &= \int_{\tau'}^{\tau}\partial_t\mathit{\Delta}\mathcal{U}_t\mathrm{d}t + \frac{\chi}{\sqrt{8\pi m}}\big[\mathrm{e}^{h_t(z)-2mV_t^2(z)}\big]_{t=\tau'}^{\tau} \\ &\qquad + \frac{\chi}{\sqrt{2\pi m}}\sum_{j=1}^{\kappa}\frac{B_{2j}}{(2j)!}m^{-2j+1}\Big[\partial_t^{2j-1}\mathrm{e}^{h_t-2mV_t^2(z)}\Big]_{t=\tau'}^{\tau} + \mathrm{O}\big(m^{-\kappa-\frac{1}{2}}\big) \\ &= \mathit{\Delta}\int_{\tau'}^{\tau}\partial_t\mathcal{U}_t\mathrm{d}t + \frac{\chi}{\sqrt{8\pi m}}\big[\mathrm{e}^{h_t(z)-2mV_t^2(z)}\big]_{t=\tau'}^{\tau} \\ &\qquad + \frac{\chi}{\sqrt{2\pi m}}\sum_{j=1}^{\kappa} m^{-2j+1}\frac{B_{2j}}{(2j)!}\Big[\partial_t^{2j-1}\mathrm{e}^{h_t-2mV_t^2(z)}\Big]_{t=\tau'}^{\tau} + \mathrm{O}\big(m^{-\kappa-\frac{1}{2}}\big).\end{aligned}$$

An application of the fundamental theorem of calculus then shows that

$$\begin{aligned}\rho_{m,n,k}(z) &= \mathit{\Delta}\mathcal{U}_\tau - \mathit{\Delta}\mathcal{U}_{\tau'} + \frac{\chi}{\sqrt{8\pi m}}\Big[\mathrm{e}^{h_t(z)-2mV_t^2(z)}\Big]_{t=\tau'}^{\tau} \\ &\quad + \frac{\chi}{\sqrt{2\pi m}}\sum_{j=1}^{\kappa} m^{-2j+1}\frac{B_{2j}}{(2j)!}\Big[\partial_t^{2j-1}\mathrm{e}^{h_t-2mV_t^2(z)}\Big]_{t=\tau'}^{\tau} + \mathrm{O}\big(m^{-\kappa-\frac12}\big).\end{aligned} \tag{5.4}$$

Next, we observe that wherever the cut-off functions χ and χ' both equal to 1, it holds that $\mathcal{U}_\tau = U_\tau$. In particular this is true on the set $\mathcal{V}\setminus\mathcal{K}$. Recall that the classical Bernstein-Walsh lemma implies that

$$\rho_{m,k}(z) \le C\mathrm{e}^{-2m(Q-\check{Q}_{\tau'})(z)}, \qquad \tau' = \frac{k}{n}. \tag{5.5}$$

By writing $\rho_{m,n,k} = \rho_{m,n} - \rho_{m,k}$ and combining (5.4) with (5.5) and (5.3), we obtain the asymptotic representation

$$\begin{aligned}\rho_{m,n}(z) &= \mathit{\Delta}\,\Omega\,\mathrm{erf}\big(2\sqrt{m}V_\tau(z)\big) + \sum_{j=0}^{\kappa} m^{-j}\mathcal{X}_{j-2,\tau} + \frac{\chi}{\sqrt{8\pi m}}\mathrm{e}^{h_\tau(z)-2mV_\tau^2(z)} \\ &\quad + \frac{\chi}{\sqrt{2\pi m}}\sum_{j=1}^{\kappa} m^{-2j+1}\frac{B_{2j}}{(2j)!}\partial_\tau^{2j-1}\mathrm{e}^{h_\tau-2mV_\tau^2(z)} + \mathrm{O}\big(m^{-\kappa-\frac12}\big),\end{aligned} \tag{5.6}$$

as $n = \tau m \to \infty$, where the error term is uniformly bounded provided that $\tau \in I_0$ and $z \in \mathbb{C}$.

