Hele-Shaw on Weakly Hyperbolic Surfaces

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ABSTRACT. We consider the Hele-Shaw flow that arises from injection of two-dimensional fluid into a point of a curved surface. The resulting fluid domains are more or less determined implicitly by a mean value property for harmonic functions. We improve on the results of Hedenmalm and Shimorin [8] and obtain essentially the same conclusions while imposing a weaker curvature condition on the surface. Incidentally, the curvature condition is the same as the one that appears in Hedenmalm and Perdomo's paper [7], where the problem of finding smooth area minimizing surfaces for a given curvature form under a natural normalizing condition was considered. Probably there are deep reasons behind this coincidence.

1. INTRODUCTION

Let Ω be a simply connected Riemann surface with a C^∞-smooth metric $d\mathbf{s}$. We consider the case when the surface Ω is conformally equivalent to a simply connected planar domain Ω, which we pick to be the unit disk $\Omega = \mathbb{D}$, and the metric $d\mathbf{s}$ is given by

$$(1.1) \qquad d\mathbf{s}(z)^2 = \omega(z) \, |dz|^2, \quad z \in \Omega,$$

for some positive C^∞-smooth weight function ω in Ω. This means that we use *isothermal coordinates*, which is basically always possible; the only real restriction imposed is that we do not allow Ω to be conformally equivalent to the sphere or the plane. The *area form* of the surface Ω is (with suitable normalization)

$$d\Sigma(z) = \omega(z) \, d\Sigma(z), \quad z \in \Omega = \mathbb{D},$$

where

$$d\Sigma(z) = \frac{dx\,dy}{\pi}, \quad z = x + iy,$$

is the normalized area element in the plane.

The Gaussian curvature function associated to the metric ds given by (1.1) is the function κ defined by

$$\kappa(z) = -\frac{2}{\omega(z)}\Delta(\log\omega)(z), \quad z \in \mathbb{D},$$

where

$$\Delta = \Delta_z = \frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right), \quad z = x + iy,$$

is the (normalized) Laplacian in the plane. The curvature form associated to the metric ds given by (1.1) is the 2-form $\mathbf{K}$ defined by

$$\mathbf{K}(z) = \kappa(z)\,d\Sigma(z) = -2\,\Delta(\log\omega)(z)\,d\Sigma(z), \quad z \in \mathbb{D}.$$

In the recent paper [7], Hedenmalm and Perdomo studied the following problem: Suppose the curvature form $\mathbf{K}$ is given and C^∞-smooth on the closed disk $\bar{\mathbb{D}}$, while the metric ds is to be found. If we ask of this metric which we are looking for that it is C^∞-smooth on $\bar{\mathbb{D}}$ and has the *mean value property*

$$(1.2) \qquad h(0) = \int_{\mathbb{D}} h(z)\,d\Sigma(z) = \int_{\mathbb{D}} h(z)\,\omega(z)\,d\Sigma(z)$$

for all bounded harmonic functions h in $\mathbb{D}$, can we then find such a metric, and is it unique? It turns out that this is indeed so if the condition

$$(1.3) \qquad \mathbf{K}(z) + \frac{1}{2}\mathbf{K}_{\mathbb{H}}(z) \le 0, \quad z \in \mathbb{D},$$

is met, where $\mathbf{K}_{\mathbb{H}}$ denotes the curvature form for the surface $\mathbb{D}$ when equipped with the (Poincaré) metric of constant curvature -1:

$$\mathbf{K}_{\mathbb{H}}(z) = -\frac{4\,d\Sigma(z)}{(1-|z|^2)^2}, \quad z \in \mathbb{D}.$$

Moreover, the result is fairly sharp, in the following sense: If the number $\frac{1}{2}$ is replaced by the slightly larger number 0.52 in condition (1.3), the resulting condition is too weak to force the existence of a C^∞-smooth bordered surface $\Omega = \langle \mathbb{D}, ds \rangle = \langle \mathbb{D}, \omega \rangle$ with the mean value property for harmonic functions and the given curvature form $\mathbf{K}$. It was conjectured in [7] that the numerically obtained number 0.52 could be replaced by $\frac{1}{2} + \varepsilon$, for any positive ε. We shall at times refer to (1.3) as the condition of *weak hyperbolicity*.

It is a natural question to ask whether there are subdomains $D(t)$, depending on the parameter t, $0 < t < 1$, with the modified mean value property

(1.4) $$t h(0) = \int_{D(t)} h(z) \, d\Sigma(z) = \int_{D(t)} h(z) \omega(z) \, d\Sigma(z)$$

for all bounded harmonic functions h in $D(t)$. We make the *a priori* assumption that the domains $D(t)$ are simply connected with C^∞-smooth boundaries. Then, if we look at the curvature condition (1.3) and use the fact that the curvature form for the Poincaré metric decreases as the domain gets smaller, we see that the analogous curvature condition for $\Omega(t) = \langle D(t), \omega \rangle$ is fulfilled, so by the Hedenmalm-Perdomo theorem, the surfaces $\Omega(t)$ exist as abstract surfaces, being determined uniquely by the given curvature form and the mean value property (1.4). However, this does not tell us that the abstract surface $\Omega(t)$ forms a subregion of Ω. It would be desirable to know that this is so. If this works, then perhaps it will eventually be possible to understand the Hedenmalm-Perdomo theorem in terms of the following growth process:

(1) for t close to 0, we form the surface $\Omega(t)$, by fairly elementary means;
(2) as t increases, we iteratively add abstractly infinitesimally thin boundary layers to the surface $\Omega(t)$, while checking that the curvature condition (1.3) permits us to do so.

