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Norm expansion along a zero variety ☆

Håkan Hedenmalm *, Serguei Shimorin, Alan Sola

Department of Mathematics, The Royal Institute of Technology, S-100 44 Stockholm, Sweden

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Abstract

The reproducing kernel function of a weighted Bergman space over domains in $\mathbb{C}^d$ is known explicitly in only a small number of instances. Here, we introduce a process of orthogonal norm expansion along a subvariety of (complex) codimension 1, which also leads to a series expansion of the reproducing kernel in terms of reproducing kernels defined on the subvariety. The problem of finding the reproducing kernel is thus reduced to the same kind of problem when one of the two entries is on the subvariety. A complete expansion of the reproducing kernel may be achieved in this manner. We carry this out in dimension $d = 2$ for certain classes of weighted Bergman spaces over the bidisk (with the diagonal $z_1 = z_2$ as subvariety) and the ball (with $z_2 = 0$ as subvariety), as well as for a weighted Bargmann–Fock space over $\mathbb{C}^2$ (with the diagonal $z_1 = z_2$ as subvariety).
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1. Introduction

The general setup. Let Ω be an open connected set in $\mathbb{C}^d$ ($d = 1, 2, 3, \ldots$). A separable Hilbert space $\mathcal{H}(\Omega)$ (over the complex field $\mathbb{C}$) of holomorphic functions on Ω is given, such that the point evaluations at points of Ω are bounded linear functionals on $\mathcal{H}(\Omega)$. By a standard result in

☆ Research supported by the Göran Gustafsson Foundation.
* Corresponding author.
E-mail addresses: haakanh@math.kth.se (H. Hedenmalm), shimorin@math.kth.se (S. Shimorin), alansola@math.kth.se (A. Sola).

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Hilbert space theory, then, to each point $w \in \Omega$, there corresponds an element $k_w \in \mathcal{H}(\Omega)$ such that

$$f(w) = \langle f, k_w \rangle_{\mathcal{H}(\Omega)}, \quad f \in \mathcal{H}(\Omega).$$

Usually, we write $k(z, w) = k_w(z)$, and when we need to emphasize the space, we write $k^{\mathcal{H}(\Omega)}$ in place of k. The function $k^{\mathcal{H}(\Omega)}$ is the *reproducing kernel* of $\mathcal{H}(\Omega)$. It is in general a difficult problem to calculate the reproducing kernel explicitly. Of course, in terms of an orthonormal basis $e_1, e_2, e_3, \ldots$ for $\mathcal{H}(\Omega)$, the answer is easy:

$$k(z, w) = \sum_{n=1}^{+\infty} e_n(z)\bar{e}_n(w).$$

In most situations where no obvious orthogonal basis is present, this requires application of the rather complicated Gram–Schmidt orthogonalization procedure. Here, we introduce a method which has the potential to supply the reproducing kernel in a more digestible form. The method also supplies an expansion of the norm in $\mathcal{H}(\Omega)$ in terms of norms of "generalized restrictions" along an analytic variety of codimension 1.

Let $p : \mathbb{C}^d \to \mathbb{C}$ be a nontrivial polynomial of d variables, and let Z_p be the variety

$$\mathrm{Z}_p = \big\{ z \in \Omega : \ p(z) = 0 \big\},$$

which we assume to be nonempty. We also assume that p has nonvanishing gradient along Z_p. This assures us that a holomorphic function in Ω that vanishes on Z_p is analytically divisible by p in Ω. The assumptions made here are excessive, and may be relaxed substantially without substantially altering the assertions made in the sequel. For instance, Ω might instead be a d-dimensional complex manifold, and p an arbitrary analytic function on Ω with nonvanishing gradient along its zero set. For $N = 0, 1, 2, 3, \ldots$, the subspace of $\mathcal{H}(\Omega)$ consisting functions holomorphically divisible in Ω by p^N is denoted by $\mathcal{N}_N(\Omega)$; it is easy to show that $\mathcal{N}_N(\Omega)$ is a closed subspace of $\mathcal{H}(\Omega)$. We also need the difference space

$$\mathcal{M}_N(\Omega) = \mathcal{N}_N(\Omega) \ominus \mathcal{N}_{N+1}(\Omega),$$

which is a closed subspace of $\mathcal{N}_N(\Omega)$. Let P_N stand for the orthogonal projection $\mathcal{H}(\Omega) \to \mathcal{N}_N(\Omega)$, while Q_N is the orthogonal projection $\mathcal{H}(\Omega) \to \mathcal{M}_N(\Omega)$. Let $\mathcal{H}_N(\Omega)$ be the Hilbert space of analytic functions f on Ω such that $p^N f \in \mathcal{H}(\Omega)$, with norm

$$\|f\|_{\mathcal{H}_N(\Omega)} = \big\|p^N f\big\|_{\mathcal{H}(\Omega)}.$$

Clearly, the operator M_p^N of multiplication by p^N is an isometric isomorphism $\mathcal{H}_N(\Omega) \to \mathcal{N}_N(\Omega)$.