It remains to relate the expansion (5.6) to the full Bergman density ρ_m. To that end, observe that on fixed compact subsets of $\mathcal{S}_\tau^\circ$, the polynomial truncation of the Bergman density has essentially no effect. More precisely, using Hörmander's estimate for the $\bar\partial$-operator, it is easy see that the asymptotic expansions of $\rho_{m,n}(z)$ and $\rho_m(z)$ agree on compact subsets of $\mathcal{S}_\tau$. In particular for $z \in \mathcal{S}_{\tau_0-\epsilon}\setminus\mathcal{K}$ we have

$$\rho_m(z) = \rho_{m,n}(z) + \mathrm{O}(m^{-\kappa-1}) = \mathit{\Delta}\Omega(z) + \mathrm{O}(m^{-\kappa-1}),$$

which implies that

$$\rho_m(z) = \mathit{\Delta}\Omega + \mathrm{O}(m^{-\kappa-1}), \qquad z \in \mathcal{S}_{\tau_0-\epsilon}\setminus\mathcal{K}.$$

As $\rho_m(z)$ admits a full asymptotic expansion throughout the plane with real-analytic coefficients, we find that this must hold for $\mathit{\Delta}\Omega(z)$ as well, and the coefficients must agree. The theorem thus follows by calculating the ∂_τ-derivatives in (5.6) using Lebniz' formula. This completes the proof. □

Remark 5.4. (a) While initially designed as a method for dealing with boundary effects, the technique presented here gives a genuinely new algorithm for computing the bulk asymptotics at a point z, at least under the given conditions on the spectral boundaries for τ near $\tau(z)$. Note however that in contrast to e.g. [8], our method is specific to one complex variable and requires non-local assumptions.

(b) In the above proof, we found in the last step that $\varDelta\Omega$ equals ρ_m up to an error of order $\mathrm{O}(m^{-\kappa-1})$. It should be possible to implement this at an earlier stage, thus bypassing some steps in the main algorithm. This minor shortcut may be useful for computational purposes in the upcoming analysis of the free energy for determinantal planar Coulomb gas ensembles.

6 PLANAR SZEGŐ LIMIT THEOREMS AND STATISTICAL MECHANICS

Recall that for an absolutely continuous measure $\mu = \rho\,\mathrm{ds}$ on the unit circle $\mathbb{T}$ with $\log\rho \in L^1(\mathbb{T}, \mathrm{ds})$, the classical Szegő limit theorems descibe the asymptotic behavior of the associated *Toeplitz determinants* $\det T_m(\rho)$, where

$$T_m(\rho) = \Big(\int_{\mathbb{T}} t^{j-k}\rho(t)\,\mathrm{ds}(t)\Big)_{0\le j,k\le m-1}.$$

Under the additional assumption that ρ' is Hölder continuous, Szegő proved in his *strong limit theorem* that

$$\det T_m(\rho) = \exp\Big(m\int_{\mathbb{T}}\log\rho\,\mathrm{ds} + E(\rho) + \mathrm{o}(1)\Big)$$

as $m \to +\infty$, where $E(\rho)$ is the squared $H^{1/2}(\mathbb{T})$-norm of $\log\rho$.

For a potential Q as above, we consider instead the Gram matrices of complex moments

$$G_{m,n}(Q) = \Big(\int_{\mathbb{C}} z^j\bar{z}^k \mathrm{e}^{-2mQ(z)}\mathrm{dA}\Big)_{0\le j,k\le n-1}$$

and the large m-asymptotics of the Gram determinants $\det G_{m,n}(Q)$. If Q_λ is a smoothly varying family of potentials and $\rho^\lambda_{m,n}$ denotes the associated polynomial densities, we have the remarkable variational identity

$$\partial_\lambda \log\det G_{m,n}(Q_\lambda) = -2m^2\int_{\mathbb{C}}\partial_\lambda Q_\lambda(z)\rho^\lambda_{m,n}(z)\mathrm{dA}(z), \tag{6.1}$$