Here, we shall obtain the existence of the smooth simply connected subdomains $D(t)$ of $\mathbb{D}$ with (1.4) under the weak hyperbolicity condition (1.3) in the category of *real-analytic surfaces*. Moreover, we shall prove that the boundary curves $\partial D(t)$ are real-analytically smooth, and that they vary real-analytically with the parameter t. The growth of the domains $D(t)$ is a certain kind of *Hele-Shaw flow*, modelling the growth of a patch of two-dimensional fluid as fluid is injected at the origin. We extend the Hele-Shaw exponential mapping theorem obtained by Hedenmalm and Shimorin in [8] to weakly hyperbolic surfaces. From the point of view of differential geometry, it is perhaps surprising that this is possible, because it does not work globally for the ordinary exponential mapping. After all, it is rather easy to construct a real-analytically smooth surface $\Omega = \langle \mathbb{D}, \omega \rangle$ whose curvature form $\mathbf{K}$ satisfies (1.3) while two geodesics emanating from the origin meet at some other point; we supply an example of such a situation in Section 7. Probably this discrepancy can be explained by the viscosity of the fluid as compared with the lack thereof in the photonic gas, which is the physical manifestation of the metric flow.

The techniques used in the paper are basically similar to what was used by Hedenmalm-Shimorin in [8]. The main difference is that the Green function Γ_1 for the weighted biharmonic operator $\Delta(1 - |z|^2)^{-1}\Delta$ is used in place of the biharmonic Green function Γ. We suspect that the main results of this paper are false if the constant $\frac{1}{2}$ in the condition (1.3) is replaced by $\frac{1}{2} + \varepsilon$, for any fixed positive ε. In another vein, we suspect that the results obtained here for real-analytic surfaces will remain valid for C^∞-smooth surfaces as well.

2. Preliminaries on Hele-Shaw Domains

Let ω be a C^∞-smooth weight function on $\bar{\mathbb{D}}$ which is strictly positive (that is, it is positive at all points). We denote by G the Green function for the Laplacian Δ on $\mathbb{D}$:

$$G(z,\zeta) = \log\left|\frac{z-\zeta}{1-\bar{\zeta}z}\right|^2, \quad (z,\zeta) \in \bar{\mathbb{D}} \times \mathbb{D}.$$

For $0 < t < +\infty$, we consider the function

$$V_t(z) = tG(z,0) - \int_{\mathbb{D}} G(z,\zeta)\omega(\zeta)\,d\Sigma(\zeta), \quad z \in \mathbb{D}.$$

This function V_t is smooth in $\mathbb{D}\setminus\{0\}$, vanishes on the unit circle $\mathbb{T} = \partial\mathbb{D}$, and has a logarithmic singularity at the origin. We denote by $\hat{V}_t$ the smallest superharmonic majorant of V_t in $\mathbb{D}$. The set $D(t)$ is now defined by

$$D(t) = D(t;\omega) = \{z \in \mathbb{D} \mid V_t(z) < \hat{V}_t(z)\}.$$

The sets $D(t)$ are known to have a number of basic properties (see [8]). For instance, the sets $D(t)$ are open, connected and increasing in the positive parameter t.

   Associated to the domains $D(t)$, we consider the number

$$T = T(\omega) = \sup\{t \in \,]0, +\infty[\mid D(t) \Subset \mathbb{D}\},$$

the *termination time* for the Hele-Shaw flow. It is known that $0 < T \le +\infty$ (see [8, Proposition 2.8]). For $0 < t < T$, the domain $D(t)$ is called a *Hele-Shaw domain*. For $t \ge T$, the domain $D(t)$ is sometimes referred to as a *generalized Hele-Shaw domain*.

3. A Weighted Biharmonic Green Function

In this section we shall need some properties of the Green function for the weighted biharmonic operator $\Delta(1 - |z|^2)^{-1}\Delta$. We review the results needed and refer to [7] for details.

   The Green function for the weighted biharmonic operator $\Delta(1 - |z|^2)^{-1}\Delta$ is the function Γ_1 on $\bar{\mathbb{D}} \times \mathbb{D}$ solving (in an appropriate sense) for fixed $\zeta \in \mathbb{D}$ the boundary value problem

(3.1)
$$\begin{cases} \Delta_z(1 - |z|^2)^{-1}\Delta_z\Gamma_1(z,\zeta) = \delta_\zeta(z) & z \in \mathbb{D}, \\ \Gamma_1(z,\zeta) = 0 & z \in \mathbb{T}, \\ \nabla_z\Gamma_1(z,\zeta) = 0 & z \in \mathbb{T}; \end{cases}$$

here δ_ζ is the (unit) Dirac mass at ζ and $\mathbb{T} = \partial\mathbb{D}$ is the unit circle. The Green function Γ_1 has the explicit expression

$$(3.2) \quad \Gamma_1(z,\zeta) = \left\{|z-\zeta|^2 - \frac{1}{4}|z^2-\zeta^2|^2\right\} G(z,\zeta) + \frac{1}{8}(1-|z|^2)(1-|\zeta|^2)$$
$$\times \left\{7 - |z|^2 - |\zeta|^2 - |z\zeta|^2 - 4\,\Re(z\bar\zeta) - 2(1-|z|^2)(1-|\zeta|^2)\frac{1-|z\zeta|^2}{|1-z\bar\zeta|^2}\right\},$$

where

$$G(z,\zeta) = \log\left|\frac{z-\zeta}{1-\bar\zeta z}\right|^2, \quad (z,\zeta) \in \bar{\mathbb{D}} \times \mathbb{D},$$

is the Green function for the Laplacian in $\mathbb{D}$. We mention that the above explicit expression for Γ_1 first appeared in [5]. A derivation of formula (3.2) can be found in [12].

It is known that the Green function Γ_1 is positive in the bidisk:

$$\Gamma_1(z,\zeta) > 0, \quad (z,\zeta) \in \mathbb{D} \times \mathbb{D};$$

see [5, Proposition 3.4] or [7, Lemma 2.2].

