RESEARCH ARTICLE

Soft Riemann-Hilbert problems and planar orthogonal polynomials

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Abstract

Riemann-Hilbert problems are jump problems for holomorphic functions along given interfaces. They arise in various contexts, for example, in the asymptotic study of certain nonlinear partial differential equations and in the asymptotic analysis of orthogonal polynomials. Matrix-valued Riemann-Hilbert problems were considered by Deift et al. in the 1990s with a noncommutative adaptation of the steepest descent method. For orthogonal polynomials on the line or on the circle with respect to exponentially varying weights, this led to a strong asymptotic expansion in the given parameters. For orthogonal polynomials with respect to exponentially varying weights in the plane, the corresponding asymptotics was obtained by Hedenmalm and Wennman (2017), based on the technically involved construction of an invariant foliation for the orthogonality. Planar orthogonal polynomials are characterized in terms of a certain matrix $\bar{\partial}$-problem (Its, Takhtajan), which we refer to as a *soft Riemann-Hilbert problem*. Here, we use this perspective to offer a simplified approach based not on foliations but instead on the ad hoc insertion of an algebraic ansatz for the Cauchy potential in the soft Riemann-Hilbert problem. This allows the problem to decompose into a hierarchy of scalar Riemann-Hilbert problems along the interface (the free boundary for a related obstacle problem). Inspired by microlocal analysis, the method allows for

control of the solution in such a way that for real-analytic weights, the asymptotics holds in the L^2 sense with error $O(e^{-\delta\sqrt{m}})$ in a fixed neighborhood of the closed exterior of the interface, for some constant $\delta > 0$, where $m \to +\infty$. Here, m is the degree of the polynomial, and in terms of pointwise asymptotics, the expansion dominates the error term in the exterior domain and across the interface (by a distance proportional to $m^{-\frac{1}{4}}$). In particular, the zeros of the orthogonal polynomial are located in the interior of the spectral droplet, away from the droplet boundary by a distance at least proportional to $m^{-\frac{1}{4}}$.

1 | THE RIEMANN-HILBERT PROBLEM FOR ORTHOGONAL POLYNOMIALS

1.1 | Notation

We write $\mathbb{C}$ for the complex plane, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ for the open unit disk, $\mathbb{D}_e := \{z \in \mathbb{C} : |z| > 1\}$ for the exterior disk, and $\mathbb{T} = \partial\mathbb{D}$ for the unit circle. We let dA and ds stand for the normalized area and length measures: $dA(z) = \pi^{-1}dxdy$ and $ds(z) = (2\pi)^{-1}|dz|$, for $z = x + iy$. Moreover, we use the complex derivative notation

$$\partial_z = \tfrac{1}{2}(\partial_x - i\partial_y), \quad \bar{\partial}_z := \tfrac{1}{2}(\partial_x + i\partial_y)$$

where, for example ∂_x stands for the partial derivative with respect to x. Also, we use $\Delta_z = \partial_z\bar{\partial}_z = \tfrac{1}{4}(\partial_x^2 + \partial_y^2)$ for the Laplacian. When it is not needed for clarity, the subscript z may be omitted. We talk about the standard classes of smooth functions, C^k, C^∞, C^ω, for k times differentiable, infinitely differentiable, and real-analytically smooth functions, respectively. Occasionally, we will meet the classes $C^{k,\alpha}$ as well, where k is a nonnegative integer and $0 < \alpha \leq 1$. Functions in $C^{k,\alpha}$ have all their partial derivatives up to and including order k in the Hölder class with exponent α (locally).

We use $A \ll B$ to denote that $A = O(B)$ in some limit procedure, and if both $A = O(B)$ and $B = O(A)$ then we write $A \asymp B$.

We will be dealing with asymptotic expansions in the variable m^{-1}, and write $A \sim A_0 + m^{-1}A_1 + m^{-2}A_2 + \cdots$ as a shorthand for expressing that $A = A_0 + m^{-1}A_1 + m^{-2}A_2 + m^{-k}A_k + O(m^{-k-1})$ holds for each fixed $k = 1, 2, 3, ...$, where the coefficients A_j, $j = 0, 1, 2, ...$, are all assumed independent of m.

1.2 | The confining potential Q

Let $Q : \mathbb{C} \to \mathbb{R}$ be a C^2-smooth function with at least logarithmic growth:

$$Q(z) \geq (1 + \varepsilon_0) \log|z| + O(1) \quad \text{as} \quad |z| \to +\infty, \tag{1.2.1}$$

for some small positive constant ε_0. We consider the cone $\mathrm{SH}(\mathbb{C})$ of all subharmonic functions $q : \mathbb{C} \to \mathbb{R} \cup \{-\infty\}$, and, given a real parameter τ with $0 < \tau < +\infty$, the convex subset $\mathrm{Subh}_\tau(\mathbb{C})$ of functions $q \in \mathrm{Subh}(\mathbb{C})$ with

$$q(z) = \tau \log |z| + \mathrm{O}(1) \quad \text{as} \ |z| \to +\infty.$$

For $0 < \tau < 1 + \varepsilon_0$, we consider the obstacle problem with Q being the obstacle from above, while we optimize over functions in $\mathrm{Subh}_\tau(\mathbb{C})$ which are dominated pointwise by the obstacle. More precisely, we consider the function

$$\check{Q}_\tau(z) := \sup\{q(z) : \ q \le Q \ \text{on} \ \mathbb{C}, \ q \in \mathrm{Subh}_\tau(\mathbb{C})\}.$$

As a matter of definition, $\check{Q}_\tau \le Q$ holds everywhere. It is a consequence of the general methods of obstacle theory that $\check{Q}_\tau$ is $C^{1,1}$-smooth, which means that its second order partial derivatives are all locally bounded in the sense of distribution theory. Moreover, $\check{Q}_\tau$ is in $\mathrm{Subh}_\tau(\mathbb{C})$ as well. The *contact set*

$$S_\tau := \big\{z \in \mathbb{C} : \ \check{Q}_\tau(z) = Q(z)\big\}$$

is compact, and off the contact set, $\check{Q}_\tau$ is harmonic. The contact set S_τ will also be referred to as the *spectral droplet*, and it grows with the parameter τ:

$$S_\tau \subset S_{\tau'}, \qquad 0 < \tau < \tau' < 1 + \varepsilon_0.$$

Moreover, in the sense of distribution theory, we have the equality

$$\Delta \check{Q}_\tau = 1_{S_\tau} \Delta Q,$$

where $1_{\mathcal{E}}$ denotes the indicator function of $\mathcal{E} \subset \mathbb{C}$, which equals 1 on $\mathcal{E}$ and vanishes elsewhere. In particular, $\Delta Q \ge 0$ holds on the spectral droplet S_τ. The growth of S_τ with the parameter τ is called *weighted Laplacian growth*, as the increase in mass in $1_{S_\tau} \Delta Q$ is infinitesimally given by harmonic measure for the point at infinity along the interface $\Gamma_\tau := \partial S_\tau$. For background material, we refer the reader to, for example [17–19], and the book [15].

We shall use the potential Q to form the family of *exponentially varying weights* e^{-2mQ}, where m is a positive real parameter, and we are interested in asymptotic properties as $m \to +\infty$. We form the associated Hilbert space L^2_{mQ} of equivalence classes of Borel measurable functions with

$$\|f\|_{mQ} := \langle f, f \rangle_{mQ}^{\frac{1}{2}} < +\infty, \tag{1.2.2}$$

supplied with corresponding sesquilinear inner product

$$\langle f, g \rangle_{mQ} := \int_{\mathbb{C}} f \bar{g} \, \mathrm{e}^{-2mQ} \mathrm{dA}. \tag{1.2.3}$$

For $n = 0, 1, 2, \ldots$, the closed finite-dimensional subspace $\mathrm{Pol}_{mQ,n}$ of L^2_{mQ} consisting of polynomials of degree $\le n$ is of interest. If all polynomials of degree $\le n$ are represented in $\mathrm{Pol}_{mQ,n}$, that is, if

$$\int_{\mathbb{C}} (1 + |z|^2)^n \mathrm{e}^{-2mQ(z)} \mathrm{dA}(z) < +\infty, \tag{1.2.4}$$

which is a consequence of (1.2.1) for $n < (1 + \varepsilon_0)m - 1$, the reproducing kernel of $\mathrm{Pol}_{mQ,n}$ can be used to model a system of $n + 1$ particles that repel each other by Coulomb interaction, while at the same time being confined by the potential Q. To be more precise, the determinantal point process induced by the *correlation kernel* of $\mathrm{Pol}_{mQ,n}$ (a variant of the reproducing kernel) models 2D Coulomb gas with a special value of the temperature parameter in the Gibbs model. For details, see, for example [17]. The reproducing kernel of $\mathrm{Pol}_{mQ,n}$, call it $K_{m,n}$, may be written in the form

$$K_{m,n}(z,w) = \sum_{j=0}^{n} e_j(z)\overline{e_j(w)},$$

where $e_0, \dots, e_n$ constitutes an orthonormal basis of the space $\mathrm{Pol}_{mQ,n}$. A particular orthonormal basis is the one that consists of the orthogonal polynomials, normalized so that each one has norm equal to 1. Let P_k be the monic orthogonal polynomial of degree k, which means that the leading coefficient equals 1:

$$P_k(z) = z^k + \mathrm{O}(z^{k-1}), \quad \text{as } |z| \to +\infty,$$

while P_k is orthogonal to the polynomials of lower degree,

$$\langle p, P_k \rangle_{mQ} = 0, \qquad p \in \mathrm{Pol}_{mQ,k-1}.$$

Then the connection between the orthogonal polynomials and the polynomial reproducing kernel becomes

$$K_{m,n}(z,w) = \sum_{j=0}^{n} \|P_j\|_{mQ}^{-2} P_j(z)\overline{P_j(w)}. \tag{1.2.5}$$

In recent work with Wennman [20], an asymptotic expansion formula for the orthogonal polynomial P_k was found, in the instance when k grows in proportion with m as $m \to +\infty$. This led to the universal appearance of error function transition for the Coulomb gas along the smooth interface $\Gamma_\tau = \partial S_\tau$ for $\tau = k/m$ (see [20, 21]). For background material, we refer to, for example [1, 2], and [17]. For related work on the real line in the determinantal case, when orthogonal polynomials are relevant and enjoy a three-term recursion, we refer to [10, 12, 13], and the book [9]. The proof of the expansion formula in [20] is based on the idea of a foliation of a neighborhood of the interface Γ_τ by smooth curves where we have (approximately) the required orthogonality along each curve in the foliation. While that approach has its merits, it may be difficult to expand it beyond the given setting, especially given that the algorithm which provides the foliation is rather unwieldy. Here, we supply an alternative approach, which avoids foliations and instead finds appropriate ad-hoc algebraic formulae for the solution of a matrix $\bar{\partial}$-problem. In so doing, we obtain a simple algebraic framework for the terms of the asymptotic expansion, expressed in terms of a Neumann series. This permits us to control the growth of the terms in the asymptotic expansion using a scale of Banach spaces, analogous to the Nishida-Nirenberg approach to the Cauchy-Kovalevskaya theorem.

1.3 | A soft Riemann-Hilbert problem for orthogonal polynomials

We consider a 1×2 matrix-valued function $Y = Y(z)$ of the form

$$Y := (P, \quad \Psi) \tag{1.3.1}$$

where P is an entire function and Ψ is C^1-smooth on $\mathbb{C}$. Then

$$\bar{\partial} Y = (0, \quad \bar{\partial} \Psi). \tag{1.3.2}$$

Next, let W_{mQ} be the *soft jump matrix*

$$W_{mQ} := \begin{pmatrix} 0 & \mathrm{e}^{-2mQ} \\ 0 & 0 \end{pmatrix}$$

and calculate

$$\bar{Y} W_{mQ} = (\bar{P}, \quad \bar{\Psi}) \begin{pmatrix} 0 & \mathrm{e}^{-2mQ} \\ 0 & 0 \end{pmatrix} = (0, \quad \bar{P} \, \mathrm{e}^{-2mQ}).$$

Here, we are interested in the asymptotics as $m \to +\infty$. From the physical point of view, the reciprocal m^{-1} plays the role of Planck's constant. The corresponding *soft Riemann-Hilbert problem*

$$\bar{\partial} Y = \bar{Y} W_{mQ} \tag{1.3.3}$$

then amounts to having

$$\bar{\partial} \Psi = \bar{P} \, \mathrm{e}^{-2mQ}. \tag{1.3.4}$$

We would like to couple the soft Riemann-Hilbert problem (1.3.3) with the following asymptotics at infinity, for a given nonnegative integer n:

$$Y = \left(z^n + \mathrm{O}(|z|^{n-1}), \quad \mathrm{O}(|z|^{-n-1}) \right). \tag{1.3.5}$$

In connection with the Riemann-Hilbert analysis of orthogonal polynomials on the real line, two rows are traditionally used instead of one, and then the asymptotic requirement (1.3.5) may be written in a more suggestive form (omitted here). The condition (1.3.5) involves asymptotics of both P and Ψ:

$$P(z) = z^n + \mathrm{O}(|z|^{n-1}), \qquad \Psi(z) = \mathrm{O}(|z|^{-n-1}), \tag{1.3.6}$$

as $|z| \to +\infty$.

Remark 1.1. A word should be said about the terminology. After all, Riemann-Hilbert problems are jump problems across smooth interfaces, and not $\bar{\partial}$-problems like (1.3.3). What speaks in favor of using this terminology is the fact that a holomorphic jump problem may be viewed as a $\bar{\partial}$-problem in the sense of distribution theory. Indeed, if we apply $\bar{\partial}$ to a holomorphic function with a jump along an interface, the result is, in the sense of distribution theory, a measure supported along the interface which amounts to the jump. More importantly, in fact, our method of analysis actually decomposes the problem at hand (given by (1.3.3) and (1.3.5)) into a hierarchy of actual Riemann-Hilbert problems, see [20–22].

The appearance of the complex conjugation of the matrix Y on the right-hand of (1.3.3) results from the need to characterize the monic orthogonal polynomial P of degree n with respect to the inner product of the Hilbert space L^2_{mQ} given by (1.2.3). The complex conjugation does not make the $\bar{\partial}$-problem any easier, as already the scalar equation $\bar{\partial}u = W\bar{u}$ is notoriously difficult, see for example [24]. The 2×2 matrix analogue of the combined problem (1.3.3)–(1.3.5) was considered by Its and Takhtajan [23], and shown to be a way to characterize the orthogonal polynomials. We briefly recapture the basic argument. Given that P is entire, the first condition in (1.3.6) says that P is a monic polynomial of degree n. Any solution Ψ to (1.3.4) with the decay $\Psi(z) = O(|z|^{-1})$ at infinity must be of the form of the *Cauchy potential*

$$\Psi(z) = \int_{\mathbb{C}} \frac{\bar{P}(\xi)\,e^{-2mQ(\xi)}}{z - \xi}\,dA(\xi) \tag{1.3.7}$$

where dA is normalized area measure, provided that the integral converges. The growth condition (1.2.1) gives that

$$|P(z)|\,e^{-2mQ(z)} = O(|z|^{n-2(1+\varepsilon_0)m}) \quad \text{as } |z| \to +\infty, \tag{1.3.8}$$

so the condition on n to guarantee the convergence of the integral (1.3.7) is that $n < 2(1 + \varepsilon_0)m - 1$. Next, we apply the finite geometric series expansion

$$\frac{1}{z - \xi} = \frac{1}{z} + \frac{\xi}{z^2} + \cdots + \frac{\xi^n}{z^{n+1}} + \frac{\xi^{n+1}}{z^{n+1}(z - \xi)},$$

to obtain that

$$\begin{aligned}
\Psi(z) &= \int_{\mathbb{C}} \frac{\bar{P}(\xi)\,e^{-2mQ(\xi)}}{z - \xi}\,dA(\xi) \\
&= \sum_{j=0}^{n} z^{-j-1} \int_{\mathbb{C}} \xi^j \bar{P}(\xi)\,e^{-2mQ(\xi)}dA(\xi) + z^{-n-1} \int_{\mathbb{C}} \frac{\xi^{n+1}\bar{P}(\xi)\,e^{-2mQ(\xi)}}{z - \xi}\,dA(\xi).
\end{aligned} \tag{1.3.9}$$

For this formula to be correct we need all the involved integrals to converge. In view of (1.3.7), this is the case provided that $n < (1 + \varepsilon_0)m - 1$. We also need to estimate the integral

$$\int_{\mathbb{C}} \frac{\xi^{n+1}\bar{P}(\xi)\,e^{-2mQ(\xi)}}{z - \xi}\,dA(\xi)$$

appearing on the right-hand side of (1.3.9). If $n \le (1 + \varepsilon_0)m - 2$ we can estimate

$$|\xi^{n+1}P(\xi)|\,e^{-2mQ(\xi)} \le C(1 + |\xi|^2)^{-\frac{3}{2}}$$

for some positive constant C, so that

$$\left| \int_{\mathbb{C}} \frac{\xi^{n+1}\bar{P}(\xi)\,e^{-2mQ(\xi)}}{z - \xi}\,dA(\xi) \right| \le C \int_{\mathbb{C}} \frac{(1 + |\xi|^2)^{-\frac{3}{2}}}{|z - \xi|}\,dA(\xi) \le 4C. \tag{1.3.10}$$

So, we assume that $n \leq (1 + \varepsilon_0)m - 2$ holds. Identifying terms in (1.3.9), it now follows that the second condition in (1.3.6) amounts to having

$$\int_{\mathbb{C}} \xi^j \bar{P}(\xi) \, e^{-2mQ(\xi)} = 0, \qquad j = 0, \dots, n-1, \tag{1.3.11}$$

so that P is orthogonal to all polynomials of degree $\leq n - 1$. It is now uniquely determined by the first condition in (1.3.6) as a monic polynomial of degree n. In conclusion, the solution Y to (1.3.1) with asymptotics (1.3.5) exists and is uniquely determined: it is given in terms of P and Ψ, where Ψ is given by (1.3.7), and $P = P_n$ is the monic orthogonal polynomial of degree n with respect to the inner product of L^2_{mQ}.