(see the proof of Theorem 1.4 below). This connects the Gram determinants with the polynomial Bergman densities, and we may apply Theorem 1.2 to extract asymptotic information. More specifically, if we can find a smooth deformation of a given potential Q into the radial potential $Q_0(z) = \frac{1}{2}|z|^2$, we may apply Theorem 1.2 to each $\rho^\lambda_{m,n}$ and then integrate in λ to obtain a the strong Szegő limit theorem for Q.

Already at this stage, it follows without much effort that the right-hand side of (6.1) admits a full asymptotic expansion in negative powers of m. Hence, for *some* coefficients $A_{j,\tau} = A_{j,\tau}(Q)$, this gives

$$\log\det G_{m,n}(Q_1) - \log\det G_{m,n}(Q_\lambda) = \sum_{j\ge -2} A_{j,\tau}m^{-j} + \mathrm{O}(m^{-\kappa-1}). \tag{6.2}$$

However, making this representation explicit involves non-trivial challenges. We hope to carry out this program in future work.

Before we proceed, we recall an interpretation of $\det G_{m,n}(Q)$ in terms of statistical mechanics. For parameters m, n and Q as above, we consider the random n-point process with joint density on $\mathbb{C}^n$ given by

$$f_{m,n}(z_1, \ldots, z_n) = \frac{1}{Z_{m,n}} \prod_{1\le j<k\le n} |z_j - z_k|^2 \mathrm{e}^{-2m\sum_{j=1}^n Q(z_j)}.$$

Here, $Z_{m,n} = Z_{m,n}(Q)$ is the normalizing constant given by

$$Z_{m,n} = \int_{\mathbb{C}^n} \prod_{1\le j<k\le n} |z_j - z_k|^2 \mathrm{e}^{-2m\sum_{j=1}^n Q(z_j)} \mathrm{dA}(z_1)\cdots \mathrm{dA}(z_n).$$

This point process has the interpretation of n charged particles in the plane at a particular "inverse temperature", interacting with Coulomb's law (in 2D) and confined by the external field mQ. The connection with Gram determinants is the well-known identity

$$Z_{m,n}(Q) = n! \det G_{m,n}(Q).$$

The asymptotic expansion (6.2) thus states that, disregarding the combinatorial constant $n!$, the free energy $\log Z_{m,n}$ admits a full asymptotic expansion, starting with terms of order m^2. While the literature on partition functions is too extensive to review here, we mention that asymptotic expansions of $\log Z_{m,n}(Q)$ are known in explicit form for radially symmetric potentials up to and including the $\mathrm{O}(1)$-term. This includes cases when the droplet $\mathcal{S}_\tau$ has non-trivial topology [1] as well as hard-wall constraints [2]. In these cases, the expansions exhibit different behavior including an oscillatory term in the former case, and terms of non-integer order in the latter. For regular but non-radially symmetric potentials with regular simply connected droplet, the first two terms in the Gram determinant expansion are known (see, e.g., [7, 21] which pertain to general β), and they coincide with the weighted logarithmic energy and the entropy of the equilibrium measure μ_Q, respectively. The full expansion was predicted in [27], including an explicit formula for the very interesting constant term in terms of ζ-regularized determinants of a certain Laplace-Beltrami operator associated to Q. The expansion up to the constant term was recently obtained in [10] for the non-radial potential $Q(z) = \frac{1}{2}|z|^2 - c\log|z-a|$, using connections to non-Hermitian orthogonality established in [6].