The function H_1 defined by

$$H_1(z,\zeta) = (1-|\zeta|^2)\left\{\frac{1}{2}(3-|\zeta|^2)\frac{1-|z\zeta|^2}{|1-z\bar\zeta|^2} + (1-|\zeta|^2)\,\Re\left[\frac{z\bar\zeta}{(1-z\bar\zeta)^2}\right]\right\},$$

for $(z,\zeta) \in \bar{\mathbb{D}} \times \bar{\mathbb{D}}$ with $z \neq \zeta$, has been coined the *harmonic compensator*; it is positive on $\bar{\mathbb{D}} \times \bar{\mathbb{D}}$. Indeed, by harmonicity in z, it is enough to consider $(z,\zeta) \in \mathbb{T} \times \mathbb{D}$, in which case we have that

$$H_1(z,\zeta) \geq (1-|\zeta|^2)\left\{\frac{1}{2}(3-|\zeta|^2)\frac{1-|\zeta|^2}{|1-z\bar\zeta|^2} - (1-|\zeta|^2)\frac{|\zeta|}{|1-z\bar\zeta|^2}\right\}$$
$$= \frac{1}{2}\frac{(1-|\zeta|^2)^2}{|1-z\bar\zeta|^2}(1-|\zeta|)(3+|\zeta|) > 0.$$

The functions Γ_1 and H_1 are related by the formula

$$(3.3) \qquad \Delta_z\Gamma_1(z,\zeta) = (1-|z|^2)(G(z,\zeta) + H_1(z,\zeta)), \quad (z,\zeta) \in \bar{\mathbb{D}} \times \mathbb{D},$$

which can be verified by straightforward computation (see [7]).

The following lemma establishes an integral representation in terms of the functions Γ_1 and H_1.

Lemma 3.1. *Let u be a smooth function on $\bar{\mathbb{D}}$ such that $u = 0$ on $\mathbb{T}$. Then u admits the representation*

$$u(\zeta) = \int_{\mathbb{D}} \Gamma_1(z,\zeta)\Delta(1-|z|^2)^{-1}\Delta u(z)\,d\Sigma(z)$$
$$+ \frac{1}{2}\int_{\mathbb{T}} H_1(z,\zeta)\,\partial_n u(z)\,d\sigma(z), \quad \zeta \in \mathbb{D},$$

where ∂_n denotes differentiation in the inward normal direction and $d\sigma$ is normalized arc length measure on $\mathbb{T}$.

A more general representation formula for radially weighted biharmonic functions will appear in a forthcoming paper by Olofsson. Therefore we omit the proof of Lemma 3.1. The lemma can also be obtained by repeated applications of Green's formula. We recall that with our normalizations Green's formula (Green's second identity) takes the form

$$(3.4) \qquad \int_{\Omega}(u\Delta v - v\Delta u)\,d\Sigma = \frac{1}{2}\int_{\partial\Omega}(v\,\partial_n u - u\,\partial_n v)\,d\sigma,$$

where $d\sigma = |dz|/2\pi$.

The following lemma is a simple consequence of the fact that Γ_1 is positive. The technique is analogous to that which was developed for the biharmonic Green function Γ in [2]; see also [3, 4, 7].

Lemma 3.2. *Let v be a smooth function on $\bar{\mathbb{D}}$ such that*

$$z \mapsto \frac{v(z)}{1-|z|^2}$$

is subharmonic in $\mathbb{D}$. Assume also that v is reproducing at the origin in the sense that

$$h(0) = \int_{\mathbb{D}} h(z)\,v(z)\,d\Sigma(z)$$

holds for every harmonic polynomial h. Then the inequality

$$\int_{\mathbb{D}} u(z)2(1-|z|^2)\,d\Sigma(z) \le \int_{\mathbb{D}} u(z)v(z)\,d\Sigma(z)$$

holds for every $u \in C^{\infty}(\bar{\mathbb{D}})$ which is subharmonic in $\mathbb{D}$.

Proof. We consider the function Φ defined by

$$\Phi(z) = \int_{\mathbb{D}} G(z,\zeta)[v(\zeta) - 2(1-|\zeta|^2)]\,d\Sigma(\zeta), \quad z \in \mathbb{D},$$

where G is the Green function for the Laplacian in $\mathbb{D}$. By construction $\Phi|_{\mathbb{T}} = 0$. Since $\Delta\Phi = \nu - 2(1 - |z|^2)$ annihilates harmonic polynomials, an application of Green's formula (3.4) shows that also $\partial_n\Phi|_{\mathbb{T}} = 0$. An alternative route to the same conclusion is to differentiate the formula which defines Φ, and to use the fact that the harmonic polynomials are dense in $L_h^1(\mathbb{D})$, the closed subspace of $L^1(\mathbb{D})$ consisting of harmonic functions. By Lemma 3.1, the function Φ has the representation

$$\Phi(z) = \int_{\mathbb{D}} \Gamma_1(z, \zeta)\Delta_\zeta(1 - |\zeta|^2)^{-1}\Delta_\zeta\Phi(\zeta)\,d\Sigma(\zeta)$$

$$= \int_{\mathbb{D}} \Gamma_1(z, \zeta)\Delta_\zeta\left[\frac{\nu(\zeta)}{1 - |\zeta|^2}\right]d\Sigma(\zeta) \geq 0.$$

Another application of Green's formula now shows that

$$\int_{\mathbb{D}} u(z)[\nu(z) - 2(1 - |z|^2)]\,d\Sigma(z)$$

$$= \int_{\mathbb{D}} u(z)\Delta\Phi(z)\,d\Sigma(z) = \int_{\mathbb{D}} \Delta u(z)\Phi(z)\,d\Sigma(z) \geq 0,$$

which concludes the proof. ☐

In the proof of the next lemma, we shall use some easy properties of Bergman kernel functions. Let us denote by $A_\alpha^2(\mathbb{D})$, for $-1 < \alpha < \infty$, the weighted Bergman space of all analytic functions in $\mathbb{D}$ that are square integrable with respect to the measure

$$d\Sigma_\alpha(z) = \omega_\alpha(z)\,d\Sigma(z),$$

where the weight ω_α is as follows:

$$\omega_\alpha(z) = (1 + \alpha)(1 - |z|^2)^\alpha, \quad z \in \mathbb{D}.$$

The Bergman kernel function for the space $A_\alpha^2(\mathbb{D})$ is the function

$$K_\alpha(z, \zeta) = \frac{1}{(1 - z\bar{\zeta})^{2+\alpha}}, \quad (z, \zeta) \in \mathbb{D} \times \mathbb{D};$$

see [6, Proposition 1.1.4]. The reproducing property of the kernel function K_α asserts that

$$f(z) = \int_{\mathbb{D}} K_\alpha(z, \zeta)f(\zeta)\omega_\alpha(z)\,d\Sigma(\zeta), \quad z \in \mathbb{D},$$

for $f \in A_\alpha^2(\mathbb{D})$.