1.4 | Further assumptions on the confining potential Q

If we assume $0 < \tau < 1 + \varepsilon_0$, and that Q is real-analytically smooth in a neighborhood of the droplet boundary $\Gamma_\tau = \partial S_\tau$, while $\Delta Q > 0$ holds on Γ_τ, it follows from work of Sakai that Γ_τ consists of real-analytically smooth arcs with certain types of possible singularities and isolated points.

To simplify the presentation, we shall focus on the parameter value $\tau = 1$. At $\tau = 1$, we now make the following assumptions:

(i) Γ_1 is a real-analytically smooth Jordan curve, and hence S_1 is the simply connected region enclosed by Γ_1,

(ii) Q is real-analytically smooth with $\Delta Q > 0$ in a neighborhood of Γ_1.

Then $\check{Q}_1$ equals Q on S_1 and is harmonic in the complement $\mathbb{C} \setminus S_1$, with a $C^{1,1}$-smooth transition across the interface Γ_1. Moreover, if we let $\check{Q}_1$ denote the restriction $\check{Q}_1|_{\mathbb{C} \setminus S_1}$, this harmonic function extends harmonically across Γ_1, so that $\check{Q}_1$ defines a harmonic function in $\mathbb{C} \setminus \mathcal{K}$, for some compact subset $\mathcal{K}$ of the interior of the spectral droplet S_1. We then form the function $R = R_1 := Q - \check{Q}_1$, which is well-defined and real-valued in $\mathbb{C} \setminus \mathcal{K}$. The function R vanishes along with its normal derivative along the interface Γ_1 while $\Delta R = \Delta Q > 0$ holds on Γ_1. The real-analytic smoothness gives that R is real-analytically smooth near Γ_1 and we know that $R > 0$ on $\mathbb{C} \setminus S_1$. Putting this information together, we see that $R(z) \asymp [\mathrm{dist}_{\mathbb{C}}(z, \Gamma_1)]^2$ near Γ_1, and that there exists a real-analytically smooth function $\hat{R}$ in a neighborhood of Γ_1 which is a square root of R (so that $\hat{R}^2 = R$), with $\hat{R} > 0$ on the exterior side of Γ_1 while $\hat{R} < 0$ holds on the interior side. This function has a natural C^2-smooth extension to $\mathbb{C} \setminus \mathcal{K}$, also denoted by $\hat{R}$, which keeps the property $\hat{R}^2 = R$, provided the compact subset $\mathcal{K} \subset \mathrm{int}\, S_1$ is big enough. The extended function is positive exteriorly to Γ_1 and negative interiorly, while it vanishes precisely along Γ_1.

1.5 | Approach to the solution of the soft Riemann-Hilbert problem

The approach in [20] is based on two things: (i) the collapsed orthogonality equations, and (ii) the notion of the *orthogonal foliation flow*. The collapsed orthogonality equations supply algorithmically the successive terms of the expansion formula for the orthogonal polynomial P, but are unable to show that the expansion formula is valid in the first place. The orthogonal

foliation flow resolves that difficulty by supplying an invariant foliation of a neighborhood of Γ_1 by curves on which approximate orthogonality to lower order polynomials holds, and integration over the flow parameter gives approximate orthogonality with respect to $e^{-2mQ}dA$ near Γ_1. Appropriate gluing procedures, supported in part by Hörmander's $\bar{\partial}$-estimates, then show that the approximate orthogonal polynomial is close to the actual orthogonal polynomial P. The argument is quite involved and moreover does not connect P with the $\bar{\partial}$-potential Ψ in (1.3.1). Here, we propose a *novel approach* which calculates directly candidates for P and Ψ from a hierarchy of Riemann-Hilbert problems along Γ_1, based on an ad-hoc hypothesis on the shape of P and Ψ. Once the guess is accurate enough, the algorithm kicks in and produces the successive terms of the asymptotic expansion. The algorithm also allows for control of the growth of the terms in the expansions as a Neumann series, which gives asymptotic control of the error terms, and shows that the expansions are valid in a mesoscopic band of width proportional to $m^{-\frac{1}{4}}$ around Γ_1, which is substantially wider than the microscopic basic length scale $m^{-\frac{1}{2}}$. The width established in [20] was only $Am^{-\frac{1}{2}}\log^{\frac{1}{2}}m$ for an arbitrarily large positive constant A. This should have consequences for the fine resolution of the polynomial reproducing kernel (1.2.5) near the droplet boundary.

1.6 | The ad-hoc shape of P, Ψ

We let $\mathbb{Q} = \mathbb{Q}_1$ denote the bounded holomorphic function in $\mathbb{C} \setminus S_1$ with $\mathrm{Re}\mathbb{Q}_1 = Q$ on Γ_1 and $\mathrm{Im}\mathbb{Q}_1(\infty) = 0$. Given the real-analytic smoothness of Q near Γ_1 and the smoothness of Γ_1, $\mathbb{Q}_1$ extends analytically across Γ_1. If

$$\mathbb{D}_e := \{z \in \mathbb{C} : |z| > 1\}$$

is the punctured exterior disk, and $\phi = \phi_1 : \mathbb{C} \setminus S_1 \to \mathbb{D}_e$ is the conformal mapping which is onto and preserves the point at infinity, with $\phi_1'(\infty) > 0$, then

$$\check{Q}_1 = \mathrm{Re}\mathbb{Q}_1 + \log|\phi_1|. \tag{1.6.1}$$

In the sequel, we frequently suppress the subscript 1 and prefer to write ϕ in place of ϕ_1. We solve the soft Riemann-Hilbert problem (1.3.3) approximately in an iterative fashion. The algorithm begins with assuming that P is of the form

$$P \sim c_{m,n}\phi^n e^{m\mathbb{Q}}F, \qquad F = F_0 + m^{-1}F_1 + m^{-2}F_2 + \cdots,$$

where $c_{m,n}$ is a constant, which is fixed by the requirement that $F(\infty) = \phi'(\infty)$. In view of the requirement (1.3.6), the value of the constant is then

$$c_{m,n} := (\phi'(\infty))^{-n-1}e^{-m\mathbb{Q}(\infty)} > 0. \tag{1.6.2}$$

We keep the ratio $\tau = n/m$ fixed, and since we focus on the parameter value $\tau = 1$, we take $n = m$, so that m is a positive integer which tends to infinity:

$$P \sim c_m \phi^m e^{m\mathbb{Q}}F, \qquad F = F_0 + m^{-1}F_1 + m^{-2}F_2 + \cdots, \tag{1.6.3}$$

where we use the shorthand notation $c_m := c_{m,m}$. For $j = 0, 1, 2, ...$, the functions F_j are all supposed to be independent of the parameter m and as functions bounded and holomorphic in a domain $\mathbb{C} \setminus \mathcal{K}$, on which $\mathbb{Q}$ and ϕ are well-defined (holomorphic and conformal, respectively), for some compact subset $\mathcal{K} \subset \operatorname{int} \mathcal{S}_1$. In the representation (1.6.3), we do not really care what happens in the compact $\mathcal{K}$, which may seem odd at first glance. However, this is actually a typical feature of the asymptotics of unweighted Bergman polynomials studied by Carleman [5] (see also [6]), following the lead of Szegő [31] (see also [32]). The analogous instance of a fixed weight was only recently analyzed in terms of asymptotic expansion by Hedenmalm and Wennman [22]. We should mention that in (1.6.3) the series for F need not converge. Instead, the precise statement would be to say that for any integer $k = 1, 2, 3, ...$,

$$F = F_0 + m^{-1}F_1 + \cdots + m^{-k}F_k + \mathrm{O}(m^{-k-1}), \tag{1.6.4}$$

with a uniform error bound in $\mathbb{C} \setminus \mathcal{K}$. In fact, we shall obtain rather strong control of the growth of the error term which in turn tells us how big is the optimal k in the expansion (1.6.4), which gives an expansion with k proportional to m and rapidly decaying error $\mathrm{O}(\mathrm{e}^{-\epsilon \sqrt{m}})$ for some small positive ϵ. The complete analysis requires an analysis of the function Ψ as well. We look for Ψ of the form

$$\Psi = c_m \, m^{-\frac{1}{2}} \phi^{-m} \mathrm{e}^{-m\mathbb{Q}} \left\{ A \operatorname{erf}(2m^{\frac{1}{2}}V) + (2\pi m)^{-\frac{1}{2}} B \chi_1 \, \mathrm{e}^{-2mR} \right\}, \tag{1.6.5}$$

where V is a regularized version of $\hat{R}$: we require that $V = \hat{R}$ near Γ_1 but allow for V to be different further away. Here, we use the notation

$$\operatorname{erf}(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-t^2/2} \mathrm{d}t$$

for the standard Gaussian error function. Like the function F, the functions A and B are understood in terms of asymptotic expansion,

$$A = A_0 + \cdots + m^{-k}A_k + \mathrm{O}(m^{-k-1}), \quad B = B_0 + \cdots + m^{-k}B_k + \mathrm{O}(m^{-k-1}). \tag{1.6.6}$$

Like the F_j, the functions A_j are all defined and bounded holomorphic in $\mathbb{C} \setminus \mathcal{K}$, with asymptotics $\mathrm{O}(|z|^{-1})$ at infinity, but the B_j are only smooth and also have a more fleeting existence: they need only be well-defined in a fixed neighborhood of the loop Γ_1. The cut-off function χ_1 of (1.6.5) solves this issue. It has $0 \leq \chi_1 \leq 1$ globally, while $\chi_1 = 1$ on a fixed small neighborhood of Γ_1 and $\chi_1 = 0$ off a slightly bigger neighborhood of Γ_1. We make sure the latter neighborhood is contained strictly inside the region where B is smoothly defined. This way we get a globally well-defined smooth function $B\chi_1$ by declaring that it vanishes wherever B is undefined.

The final step is to show that the thus defined approximate solution is close to the true solution $Y = (P, \Psi)$ where P is the monic orthogonal polynomial of degree n and Ψ is given by (1.3.7).

A first hint that the ansatz (1.6.5) could be the right one came from a rough calculation based on the orthogonal foliation flow of [20]. The ansatz is also inspired by the fact that the class of univariate functions of the form

$$f(x) = p(x) \operatorname{erf}(x) + q(x) \, \mathrm{e}^{-x^2/2},$$

where p and q are polynomials, is closed under both differentiation and its inverse, the latter in the sense that to each such f there exists another function F of the same form with $F' = f$.

1.7 | Glimpses of the algorithm

Assuming that Ψ is given by the formula (1.6.5), where A is holomorphic, its $\bar{\partial}$-derivative equals

$$\bar{\partial}\Psi = (2/\pi)^{\frac{1}{2}} c_m \phi^{-m} e^{-mQ} \left(A\bar{\partial}\hat{R} - B^1\bar{\partial}R + \frac{1}{2}m^{-1}\bar{\partial}B^1 \right) e^{-2mR} \qquad (1.7.1)$$

in a neighborhood of Γ_1, where we agree to the notation $B^1 := B\chi_1$. On the other hand, if P is given by (1.6.3), we may use the fact (1.6.1):

$$\bar{P} e^{-2mQ} \sim c_m \bar{F} \phi^{-m} e^{-mQ} e^{-2mR}. \qquad (1.7.2)$$

The equation (1.3.4) requires the expressions (1.7.1) and (1.7.2) to coincide, which reduces to the equation

$$A\bar{\partial}\hat{R} - 2B\hat{R}\,\bar{\partial}\hat{R} + \frac{1}{2}m^{-1}\bar{\partial}B = (\pi/2)^{\frac{1}{2}}\bar{F} \qquad (1.7.3)$$

in a small neighborhood of Γ_1, since $B^1 = B\chi_1 = B$ there. Note that we used the fact that $\bar{\partial}R = \bar{\partial}(\hat{R})^2 = 2\hat{R}\bar{\partial}\hat{R}$. We restrict to Γ_1 and use that $\hat{R}$ vanishes on the curve:

$$A\bar{\partial}\hat{R} + \frac{1}{2}m^{-1}\bar{\partial}B = (\pi/2)^{\frac{1}{2}}\bar{F} \quad \text{on} \quad \Gamma_1. \qquad (1.7.4)$$

Taken together with the asymptotics (1.3.6), which amounts to having the asymptotics as $|z| \to +\infty$

$$F(z) = \phi'(\infty) + O(|z|^{-1}), \quad A(z) = O(|z|^{-1}), \qquad (1.7.5)$$

the equation (1.7.4) amounts to a Riemann-Hilbert problem in an alternative coordinate chart (where the unit circle $\mathbb{T}$ replaces Γ_1). We explain the details of this step in Section 3 below. For the moment we observe that the equation involves an unknown B, but luckily it is of higher order, so it does not influence the first terms A_0 and F_0 which get determined right away by the Riemann-Hilbert problem. The original equation (1.7.3) contains more information than (1.7.4) alone. Suppose for the moment we were able to solve the equation (1.7.4). Then we may solve for B using (1.7.3):

$$B = \frac{A\bar{\partial}\hat{R} + \frac{1}{2}m^{-1}\bar{\partial}B - (\pi/2)^{\frac{1}{2}}\bar{F}}{2\hat{R}\,\bar{\partial}\hat{R}}. \qquad (1.7.6)$$

Since $\hat{R}$ vanishes only to degree 1 with $\bar{\partial}\hat{R} \neq 0$ along Γ_1, the division produces a smooth function, and if the numerator is real-analytic across Γ_1, so is the ratio and hence B. The combination of the Riemann-Hilbert "jump" problem (1.7.4) and the smooth division problem (1.7.6) supplies the

full algorithm. We may liken it to the Newton algorithm for finding the zeros of polynomials: once we are in the ballpark the algorithm gets us ever closer to the solution. We use (1.7.4) with $B = 0$ to get the initial A and F. Next, we apply (1.7.6) with the previous choices of A, F, and B to get an updated choice for B. This new B is then implemented into (1.7.4) to get improved A and F. Proceeding iteratively we obtain the full asymptotic expansion.

Our algorithm only gives the asymptotic expansion for the functions A, B, F, and not the functions themselves, with Gevrey class growth of the coefficients. To pass from asymptotic expansion to actual functions with the indicated asymptotic behavior, we may, for example, employ the methods of [16]. This gives us smooth functions which are holomorphic whenever the asymptotic expansions are.

1.8 | Main results

Apart from the approach based on the ad-hoc shape of the solution as outlined in the prior two subsections, which is interesting per se and may carry over in other situations, it is valuable that we may achieve better control of the error in our approximations than what was available earlier. If we write

$$F^{\approx} = F^{\langle \kappa \rangle} := F_0 + m^{-1}F_1 + \cdots + m^{-\kappa}F_\kappa,$$

where $\kappa = \kappa_*(m) \asymp m^{\frac{1}{2}}$ is appropriately chosen, and the coefficient functions F_j come from the algorithm presented in Subsection 1.7, we have the following.