6.1 Gram determinant asymptotics for Hele-Shaw potentials

We conclude this paper with a proof of Theorem 1.4, which supplies a simple expansion formula for Gram determinants associated to *Hele-Shaw potentials* of the form

$$Q(z) = Q_\mu(z) = \frac{1}{2}|z|^2 - U^\mu(z),$$

where μ is a finite (possibly signed) measure. For this class, a canonical deformation chain is given by

$$Q^\lambda(z) = \frac{1}{2}|z|^2 - \lambda U^\mu(z)$$

for $0 \le \lambda \le 1$. We will denote objects pertaining to the potential Q^λ with the upper subscript λ, so thath e.g. the harmonic coefficient functions from Section 2.4 corresponding to Q^λ are denoted by $\Lambda^\lambda_{j,\tau}$.

Recall that we assume that Q^λ remains admissible and $\mathcal{S}^\lambda_\tau$ remains bounded away from $\mathrm{supp}(\mu)$ throughout the deformation chain. Theorem 1.4, then asserts that, for any fixed accuracy parameter $\kappa \in \mathbb{Z}_{>0}$, we have the asymptotics

$$\log\det G_{m,n}(Q) = \log\det G_{m,n}(\tfrac{1}{2}|z|^2) + 2m^2 \int_0^1 \int_{\mathbb{C}} \Upsilon^\lambda_\tau(z) \mathrm{d}\mu(z) \mathrm{d}\lambda + \mathrm{O}(m^{-\kappa-1})$$

as $n = \tau m \to +\infty$, where Υ^λ_τ is defined by (1.8).

Sketch of proof of Theorem 1.4. We first prove the variational formula (6.1). To that end, it is more convenient to use the Coulomb gas formalism. The logarithmic derivative of $G_{m,n}(Q^\lambda)$ coincides with that of $Z^\lambda_{m,n}$. Computing the latter, we find that

$$\partial_\lambda \log Z^\lambda_{m,n} = \frac{\partial_\lambda Z^\lambda_{m,n}}{Z^\lambda_{m,n}} = -2m \sum_{j=1}^n \int_{\mathbb{C}^n} \partial_\lambda Q^\lambda(z_j) f_{m,n}(z_1, \ldots, z_n) \mathrm{dA}(z_1) \cdots \mathrm{dA}(z_n).$$

We recognize the right-hand side as $-2m$ times the expected value of the linear statistic

$$\sum_{j=1}^n \partial_\lambda Q^\lambda(z_j),$$

which by the interpretation of $n\rho_{m,n}$ as the one-point function for the Coulomb gas ensemble is given by

$$\mathbb{E}\Big[\sum_{j=1}^n \partial_\lambda Q^\lambda(z_j)\Big] = m \int_{\mathbb{C}} \partial_\lambda Q^\lambda(z)\, \rho^\lambda_{m,n}(z)\, \mathrm{dA}(z).$$

The first claim follows.

We turn to the claim of the theorem. The simple structure of the deformation chain shows that $\partial_\lambda Q^\lambda = -U^\mu$. We combine this with (6.1) to obtain

$$\begin{aligned}\partial_\lambda \log G^\lambda_{m,n} &= 2m^2 \int_{\mathbb{C}} U^\mu(z)\, \rho^\lambda_{m,n}(z)\, \mathrm{dA}(z) \\ &= 2m^2 \int_{\mathbb{C}} \Big(\int_{\mathbb{C}} \log|z-w|\, \rho^\lambda_{m,n}(z)\, \mathrm{dA}(z) \Big) \mathrm{d}\mu(w),\end{aligned}$$

where the change in the order of integration is justified by the regularity of $\rho^\lambda_{m,n}$. Integrating over λ, we obtain the representation

$$\log G_{m,n}(Q) = \log G_{m,n}(Q^0) + 2m^2 \int_0^1 \int_{\mathbb{C}} U^{\rho^\lambda_{m,n}}(w)\, \mathrm{d}\mu(w)\, \mathrm{d}\lambda, \tag{6.3}$$

where we tacitly identify $\rho^\lambda_{m,n}$ with the weighted area measure $\rho^\lambda_{m,n}\mathrm{dA}$. It only remains to identify the potential on the right-hand side. Let us define

$$V^\lambda_\tau = U^\lambda_\tau + \frac{1}{2m}\partial_\tau U^\lambda_\tau + \sum_{\ell=1}^{\lceil\frac{\kappa}{2}\rceil} m^{-2\kappa}\frac{B_{2\ell}}{(2\ell)!}\partial_\tau^{2\ell}U^\ell_\tau.$$