The following lemma is inspired by [7, Proposition 3.2].

Lemma 3.3. *Let v be a nonnegative smooth weight function satisfying the assumptions of Lemma 3.2. Let $e^{i\theta} \in \mathbb{T}$. Then either $v(e^{i\theta}) > 0$, or $v(e^{i\theta}) = 0$ and $\partial_n v(e^{i\theta}) > 0$.*

Proof. Without loss of generality, we may assume that $e^{i\theta} = 1$. To reach a contradiction, we assume that the conclusion of the lemma does not hold. Then there is a positive constant C such that

$$v(z) \leq C|z-1|^2, \quad z \in \mathbb{D}.$$

Fix r in the interval $0 < r < 1$. By Lemma 3.2 applied to the function

$$u(z) = |K_1(z,r)|^2 = \frac{1}{|1-rz|^6},$$

we have that

(3.5)
$$\frac{1}{(1-r^2)^3} = \int_{\mathbb{D}} |K_1(z,r)|^2 2(1-|z|^2)\, d\Sigma(z)$$

$$\leq C \int_{\mathbb{D}} |K_1(z,r)|^2 |z-1|^2\, d\Sigma(z).$$

The leftmost equality in (3.5) holds because of the reproducing property of K_1. We turn to estimating the right hand side in (3.5). By the elementary inequality

$$r^2|z-1|^2 \leq |1-rz|^2, \quad z \in \mathbb{D},$$

we have that

$$\int_{\mathbb{D}} |K_1(z,r)|^2 |z-1|^2\, d\Sigma(z) \leq \frac{1}{r^2} \int_{\mathbb{D}} \frac{1}{|1-rz|^4}\, d\Sigma(z)$$

$$= \frac{1}{r^2(1-r^2)^2},$$

where the last equality follows from the reproducing property of K_0. Thus, by (3.5), we have that

$$\frac{1}{(1-r^2)^3} \leq \frac{C}{r^2(1-r^2)^2},$$

which is clearly impossible for r close to 1. $\qquad\square$

The next lemma is analogous to [8, Lemma 5.2].

Lemma 3.4. *Let u be a smooth real-valued function in $\bar{\mathbb{D}} \setminus \{0\}$ with a logarithmic singularity at the origin such that*

$$\Delta(1-|z|^2)^{-1}\Delta u = \Delta\delta_0 - \mu \quad \text{in } \mathbb{D},$$

where $0 \leq \mu \in C^\infty(\mathbb{D})$. Suppose that $u|_\mathbb{T} = 0$ and $\partial_n u \leq 0$ on $\mathbb{T}$. Then

$$u(z) \leq \log|z|^2 + \frac{3}{2} - 2|z|^2 + \frac{1}{2}|z|^4 < 0, \quad z \in \mathbb{D}.$$

Proof. Recall that the kernels Γ_1 and H_1 are both positive. By the representation formula in Lemma 3.1 we have that

$$u(\zeta) \leq \int_\mathbb{D} \Gamma_1(z, \zeta) \, \Delta \delta_0(z) \, d\Sigma(z)$$

$$= \Delta_z \Gamma_1(0, \zeta) = \log|\zeta|^2 + \frac{3}{2} - 2|\zeta|^2 + \frac{1}{2}|\zeta|^4,$$

where the integration is to be interpreted in the usual sense of distribution theory. The proof is complete. ∎

4. Simply Connectedness of Hele-Shaw domains

In this section we shall show that each Hele-Shaw domain $D(t)$, $0 < t < T$, is simply connected and has real-analytic Jordan boundary. First we need a lemma.

Lemma 4.1. *Let v be a strictly positive smooth function on $\bar{\mathbb{D}}$ such that the function*

$$z \mapsto \frac{v(z)}{1 - |z|^2}$$

is subharmonic in $\mathbb{D}$. Let

$$W(z) = \log|z|^2 - \int_\mathbb{D} G(z, \zeta) v(\zeta) \, d\Sigma(\zeta), \quad z \in \mathbb{D},$$

and denote by $\widehat{W}$ the smallest superharmonic majorant of W in $\mathbb{D}$. Assume that the coincidence set

$$\{z \in \mathbb{D} \mid W(z) = \widehat{W}(z)\}$$

is a compact subset of $\mathbb{D}$. Then

$$W(z) \leq \log|z|^2 + \frac{3}{2} - 2|z|^2 + \frac{1}{2}|z|^4 < 0, \quad z \in \mathbb{D};$$

in particular, the coincidence set is empty.

Proof. We first introduce some preliminary notation. We consider the functions

$$W_r(z) = r \log|z|^2 - \int_\mathbb{D} G(z, \zeta) v(\zeta) \, d\Sigma(\zeta), \quad z \in \mathbb{D},$$

for $1 \leq r < \infty$, and note that $W_1 = W$. By construction, W_r vanishes on $\mathbb{T}$. Denote by $\widehat{W}_r$ the smallest superharmonic majorant of W_r in $\mathbb{D}$, and write

$$B(r) = \{z \in \mathbb{D} \mid W_r(z) < \widehat{W}_r(z)\}.$$

The set $B(r)$ gets bigger as r increases; see [8, Proposition 2.7]. If $W_r \leq 0$ in $\mathbb{D}$, then $\partial_n W_r \leq 0$ on $\mathbb{T}$. Conversely, if $\partial_n W_r \leq 0$ on $\mathbb{T}$, then Lemma 3.4 applied to the function $r^{-1} W_r$ shows that

$$(4.1) \qquad W_r(z) \leq r \left(\log |z|^2 + \frac{3}{2} - 2|z|^2 + \frac{1}{2}|z|^4 \right) < 0, \quad z \in \mathbb{D}.$$

If $\partial_n W_1 \leq 0$ on $\mathbb{T}$, then we are done. Assume, to reach a contradiction, that $\max_{\mathbb{T}} \partial_n W_1$ is strictly positive. In view of the definition of W_r, we have that

$$\partial_n W_r(z) = -2(r-1) + \partial_n W_1(z), \quad z \in \mathbb{T}.$$

This formula makes evident that

$$\max_{\mathbb{T}} \partial_n W_r > 0, \quad \text{for } 1 \leq r < r_1 = 1 + \max_{\mathbb{T}} \partial_n W_1/2,$$

while $\max_{\mathbb{T}} \partial_n W_r \leq 0$ for $r \geq r_1$.