Theorem 1.2. *We assume that the properties (i)–(ii) of Subsection 1.4 hold, and put $P^{\approx} :=$ $c_m \phi^m e^{mQ} F^{\approx}$, where $F^{\approx}$ is as above. Then each F_j extends holomorphically to a fixed neighborhood of $\mathbb{C} \setminus \text{int } S_1$ for each $j = 0, \ldots, \kappa_*(m)$, while $\log F_0$ is bounded and holomorphic in the same neighborhood. Then there exists a fixed C^∞-smooth cut-off function $\chi_{1,1}$ on $\mathbb{C}$, with $0 \le \chi_{1,1} \le 1$, which equals 1 in an open neighborhood of the closure of $\mathbb{C} \setminus S_1$, and vanishes off a slightly larger neighborhood, such that $\chi_{1,1}P^{\approx}$ becomes globally well-defined. Moreover, it is close to the monic orthogonal polynomial P of degree $n = m$ in L^2_{mQ}:*

$$\|P - \chi_{1,1}P^{\approx}\|_{mQ} = \mathrm{O}\left(c_m m^{\frac{3}{4}} e^{-\epsilon\sqrt{m}}\right), \quad \text{where } \|\chi_{1,1}P^{\approx}\|_{mQ} \asymp c_m m^{-\frac{1}{4}}.$$

Here, $c_m = c_{m,m}$ is given by (1.6.2) and $\epsilon > 0$ is a constant that only depends on Q. It follows that

$$P = c_m \phi^m e^{mQ}\left(F^{\approx} + \mathrm{O}\left(m^{\frac{5}{4}} e^{-\frac{1}{2}\epsilon\sqrt{m}}\right)\right), \quad \text{on } D_m,$$

holds in the uniform norm as $m \to +\infty$, where D_m is the union of $\mathbb{C} \setminus S_1$ and a certain band of width $\asymp m^{-\frac{1}{4}}$ around the loop $\Gamma_1 = \partial S_1$.

The approximation shows that for m big enough, the zeros of P are confined to the complement $\mathbb{C} \setminus D_m$. See Remark 4.1 below for details.

Remark 1.3. The algorithm also calculates the asymptotic expansion of the norm $\|P\|^2_{mQ}$ with an analogously small error term, as we may see from the relations (4.3.6) and (4.3.2), together with the observation that we may replace $\chi_{0,1}$ in (4.3.2) by its square without any changes in the estimates.

1.9 | Organization of the paper

In Section 2, we present useful background material: the Bernstein-Walsh lemma, Hörmander estimates for the $\bar{\partial}$-equation, as well as Szegő projections and the exterior Herglotz operator. Then in Section 3, we present the algorithm which produces the successive approximations, at first intuitively, and later with the inclusion of error terms that come from an approximate solution of a partial differential equation for a function we denote by B. The algorithm dismisses what happens inside a fixed compact part of interior of the droplet S_1, so to justify that, we apply the method of C^∞-smooth cut-off functions in Section 4, and solve $\bar{\partial}$-equations with growth control to obtain polynomials. The proof of the main theorem is also supplied in Section 4, but it requires certain estimates that are only developed in the following section, Section 5. The method in Section 5 involves sequences of nested Banach spaces of real-analytic functions in neighborhoods of the unit circle $\mathbb{T}$, and resembles the Nishida-Nirenberg approach to the Cauchy-Kovalevskaya theorem.

1.10 | Remarks on other related work

Determinantal processes are used to model the repulsion of fermions. This applies to finite as well as infinite particle system, see, for example [26]. The process is induced by a correlation kernel, often a Bergman kernel, or a polynomial Bergman kernel. These Bergman kernels enjoy local asymptotic expansions, following work of Tian et al. (see, for example [33]). A version which relies on microlocal analysis is supplied by Berman et al. [4] (compare with [7]), while a stronger expansion theorem with exponentially decaying error term was recently obtained by Rouby et al. [29]. It should be mentioned that while these earlier works are somewhat related to the theme of the present paper, the approaches are strictly local and cannot touch the nonlocal phenomena studied here. We also point out that the literature on local Bergman kernel expansion is extensive and that we only supply a sample of the possible references.

A finite particle determinantal process may be thought of in terms of a corresponding Gibbs model with a specific thermodynamical inverse temperature β. As the temperature freezes to 0, the determinantal property is lost, but we expect crystallization to energy minimizing Fekete point configurations. Here, we should mention the *Abrikosov conjecture*, which maintains that the energetically optimal configuration is achieved in the many particle limit by the hexagonal lattice (see, for example [30]; a complete solution is available in the higher dimensional spaces $\mathbb{R}^8$ and $\mathbb{R}^{24}$, see [8]). We also mention that a higher-dimensional complex manifold notion of Fekete points was considered by Berman et al. [3].

2 | PRELIMINARIES

We need two basic estimates, the growth estimate which comes from a polynomial growth assumption, and an estimate of solutions to $\bar{\partial}$-problems with polynomial growth, which comes

from application of Hörmander's L^2-method. We prefer to formulate both results for a flexible parameter τ rather than just for $\tau = 1$. We also introduce the Herglotz operator, which solves a boundary value problem.

2.1 | A weighted Bernstein-Walsh lemma

We recall Proposition 2.2.2 in [20]. The formulation needs the space $L^2_{mQ}(\Omega)$, which for an open subset $\Omega \subset \mathbb{C}$ is the standard L^2-space with finite norm

$$\|f\|^2_{mQ,\Omega} := \int_\Omega |f|^2 e^{-2mQ} dA < +\infty.$$

Proposition 2.1. *Let $\tau = n/m$ be fixed with $0 < \tau < 1 + \varepsilon_0$, and suppose that $\mathcal{K}$ is a compact subset of S_τ. Also, fix a positive real parameter δ. Then there exists a positive constant C such that for any $u \in L^2_{mQ,\mathbb{C}\setminus\mathcal{K}}$ which is holomorphic on $\mathbb{C} \setminus \mathcal{K}$ with polynomial growth control $u(z) = O(|z|^n)$ as $|z| \to +\infty$, we have that*

$$|u(z)| \leq Cm^{\frac{1}{2}}\|u\|_{mQ,\mathbb{C}\setminus\mathcal{K}} e^{m\check{Q}_\tau(z)}, \qquad \mathrm{dist}_\mathbb{C}(z,\mathcal{K}) \geq \delta m^{-\frac{1}{2}}.$$

2.2 | Hörmander's estimate for polynomial $\bar\partial$-problems

As our method relies on neglecting the behavior of the function we are looking for in certain regions of the plane, it is important to be able to reconstruct a globally defined function. We do this by solving the appropriate $\bar\partial$-problem, while appealing to the following basic estimate, which follows from Proposition 2.4.1 in [20].

Proposition 2.2. *Let $\tau = n/m$ be fixed with $0 < \tau < 1 + \varepsilon_0$, and suppose that $f \in L^\infty(\mathbb{C})$ vanishes off S_τ. Then there exists a solution u to $\bar\partial u = f$ with $u(z) = O(|z|^{n-1})$ as $|z| \to +\infty$ with*

$$\int_\mathbb{C} |u|^2 e^{-2mQ} dA \leq \frac{1}{2m} \int_{S_\tau} |f|^2 e^{-2mQ} \frac{dA}{\Delta Q},$$

provided that the right-hand side is finite.

2.3 | Projections and the Herglotz operator

We shall need the Szegő projection

$$\mathbf{P}_{H^2} f(z) := \int_\mathbb{T} \frac{f(\zeta)}{1 - z\bar\zeta} ds(\zeta), \qquad z \in \mathbb{D}.$$

It is the orthogonal projection $L^2(\mathbb{T}) \to H^2$, where H^2 is the standard Hardy space in the unit disk, which may be understood both as a space of holomorphic functions in the disk $\mathbb{D}$ and as a subspace

of $L^2(\mathbb{T})$ via nontangential boundary values. For the details, see any book on Hardy spaces, for example [11, 14, 25]. We also need the orthogonal projections to the Hardy subspaces H_0^2, H_-^2, and $H_{-,0}^2$, denoted by $\mathbf{P}_{H_0^2}$, $\mathbf{P}_{H_-^2}$, and $\mathbf{P}_{H_{-,0}^2}$. Here, the space H_0^2 is the codimension 1 subspace of H^2 consisting of all functions that vanish at the origin. The space H_-^2 may be thought of as a space of conjugate-holomorphic functions in the disk, or, alternatively, by Schwarz reflection, as a space of holomorphic functions in the exterior disk $\mathbb{D}_e$. Then $f \in H_-^2$ if and only if the reflected function $\overline{f(1/\bar{z})}$ is in H^2. Moreover, the space $H_{-,0}^2$ has codimension 1 in H_-^2, and consists of all functions in H_-^2 that vanish at the point at infinity. The space $L^2(\mathbb{T})$ splits orthogonally as a direct sum in two ways:

$$L^2(\mathbb{T}) = H_0^2 \oplus H_-^2 = H^2 \oplus H_{-,0}^2.$$

The corresponding exterior Szegő projections $\mathbf{P}_{H_-^2}$ and $\mathbf{P}_{H_{-,0}^2}$ may be expressed as integrals as well:

$$\mathbf{P}_{H_-^2} f(z) = z \int_{\mathbb{T}} \frac{f(\zeta)}{z - \zeta} \mathrm{d}s(\zeta), \qquad \mathbf{P}_{H_{-,0}^2} f(z) = \int_{\mathbb{T}} \frac{\zeta f(\zeta)}{z - \zeta} \mathrm{d}s(\zeta),$$

for $z \in \mathbb{D}_e$. Analogously, we find that

$$\mathbf{P}_{H_0^2} f(z) = z \int_{\mathbb{T}} \frac{\bar{\zeta} f(\zeta)}{1 - z\bar{\zeta}} \mathrm{d}s(\zeta), \qquad z \in \mathbb{D}.$$

We shall also be interested in the exterior *Herglotz transform*

$$\mathbf{H}_{\mathbb{D}_e} f(z) := \int_{\mathbb{T}} \frac{z + \zeta}{z - \zeta} f(\zeta) \mathrm{d}s(\zeta), \qquad z \in \mathbb{D}_e.$$

Since

$$\frac{z + \zeta}{z - \zeta} = \frac{2\zeta}{z - \zeta} + 1,$$

it follows that

$$\mathbf{H}_{\mathbb{D}_e} f = \mathbf{P}_{H_-^2} f + \mathbf{P}_{H_{-,0}^2} f = 2\mathbf{P}_{H_{-,0}^2} + \langle f \rangle_{\mathbb{T}},$$

where we introduce the notation for the mean value on the unit circle,

$$\langle f \rangle_{\mathbb{T}} := \int_{\mathbb{T}} f(\zeta) \mathrm{d}s(\zeta). \tag{2.3.1}$$

If f is a real-valued function on the circle, say in $L^2(\mathbb{T})$, then the Herglotz transform obtains a solution $u = \mathbf{H}_{\mathbb{D}_e} f$ to the problem $\operatorname{Re} u = f$ on the circle $\mathbb{T}$, with u holomorphic in $\mathbb{D}_e$. As such, it is uniquely determined by the requirements that $u \in H_-^2$ and $\operatorname{Im} u(\infty) = 0$.

We may use the Herglotz operator (or the exterior Szegő projection) to solve jump problems. We need a simplified version of Proposition 2.5.1 in [20].

Proposition 2.3. *Let G be a function in $L^2(\mathbb{T})$. Then f satisfies*

$$f \in H^2_- \cap (G + H^2)$$

if and only if

$$f = C + \mathbf{P}_{H^2_{-,0}} G$$

for some constant C.

At times we shall also need the harmonic extension to the exterior disk $\mathbb{D}_e$ given by the Poisson integral

$$\mathbf{U}_{\mathbb{D}_e} f(z) := (|z|^2 - 1) \int_{\mathbb{T}} \frac{f(\zeta)}{|z - \zeta|^2} \mathrm{ds}(\zeta), \qquad z \in \mathbb{D}_e. \tag{2.3.2}$$

If f is continuous, then $\mathbf{U}_{\mathbb{D}_e} f$ extends continuously to $\bar{\mathbb{D}}_e$, while if f is real-analytic, then $\mathbf{U}_{\mathbb{D}_e} f$ extends harmonically across $\mathbb{T}$.

3 | PRESENTATION OF THE ALGORITHM

We first use the conformal mapping $\phi = \phi_1 : \mathbb{C} \setminus S_1 \to \mathbb{D}_e$ and its inverse $\varphi = \varphi_1 = \phi_1^{-1} : \mathbb{D}_e \to \mathbb{C} \setminus S_1$ to turn our interface loop $\Gamma_1 = \partial S_1$ into the unit circle $\mathbb{T}$. Since φ remains conformal on a slightly bigger external disk

$$\mathbb{D}_e(0, \rho) := \{z \in \mathbb{C} : |z| > \rho\}$$

with $0 < \rho < 1$, this means that we discard what happens inside some compact subset $\mathcal{K}$ contained in the interior of the droplet S_1 (we may think of $\mathcal{K}$ as the complement of $\varphi(\mathbb{D}_e(0, \rho))$). Later, we justify this by using cut-off functions to get a globally defined function, and then we apply Hörmander's $\bar{\partial}$-estimate to get a polynomial solution which is close to the function we started with.

3.1 | Transfer to the unit circle and the idea of the algorithm

We write $\varphi = \varphi_1 := \phi_1^{-1} : \mathbb{D}_e \to \mathbb{C} \setminus S_1$ for the indicated conformal mapping, tacitly extended across $\mathbb{T}$, and consider

$$\mathrm{R} := R \circ \varphi, \quad \hat{\mathrm{R}} := \hat{R} \circ \varphi,$$

and the associated functions

$$\mathrm{A} := A \circ \varphi, \quad \mathrm{B} := B \circ \varphi, \quad \mathrm{F} := \varphi' F \circ \varphi.$$

Here, A and F are holomorphic functions in a neighborhood of $\bar{\mathbb{D}}_e$ with asymptotics in accordance with (1.7.5):

$$F(z) = 1 + O(|z|^{-1}), \quad A(z) = O(|z|^{-1}), \tag{3.1.1}$$

as $|z| \to +\infty$. In particular, F is bounded with a positive value at infinity, while A is bounded and vanishes at infinity. In terms of these functions, the equation (1.7.3) reads

$$A\bar{\partial}\hat{R} - 2B\hat{R}\,\bar{\partial}\hat{R} + \frac{1}{2}m^{-1}\bar{\partial}B = (\pi/2)^{\frac{1}{2}}\bar{F}. \tag{3.1.2}$$

Just as before, we split the equation in two separate steps:

$$A\bar{\partial}\hat{R} + \frac{1}{2}m^{-1}\bar{\partial}B = (\pi/2)^{\frac{1}{2}}\bar{F} \quad \text{on} \quad \mathbb{T}, \tag{3.1.3}$$

and

$$B = \frac{A\bar{\partial}\hat{R} + \frac{1}{2}m^{-1}\bar{\partial}B - (\pi/2)^{\frac{1}{2}}\bar{F}}{2\hat{R}\,\bar{\partial}\hat{R}}. \tag{3.1.4}$$

We will refer to these as STEPS I and II, which we analyze below in some detail. Local analysis around the circle $\mathbb{T}$ shows that

$$\bar{\partial}\hat{R} = 2^{-\frac{1}{2}}(\Delta R)^{\frac{1}{2}}\zeta \quad \text{on} \quad \mathbb{T}. \tag{3.1.5}$$

Here and in the sequel, ζ stands for the coordinate function $\zeta(z) = z$. We recall the notation $\mathbf{H}_{\mathbb{D}_e}$ for the exterior Herglotz operator from Subsection 2.3. We introduce the function H_R given by

$$H_R := \pi^{-\frac{1}{4}}\exp\left(\frac{1}{4}\mathbf{H}_{\mathbb{D}_e}[\log\Delta R]\right),$$

which is a bounded (and bounded away from 0) holomorphic function in $\mathbb{D}_e$ with holomorphic extension across $\mathbb{T}$. Then $|H_R|^2 = \pi^{-\frac{1}{2}}(\Delta R)^{\frac{1}{2}}$ holds on $\mathbb{T}$, and the value at infinity equals

$$H_R(\infty) = \pi^{-\frac{1}{4}}\exp\left(\frac{1}{4}\langle\log\Delta R\rangle_{\mathbb{T}}\right) > 0.$$

It now follows from (3.1.5) that

$$\bar{\partial}\hat{R} = 2^{-\frac{1}{2}}\zeta(\Delta R)^{\frac{1}{2}} = (\pi/2)^{\frac{1}{2}}\zeta|H_R|^2 \quad \text{on} \quad \mathbb{T}. \tag{3.1.6}$$

STEP I: In view of (3.1.6), the equation (3.1.3) may be written as

$$\frac{F}{H_R} - \overline{\zeta A H_R} = (2\pi)^{-\frac{1}{2}}m^{-1}\frac{\partial\bar{B}}{H_R} \quad \text{on} \quad \mathbb{T}. \tag{3.1.7}$$

From the data (1.7.5), we see that in (3.1.7), F/H_R s bounded and holomorphic in $\bar{\mathbb{D}}_e$, so that $F/H_R \in H_-^2$, whereas $\overline{\zeta A H_R}$ extends by Schwarz reflection to a bounded holomorphic function

on $\bar{\mathbb{D}}$, so that $\overline{\zeta A H_R} \in H^2$. This means that (3.1.7) is a Riemann-Hilbert problem with jump $(2\pi)^{-\frac{1}{2}} m^{-1} \partial \bar{B}/H_R$, a situation covered by Proposition 2.3. By that result, we may solve for F/H_R,

$$\frac{F}{H_R} = a_1 + (2\pi)^{-\frac{1}{2}} m^{-1} \mathbf{P}_{H^2_{-,0}} \left[\frac{\partial \bar{B}}{H_R} \right], \tag{3.1.8}$$

where a_1 is a constant. At the same time, by taking complex conjugates, we find that

$$\zeta A H_R = \bar{a}_1 - (2\pi)^{-\frac{1}{2}} m^{-1} \mathbf{P}_{H^2_-} \left[\frac{\bar{\partial} B}{\bar{H}_R} \right], \tag{3.1.9}$$

while the constant a_1 is given by

$$a_1 = \frac{F(\infty)}{H_R(\infty)} = \frac{1}{H_R(\infty)} = \pi^{\frac{1}{4}} \exp \left(-\frac{1}{4} \langle \log \Delta R \rangle_{\mathbb{T}} \right) > 0, \tag{3.1.10}$$

since $F(\infty) = 1$ is our assumed normalization and $H_R(\infty)$ is known.