It is then readily verified that ΔV^λ_τ coincides up to order $\mathrm{O}(m^{-\kappa-\frac{1}{2}})$ with the right-hand side of the asymptotic expansion of $\rho^\lambda_{m,n}$, obtained by applying in (5.6) to $Q = Q^\lambda$. In other words,

$$\Delta V^\lambda_\tau = \rho^\lambda_{n,m} + \mathrm{O}(m^{-\kappa-\frac{1}{2}}).$$

Moreover, on closed subsets of $\mathcal{S}^c_\tau$ we have that

$$V^\lambda_\tau = \Upsilon^\lambda_\tau + \mathcal{C}^\lambda_\tau, \tag{6.4}$$

where Υ^λ_τ and $\mathcal{C}^\lambda_\tau$ were defined in connection the statement of the theorem. It follows from the definition of $\mathcal{C}^\lambda_\tau$ that the logarithmic potential of $\rho^\lambda_{m,n}$ is given by

$$U^{\rho^\lambda_{m,n}} = V^\lambda_\tau - \mathcal{C}^\lambda_\tau + \mathrm{O}(m^{-\kappa-\frac{1}{2}}). \tag{6.5}$$

The claim then follows by combining (6.3), (6.4) and (6.5) with the assumption that μ is supported away from $\mathcal{S}^\lambda_\tau$. □

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REFERENCES

[1] Y. Ameur, C. Charlier, and J. Cronvall, *The two-dimensional coulomb gas: fluctuations through a spectral gap*, arXiv:2210.13959 (2022).
[2] Y. Ameur, C. Charlier, J. Cronvall, and J. Lenells, *Exponential moments for disk counting statistics at the hard edge of random normal matrices*, J. Spectr. Theory **13** (2023), no. 3, 841–902. MR4670346
[3] Y. Ameur and J. Cronvall, *Szego type asymptotics for the reproducing kernel in spaces of full-plane weighted polynomials*, Comm. Math. Phys. **48** (2022), 1–58.
[4] Y. Ameur, H. Hedenmalm, and N. Makarov, *Berezin transform in polynomial Bergman spaces*, Comm. Pure Appl. Math. **63** (2010), no. 12, 1533–1584. MR2742007
[5] ———, *Fluctuations of eigenvalues of random normal matrices*, Duke Math. J. **159** (2011), no. 1, 31–81. MR2817648
[6] F. Balogh, M. Bertola, S.-Y. Lee, and K. D. T-R McLaughlin, *Strong asymptotics of the orthogonal polynomials with respect to a measure supported on the plane*, Comm. Pure Appl. Math. **68** (2015), no. 1, 112–172. MR3280250

[7] R. Bauerschmidt, P. Bourgade, M. Nikula, and H.-T. Yau, *The two-dimensional Coulomb plasma: quasi-free approximation and central limit theorem*, Adv. Theor. Math. Phys. **23** (2019), no. 4, 841–1002. MR4063572

[8] R. Berman, B. Berndtsson, and J. Sjöstrand, *A direct approach to Bergman kernel asymptotics for positive line bundles*, Ark. Mat. **46** (2008), no. 2, 197–217. MR2430724

[9] R. J. Berman, *Determinantal point processes and fermions on polarized complex manifolds: bulk universality*, Algebraic and analytic microlocal analysis, 2018. MR3903320

[10] S. Byun and M. Yang, *Free energy expansions of a conditional GinUE and large deviations for the smallest eigenvalue of the LUE*, arXiv:2402.18983 (2024).