For $1 \leq r < r_1$, the function W_r attains a positive maximum at some point $z(r) \in \mathbb{D}$. Clearly, $z(r) \in \mathbb{D} \setminus B(r) \subset \mathbb{D} \setminus B(1)$. Let $\{r_j\}_{j=2}^{\infty}$ be a sequence with $1 \leq r_j < r_1$ and $r_j \to r_1$ as $j \to +\infty$. In view of the assumed compactness of $\mathbb{D} \setminus B(1)$, we may by passing to a subsequence assume that $z(r_j)$ converges to a point $z_1 \in \mathbb{D} \setminus B(1)$ as $j \to +\infty$. Since $W_{r_j}(z(r_j)) > 0$, we obtain in the limit that $W_{r_1}(z_1) \geq 0$, which contradicts (4.1) for $r = r_1$. It follows that $B(1) = \mathbb{D}$, and that the estimate (4.1) holds for all r, $1 \leq r < +\infty$. ☐

We may now derive the asserted properties of $D(t)$. We recall the definition of the termination time $T = T(\omega)$ for the Hele-Shaw flow.

Theorem 4.2. *Let $\Omega = \langle \mathbb{D}, \omega \rangle$ be a simply connected Riemann surface with a metric $\mathrm{ds}^2 = \omega|\mathrm{d}z|^2$. Assume that the weight function ω is real-analytic and strictly positive in $\mathbb{D}$ and that the curvature condition*

$$(4.2) \qquad \mathbf{K}(z) + \frac{1}{2}\mathbf{K}_{\mathbb{H}}(z) \leq 0, \quad z \in \mathbb{D},$$

is satisfied. Then, for $0 < t < T$, the Hele-Shaw domain $D(t) = D(t; \omega)$ is simply connected.

Proof. Without loss of generality, we can assume that ω is smooth up to the boundary. It is known that $\partial D(t)$ has a local Schwarz function at every point and

that this implies that $D(t)$ can have at most finitely many holes (see [8, Section 4]; the argument uses the work of Sakai [11]). Let $D_\bullet(t)$ be the simply connected domain obtained from $D(t)$ by adding all the interior holes. Let $\varphi : \mathbb{D} \to D_\bullet(t)$ be a Riemann map with $\varphi(0) = 0$. In view of the regularity of $\partial D(t)$ implied by the existence of a local Schwarz function, the map φ extends analytically to a neighborhood of the closed disk $\bar{\mathbb{D}}$. Let $B = \varphi^{-1}(D(t))$, and note that $\mathbb{D} \setminus B$ is a compact subset of $\mathbb{D}$. Introduce the function

$$(4.3) \qquad v(z) = \frac{1}{t}(\omega \circ \varphi)(z)|\varphi'(z)|^2, \quad z \in \mathbb{D}.$$

By the curvature assumption (4.2), the function

$$z \mapsto \frac{v(z)}{1 - |z|^2}$$

is logarithmically subharmonic in $\mathbb{D}$ and, hence, subharmonic there.

The set B can be interpreted as the non-coincidence set of an obstacle problem. A computation shows that

$$\Delta[V_t \circ \varphi(z)] = t\delta_0(z) - \omega \circ \varphi(z)|\varphi'(z)|^2, \quad z \in \mathbb{D},$$

where the potential function V_t is as in Section 2. Let W be defined in terms of v as in Lemma 4.1. It is easy to see that

$$W(z) = \frac{1}{t}(V_t \circ \varphi(z) - P[V_t \circ \varphi|_{\mathbb{T}}](z)), \quad z \in \mathbb{D},$$

where $P[\cdot]$ denotes the usual Poisson integral in $\mathbb{D}$. By [8, Proposition 2.9(a)], the function $\hat{V}_t$ is also the smallest superharmonic majorant for V_t in $D_\bullet(t)$. By conformal invariance, the function $\hat{V}_t \circ \varphi$ is the smallest superharmonic majorant for $V_t \circ \varphi$ in $\mathbb{D}$, and we have that

$$\widehat{W}(z) = \frac{1}{t}(\hat{V}_t \circ \varphi(z) - P[V_t \circ \varphi|_{\mathbb{T}}](z)), \quad z \in \mathbb{D}.$$

It is now clear that
$$B = \{z \in \mathbb{D} \mid W(z) < \widehat{W}(z)\}.$$

Lemma 4.1 shows that $B = \mathbb{D}$, which means that $D(t) = D_\bullet(t)$ is simply connected. The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad \square$

Theorem 4.3. *Let $\Omega = \langle \mathbb{D}, \omega \rangle$ be as in Theorem 4.2. Then the boundary $\partial D(t)$ of the Hele-Shaw domain is a real-analytic Jordan curve for $0 < t < T$.*

Proof. In view of the apriori regularity of $\partial D(t)$ (see [8, Section 4]) which follows from Sakai's theorem [11], as well as from Theorem 4.2, the boundary $\partial D(t)$ is real-analytic with the exception of at most finitely many cusp or contact points (see [8] for the terminology). Let $\varphi : \mathbb{D} \to D(t)$ be a Riemann map with $\varphi(0) = 0$. The regularity of $\partial D(t)$ afforded by the existence of a local Schwarz function shows that the map φ extends analytically to a neighborhood of $\bar{\mathbb{D}}$. The cusp points of $\partial D(t)$ correspond to points $z \in \mathbb{T}$ such that $\varphi'(z) = 0$. Arguing as in [8, Subsection 5.3] we see that the mean value identity

$$h(0) = \int_{\mathbb{D}} h(z)v(z)\,\mathrm{d}\Sigma(z)$$

holds for harmonic polynomials h, where v is given by (4.3). The non-vanishing of φ' on $\mathbb{T}$ now follows by Lemma 3.3.