This ends our analysis of STEP I.

STEP II: We turn to the analysis of the next step, based on the division formula (3.1.4). To this end, we apply first (3.1.6), to see that we may write

$$\bar{\partial} \hat{R} = (\pi/2)^{\frac{1}{2}} \zeta |H_R|^2 + 2 W_R \hat{R} \bar{\partial} \hat{R}, \tag{3.1.11}$$

where the expression W_R is real-analytic near $\mathbb{T}$. We recall that $\mathbf{U}_{\mathbb{D}_e}$ stands for the harmonic extension to $\mathbb{D}_e$ of the restriction to $\mathbb{T}$ of a given smooth function. It is an immediate consequence of (3.1.7) that

$$\zeta A H_R = \mathbf{U}_{\mathbb{D}_e}[\zeta A H_R] = \frac{\bar{F}}{\bar{H}_R} - (2\pi)^{-\frac{1}{2}} m^{-1} \mathbf{U}_{\mathbb{D}_e} \left[\frac{\bar{\partial} B}{\bar{H}_R} \right] \tag{3.1.12}$$

holds in a neighborhood of the closed exterior disk $\bar{\mathbb{D}}_e$, in the sense that each term extends harmonically across $\mathbb{T}$. It now follows from (3.1.11) and (3.1.12) that

$$A \bar{\partial} \hat{R} + \frac{1}{2} m^{-1} \bar{\partial} B - (\pi/2)^{\frac{1}{2}} \bar{F} = \frac{1}{2} m^{-1} \left(\bar{\partial} B - \bar{H}_R \mathbf{U}_{\mathbb{D}_e} (\bar{\partial} B / \bar{H}_R) \right) + 2 A \hat{R} \bar{\partial} \hat{R} \, W_R,$$

so that (3.1.4) asserts that

$$B = \frac{1}{2} m^{-1} \frac{\bar{\partial} B - \bar{H}_R \mathbf{U}_{\mathbb{D}_e} (\bar{\partial} B / \bar{H}_R)}{2 \hat{R} \bar{\partial} \hat{R}} + A W_R. \tag{3.1.13}$$

3.2 | The equation for the function B alone

In view of the expression for the solution to the Riemann-Hilbert problem (3.1.9) from STEP I and the relation (3.1.10), the expression for B may be written as

$$B = \frac{1}{2} m^{-1} \frac{\bar{\partial} B - \bar{H}_R \mathbf{U}_{\mathbb{D}_e} (\bar{\partial} B / \bar{H}_R)}{2 \hat{R} \bar{\partial} R} + \frac{W_R}{\zeta H_R} \left(a_1 - (2\pi)^{-\frac{1}{2}} m^{-1} \mathbf{P}_{H^2_-} \left[\frac{\bar{\partial} B}{\bar{H}_R} \right] \right),$$

since $a_1 > 0$, which we simplify to

$$B = a_1 \frac{W_R}{\zeta H_R} + m^{-1} \left(\frac{\bar{\partial}B - \bar{H}_R \mathbf{U}_{\mathbb{D}_e}[\bar{\partial}B/\bar{H}_R]}{4\hat{R}\bar{\partial}\hat{R}} - (2\pi)^{-\frac{1}{2}} \frac{W_R}{\zeta H_R} \mathbf{P}_{H^2} \left[\frac{\bar{\partial}B}{\bar{H}_R} \right] \right). \qquad (3.2.1)$$

Note that in (3.2.1), we have merged STEPS I and II in a single equation which only concerns the function B (the function A no longer appears). This is an operator equation of the type

$$B = B_0 + m^{-1}\mathbf{T}[B], \quad \text{where} \quad B_0 := a_1 \frac{W_R}{\zeta H_R} \quad \text{and} \quad \mathbf{T}[B] := \mathbf{L}[\bar{\partial}B/\bar{H}_R], \qquad (3.2.2)$$

and the operator $\mathbf{L}$ is given by

$$\mathbf{L}[f] := \bar{H}_R \frac{f - \mathbf{U}_{\mathbb{D}_e}[f]}{4\hat{R}\bar{\partial}\hat{R}} - (2\pi)^{-\frac{1}{2}} \frac{W_R}{\zeta H_R} \mathbf{P}_{H^2_-}[f]. \qquad (3.2.3)$$

We may attempt to solve the equation (3.2.2) by iteration, that is, by finite Neumann series approximation.

3.3 | Quantitative work: approximate solutions $A^{\approx}$, $B^{\approx}$, $F^{\approx}$

In the preceding subsection, we were a little naive, believing that the equation (3.2.2) would have an exact solution. As for now, we only expect the equation (3.2.2) to be approximately solvable. So, we suppose we have found an approximate solution $B^{\approx}$, with

$$B^{\approx} = B_0 + m^{-1}\mathbf{T}[B^{\approx}] + \mathcal{E}_m. \qquad (3.3.1)$$

where $\mathcal{E}_m$ measures the error in solving the equation (3.2.2). Later on, in Section 5, we will show that there exists a $B^{\approx}$ such $\mathcal{E}_m$ decays rapidly as $m \to +\infty$. Naturally, $B^{\approx}$ also depends on m, but it is more important that the error term reflects this. Based on our previously derived equations (3.1.8) and (3.1.9), we put accordingly

$$A^{\approx} := \frac{a_1}{\zeta H_R} - m^{-1} \frac{(2\pi)^{-\frac{1}{2}}}{\zeta H_R} \mathbf{P}_{H^2} \left[\frac{\bar{\partial}B^{\approx}}{\bar{H}_R} \right] \qquad (3.3.2)$$

and

$$F^{\approx} := a_1 H_R + (2\pi)^{-\frac{1}{2}} m^{-1} H_R \mathbf{P}_{H^2_{-,0}} \left[\frac{\partial\bar{B}^{\approx}}{H_R} \right]. \qquad (3.3.3)$$

In particular, it follows that $F^{\approx}$ is properly normalized at infinity, as

$$F^{\approx}(\infty) = a_1 H_R(\infty) = 1,$$

in view of (3.1.10). It follows from the relations (3.3.2) and (3.3.3) as well as (3.1.10) that

$$\frac{\bar{F}^{\approx}}{\bar{H}_R} - \zeta H_R A^{\approx} = (2\pi)^{-\frac{1}{2}} m^{-1} \mathbf{U}_{\mathbb{D}_e} \left[\frac{\bar{\partial}B^{\approx}}{\bar{H}_R} \right], \qquad (3.3.4)$$

so that

$$\zeta |H_R|^2 A^{\approx} = \bar{F}^{\approx} - (2\pi)^{-\frac{1}{2}} m^{-1} \bar{H}_R \mathbf{U}_{\mathbb{D}_e} \left[\frac{\bar{\partial} B^{\approx}}{\bar{H}_R} \right]. \tag{3.3.5}$$

We see from this and (3.1.11) that

$$A^{\approx} \bar{\partial} \hat{R} = (\pi/2)^{\frac{1}{2}} \zeta |H_R|^2 A^{\approx} + 2W_R \hat{R} \bar{\partial} \hat{R} A^{\approx}$$

$$= (\pi/2)^{\frac{1}{2}} \bar{F}^{\approx} - \frac{1}{2} m^{-1} \bar{H}_R \mathbf{U}_{\mathbb{D}_e} \left[\frac{\bar{\partial} B^{\approx}}{\bar{H}_R} \right] + 2W_R \hat{R} \bar{\partial} \hat{R} A^{\approx},$$

so that

$$A^{\approx} \bar{\partial} \hat{R} + \frac{1}{2} m^{-1} \bar{\partial} B^{\approx}$$

$$= (\pi/2)^{\frac{1}{2}} \bar{F}^{\approx} + \frac{1}{2} m^{-1} \left(\bar{\partial} B^{\approx} - \bar{H}_R \mathbf{U}_{\mathbb{D}_e} \left[\frac{\bar{\partial} B^{\approx}}{\bar{H}_R} \right] \right) + 2W_R \hat{R} \bar{\partial} \hat{R} A^{\approx}.$$

Finally, if we apply the relation (3.3.1) which defines the error $\mathcal{E}_m$, we find that

$$A^{\approx} \bar{\partial} \hat{R} + \frac{1}{2} m^{-1} \bar{\partial} B^{\approx} - B^{\approx} \bar{\partial} R$$

$$= (\pi/2)^{\frac{1}{2}} \bar{F}^{\approx} + \frac{1}{2} m^{-1} \left(\bar{\partial} B^{\approx} - \bar{H}_R \mathbf{U}_{\mathbb{D}_e} \left[\frac{\bar{\partial} B^{\approx}}{\bar{H}_R} \right] \right)$$

$$+ 2W_R \hat{R} \bar{\partial} \hat{R} A^{\approx} - B_0 \bar{\partial} R - m^{-1} (\bar{\partial} R) \mathbf{T}[B^{\approx}] - \mathcal{E}_m \bar{\partial} R. \tag{3.3.6}$$

Since $\bar{\partial} R = 2\hat{R} \bar{\partial} \hat{R}$, we read off the definition of $\mathbf{T}$ that

$$(\bar{\partial} R) \mathbf{T}[B^{\approx}] = \frac{1}{2} \left(\bar{\partial} B^{\approx} - \bar{H}_R \mathbf{U}_{\mathbb{D}_e} \left[\frac{\bar{\partial} B^{\approx}}{\bar{H}_R} \right] \right) - (2\pi)^{-\frac{1}{2}} \frac{W_R \bar{\partial} R}{\zeta H_R} \mathbf{P}_{H^2} \left[\frac{\bar{\partial} B^{\approx}}{\bar{H}_R} \right],$$

so that (3.3.6) simplifies to

$$A^{\approx} \bar{\partial} \hat{R} + \frac{1}{2} m^{-1} \bar{\partial} B^{\approx} - B^{\approx} \bar{\partial} R$$

$$= (\pi/2)^{\frac{1}{2}} \bar{F}^{\approx} + (2\pi)^{-\frac{1}{2}} m^{-1} \frac{W_R \bar{\partial} R}{\zeta H_R} \mathbf{P}_{H^2} \left[\frac{\bar{\partial} B^{\approx}}{\bar{H}_R} \right] + W_R \bar{\partial} R A^{\approx} - B_0 \bar{\partial} R - \mathcal{E}_m \bar{\partial} R$$

$$= (\pi/2)^{\frac{1}{2}} \bar{F}^{\approx} - \mathcal{E}_m \bar{\partial} R, \tag{3.3.7}$$

where in the last step we used the definitions of B_0 and $A^{\approx}$.

4 | THE USEFULNESS OF CUT-OFF FUNCTIONS

The method outlined in the previous subsection only supplies a tentative solution to the soft Riemann-Hilbert problem of (1.3.3) and (1.3.5). So far, however, we have not even been able to

assert much rigorously. For instance, it is not clear that the approximate solution $Y = (P, \Psi)$ supplied in terms of the triple $(A^\approx, B^\approx, F^\approx)$ is even close to the actual solution. We will explain why the function $F^\approx$ defines a good approximation $P^\approx$ of the orthogonal polynomial P.

4.1 | Cut-off functions and a modification of the potential $\hat{R}$

Let χ_0 be a *radial* cut-off function, which is C^∞-smooth in the plane with $0 \leq \chi_0 \leq 1$ globally: $\chi_0 = 1$ on a fixed (closed) annular neighborhood $\mathcal{A}_1$ of $\mathbb{T}$, while $\chi_0 = 0$ off a slightly larger fixed open annular neighborhood $\mathcal{A}_0$. We ask of χ_0 furthermore that $|\bar{\partial}\chi_0|^2 \leq C_1 \chi_0$ holds for some large positive constant C_1. The closure clos $\mathcal{A}_0$ is the support of χ_0, and $\mathcal{A}_1 \subset \mathcal{A}_0 \subset \text{clos } \mathcal{A}_0$. We let $V \geq 0$ be a smooth potential in the plane such that $V = \hat{R}$ holds on clos $\mathcal{A}_0$, the support of χ_0, and that $V = 0$ holds only along the circle $\mathbb{T}$, and V is Lipschitz continuous in the plane, with minimal growth $V(z) \geq \log|z|$ for $|z| \gg 1$. It is not difficult to see that

$$\int_{\mathbb{C}} (1 - \chi_0)|\bar{\partial}V| \, e^{-2mV^2 + mV_-^2} dA \leq C_2 m^{-1} e^{-\varepsilon_1 m}, \tag{4.1.1}$$

for some positive constants C_2 and ε_1 and big enough m. Here, $V_-^2 := 1_{\mathbb{D}} V^2$ equals V^2 in the disk $\mathbb{D}$ while it vanishes outside. The property (4.1.1) might not hold with V replaced by $\hat{R}$, and it is the reason why we need to introduce V in the first place. It is not too difficult to construct a viable V; the necessary details are left to the interested reader. We put

$$\Psi^\approx = A^\approx \, \text{erf}(2\sqrt{m}V) + (2\pi m)^{-\frac{1}{2}} \chi_0 \, B^\approx \, e^{-2mR}, \tag{4.1.2}$$

and use the function $\Psi^\approx$ to define the corresponding approximate function

$$\Psi^\approx := c_m m^{-\frac{1}{2}} \phi^{-m} e^{-m\mathbb{Q}} \Psi^\approx \circ \phi,$$

which is then well-defined and smooth in the whole exterior domain to the curve Γ_1 as well as across the interface Γ_1 (so that it is smooth in a neighborhood of Γ_1). We take a radius ρ_1 with $0 < \rho_1 < 1$ and the associated exterior disk

$$\mathbb{D}_e(0, \rho_1) := \{z \in \mathbb{C} : \rho_1 < |z| < +\infty\}.$$

We require the support clos $\mathcal{A}_0$ of the cut-off function χ_0 to be contained inside the exterior disk $\mathbb{D}_e(0, \rho_1)$, and moreover, that ρ_1 is so close to 1 that $\hat{R}$ is smooth and well-defined in the annulus $\{z \in \mathbb{C} : \rho_1 \leq |z| \leq 1\}$. In addition, we require of V that $V = \hat{R}$ holds there in addition to the annulus clos $\mathcal{A}_0$.