[11] D. Catlin, *The Bergman kernel and a theorem of Tian*, Analysis and geometry in several complex variables (Katata, 1997), 1999, pp. 1–23. MR1699887

[12] L. Charles, *Analytic Berezin-Toeplitz operators*, Math. Z. **299** (2021), no. 1-2, 1015–1035. MR4311628

[13] A. S. Fokas, A. R. Its, and A. V. Kitaev, *The isomonodromy approach to matrix models in 2d quantum gravity*, Comm. Math. Phys. **147** (1992), no. 2, 395–430. MR1174420

[14] H. Hedenmalm, *Soft Riemann-Hilbert problems and planar orthogonal polynomials*, Comm. Pure Appl. Math. (2023).

[15] H. Hedenmalm and N. Makarov, *Coulomb gas ensembles and Laplacian growth*, Proc. Lond. Math. Soc. (3) **106** (2013), no. 4, 859–907. MR3056295

[16] H. Hedenmalm and S. Shimorin, *Hele-Shaw flow on hyperbolic surfaces*, J. Math. Pures Appl. (9) **81** (2002), no. 3, 187–222. MR1894061

[17] H. Hedenmalm and A. Wennman, *Off-spectral analysis of Bergman kernels*, Comm. Math. Phys. **373** (2020), no. 3, 1049–1083. MR4061405

[18] ———, *Planar orthogonal polynomials and boundary universality in the random normal matrix model*, Acta Math. **227** (2021), no. 2, 309–406. MR4366416

[19] ———, *Berezin density and planar orthogonal polynomials*, Trans. Amer. Math. Soc. (to appear) (2024).

[20] A. R. Its and L. A. Takhtajan, *Normal matrix models, dbar-problem, and orthogonal polynomials on the complex plane*, arXiv:0708.3867 (2007).

[21] T. Leblé and S. Serfaty, *Fluctuations of two dimensional Coulomb gases*, Geom. Funct. Anal. **28** (2018), no. 2, 443–508. MR3788208

[22] L. Nirenberg, *An abstract form of the nonlinear Cauchy-Kowalewski theorem*, J. Differential Geometry **6** (1972), 561–576. MR322321

[23] T. Nishida, *A note on a theorem of Nirenberg*, J. Differential Geometry **12** (1977), no. 4, 629–633. MR512931

[24] J. Ross and M. Singer, *Asymptotics of partial density functions for divisors*, J. Geom. Anal. **27** (2017), no. 3, 1803–1854. MR3667411

[25] O. Rouby, J. Sjöstrand, and S. V. Ngọc, *Analytic Bergman operators in the semiclassical limit*, Duke Math. J. **169** (2020), no. 16, 3033–3097. MR4167085

[26] G. Tian, *On a set of polarized Kähler metrics on algebraic manifolds*, J. Differential Geom. **32** (1990), no. 1, 99–130. MR1064867

[27] A. Zabrodin and Paul. Wiegmann, *Large-N expansion for the 2D Dyson gas*, J. Phys. A **39** (2006), no. 28, 8933–8963. MR2240466

[28] S. Zelditch, *Szegő kernels and a theorem of Tian*, Internat. Math. Res. Notices **6** (1998), 317–331. MR1616718

[29] ———, *Interfaces in spectral asymptotics and nodal sets*, Frontiers in analysis and probability—in the spirit of the Strasbourg-Zürich meetings, 2020, pp. 383–449.

[30] S. Zelditch and P. Zhou, *Central limit theorem for spectral partial Bergman kernels*, Geom. Topol. **23** (2019), no. 4, 1961–2004. MR3981005

[31] ———, *Interface asymptotics of partial Bergman kernels on S^1-symmetric Kähler manifolds*, J. Symplectic Geom. **17** (2019), no. 3, 793–856. MR4022214

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