If $D(t)$ has a contact point, then, for $t' = t + \delta$, with $\delta > 0$ sufficiently small, the domain $D(t')$ has a hole, which is impossible by Theorem 4.2. For details of this argument, see [8, Subsection 5.3].   □

5. MODIFICATION OF A LEMMA OF KORENBLUM

The following lemma is inspired by [8, Lemma 8.3] which is due to Boris Korenblum.

Lemma 5.1. *Fix α, $0 < \alpha < +\infty$. Let ω be a nonnegative function such that the function*

(5.1)
$$z \mapsto \log \frac{\omega(z)}{(1 - |z|^2)^{2\alpha}}$$

is subharmonic in $\mathbb{D}$. Assume also that ω is area-integrable in $\mathbb{D}$ and reproducing at the origin in the sense that

$$h(0) = \int_{\mathbb{D}} h(z)\omega(z)\,\mathrm{d}\Sigma(z)$$

holds for all harmonic polynomials h. Then, for a specific positive constant $c_{p,\alpha}$ depending only on p, α, we have

(5.2)
$$\int_0^1 \omega(r)^p\,\mathrm{d}r \le c_{p,\alpha} < +\infty, \quad \text{for } 0 < p < \frac{\pi}{\pi + 2\alpha(4 - \pi)}.$$

Remark 5.2. The constant $c_{p,\alpha}$ is given by equation (5.8) below.

Proof. The proof depends on the formula

(5.3)
$$\int_0^1 \frac{1}{(1-r)^2}\chi_{\mathbb{D}(r,1-r)}(z)\,\mathrm{d}r = \frac{1 - |z|^2}{|1 - z|^2}, \quad z \in \mathbb{D},$$

where χ_D denotes the characteristic function for the set D and $\mathbb{D}(z_0, r)$ is the open disk with center z_0 and radius r. The formula (5.3) is obtained by straightforward computation.

By the reproducing property of ω, we have

$$\int_{\mathbb{D}} \frac{1 - |rz|^2}{|1 - rz|^2} \omega(z) \, \mathrm{d}\Sigma(z) = 1, \quad 0 < r < 1,$$

and an application of Fatou's lemma shows that

$$(5.4) \qquad \int_{\mathbb{D}} \frac{1 - |z|^2}{|1 - z|^2} \omega(z) \, \mathrm{d}\Sigma(z) \leq 1.$$

We now turn to the proof of (5.2). Fix an r with $0 < r < 1$. By the subharmonicity assumption (5.1), we have that

$$\log \frac{\omega(r)}{(1 - r^2)^{2\alpha}} \leq \frac{1}{(1 - r)^2} \int_{\mathbb{D}(r, 1-r)} \log \omega(z) \, \mathrm{d}\Sigma(z)$$
$$+ \frac{2\alpha}{(1 - r)^2} \int_{\mathbb{D}(r, 1-r)} \log \frac{1}{1 - |z|^2} \, \mathrm{d}\Sigma(z).$$

By an application of the geometric-arithmetic mean inequality, we see that

$$(5.5) \qquad \frac{\omega(r)}{(1 - r^2)^{2\alpha}} \leq F(r)^{2\alpha} \frac{1}{(1 - r)^2} \int_{\mathbb{D}(r, 1-r)} \omega(z) \, \mathrm{d}\Sigma(z),$$

where we have written $F(r)$ for the quantity

$$(5.6) \qquad F(r) = \exp\left\{ \frac{1}{(1 - r)^2} \int_{\mathbb{D}(r, 1-r)} \log \frac{1}{1 - |z|^2} \, \mathrm{d}\Sigma(z) \right\}, \quad 0 < r < 1.$$

It is clear that $F(r)$ is bounded from above away from the right end-point $r = 1$. Below, we will show that $F(r)$ admits the estimate

$$(5.7) \qquad F(r) \leq \frac{1}{(1 - r)^{4/\pi + \varepsilon}}, \quad r_\varepsilon < r < 1,$$

for every positive ε, where $0 < r_\varepsilon < 1$. In view of (5.3), (5.4), and (5.5), we have that

$$\int_0^1 \frac{\omega(r)}{(1 - r^2)^{2\alpha} F(r)^{2\alpha}} \, \mathrm{d}r \leq \int_0^1 \frac{1}{(1 - r)^2} \int_{\mathbb{D}(r, 1-r)} \omega(z) \, \mathrm{d}\Sigma(z) \, \mathrm{d}r \leq 1.$$

An application of Hölder's inequality now gives that

$$(5.8) \qquad \int_0^1 \omega(r)^p \, \mathrm{d}r \leq \left\{ \int_0^1 [(1 - r^2) F(r)]^{2\alpha p/(1-p)} \, \mathrm{d}r \right\}^{1-p} = c_{p,\alpha},$$

where the rightmost equality is used to define the constant $c_{p,\alpha}$. The estimate (5.2) now follows. The finiteness of $c_{p,\alpha}$ for the asserted values of p follows by (5.7).

Finally, we turn to the proof of (5.7). By a change to polar coordinates in (5.6), we arrive at the formula

$$\log F(r) =$$
$$= \frac{2}{\pi(1-r)^2} \int_{2r-1}^1 t \log\left[\frac{1}{1-t^2}\right] \arcsin\left[\frac{\sqrt{(1-t^2)(t^2-(2r-1)^2)}}{2rt}\right] dt,$$

which is valid for $\frac{1}{2} < r < 1$. We note that by the geometric-arithmetic mean inequality, we have that

$$\frac{\sqrt{(1-t^2)[t^2-(2r-1)^2]}}{2rt} \le \frac{1-r}{t}.$$

From an elementary estimate of the arcsine, we find that

$$\log F(r) \le \frac{2}{\pi}\frac{1+\varepsilon}{1-r} \int_{2r-1}^1 \log\left[\frac{1}{1-t^2}\right] dt$$

for r, $\frac{1}{2} < r < 1$, sufficiently close to 1. A straightforward computation shows that

$$\int_{2r-1}^1 \log\left[\frac{1}{1-t^2}\right] dt = 2(1-r)\log\frac{1}{1-r} + 4(1-\log 2)(1-r) + 2r\log r.$$