4.2 | Application of the Cauchy-Green formula

Let G be a bounded holomorphic function in the exterior disk $\mathbb{D}_e(0, \rho_1) = \{z \in \mathbb{C} : |z| > \rho_1\}$, subject to the growth bound

$$|G(z)| \leq \|G\|_{mR_-} e^{mR_-(z)}, \qquad z \in \mathbb{D}_e(0, \rho_1), \tag{4.2.1}$$

where $\|G\|_{mR_-}$ denotes the smallest such constant and $R_- := 1_{\mathbb{D}}R$. We now take the $\bar{\partial}$-derivative in (4.1.2) and apply the identity (3.3.7) as well as the fact that $V = \hat{R}$ on the support of χ_0 while $A^{\approx}$ is holomorphic there:

$$
\begin{aligned}
\bar{\partial}\Psi^{\approx} &= (2m/\pi)^{\frac{1}{2}}\Big[\chi_0 A^{\approx}\bar{\partial}\hat{R} - \chi_0 B^{\approx}\bar{\partial}R + \tfrac{1}{2}m^{-1}\chi_0\bar{\partial}B^{\approx} \\
&\quad + \tfrac{1}{2}m^{-1}B^{\approx}\bar{\partial}\chi_0\Big]e^{-2mR} + (2m/\pi)^{\frac{1}{2}}(1-\chi_0)A^{\approx}\bar{\partial}V\, e^{-2mV^2} \\
&= (2m/\pi)^{\frac{1}{2}}\Big((\pi/2)^{\frac{1}{2}}\chi_0\bar{F}^{\approx} + \tfrac{1}{2}m^{-1}B^{\approx}\bar{\partial}\chi_0 - \chi_0\mathcal{E}_m\bar{\partial}R\Big)e^{-2mR} \\
&\quad + (2m/\pi)^{\frac{1}{2}}(1-\chi_0)A^{\approx}\bar{\partial}V\, e^{-2mV^2}.
\end{aligned}
\tag{4.2.2}
$$

Since $V(z)$ approaches $+\infty$ as $|z| \to +\infty$, the potential $\Psi^{\approx}$ has a limit at infinity, which is given by

$$
(\zeta\Psi^{\approx})(\infty) = (\zeta A^{\approx})(\infty) = \frac{a_1}{H_R(\infty)} - m^{-1}\frac{(2\pi)^{-\frac{1}{2}}}{H_R(\infty)}\left\langle\frac{\bar{\partial}B^{\approx}}{\bar{H}_R}\right\rangle_{\mathbb{T}}.
\tag{4.2.3}
$$

In view of the Cauchy-Green formula, we have

$$
G(\infty)(\zeta\Psi^{\approx})(\infty) - \frac{1}{2\pi i}\int_{\mathbb{T}(0,\rho_1)}G\Psi^{\approx}dz = \int_{\mathbb{D}_e(0,\rho_1)}G\bar{\partial}\Psi^{\approx}dA,
$$

which we rewrite in the form

$$
\begin{aligned}
&m^{\frac{1}{2}}\int_{\mathbb{D}_e(0,\rho_1)}G\chi_0\bar{F}^{\approx}e^{-2mR}dA \\
&= G(\infty)(\zeta\Psi^{\approx})(\infty) - \frac{1}{2\pi i}\int_{\mathbb{T}(0,\rho_1)}G\Psi^{\approx}dz - \int_{\mathbb{D}_e(0,\rho_1)}G\left(\bar{\partial}\Psi^{\approx} - m^{\frac{1}{2}}\chi_0\bar{F}^{\approx}e^{-2mR}\right)dA \\
&= G(\infty)(\zeta\Psi^{\approx})(\infty) - \frac{1}{2\pi i}\int_{\mathbb{T}(0,\rho_1)}GA^{\approx}\,\mathrm{erf}(2m^{\frac{1}{2}}\hat{R})dz \\
&\quad - (2m/\pi)^{\frac{1}{2}}\int_{\mathbb{D}_e(0,\rho_1)}G\left\{\tfrac{1}{2}m^{-1}B^{\approx}\bar{\partial}\chi_0\,e^{-2mR} + (1-\chi_0)A^{\approx}\bar{\partial}V\,e^{-2mV^2}\right\}dA \\
&\quad + (2m/\pi)^{\frac{1}{2}}\int_{\mathbb{D}_e(0,\rho_1)}G\chi_0\mathcal{E}_m\bar{\partial}R\,e^{-2mR}dA,
\end{aligned}
\tag{4.2.4}
$$

where in the last step we used that $\chi_0 = 0$ and $V = \hat{R}$ on the circle $\mathbb{T}(0,\rho_1)$, as well as the formulæ (4.1.2) and (4.2.2). We now turn to estimating the expressions on the right-hand side of (4.2.4). A first observation is that in the integral along the circle $\mathbb{T}(0,\rho_1)$, $\hat{R} < 0$, so that $\mathrm{erf}(2m^{\frac{1}{2}}\hat{R})$ is small there. To obtain a precise estimate, we may apply integration by parts to see that for $x > 0$,

$$
\mathrm{erf}(-x) = (2\pi)^{-\frac{1}{2}}x^{-1}e^{-x^2/2} - (2\pi)^{-\frac{1}{2}}\int_x^{+\infty}t^{-2}e^{-t^2/2}dt < (2\pi)^{-\frac{1}{2}}x^{-1}e^{-x^2/2},
$$

so that

$$\operatorname{erf}(2m^{\frac{1}{2}}\hat{R}) < (8m\pi)^{-\frac{1}{2}}R^{-\frac{1}{2}}e^{-2mR} \quad \text{on} \quad \mathbb{T}(0, \rho_1).$$

This gives that

$$\left| \frac{1}{2\pi i} \int_{\mathbb{T}(0,\rho_1)} GA^{\approx} \operatorname{erf}(2m^{\frac{1}{2}}\hat{R})dz \right|$$

$$\leq (8m\pi)^{-\frac{1}{2}} \|G\|_{mR_-} \|A^{\approx}\|_{L^{\infty}(\mathbb{T}(0,\rho_1))} \|R^{-\frac{1}{2}}e^{-mR}\|_{L^{\infty}(\mathbb{T}(0,\rho_1))}$$

$$= O\left(\|G\|_{mR_-} \|A^{\approx}\|_{L^{\infty}(\mathbb{T}(0,\rho_1))} m^{-\frac{1}{2}} e^{-\varepsilon_2 m} \right), \tag{4.2.5}$$

where the positive constant ε_2 may be taken as the minimum of R on the circle $\mathbb{T}(0,\rho_1)$, and the implicit "O" constant remains bounded as $m \to +\infty$. This offers an exponential decay estimate of one term on the right-hand side of (4.2.4). To handle the next term, we use the estimate

$$\left| \int_{\mathbb{D}_e(0,\rho_1)} GB^{\approx} \bar{\partial}\chi_0\, e^{-2mR} dA \right|$$

$$\leq \|G\|_{mR_-} \|B^{\approx}\|_{L^{\infty}(\mathcal{A}_0)} \|\bar{\partial}\chi_0\|_{L^{\infty}(\mathbb{C})} \int_{\mathcal{A}_0\setminus\mathcal{A}_1} e^{-mR} dA$$

$$= O\left(\|G\|_{mR_-} \|B^{\approx}\|_{L^{\infty}(\mathcal{A}_0)} e^{-\varepsilon_3 m} \right), \tag{4.2.6}$$

where the positive constant ε_3 may be taken as the infimum of R on the set $\mathcal{A}_0 \setminus \mathcal{A}_1$, and the implicit "O" constant remains bounded as $m \to +\infty$. The following term is handled similarly:

$$\left| \int_{\mathbb{D}_e(0,\rho_1)} (1-\chi_0)GA^{\approx} \bar{\partial}V\, e^{-2mV^2} dA \right|$$

$$\leq \|G\|_{mR_-} \|A^{\approx}\|_{L^{\infty}(\mathbb{T}(0,\rho_1))} \int_{\mathbb{D}_e(0,\rho_1)} (1-\chi_0)|\bar{\partial}V|e^{-2mV^2+mV_-^2} dA$$

$$\leq C_2 \|G\|_{mR_-} \|A^{\approx}\|_{L^{\infty}(\mathbb{T}(0,\rho_1))} m^{-1}e^{-\varepsilon_1 m}, \tag{4.2.7}$$

in view of (4.1.1) and the maximum principle applied to the bounded holomorphic function $A^{\approx}$ on $\mathbb{D}_e(0,\rho_1)$. The last term on the right-hand side of (4.2.4) involves the error term $\mathcal{E}_m$, and we estimate it as follows:

$$\left| \int_{\mathbb{D}_e(0,\rho_1)} G\chi_0\mathcal{E}_m\bar{\partial}R\, e^{-2mR} dA \right|$$

$$\leq \|G\|_{mR_-} \|\mathcal{E}_m\|_{L^{\infty}(\mathcal{A}_0)} \int_{\mathcal{A}_0} |\bar{\partial}R|\, e^{-2mR+mR_-} dA$$

$$= O\left(m^{-\frac{1}{2}} \|G\|_{mR_-} \|\mathcal{E}_m\|_{L^{\infty}(\mathcal{A}_0)} \right), \tag{4.2.8}$$

which tells us about the need to estimate the error $\mathcal{E}_m$ uniformly on the annulus $\mathcal{A}_0$. For the bookkeeping, we apply the obtained estimates (4.2.5), (4.2.6), (4.2.7), and (4.2.8) in the context of (4.2.4), to obtain that

$$m^{\frac{1}{2}} \int_{\mathbb{D}_e(0,\rho_1)} G\chi_0 \bar{F}^{\approx} e^{-2mR} dA$$

$$= a_1 \frac{G(\infty)}{H_R(\infty)} - \frac{(2\pi)^{-\frac{1}{2}} G(\infty)}{m H_R(\infty)} \left\langle \frac{\bar{\partial} B^{\approx}}{\bar{H}_R} \right\rangle_{\mathbb{T}} + O\left(\|G\|_{mR_-} \|A^{\approx}\|_{L^\infty(\mathbb{T}(0,\rho_1))} e^{-\varepsilon_4 m} \right)$$

$$+ O\left(\|G\|_{mR_-} \|B^{\approx}\|_{L^\infty(\mathcal{A}_0)} m^{-\frac{1}{2}} e^{-\varepsilon_4 m} \right) + O\left(\|G\|_{mR_-} \|\mathcal{E}_m\|_{L^\infty(\mathcal{A}_0)} \right), \tag{4.2.9}$$

where ε_4 stands for the least of the positive constants $\varepsilon_1, \varepsilon_2, \varepsilon_3$, and we have applied the identity (4.2.3). If $G(\infty) = 0$, and if the norm of $\mathcal{E}_m$ decays quickly to 0 as $m \to +\infty$, it follows from (4.2.9) that $\chi_0 F^{\approx}$ is approximately orthogonal to G in the space $L^2(\mathbb{D}_e(0,\rho_1), e^{-2mR} dA)$. Finally, we would like to turn this into an approximate orthogonality between the functions $F^{\approx}$ and G, but involving a related smooth cut-off function which avoids making a smooth cut-off in the exterior disk $\mathbb{D}_e$. The cut-off function χ_0 factors $\chi_0 = \chi_{0,0} \chi_{0,1}$, where $\chi_{0,j}$ are radial and C^∞-smooth with $0 \le \chi_{0,j} \le 1$, and $\chi_{0,0}$ decreases as the radius increases, whereas $\chi_{0,1}$ instead increases. Then $\chi_{0,0} = \chi_{0,1} = 1$ on the annulus $\mathcal{A}_1$, so that in particular $\chi_{0,1} = 1$ holds on $\mathbb{D}_e$ while $\chi_{0,0} = 1$ on $\mathbb{D}$. Moreover, for $j = 0, 1$, $|\bar{\partial}\chi_{0,j}|^2 \le C_1 \chi_{0,j}$ holds with the same constant as for χ_0. The identity

$$\int_{\mathbb{D}_e(0,\rho_1)} G\chi_{0,1} \bar{F}^{\approx} e^{-2mR} dA$$

$$= \int_{\mathbb{D}_e(0,\rho_1)} G\chi_0 \bar{F}^{\approx} e^{-2mR} dA + \int_{\mathbb{D}_e(0,\rho_1)} G(1 - \chi_{0,0})\chi_{0,1} \bar{F}^{\approx} e^{-2mR} dA \tag{4.2.10}$$

together with the estimate

$$\int_{\mathbb{D}_e(0,\rho_1)} G(1 - \chi_{0,0})\chi_{0,1} \bar{F}^{\approx} e^{-2mR} dA$$

$$= O\left(\|G\|_{mR_-} \|F^{\approx}\|_{L^\infty(\mathbb{T}(0,\rho_1))} m^{-\frac{1}{2}} e^{-\varepsilon_5 m} \right), \tag{4.2.11}$$

tells us that we may think about $\chi_{0,1} F^{\approx}$ in place of $\chi_0 F^{\approx}$ as being approximately orthogonal to G provided that $G(\infty) = 0$. The estimate (4.2.11) holds for some $\varepsilon_5 > 0$ because R is bounded below by a positive constant on $\mathbb{D}_e(0,\rho_1) \setminus \mathcal{A}_1$, and because R has some minimal logarithmic growth at infinity.

4.3 | Control of the error terms

By a combination of (4.2.9), (4.2.10), and (4.2.11), we obtain that

$$m^{\frac{1}{2}} \int_{\mathbb{D}_e(0,\rho_1)} G\chi_{0,1} \bar{F}^{\approx} e^{-2mR} dA$$

$$= \frac{G(\infty)}{H_R(\infty)^2} - \frac{(2\pi)^{-\frac{1}{2}} G(\infty)}{m H_R(\infty)} \left\langle \frac{\bar{\partial} B^{\approx}}{\bar{H}_R} \right\rangle_{\mathbb{T}} + O\left(\|G\|_{mR_-} \|A^{\approx}\|_{L^\infty(\mathbb{T}(0,\rho_1))} e^{-\varepsilon_6 m} \right)$$

$$+ O\left(\|G\|_{mR_-} \|B^{\approx}\|_{L^{\infty}(\mathcal{A}_0)} m^{-\frac{1}{2}} e^{-\varepsilon_6 m} \right) + O\left(\|G\|_{mR_-} \|F^{\approx}\|_{L^{\infty}(\mathbb{T}(0,\rho_1))} e^{-\varepsilon_6 m} \right)$$

$$+ O\left(\|G\|_{mR_-} \|\mathcal{E}_m\|_{L^{\infty}(\mathcal{A}_0)} \right), \tag{4.3.1}$$

if $\varepsilon_6 > 0$ is the minimum of ε_4 and ε_5. Here, we applied the formula for a_1 of (3.1.10). We may apply the identity with $G = F^{\approx}$, while recalling the normalization $F^{\approx}(\infty) = 1$, and since, by the maximum principle,

$$\|G\|_{mR_-} \leq \|G\|_{L^{\infty}(\mathbb{T}(0,\rho_1))},$$

it follows from (4.3.1) that

$$m^{\frac{1}{2}} \int_{\mathbb{D}_e(0,\rho_1)} \chi_{0,1} |F^{\approx}|^2 e^{-2mR} dA$$

$$= \frac{1}{H_R(\infty)^2} - \frac{(2\pi)^{-\frac{1}{2}}}{mH_R(\infty)} \left\langle \frac{\bar{\partial}B^{\approx}}{\bar{H}_R} \right\rangle_{\mathbb{T}} + O\left(\|F^{\approx}\|_{L^{\infty}(\mathbb{T}(0,\rho_1))} \|A^{\approx}\|_{L^{\infty}(\mathbb{T}(0,\rho_1))} e^{-\varepsilon_6 m} \right)$$

$$+ O\left(\|F^{\approx}\|_{L^{\infty}(\mathbb{T}(0,\rho_1))} \|B^{\approx}\|_{L^{\infty}(\mathcal{A}_0)} m^{-\frac{1}{2}} e^{-\varepsilon_6 m} \right) + O\left(\|F^{\approx}\|^2_{L^{\infty}(\mathbb{T}(0,\rho_1))} e^{-\varepsilon_6 m} \right)$$

$$+ O\left(\|F^{\approx}\|_{L^{\infty}(\mathbb{T}(0,\rho_1))} \|\mathcal{E}_m\|_{L^{\infty}(\mathcal{A}_0)} \right), \tag{4.3.2}$$

which tells us how to compute the left-hand side norm once we may control the error term $\mathcal{E}_m$. Incidentally, it follows that up to small error, the mean value

$$\left\langle \frac{\bar{\partial}B^{\approx}}{\bar{H}_R} \right\rangle_{\mathbb{T}}$$

must be real-valued. We recall that $\phi = \varphi^{-1}$, and write $G := \phi' G \circ \phi$ and $F^{\approx} := \phi' F^{\approx} \circ \phi$, which means that we think that the two functions transform as $(1,0)$-forms. Moreover, we put $\chi_{1,1} := \chi_{0,1} \circ \phi$, and observe that the change-of-variables formula gives that

$$\int_{\mathbb{D}_e(0,\rho_1)} G \chi_{0,1} \bar{F}^{\approx} e^{-2mR} dA = \int_{\varphi(\mathbb{D}_e(0,\rho_1))} G \chi_{1,1} \bar{F}^{\approx} e^{-2mR} dA \tag{4.3.3}$$

and that

$$\int_{\mathbb{D}_e(0,\rho_1)} \chi_{0,1} |F^{\approx}|^2 e^{-2mR} dA = \int_{\varphi(\mathbb{D}_e(0,\rho_1))} \chi_{1,1} |F^{\approx}|^2 e^{-2mR} dA. \tag{4.3.4}$$

We note that the integrals on the right-hand sides of (4.3.3) and (4.3.4) may be thought to extend to all of $\mathbb{C}$ by declaring that the integrand vanishes in the complement of $\varphi(\mathbb{D}_e(0,\rho_1))$. The extended integrands are then C^{∞}-smooth on $\mathbb{C}$. In terms of $F^{\approx} = \phi' F^{\approx} \circ \phi$, we write in accordance with (1.6.3) that

$$P^{\approx} := c_m \phi^m e^{m\mathbb{Q}} F^{\approx}, \quad g := \phi^m e^{m\mathbb{Q}} G,$$

which we think of as the approximate monic orthogonal polynomial of degree m and a general approximate polynomial of degree $\leq m$, respectively. We find that by (1.6.1) and the identity $R = Q - \check{Q}_1$ (cf. Subsection 1.4), we have

$$\int_{\mathbb{D}_e(0,\rho_1)} G\, \chi_{0,1}\, \bar{F}^{\approx} e^{-2mR} dA = c_m^{-1} \int_{\mathbb{C}} g\, \chi_{1,1}\, \bar{P}^{\approx} e^{-2mQ} dA, \tag{4.3.5}$$

while

$$\int_{\mathbb{D}_e(0,\rho_1)} \chi_{0,1}^2\, |F^{\approx}|^2 e^{-2mR} dA = c_m^{-2} \int_{\mathbb{C}} \chi_{1,1}^2\, |P^{\approx}|^2 e^{-2mQ} dA. \tag{4.3.6}$$

Note that in (4.3.6) we square the cut-off function, which is all right since the square shares all the essential features with the cut-off function itself. Most importantly, (4.3.2) holds with $\chi_{0,1}$ replaced by its square $\chi_{0,1}^2$. We shall apply the identity (4.3.5) to an actual polynomial g of degree $\leq m - 1$. In this case, $G(\infty) = 0$, so that the first term on the right-hand side of (4.3.1) vanishes and all that remains are the error terms. To simplify the further discussion we recall the notation $\mathrm{Pol}_n = \mathrm{Pol}_{mQ,n}$ for the subspace of all polynomials with degree $\leq n$ in the Hilbert space $L_{mQ}^2 = L^2(\mathbb{C}, e^{-2mQ})$. We normalize g to have norm 1 in the space $\mathrm{Pol}_{mQ,m-1}$. The Bernstein-Walsh estimate of Proposition 2.1 applies in particular to polynomials of degree $\leq m$, and reads

$$|h(z)| \leq C_Q \|h\|_{mQ} m^{\frac{1}{2}} e^{m\check{Q}_1(z)}, \qquad z \in \mathbb{C}, \tag{4.3.7}$$

for some positive constant C_Q and every $h \in \mathrm{Pol}_{mQ,m}$. As we apply this estimate to $g \in \mathrm{Pol}_{mQ,m-1}$, it follows that

$$|G| \leq C_Q m^{\frac{1}{2}} |\varphi'| e^{mR_-}.$$

If $\rho_1 < 1$ is close enough to 1, the derivative φ' is uniformly bounded in $\mathbb{D}_e(0, \rho_1)$, so that in view of definition of the norm $\|\cdot\|_{mR_-}$ in (4.2.1), we find that

$$\|G\|_{mR_-} = O(m^{\frac{1}{2}}) \quad \text{as } m \to +\infty. \tag{4.3.8}$$

By classical duality, we have that

$$\sup_{g \in \mathrm{Pol}_{m-1} : \|g\|_{mQ}=1} \left| \int_{\mathbb{C}} g\, \chi_{1,1}\, \bar{P}^{\approx} e^{-2mQ} dA \right| = \inf_{h \in (\mathrm{Pol}_{m-1})^{\perp}} \|\chi_{1,1} P^{\approx} - h\|_{mQ},$$

and if we combine this identity with the estimate (4.3.8) together with the relations (4.3.1) and (4.3.5), we arrive at

$$\inf_{h \in (\mathrm{Pol}_{m-1})^{\perp}} \|\chi_{1,1} P^{\approx} - h\|_{mQ} = O\left(c_m \|A^{\approx}\|_{L^{\infty}(\mathbb{T}(0,\rho_1))} m^{\frac{1}{2}} e^{-\varepsilon_6 m}\right)$$

$$+ O\left(c_m \|B^{\approx}\|_{L^{\infty}(\mathcal{A}_0)} e^{-\varepsilon_6 m}\right) + O\left(c_m \|F^{\approx}\|_{L^{\infty}(\mathbb{T}(0,\rho_1))} m^{\frac{1}{2}} e^{-\varepsilon_6 m}\right)$$

$$+ O\left(c_m m^{\frac{1}{2}} \|\mathcal{E}_m\|_{L^{\infty}(\mathcal{A}_0)}\right). \tag{4.3.9}$$

4.4 | Application of Hörmander's $\bar{\partial}$-estimate

The next step follows the lines of [20]. We want to turn the approximate polynomial $P^{\approx}$ into an actual polynomial. We put

$$P^* := \chi_{1,1} P^{\approx} - u_1, \tag{4.4.1}$$

where u_1 solves the $\bar{\partial}$-problem

$$\bar{\partial} u_1 = P^{\approx} \bar{\partial} \chi_{1,1}.$$

As before, $\chi_{1,1} P^{\approx}$ is extended to vanish wherever $P^{\approx}$ gets undefined, and hence defines a C^{∞}-smooth function in the complex plane $\mathbb{C}$. Clearly, $\bar{\partial} P^* = 0$ holds throughout and hence P^* defines an entire function. By Proposition 2.2, based on Hörmander's classical $\bar{\partial}$-estimate (which in itself is a dualized version of the corresponding Carleman estimate), there exists a u_1 of growth $u_1(z) = \mathrm{O}(|z|^{n-1})$ at infinity, such that

$$\|u_1\|_{mQ}^2 \leq \frac{1}{2m} \int_{S_1} |P^{\approx}|^2 |\bar{\partial} \chi_{1,1}|^2 \frac{\mathrm{e}^{-2mQ}}{\Delta Q} \mathrm{dA}. \tag{4.4.2}$$

In view of the growth bound on u_1, P^* must be a polynomial of degree $n = m$, and its leading coefficient equals 1, since $F^{\approx}(\infty) = \phi'(\infty)$ and hence, by the definition of $c_m = c_{m,m}$ in (1.6.2),

$$\lim_{|z| \to +\infty} z^{-n} P^{\approx}(z) = c_m \phi'(\infty)^m \mathrm{e}^{m\mathbb{Q}(\infty)} F^{\approx}(\infty) = 1.$$

After a little bit of rewriting, we find that

$$\int_{S_1} |P^{\approx}|^2 |\bar{\partial} \chi_{1,1}|^2 \frac{\mathrm{e}^{-2mQ}}{\Delta Q} \mathrm{dA} = c_m^2 \int_{\mathbb{D}_e(0,\rho_1)} |F^{\approx}|^2 |\bar{\partial} \chi_{0,1}|^2 \frac{\mathrm{e}^{-2mR}}{\Delta R} \mathrm{dA}.$$

This expression decays exponentially as $m \to +\infty$, as R is strictly positive on the support of $\bar{\partial} \chi_{0,1}$. It now follows from (4.4.2) that

$$\|u_1\|_{mQ} = \mathrm{O}\big(c_m \|F^{\approx}\|_{L^{\infty}(\mathbb{T}(0,\rho_1))} m^{-1} \mathrm{e}^{-\varepsilon_7 m}\big) \tag{4.4.3}$$

for a positive constant $\varepsilon_7 > 0$. This procedure gives us a monic polynomial P^* of degree n with

$$\inf_{h \in (\mathrm{Pol}_{m-1})^{\perp}} \|P^* - h\|_{mQ} = \mathrm{O}\left(c_m \|A^{\approx}\|_{L^{\infty}(\mathbb{T}(0,\rho_1))} m^{\frac{1}{2}} \mathrm{e}^{-\varepsilon_8 m}\right)$$

$$+ \mathrm{O}\big(c_m \|B^{\approx}\|_{L^{\infty}(\mathcal{A}_0)} \mathrm{e}^{-\varepsilon_8 m}\big) + \mathrm{O}\left(c_m \|F^{\approx}\|_{L^{\infty}(\mathbb{T}(0,\rho_1))} m^{\frac{1}{2}} \mathrm{e}^{-\varepsilon_8 m}\right)$$

$$+ \mathrm{O}\left(c_m m^{\frac{1}{2}} \|\mathcal{E}_m\|_{L^{\infty}(\mathcal{A}_0)}\right), \tag{4.4.4}$$

if ε_8 is the least of ε_6 and ε_7. The minimizing element $h \in (\mathrm{Pol}_{m-1})^{\perp}$ must now be a constant multiple of the *true monic orthogonal polynomial P*, that is,

$$\inf_{h \in (\mathrm{Pol}_{m-1})^{\perp}} \left\| P^* - h \right\|_{mQ} = \left\| P^* - \alpha P \right\|_{mQ}, \quad \text{where} \quad \alpha = \|P\|_{mQ}^{-2} \langle P^*, P \rangle_{mQ},$$

simply as a result of the Pythagorean theorem. Later on, in Subsections 5.8 and 5.9, we shall see that it is possible to arrange that

$$\|A^{\approx}\|_{L^{\infty}(\mathbb{T}(0,\rho_1))} = \mathrm{O}(1), \quad \|F^{\approx}\|_{L^{\infty}(\mathbb{T}(0,\rho_1))} = \mathrm{O}(1),$$

$$\|B^{\approx}\|_{L^{\infty}(\mathcal{A}_0)} = \mathrm{O}(1), \quad \|\mathcal{E}_m\|_{L^{\infty}(\mathcal{A}_0)} = \mathrm{O}(e^{-\varepsilon\sqrt{m}}) \tag{4.4.5}$$

as $m \to +\infty$, for some constant $\varepsilon > 0$ which only depends on R. This assertion is contained in Theorem 5.3, and also expressed explicitly in (5.8.4), (5.8.5), and (5.9.6), provided that the annulus $\mathcal{A}_0$ is thin enough and $\rho_1 < 1$ is sufficiently close to 1 (the control of the norm in $\mathscr{H}_{\frac{1}{2}\sigma_0}$ implies uniform control on an annulus about $\mathbb{T}$). Since the rate of decay $e^{-\varepsilon\sqrt{m}}$ is slower than exponential decay, we conclude from (4.4.4) and (4.4.5) that

$$\left\| P^* - \alpha P \right\|_{mQ} = \mathrm{O}(c_m \, m^{\frac{1}{2}} e^{-\varepsilon\sqrt{m}}) \tag{4.4.6}$$

where $\alpha = \|P\|_{mQ}^{-2} \langle P^*, P \rangle_{mQ}$. Next, according to the Bernstein-Walsh estimate (4.3.7), it follows that

$$|P^*(z) - \alpha P(z)| = \mathrm{O}\left(c_m m \, e^{-\varepsilon\sqrt{m}} e^{m\check{Q}_1(z)} \right).$$

uniformly in z as $m \to +\infty$. Since both P^* and P are monic, we find that

$$|1 - \alpha| = \lim_{|z| \to +\infty} \left| \frac{P^*(z) - \alpha P(z)}{z^m} \right| = \mathrm{O}(m \, e^{-\varepsilon\sqrt{m}}),$$

which means that α is very close to 1. Finally, to appreciate the value of (4.4.6), we should have some general idea of the size of the norm $\|P^*\|_{mQ}$. In view of (3.3.3), we have that $F^{\approx} = a_1 H_R + \mathrm{O}(m^{-1})$, a fact which relies on the properties of $B^{\approx}$. Actually, this is asserted by Theorem 5.3 in the Subsection 5.9 below (cf. equation (5.9.5)). Using the norm identity (4.3.6), this information gives that

$$\|\chi_{1,1} P^{\approx}\|_{mQ} \asymp c_m m^{-\frac{1}{4}}, \tag{4.4.7}$$

and if we use the estimate (4.4.3), it follows that

$$\|P^*\|_{mQ} \asymp c_m m^{-\frac{1}{4}}.$$

The formula

$$P = \alpha^{-1} P^* - \alpha^{-1}(P^* - \alpha P)$$

now shows that

$$\|P - P^*\|_{mQ} = O\left(c_m m^{\frac{3}{4}} e^{-\epsilon\sqrt{m}}\right)$$

as $m \to +\infty$. We can easily turn this into a pointwise estimate by applying the Bernstein-Walsh inequality (4.3.7). However, it is more appealing to relate P directly to the approximate polynomial $P^{\approx}$ which the algorithm provides. In view of the exponential decay estimate (4.4.3) together with the basic estimate (4.4.5), we have by (4.4.1) that

$$\|P - \chi_{1,1}P^{\approx}\|_{mQ} = O\left(c_m m^{\frac{3}{4}} e^{-\epsilon\sqrt{m}}\right) \tag{4.4.8}$$

as $m \to +\infty$. For instance, this means that in the sense of the L^2_{mQ}-norm, the polynomial P is small where $\chi_{1,1} = 0$. The Bernstein-Walsh estimate of Proposition 2.2.2 in [20] applies here as well, although $P - \chi_{1,1}P^{\approx}$ is not a polynomial. It gives that

$$|P(z) - P^{\approx}(z)| = O\left(c_m m^{\frac{5}{4}} e^{-\epsilon\sqrt{m}} e^{m\check{Q}_1(z)}\right) \tag{4.4.9}$$

holds uniformly as $m \to +\infty$ over the whole region where $\chi_{1,1} = 1$ (this is the image under φ of a closed exterior disk with a radius ρ_2 such that $\rho_1 < \rho_2 < 1$), except that we would need to take a step away from the boundary (of the region where $\chi_{1,1} = 1$) of size at least $m^{-\frac{1}{2}}$. We rewrite (4.4.9) in the form

$$P(z) = P^{\approx}(z) + O\left(c_m m^{\frac{5}{4}} e^{-\epsilon\sqrt{m}} e^{m\check{Q}_1(z)}\right)$$

$$= c_m \phi^m e^{mQ}\left(F^{\approx}(z) + O\left(m^{\frac{5}{4}} e^{-\epsilon\sqrt{m}} e^{mR_-}\right)\right), \tag{4.4.10}$$

which holds uniformly on the same region where (4.4.9) holds, where $R_- = R_-\circ\phi = \check{Q}_1 - \check{Q}_1$. If we consider the region

$$D_m := \left\{z \in \mathbb{D}_e(0, \rho_2): \ R_-(z) \le \frac{1}{2}\epsilon m^{-\frac{1}{2}}\right\}, \tag{4.4.11}$$

which extends inward from the exterior disk $\mathbb{D}_e$ a distance proportional to $m^{-\frac{1}{4}}$, we obtain from (4.4.10) that

$$P(z) = c_m \phi^m e^{mQ}\left(F^{\approx}(z) + O\left(m^{\frac{5}{4}} e^{-\frac{1}{2}\epsilon\sqrt{m}}\right)\right), \qquad z \in D_m, \tag{4.4.12}$$

uniformly as $m \to +\infty$, where $D_m = \varphi(D_m)$ and $\varphi = \phi^{-1}$.

Remark 4.1. According to the estimate (5.9.5) below, $F^{\approx} = a_1 H_R + O(m^{-1})$ holds in a fixed neighborhood of $\bar{\mathbb{D}}_e$, where the positive constant a_1 is given by (3.1.10). If P were to have a zero

$z_0 = z_0(m)$ in the domain $D_m = \varphi(\mathrm{D}_m)$, then (4.4.12) tells us that the decay $F^{\approx}(z_0(m)) = \mathrm{O}(m^{\frac{5}{4}} \mathrm{e}^{-\frac{1}{2}\epsilon\sqrt{m}})$ must hold, and consequently, $F^{\approx}(w_0(m)) = \mathrm{O}(m^{\frac{5}{4}} \mathrm{e}^{-\frac{1}{2}\epsilon\sqrt{m}})$, where $w_0(m) = \phi(z_0(m)) \in \mathrm{D}_m$. But then, since $w_0(m) \in \mathrm{D}_m$ is within distance $\asymp m^{-\frac{1}{4}}$ to the closed exterior disk $\bar{\mathbb{D}}_{\mathrm{e}}$, and $a_1 \mathrm{H}_{\mathrm{R}}$ is uniformly bounded away from 0 there, the approximation $F^{\approx}(w_0(m)) = a_1 \mathrm{H}_{\mathrm{R}}(w_0(m)) + \mathrm{O}(m^{-1})$ of the estimate (5.9.5) below shows that the required decay is impossible. In other words, for big enough m, P has no zeros in D_m. Moreover, the domain D_m consists of points of with distance at most $\asymp m^{-\frac{1}{4}}$ away from the complement of the droplet $\mathbb{C} \setminus S_1$.

4.5 | The proof of the main theorem

We are now prepared to obtain Theorem 1.2. Please note that the proof relies crucially on Theorem 5.3, which is established in Section 5 below.