Using the elementary inequality $\log r \le r - 1$, we see that

$$\int_{2r-1}^1 \log\left[\frac{1}{1-t^2}\right] dt \le 2(1-r)\log\frac{1}{1-r},$$

for $2(1-\log 2) \le r < 1$. As we return to the function F, we see that

$$\log F(r) \le \frac{4}{\pi}(1+\varepsilon)\log\frac{1}{1-r}$$

for r close to 1, which yields (5.7). □

Remark 5.3. Assume that ω is logarithmically subharmonic in $\mathbb{D}$ and reproducing in the sense of Lemma 5.1. Letting $\alpha \to 0$ in (5.2), we see that

$$\int_0^1 \omega(r)^p \, dr \le 1, \quad 0 < p < 1.$$

By monotone convergence, we conclude that

$$\int_0^1 \omega(r)\,dr \leq 1.$$

In [8, Lemma 8.3], this last inequality was shown to hold true under the slightly weaker assumption that ω is subharmonic and reproducing in $\mathbb{D}$.

In terms of the curvature form $\mathbf{K}$ of $\Omega = \langle \mathbb{D}, \omega \rangle$, the assumption (5.1) means that

$$(5.9) \qquad\qquad \mathbf{K}(z) + \alpha \mathbf{K}_{\mathbb{H}}(z) \leq 0, \quad z \in \mathbb{D},$$

as is easily verified by straightforward computation. In this context, we recall that the Poincaré metric of constant curvature -1 on $\mathbb{D}$ corresponds to the weight function

$$\omega_{\mathbb{H}}(z) = \frac{4}{(1 - |z|^2)^2}, \quad z \in \mathbb{D}.$$

We shall only need Lemma 5.1 for the parameter value $p = \frac{1}{2}$. In geometric language, the estimate in the lemma then says that the length of the radial segment $[0, 1[$ in the metric (1.1) is less than or equal to $c_\alpha = c_{1/2,\alpha}$. By a conformal mapping argument, we arrive at the proposition below. The notation $\mathbf{B}(z_0, r)$ stands for a metric disk of radius r about the point $z_0 \in \mathbb{D}$; also, as before, the parameter $T = T(\omega)$ is the termination time for the Hele-Shaw flow emanating from the origin.

Proposition 5.4. *Let $\Omega = \langle \mathbb{D}, \omega \rangle$ be a simply connected Riemann surface with a metric $ds^2 = \omega\,|dz|^2$. Assume that the weight function ω is C^2-smooth and strictly positive in $\mathbb{D}$ and that the curvature condition*

$$\mathbf{K}(z) + \alpha \mathbf{K}_{\mathbb{H}}(z) \leq 0, \quad z \in \mathbb{D},$$

is satisfied for some α with $0 \leq \alpha < \pi/(8 - 2\pi)$. Let $c_\alpha = c_{1/2,\alpha}$ be as in Lemma 5.1. If for some t, $0 < t < T$, the Hele-Shaw domain $D(t)$ is simply connected and $\partial D(t)$ is a Jordan curve, then $D(t) \subset \mathbf{B}(0, c_\alpha \sqrt{t})$.

Proof. Let $z_0 \in D(t)$, and let $\varphi : \mathbb{D} \to D(t)$ be the conformal map such that $\varphi(0) = 0$ and $\varphi^{-1}(z_0) \in [0, 1[$. Let ν be defined by (4.3). We shall apply Lemma 5.1 to the weight ν. It is straightforward to see that ν satisfies the subharmonicity assumption in Lemma 5.1. We proceed to check the reproducing property of ν.

A classical result of Torsten Carleman [1, Section 1] asserts that the space of analytic polynomials is dense in the Bergman space $A^p(D)$ ($0 < p < +\infty$) of all p-th power area integrable analytic functions in D if the domain D is simply connected with Jordan boundary. Let h be a harmonic polynomial. In view of

Carleman's theorem, we can find a sequence $\{h_j\}$ of functions harmonic in a neighbourhood of $\bar{D}(t)$ such that $h_j \to h \circ \varphi^{-1}$ in $L^1(D(t))$. By the mean value property of $D(t)$ (see [8, Theorem 2.3]), we have

$$th_j(0) = \int_{D(t)} h_j(z)\omega(z)\,d\Sigma(z),$$

and a passage to the limit yields that

$$th(0) = \int_{D(t)} h \circ \varphi^{-1}(z)\omega(z)\,d\Sigma(z).$$

An obvious change of variables now gives the reproducing property of ν.

An application of Lemma 5.1 gives that

$$\int_0^1 \sqrt{\nu(r)}\,dr \le c_\alpha.$$

This last inequality shows that the geodesic distance in the metric (1.1) from the origin to the point z_0 is less than $c_\alpha\sqrt{t}$. We conclude that $D(t) \subset \mathbf{B}(0, c_\alpha\sqrt{t})$. $\quad\square$

We remark that for $\alpha = 0$ the above proposition recovers [8, Proposition 8.2]. For a complete surface, we have the following corollary.

Corollary 5.5. *Let $\Omega = \langle \mathbb{D}, \omega \rangle$ be as in Theorem 4.2 and assume in addition that Ω is complete. Then $T = +\infty$. This in turn implies that the Hele-Shaw flow covers all of Ω, that is, $\mathbb{D} = \bigcup_{t>0} D(t)$.*

Proof. Since Ω is complete, a famous theorem of Hopf, Rinow and de Rham asserts that the metric disks $\mathbf{B}(0, r)$, $0 < r < +\infty$, are all precompact in $\Omega = \langle \mathbb{D}, \omega \rangle$ (see [9, Theorem I.10.3]). In view of Theorems 4.2 and 4.3, the Hele-Shaw domains $D(t)$ are simply connected with real-analytic Jordan curve boundaries for as long as $0 < t < T$, that is, $D(t) \Subset \mathbb{D}$. Moreover, Proposition 5.4 guarantees that the growth of the Hele-Shaw domains is essentially no faster than that of the metric disks. The conclusion is that $D(t)$ remains precompact in $\mathbb{D}$ for all t, $0 < t < +\infty$, so that $T = +\infty$. In addition, an obstacle problem argument shows that the Hele-Shaw flow covers all of Ω (see [8, pp. 219-220]). $\quad\square$

6. THE HELE-SHAW EXPONENTIAL MAPPING

Just as in [8], we have the following existence result regarding the so called *Hele-Shaw exponential mapping*.