Proof of Theorem 1.2. The asserted L^2_{mQ}-norm approximation of P by $\chi_{1,1} P^{\approx}$ is expressed in the estimate (4.4.8), while the uniform control along shrinking domains is asserted in (4.4.12). The approximate size of the norm of $\chi_{1,1} P^{\approx}$ is found in (4.4.7). □

5 | ASYMPTOTIC GROWTH ANALYSIS

5.1 | Discrete analysis of Nishida-Nirenberg type

If we write $s := m^{-1}$, which we think of as tending to 0^+, the equation (3.2.2) assumes the form

$$\mathrm{B} = \mathrm{B}_0 + s\mathbf{T}[\mathrm{B}], \tag{5.1.1}$$

which we may think of as a discrete one time-step Cauchy-Kovalevskaya evolution equation, where s is the time step. The continuous Cauchy-Kovalevskaya evolution was analyzed by Nirenberg [27] and Nishida [28]. Nishida uses a continuous Banach scale for his solution, which suggests we should employ a discrete collection of Banach spaces for our one-step problem. The operators $\mathbf{T}$ and $\mathbf{L}$ do not involve the parameter s, and iteration of (5.1.1) would give that

$$\mathrm{B} = \mathrm{B}_0 + s\mathbf{T}[\mathrm{B}_0] + \cdots + s^N \mathbf{T}^N[\mathrm{B}_0] + s^{N+1}\mathbf{T}^{N+1}[\mathrm{B}], \tag{5.1.2}$$

which we think of as saying that

$$\mathrm{B} = \mathrm{B}_0 + s\mathbf{T}[\mathrm{B}_0] + \cdots + s^N \mathbf{T}^N[\mathrm{B}_0] + \mathrm{O}(s^{N+1}),$$

that is, an asymptotic expansion of B in $s = m^{-1}$. Once the asymptotic expansion is found, a suitable function B may be concocted from the (possibly divergent) expansion, as in [16]. The functions involved, H_{R}, $\hat{\mathrm{R}}$, and W_{R} are all real-analytic in a neighborhood of $\mathbb{T}$, and hence the same applies to B_0. To know better how to forge an approximate solution $\mathrm{B}^{\approx}$, we first analyze carefully the operators $\mathbf{T}$ and $\mathbf{L}$.

5.2 │ Polarization and the scale of Banach spaces

We follow the approach in [20], and analyze real-analytic functions f in a neighborhood of $\mathbb{T}$ in terms of polarized extensions $f^\diamond(z, w)$, where z, w are both near the circle $\mathbb{T}$ and close to one another. The polarization $f^\diamond(z, w)$ is holomorphic in $(z, \bar{w})$, and the defining property is that $f^\diamond(z, z) = f(z)$. For $0 < \rho < 1$, we consider annuli of the form

$$\mathbb{A}(\rho) := \{z \in \mathbb{C} : \rho < |z| < \rho^{-1}\},$$

as well as polarized neighborhoods of $\mathbb{T}$ of the form

$$\hat{\mathbb{A}}(\rho, \sigma) := \{(z, w) \in \mathbb{C}^2 : z, w \in \mathbb{A}(\rho), |z - w| < 2\sigma\},$$

where $0 < \sigma < 1$. We shall let ρ be connected to the parameter σ via

$$\rho = \rho(\sigma) := \frac{1}{\sigma + \sqrt{1 + \sigma^2}} < 1, \tag{5.2.1}$$

so that asymptotically as $\sigma \to 0^+$, we have that $\rho(\sigma) = 1 - \sigma + O(\sigma^2)$. For this case, we simplify the notation for the polarized domain:

$$\hat{\mathbb{A}}(\sigma) := \hat{\mathbb{A}}(\rho(\sigma), \sigma).$$

We consider a suitable scale of Banach spaces $\mathscr{H}_\sigma^\infty$ indexed by σ. To be precise, the space $\mathscr{H}_\sigma^\infty$ consists of all L^∞ functions on $\hat{\mathbb{A}}(\sigma)$ that are holomorphic in the variables $(z, \bar{w})$, supplied with the L^∞ norm. Then $\mathscr{H}_\sigma^\infty$ is a Banach algebra, since $\|fg\|_{\mathscr{H}_\sigma^\infty} \leq \|f\|_{\mathscr{H}_\sigma^\infty} \|g\|_{\mathscr{H}_\sigma^\infty}$ for $f, g \in \mathscr{H}_\sigma^\infty$. In terms of unique analytic continuation, it is clear that the spaces $\mathscr{H}_\sigma^\infty$ get smaller as σ increases, that is, $\mathscr{H}_\sigma^\infty \subset \mathscr{H}_{\sigma'}^\infty$ holds for $0 < \sigma' < \sigma < 1$, and that the injection mapping $\mathscr{H}_\sigma^\infty \hookrightarrow \mathscr{H}_{\sigma'}^\infty$ is norm contractive. We begin with a sufficiently small σ_0 with $0 < \sigma_0 < 1$ such that the functions

$$\frac{1}{\bar{H}_R}, \quad \bar{H}_R \frac{1 - |\zeta|^2}{4\hat{R}\bar{\partial}\hat{R}}, \quad \frac{W_R}{\zeta H_R}$$

appearing in the definitions of $\mathbf{L}$ and $\mathbf{T}$ may be thought of as elements of $\mathscr{H}_{\sigma_0}^\infty$. How small σ_0 will need to be then naturally will depend on R. In view of (3.2.2), the operator $\mathbf{T}$ consists of first taking the $\bar{\partial}$-derivative, then multiplying by $\bar{H}_R^{-1}$, and finally applying the operator $\mathbf{L}$. In its turn, the main ingredients for the operator $\mathbf{L}$ are $\mathbf{P}_{H_-^2}$ and the operator $\mathbf{M}$ given by

$$\mathbf{M}[f] := \frac{f - \mathbf{U}_{\mathbb{D}_e}[f]}{1 - |\zeta|^2}.$$

Indeed, the direct relationship reads

$$\mathbf{L}[f] := \frac{(1 - |\zeta|^2)\bar{H}_R}{4\hat{R}\bar{\partial}\hat{R}} \mathbf{M}[f] - (2\pi)^{-\frac{1}{2}} \frac{W_R}{\zeta H_R} \mathbf{P}_{H_-^2}[f]. \tag{5.2.2}$$

5.3 | The basic growth estimate

The basic result in this section consists of the following couple of estimates.

Theorem 5.1. *If $\sigma_0 > 0$ is as in the preceding subsection, then there exists a positive constant M_1, depending on R, such that the following estimates hold for $f \in \mathscr{H}_{\sigma_0}^\infty$:*

$$\|(\mathbf{T}^k[f])^\diamond\|_{\mathscr{H}_{\frac{1}{2}\sigma_0}^\infty} \leq M_1^k \, k^{2k} \|f^\diamond\|_{\mathscr{H}_{\sigma_0}^\infty},$$

and

$$\|\bar{\partial}_w (\mathbf{T}^{k-1}[f])^\diamond\|_{\mathscr{H}_{\frac{1}{2}\sigma_0}^\infty} \leq 12\sigma_0^{-1} M_1^{k-1} k^{2k-1} \|f^\diamond\|_{\mathscr{H}_{\sigma_0}^\infty}.$$

In the theorem, we understand the value of k^{2k} to be equal to 1 for $k = 0$.
We proceed with the necessary steps to obtain the theorem.

5.4 | Control of the $\bar{\partial}$-derivative

In terms of polarizations, the $\bar{\partial}$-derivative is the operator $\bar{\partial}_w$, the holomorphic differentiation with respect to the variable $\bar{w}$. For a given function $f \in \mathscr{H}_\sigma^\infty$, we may think of z as fixed in the smaller annulus $\mathbb{A}(\rho')$, where $\rho' = \rho(\sigma')$ is given by (5.2.1) and $0 < \sigma' < \sigma \leq \sigma_0 < 1$. The condition that $(z, w) \in \hat{\mathbb{A}}(\sigma')$ entails that w is at distance at least $\rho' - \rho$ to the boundary of the w-slice of $\hat{\mathbb{A}}(\sigma)$ where we know that $f(z, w)$ is $\bar{w}$-holomorphic and bounded. Moreover, a simple calculation gives that

$$\rho' - \rho \geq \frac{1}{6}(\sigma - \sigma'), \tag{5.4.1}$$

and hence the standard Cauchy estimates for the derivative show that for $f \in \mathscr{H}_\sigma^\infty$,

$$\|\bar{\partial}_w f^\diamond\|_{\mathscr{H}_{\sigma'}^\infty} \leq 6 \frac{\|f\|_{\mathscr{H}_\sigma^\infty}}{\sigma - \sigma'}, \qquad 0 < \sigma' < \sigma \leq \sigma_0. \tag{5.4.2}$$

5.5 | Control of the operator L

For f with polarization $f^\diamond \in \mathscr{H}_\sigma^\infty$, the restriction of f to $\mathbb{T}$ has a bounded holomorphic extension to the annulus $\rho < |z| < \rho^{-1}$ with $\rho = \rho(\sigma)$ (see (5.2.1)) provided by the formula

$$f_\mathbb{T}(z) := f^\diamond\left(z, \frac{1}{\bar{z}}\right).$$

The Cauchy integral formula allows us to split uniquely

$$f_\mathbb{T}(z) = f_\mathbb{T}^+(z) + f_\mathbb{T}^-(z),$$

where $f_{\mathbb{T}}^+(z)$ is bounded and holomorphic in the disk $|z| < \rho^{-1}$ with $f_{\mathbb{T}}^+(0) = 0$ while $f_{\mathbb{T}}^-(z)$ is bounded and holomorphic in the exterior disk $|z| > \rho$. Indeed, we may estimate the norms:

$$\|f_{\mathbb{T}}^\pm\|_{L^\infty(\mathbb{A}(\rho))} \leq \frac{2\|f_{\mathbb{T}}\|_{L^\infty(\mathbb{A}(\rho))}}{1 - \rho^2} \leq \frac{3\|f^\diamond\|_{\mathscr{H}_\sigma^\infty}}{\sigma}, \qquad \rho = \rho(\sigma). \tag{5.5.1}$$

Next, we identify

$$\mathbf{P}_{H_-^2}[f] = f_{\mathbb{T}}^-, \tag{5.5.2}$$

and, moreover, we see that the harmonic extension $\mathbf{U}_{\mathbb{D}_e}[f]$ is given by

$$\mathbf{U}_{\mathbb{D}_e}[f](z) = f_{\mathbb{T}}^+(1/\bar{z}) + f_{\mathbb{T}}^-(z),$$

which is bounded and harmonic for $|z| > \rho$. Hence the polarization of the harmonic extension is given by

$$(\mathbf{U}_{\mathbb{D}_e}[f])^\diamond(z, w) = f_{\mathbb{T}}^+(1/\bar{w}) + f_{\mathbb{T}}^-(z),$$

so that the function

$$F(z, w) := (\mathbf{U}_{\mathbb{D}_e}[f])^\diamond(z, w) - f^\diamond(z, w) = f_{\mathbb{T}}^+(1/\bar{w}) + f_{\mathbb{T}}^-(z) - f^\diamond(z, w)$$

is in $\mathscr{H}_\sigma^\infty$ and vanishes on the complex variety $1 - z\bar{w} = 0$. In view of (5.5.1), its norm is easily controlled:

$$\|F\|_{\mathscr{H}_\sigma^\infty} \leq \frac{7\|f^\diamond\|_{\mathscr{H}_\sigma^\infty}}{\sigma}. \tag{5.5.3}$$

It follows from the Weierstrass division theorem that the function

$$G(z, w) := \frac{F(z, w)}{1 - z\bar{w}} = \frac{(\mathbf{U}_{\mathbb{D}_e}[f])^\diamond(z, w) - f^\diamond(z, w)}{1 - z\bar{w}}$$

is holomorphic in the variables $(z, \bar{w})$ on $\hat{\mathbb{A}}(\sigma)$. If $w \in \mathbb{A}(\rho')$ where $\rho' = \rho(\sigma')$ and $0 < \sigma' < \sigma$, we see that

$$|1 - z\bar{w}| \geq \rho' - \rho, \qquad |z| \in \{\rho, \rho^{-1}\}.$$

In view of (5.5.3), (5.4.1), and the maximum principle, then, we obtain that

$$\|(\mathbf{M}[f])^\diamond\|_{\mathscr{H}_{\sigma'}^\infty} = \|G\|_{\mathscr{H}_{\sigma'}^\infty} \leq \frac{42\,\|f^\diamond\|_{\mathscr{H}_\sigma^\infty}}{\sigma(\sigma - \sigma')}.$$

This is the main expression to control in the definition of the operator $\mathbf{L}$, the other being controlled by (5.5.1) in view of the identity (5.5.2). Together, these estimates show that

$$\|(\mathbf{L}[f])^\diamond\|_{\mathscr{H}_{\sigma'}^\infty} \leq \frac{L_1\|f^\diamond\|_{\mathscr{H}_\sigma^\infty}}{\sigma(\sigma - \sigma')}, \qquad 0 < \sigma' < \sigma \leq \sigma_0 < 1, \tag{5.5.4}$$

for some positive constant L_1.

5.6 | Estimation of the operator T

We think of $\bar{\partial}$ as acting $\mathscr{H}_\sigma^\infty \to \mathscr{H}_{\sigma''}^\infty$, whereas $\mathbf{L}$ acts $\mathscr{H}_{\sigma''}^\infty \to \mathscr{H}_{\sigma'}^\infty$, where $0 < \sigma' < \sigma'' < \sigma \leq \sigma_0$. A combination of the estimates (5.4.2) and (5.5.4) shows that

$$\|(\mathbf{T}[f])^\diamond\|_{\mathscr{H}_{\sigma'}^\infty} \leq \frac{L_2\|f^\diamond\|_{\mathscr{H}_\sigma^\infty}}{\sigma''(\sigma - \sigma'')(\sigma'' - \sigma')}, \qquad 0 < \sigma' < \sigma'' < \sigma \leq \sigma_0 < 1,$$

for some positive constant L_2. We may insert the choice $\sigma'' = \frac{1}{2}(\sigma + \sigma')$ and use the fact that $\sigma'' \geq \sigma'$, to obtain the estimate

$$\|(\mathbf{T}[f])^\diamond\|_{\mathscr{H}_{\sigma'}^\infty} \leq \frac{4L_2\|f^\diamond\|_{\mathscr{H}_\sigma^\infty}}{\sigma'(\sigma - \sigma')^2}, \qquad 0 < \sigma' < \sigma \leq \sigma_0 < 1. \tag{5.6.1}$$

5.7 | The proof of the basic growth estimate

We may now obtain Theorem 5.1.

Proof of Theorem 5.1. We consider a strictly decreasing finite sequence of positive reals σ_j, $j = 0, 1, \ldots, k$, where σ_0 is as before, and we require that $\sigma_k = \frac{1}{2}\sigma_0$. We apply (5.6.1) with $\sigma = \sigma_j$ and $\sigma' = \sigma_{j+1}$ and iterate. Since $\sigma_k = \frac{1}{2}\sigma_0$, we obtain successively

$$\|(\mathbf{T}^k[f])^\diamond\|_{\mathscr{H}_{\frac{1}{2}\sigma_0}^\infty} \leq \left(\frac{8L_2}{\sigma_0}\right)^k \frac{\|f^\diamond\|_{\mathscr{H}_{\sigma_0}^\infty}}{\prod_{j=0}^{k-1}(\sigma_j - \sigma_{j+1})^2}. \tag{5.7.1}$$

By combining further with the Cauchy estimate (5.4.2), we find that

$$\|\bar{\partial}_w(\mathbf{T}^{k-1}[f])^\diamond\|_{\mathscr{H}_{\frac{1}{2}\sigma_0}^\infty} \leq 6\left(\frac{8L_2}{\sigma_0}\right)^{k-1} \frac{\|f^\diamond\|_{\mathscr{H}_{\sigma_0}^\infty}}{(\sigma_{k-1} - \sigma_k)\prod_{j=0}^{k-2}(\sigma_j - \sigma_{j+1})^2}. \tag{5.7.2}$$

We make the simplest choice possible, all the σ_j in an arithmetic progression:

$$\sigma_j = \sigma_0 - \frac{j\sigma_0}{2k}, \qquad j = 0, \ldots, k.$$

With this choice, we read off from (5.7.1) that

$$\|(\mathbf{T}^k[f])^\diamond\|_{\mathscr{H}_{\frac{1}{2}\sigma_0}^\infty} \leq M_1^k\, k^{2k}\|f^\diamond\|_{\mathscr{H}_{\sigma_0}^\infty}, \qquad M_1 := \frac{32\,L_2}{\sigma_0^3},$$

as claimed. This tells us how to pick M_1 in terms of L_2 and σ_0, and each depends only on the properties of R. Finally, as for (5.7.2), we obtain

$$\|\bar{\partial}_w(\mathbf{T}^{k-1}[f])^\diamond\|_{\mathscr{H}_{\frac{1}{2}\sigma_0}^\infty} \leq 12\sigma_0^{-1} M_1^{k-1} k^{2k-1}\|f^\diamond\|_{\mathscr{H}_{\sigma_0}^\infty},$$

which completes the proof. □

5.8 | The choice of an approximate function B

The first estimate in Theorem 5.1 says that in the Taylor expansion (5.1.2), the norm growth of the coefficients in $\mathscr{H}^\infty_{\frac{1}{2}\sigma_0}$ is such that as a function of the parameter $s = m^{-1}$, the function B behaves as if it were from the Gevrey class with exponent 3. We now make an effort to find a universal representative B. We recall from (5.1.2) that the asymptotic expansion for B should be

$$\mathrm{B} \sim \mathrm{B}_0 + s\mathbf{T}[\mathrm{B}_0] + s^2\mathbf{T}^2[\mathrm{B}_0] + \cdots,$$

where $s = m^{-1}$. The approach in [16] is based on the idea to work with the convergent series

$$\mathrm{B} \sim \sum_{k=0}^{+\infty} s^k \mu_k(s)\mathbf{T}^k[\mathrm{B}_0],$$

where the multiplier functions $\mu_k(s)$ are chosen appropriately to ensure convergence for any s, while for an individual index k, $\mu_k(s) = 1 + \mathrm{O}(s^N)$ holds for any finite positive integer N as $s \to 0$. This is natural when we would like to have some degree of smoothness in the parameter $s = m^{-1}$, for instance when considering Gevrey classes, but in our context, m is a positive integer, and s therefore discrete. A simpler approach which we employ here is to consider finite cut-offs ("abschnitts") $\mathrm{B}^{\langle\varkappa-1\rangle}$ given by

$$\mathrm{B}^{\approx} := \mathrm{B}^{\langle\varkappa-1\rangle} = \mathrm{B}_0 + s\mathbf{T}[\mathrm{B}_0] + \cdots + s^{\varkappa-1}\mathbf{T}^{\varkappa-1}[\mathrm{B}_0], \tag{5.8.1}$$

where the positive integer parameter $\varkappa$ will be taken to depend on $s = m^{-1}$. We then calculate the error term $\mathcal{E}_m$ from the equality (3.3.1) in reverse:

$$\mathcal{E}_m = \mathrm{B}^{\approx} - \mathrm{B}_0 - s\mathbf{T}[\mathrm{B}^{\approx}] = -s^\varkappa \mathbf{T}^\varkappa[\mathrm{B}_0] = -m^{-\varkappa}\mathbf{T}^\varkappa[\mathrm{B}_0].$$

Now, in view of the first estimate of Theorem 5.1, we have

$$\|\mathcal{E}_m^\diamond\|_{\mathscr{H}^\infty_{\frac{1}{2}\sigma_0}} = m^{-\varkappa}\|(\mathbf{T}^\varkappa[\mathrm{B}_0])^\diamond\|_{\mathscr{H}^\infty_{\frac{1}{2}\sigma_0}} \leq M_1^\varkappa m^{-\varkappa}\varkappa^{2\varkappa}\|\mathrm{B}_0^\diamond\|_{\mathscr{H}^\infty_{\sigma_0}}. \tag{5.8.2}$$

It is now a calculus exercise to minimize over the parameter $\varkappa$. To this end, we provide a lemma.

Lemma 5.2. *For a real parameter β, consider the function $\eta(t) := \beta t + 2t \log t$ for $0 < t < +\infty$, while $\eta(0) := 0$, and let t_0 denote the positive real number $t_0 := \mathrm{e}^{-\frac{1}{2}\beta-1}$. Then $\eta(t)$ enjoys the estimate*

$$\eta(t) \leq \begin{cases} -2t, & 0 \leq t \leq t_0, \\ -2t_0 + t_0^{-1}, & t_0 \leq t \leq t_0 + 1. \end{cases}$$

Proof. We calculate the first two derivatives:

$$\eta'(t) = \beta + 2 + 2\log t \quad \text{and} \quad \eta''(t) = 2/t.$$

In particular, it follows that $\eta(t)$ is convex. We look for critical points, and find the unique critical point $t_0 = e^{-\frac{1}{2}\beta - 1}$ with value $\eta(t_0) = -2t_0 = -2e^{-\frac{1}{2}\beta - 1}$. At the point $t = t_0$, the function $\eta(t)$ has a global minimum. We attempt to estimate the growth of $\eta(t)$ for $t > t_0$. Since $\eta''(t)$ is decreasing, the integration-by-parts formula shows that

$$\eta(t) = \eta(t_0) + \int_{t_0}^{t}(t - x)\eta''(x)\mathrm{d}x \leq \eta(t_0) + \int_{t_0}^{t}(t - x)\eta''(t_0)\mathrm{d}x$$

$$= \eta(t_0) + \frac{\eta''(t_0)}{2}(t - t_0)^2, \qquad t \geq t_0.$$

We implement this for t in the interval $[t_0, t_0 + 1]$ and obtain

$$\eta(t) \leq \eta(t_0) + \frac{\eta''(t_0)}{2} = -2t_0 + t_0^{-1}, \qquad t_0 \leq t \leq t_0 + 1.$$

Also, by convexity and the fact that $\lim \eta(t) = 0$ as $t \to 0^+$, we have the estimate

$$\eta(t) \leq \frac{\eta(t_0)}{t_0}t = -2t, \qquad 0 \leq t \leq t_0.$$

which completes the proof of the lemma. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

We shall apply the lemma to the function

$$\eta(k) = \log(M_1^k m^{-k} k^{2k}) = \beta k + 2k \log k \quad \text{with} \quad \beta = \log M_1 - \log m.$$

Then $t_0 = e^{-\frac{1}{2}\beta - 1} = e^{-1}M_1^{-\frac{1}{2}}m^{\frac{1}{2}}$ and it follows from Lemma 5.2 that if $\kappa_*(m)$ denotes the unique integer in the interval $[t_0, t_0 + 1[$,

$$\eta(\kappa_*(m)) \leq -2t_0 + t_0^{-1} = -2e^{-1}M_1^{-\frac{1}{2}}m^{\frac{1}{2}} + e\,M_1^{\frac{1}{2}}m^{-\frac{1}{2}} \leq -\epsilon m^{\frac{1}{2}} + \log 2,$$

where $\epsilon := 2e^{-1}M_1^{-\frac{1}{2}}$, provided that m is big enough, in this case $m \geq 25M_1$ suffices. After exponentiation, we find that

$$M_1^{\kappa_*(m)}m^{-\kappa_*(m)}\kappa_*(m)^{2\kappa_*(m)} \leq 2\exp(-\epsilon\sqrt{m}). \tag{5.8.3}$$

This parameter value $\kappa = \kappa_*(m)$ is the choice we insert into the definition (5.8.1). *It now follows from* (5.8.2) *and* (5.8.3) *that with $\epsilon = 2e^{-1}M_1^{-\frac{1}{2}}$,*

$$\|\mathcal{E}_m^{\diamond}\|_{\mathscr{H}_{\frac{1}{2}\sigma_0}^{\infty}} = m^{-\kappa_*(m)}\|(\mathbf{T}^{\kappa_*(m)}[\mathrm{B}_0])^{\diamond}\|_{\mathscr{H}_{\frac{1}{2}\sigma_0}^{\infty}}$$

$$\leq M_1^{\kappa_*(m)}m^{-\kappa_*(m)}\kappa_*(m)^{2\kappa_*(m)}\|\mathrm{B}_0^{\diamond}\|_{\mathscr{H}_{\sigma_0}^{\infty}} \leq 2e^{-\epsilon\sqrt{m}}\|\mathrm{B}_0^{\diamond}\|_{\mathscr{H}_{\sigma_0}^{\infty}}, \tag{5.8.4}$$

provided that $m \geq 25M_1$. This controls the error term $\mathcal{E}_m$, but we will also need to estimate $B^\approx$ itself. By the triangle inequality and the first estimate of Theorem 5.1, combined with the first estimate of Lemma 5.2 with $\beta = \log M_1 - \log m$,

$$\|(B^\approx)^\diamond\|_{\mathscr{H}^\infty_{\frac{1}{2}\sigma_0}} \leq \sum_{j=0}^{\kappa_*(m)-1} m^{-j} \|(\mathbf{T}^j[B_0])^\diamond\|_{\mathscr{H}^\infty_{\frac{1}{2}\sigma_0}} \leq \sum_{j=0}^{\kappa_*(m)-1} M_1^j m^{-j} j^{2j} \|B_0^\diamond\|_{\mathscr{H}^\infty_{\sigma_0}}$$

$$= \sum_{j=0}^{\kappa_*(m)-1} e^{\eta(j)} \|B_0^\diamond\|_{\mathscr{H}^\infty_{\sigma_0}} \leq \sum_{j=0}^{\kappa_*(m)-1} e^{-2j} \|B_0^\diamond\|_{\mathscr{H}^\infty_{\sigma_0}} \leq \frac{\|B_0^\diamond\|_{\mathscr{H}^\infty_{\sigma_0}}}{1 - e^{-2}}, \tag{5.8.5}$$

so the norm of $B^\approx$ is uniformly bounded. To control $\bar{\partial} B^\approx$ as well, we may rely instead on the second estimate of Theorem 5.1. We find that

$$\|\bar{\partial}_w (B^\approx)^\diamond\|_{\mathscr{H}^\infty_{\frac{1}{2}\sigma_0}} \leq \sum_{j=0}^{\kappa_*(m)-1} m^{-j} \|\bar{\partial}_w (\mathbf{T}^j[B_0])^\diamond\|_{\mathscr{H}^\infty_{\frac{1}{2}\sigma_0}}$$

$$\leq 12\sigma_0^{-1} \sum_{j=0}^{\kappa_*(m)-1} M_1^j m^{-j} (j+1)^{2j+1} \|B_0^\diamond\|_{\mathscr{H}^\infty_{\sigma_0}}$$

$$\leq \frac{12\,e^2}{\sigma_0} \sum_{j=0}^{\kappa_*(m)-1} (j+1) M_1^j m^{-j} j^{2j} \|B_0^\diamond\|_{\mathscr{H}^\infty_{\sigma_0}} = \frac{12\,e^2}{\sigma_0} \sum_{j=0}^{\kappa_*(m)-1} (j+1)\, e^{\eta(j)} \|B_0^\diamond\|_{\mathscr{H}^\infty_{\sigma_0}}$$

$$\leq 12\sigma_0^{-1} e^2 \sum_{j=0}^{\kappa_*(m)-1} (j+1) e^{-2j} \|B_0^\diamond\|_{\mathscr{H}^\infty_{\sigma_0}} \leq \frac{12\sigma_0^{-1} e^2}{(1 - e^{-2})^2} \|B_0^\diamond\|_{\mathscr{H}^\infty_{\sigma_0}}, \tag{5.8.6}$$

where in the middle we used the elementary estimate $(j+1)^j \leq e\, j^j$ which we recognize as $(1 + j^{-1})^j \leq e$. This estimate holds trivially for $j = 0$ since we interpret 0^0 as 1.

5.9 | The approximate functions $A^\approx$ and $F^\approx$

The equations (3.3.2) and (3.3.3) give us $A^\approx$ and $F^\approx$ in terms of $B^\approx$, which in turn is given by (5.8.1) with $\kappa = \kappa_*(m)$, where $\kappa_*(m)$ is as in the preceding subsection. The function $\bar{\partial} B^\approx$ gets estimated by (5.8.6), and division by $\bar{H}_R$ is fine since its polarization is in $\mathscr{H}^\infty_{\sigma_0}$ by assumption and it is bounded away from 0 in a fixed polarized neighborhood of $\mathbb{T}$ (so taking $\sigma_0 > 0$ small enough we may safely divide). In view of (5.5.1) and (5.8.6),

$$\left\| \mathbf{P}_{H^2_-} \left[\frac{\bar{\partial} B^\approx}{\bar{H}_R} \right] \right\|_{L^\infty(\mathbb{T}(0,\rho_*))} \leq \frac{72\,\sigma_0^{-2} e^2}{(1 - e^{-2})^2} \|B_0^\diamond\|_{\mathscr{H}^\infty_{\sigma_0}} \|(\bar{H}_R^\diamond)^{-1}\|_{\mathscr{H}^\infty_{\sigma_0}} \tag{5.9.1}$$

if the radius $\rho_* < 1$ is given by

$$\rho_* := \rho(\tfrac{1}{2}\sigma_0) = \frac{2}{\sigma_0 + \sqrt{4 + \sigma_0^2}}.$$

In the same manner, we obtain from (5.5.1) that

$$\left\| \mathbf{P}_{H_0^2} \left[\frac{\bar{\partial} B^{\approx}}{\bar{H}_R} \right] \right\|_{L^\infty(\mathbb{T}(0,\rho_*^{-1}))} \leq \frac{72\sigma_0^{-2}e^2}{(1-e^{-2})^2} \|B_0^\diamond\|_{\mathscr{H}_{\sigma_0}^\infty} \|(\bar{H}_R^\diamond)^{-1}\|_{\mathscr{H}_{\sigma_0}^\infty}, \tag{5.9.2}$$

where we recall from Subsection 2.3 that $\mathbf{P}_{H_0^2}$ denotes the orthogonal projection $L^2(\mathbb{T}) \to H_0^2$. The identity

$$\text{conj } \mathbf{P}_{H_0^2}\left[\frac{\bar{\partial} B^{\approx}}{\bar{H}_R} \right]\left(\frac{1}{\bar{z}} \right) = \mathbf{P}_{H_{-,0}^2}\left[\frac{\partial \bar{B}^{\approx}}{H_R} \right](z)$$

where "conj" denote complex conjugation, shows that (5.9.2) gives the equivalent estimate

$$\left\| \mathbf{P}_{H_{-,0}^2} \left[\frac{\partial \bar{B}^{\approx}}{H_R} \right] \right\|_{L^\infty(\mathbb{T}(0,\rho_*))} \leq \frac{72\,\sigma_0^{-2}e^2}{(1-e^{-2})^2} \|B_0^\diamond\|_{\mathscr{H}_{\sigma_0}^\infty} \|(\bar{H}_R^\diamond)^{-1}\|_{\mathscr{H}_{\frac{1}{2}\sigma_0}^\infty}. \tag{5.9.3}$$

We obtain as a consequence of (5.9.2) and the definition of $A^{\approx}$ in (3.3.2) that

$$\left\| A^{\approx} - \frac{a_1}{\zeta H_R} \right\|_{L^\infty(\mathbb{T}(0,\rho_*))} \leq m^{-1}(2\pi)^{-\frac{1}{2}} \frac{72\sigma_0^{-2}e^2}{(1-e^{-2})^2}$$

$$\times \|(\zeta H_R)^{-1}\|_{L^\infty(\mathbb{T}(0,\rho_*))} \|B_0^\diamond\|_{\mathscr{H}_{\sigma_0}^\infty} \|(\bar{H}_R^\diamond)^{-1}\|_{\mathscr{H}_{\sigma_0}^\infty} = O(m^{-1}), \tag{5.9.4}$$

and, analogously, from (5.9.3) together with the definition of $F^{\approx}$ in (3.3.3) we find that

$$\left\| F^{\approx} - a_1 H_R \right\|_{L^\infty(\mathbb{T}(0,\rho_*))} \leq m^{-1}(2\pi)^{-\frac{1}{2}} \frac{72\sigma_0^{-2}e^2}{(1-e^{-2})^2}$$

$$\times \|H_R\|_{L^\infty(\mathbb{T}(0,\rho_*))} \|B_0^\diamond\|_{\mathscr{H}_{\sigma_0}^\infty} \|(\bar{H}_R^\diamond)^{-1}\|_{\mathscr{H}_{\sigma_0}^\infty} = O(m^{-1}) \tag{5.9.5}$$

as $m \to +\infty$. In particular,

$$\|A^{\approx}\|_{L^\infty(\mathbb{T}(0,\rho_*))} = O(1) \quad \text{and} \quad \|F^{\approx}\|_{L^\infty(\mathbb{T}(0,\rho_*))} = O(1) \tag{5.9.6}$$

as $m \to +\infty$. We recall the definition of the function $B^{\approx}$:

$$B^{\approx} = B^{\langle\kappa-1\rangle} = B_0 + \cdots + m^{-\kappa+1}\mathbf{T}^{\kappa-1}B_0, \qquad \kappa := \kappa_*(m),$$

where $\kappa_*(m)$ is the unique integer in the interval $[t_0, t_0 + 1[$ with $t_0 = e^{-1}M_1^{-\frac{1}{2}}m^{\frac{1}{2}}$. Also, the functions $A^{\approx}$ and $F^{\approx}$ are given in terms of $B^{\approx}$ by the relations (3.3.2) and (3.3.3), respectively. We gather our observations in a theorem.

Theorem 5.3. *With the above choices of $B^{\approx}$, $A^{\approx}$, and $F^{\approx}$, the size of the error term $\mathcal{E}_m$ in the equality (3.3.1) is controlled by the estimate (5.8.4), whereas the size of $B^{\approx}$ itself is controlled by (5.8.5) and (5.8.6). As for the functions $A^{\approx}$ and $F^{\approx}$, their norms are uniformly bounded according to (5.9.6), and, to higher precision, controlled by (5.9.4) and (5.9.5), respectively.*

This result was implemented back in Subsection 4.4 to establish the necessary facts which support Theorem 1.2.

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