Theorem 6.1. *Let $\Omega = \langle \mathbb{D}, \omega \rangle$ be as in Theorem 4.2. Let $z_0 \in \Omega$. Then there exists a real-analytic mapping $\Phi = \mathrm{HSexp}_{z_0} : \mathbb{D}(0, \sqrt{T}) \to \Omega$ such that*

- $\Phi(0) = z_0$,

- *each ray $\{z \in \mathbb{D}(0, \sqrt{T}) \setminus \{0\} \mid \arg(z) = \theta\}$ is mapped onto a curve in Ω which points in the same direction as the ray at z_0,*
- *the map Φ maps each pair consisting of a concentric circle about the origin and a straight line passing through the origin onto a pair of orthogonal curves, and*
- *for each $0 < r < \sqrt{T}$, the domain $\Phi(\mathbb{D}(0, r))$ equals the Hele-Shaw domain $\mathbf{D}(z_0, r^2)$.*

The map Φ is uniquely determined in the class of C^1-mappings satisfying the above four properties. Furthermore, the map Φ has the asymptotics

$$\Phi(z) = z_0 + \omega(0)^{-1/2} z + O(|z|^2) \quad \text{as } |z| \to 0.$$

If the surface Ω is complete, then $T = +\infty$, which in turn implies that the Hele-Shaw flow covers all of Ω, that is, Ω is the union of all the domains $\mathbf{D}(z_0, t)$ over all positive t.

Proof. Using properties of the weighted biharmonic Green function Γ_1 we have shown that the Hele-Shaw flow domain $D(t)$, $0 < t < T$, is simply connected and has real-analytic Jordan boundary (see Section 3 and Section 4). Also, in Section 5 we showed that $T = +\infty$ if Ω is complete. Knowing this, the proof of Theorem 6.1 proceeds exactly as in [8]. We omit the details. □

7. AN EXAMPLE OF A WEAKLY HYPERBOLIC SURFACE

In this section, we give an example of a complete real-analytic metric (1.1) satisfying (5.9) for a given positive α, which, for some r, $0 < r < 1$, has the circle $r\mathbb{T}$ as a geodesic curve. By a pull-back with a conformal self-map of the unit disk (Mœbius automorphism), we obtain a metric of the kind alluded to in the introduction (for details, see below).

Let I be a bounded interval on the real line $\mathbb{R}$. The differential equation for a geodesic curve $\gamma : I \to \mathbb{D}$ in the metric (1.1) can be written

$$(7.1) \qquad \gamma''(t) + \frac{1}{\omega(\gamma(t))} \frac{\partial \omega}{\partial z}(\gamma(t))[\gamma'(t)]^2 = 0, \quad t \in I,$$

where the curve γ is to be parameterized proportionally to arc length, that is, the quantity

$$\|\gamma'(t)\|_{\gamma(t)}^2 = \omega(\gamma(t)) \, |\gamma'(t)|^2$$

is to be constant.

We now specialize to a radial weight:

$$\omega(z) = \omega_0(|z|^2), \quad z \in \mathbb{D}.$$

Let γ_r be the circle $r\mathbb{T}$ parameterized by $\gamma_r(t) = re^{it}$ for real t, $-\pi < t \leq \pi$. A straightforward computation shows that γ_r satisfies (7.1) if and only if

$$(7.2) \qquad \omega_0(r^2) + r^2 \omega_0'(r^2) = 0.$$

Fix α, $0 < \alpha < +\infty$. We introduce a positive real parameter c, and consider weights of the form

$$\omega(z) = \frac{c}{(1 - |z|^2)^2} + (1 - |z|^2)^{2\alpha}, \quad z \in \mathbb{D}.$$

By comparison with the Poincaré metric, it is clear that the metric (1.1) so obtained is complete. Since the function

$$z \mapsto \frac{\omega(z)}{(1 - |z|^2)^{2\alpha}}$$

is the sum of two logarithmically subharmonic functions, it is itself logarithmically subharmonic (see Corollary 1.6.8 of [10]). This shows that (5.9) holds.

We now show that the equation (7.2) has a solution r, $0 < r < 1$. A computation shows that

$$\omega_0(\varrho) + \varrho\,\omega_0'(\varrho) = c\frac{1 + \varrho}{(1 - \varrho)^3} + (1 - \varrho)^{2\alpha - 1}(1 - (1 + 2\alpha)\varrho).$$

Note that the second term on the right hand side is negative for $\varrho > 1/(1 + 2\alpha)$. Thus, for c small and positive, the function

$$\varrho \mapsto \omega_0(\varrho) + \varrho\,\omega_0'(\varrho)$$

attains negative values in $]0, 1[$. Taking the endpoints $\varrho = 0, 1$ also into account, we see that equation (7.2) has at least two solutions r in the interval $]0, 1[$ for small positive values of c.

We should explain in more detail how the pull-back with the Mœbius automorphism works. Let $r = r_0$, $0 < r_0 < 1$, be a root of the equation (7.2). Moreover, let $\varphi : \mathbb{D} \to \mathbb{D}$ be an involutive Mœbius automorphism (so that $\varphi \circ \varphi(z) = z$) of the disk sending 0 to r_0, and thus r_0 to 0. The weight

$$\omega_\varphi(z) = |\varphi'(z)|^2 \omega \circ \varphi(z), \quad z \in \mathbb{D},$$

is such that $\langle \mathbb{D}, \omega \rangle$ and $\langle \mathbb{D}, \omega_\varphi \rangle$ have the same geodesics after composition with φ. So, the circular geodesic

$$\mathbb{T}(0, r_0) = \{z \in \mathbb{C} \mid |z| = r_0\}$$

for $\langle \mathbb{D}, \omega \rangle$ corresponds to the circular geodesic $\varphi(\mathbb{T}(0, r_0))$ for $\langle \mathbb{D}, \omega_\varphi \rangle$; the latter then passes through the origin.

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