The norm expansion. We obtain a natural orthogonal decomposition

$$g = \sum_{N=0}^{+\infty} Q_N g, \qquad \|g\|^2_{\mathcal{H}(\Omega)} = \sum_{N=0}^{+\infty} \|Q_N g\|^2_{\mathcal{H}(\Omega)}, \tag{1.1}$$

since

$$\bigcap_{N=0}^{+\infty} \mathcal{N}_N(\Omega) = \{0\}, \tag{1.2}$$

which expresses the fact that no analytic function on Ω may be holomorphically divisible by p^N for all positive integers N unless the function vanishes identically. In other words, we have an orthogonal decomposition

$$\mathcal{H}(\Omega) = \bigoplus_{N=0}^{+\infty} \mathcal{M}_N(\Omega).$$

If we introduce the operator $R_N : \mathcal{H}(\Omega) \to \mathcal{H}_N(\Omega)$ defined by $R_N g = (Q_N g)/p^N$, it is possible to write the above decomposition in the form

$$g = \sum_{N=0}^{+\infty} p^N R_N g, \qquad \|g\|^2_{\mathcal{H}(\Omega)} = \sum_{N=0}^{+\infty} \|R_N g\|^2_{\mathcal{H}_N(\Omega)}.$$

The space of restrictions to Z_p of the functions in $\mathcal{H}_N(\Omega)$ is denoted by $\mathcal{H}_N(\mathrm{Z}_p)$. It is supplied with the induced Hilbert space norm

$$\|f\|_{\mathcal{H}_N(\mathrm{Z}_p)} = \inf\bigl\{\|g\|_{\mathcal{H}_N(\Omega)}\colon\ g \in \mathcal{H}_N(\Omega) \text{ with } g|_{\mathrm{Z}_p} = f\bigr\}.$$

Let $\mathcal{G}_N(\Omega)$ denote the closed subspace of $\mathcal{H}_N(\Omega)$ consisting of g with $p^N g \in \mathcal{M}_N(\Omega)$. Also, let $\oslash_p$ denote the operation of taking the restriction to Z_p of a function defined on Ω. It is easy to see that we have

$$\|g\|_{\mathcal{H}_N(\Omega)} = \|\oslash_p g\|_{\mathcal{H}_N(\mathrm{Z}_p)} \tag{1.3}$$

if and only if $g \in \mathcal{G}_N(\Omega)$ (we recall that $g \in \mathcal{H}_N(\Omega)$ means that $p^N g \in \mathcal{N}_N(\Omega)$). By polarizing (1.3), we find that

$$\langle f, g\rangle_{\mathcal{H}_N(\Omega)} = \langle \oslash_p f, \oslash_p g\rangle_{\mathcal{H}_N(\mathrm{Z}_p)}, \quad f, g \in \mathcal{G}_N(\Omega). \tag{1.4}$$

We may now rewrite the orthogonal decomposition in yet another guise (for $g \in \mathcal{H}(\Omega)$):

$$g = \sum_{N=0}^{+\infty} p^N R_N g, \qquad \|g\|^2_{\mathcal{H}(\Omega)} = \sum_{N=0}^{+\infty} \|\oslash_p R_N g\|^2_{\mathcal{H}_N(\mathrm{Z}_p)}. \tag{1.5}$$

In a practical situation, if we want to make use of this norm decomposition, we need to be able to characterize the restriction spaces $\mathcal{H}_N(\mathrm{Z}_p)$ in terms of a condition on Z_p (which has codimension 1), and also to characterize the operators $\widetilde{R}_N = \oslash_p R_N$. This is quite often possible.

Expansion of the reproducing kernel. The above orthogonal decomposition corresponds to a reproducing kernel decomposition

$$k^{\mathcal{H}(\Omega)}(z,w)=\sum_{N=0}^{+\infty}k^{\mathcal{M}_N(\Omega)}(z,w)=\sum_{N=0}^{+\infty}p(z)^N\,\bar{p}(w)^N k^{\mathcal{G}_N(\Omega)}(z,w). \tag{1.6}$$

Sometimes it is possible to characterize the restriction of $k^{\mathcal{H}_N(\Omega)}$ to $\Omega\times\mathrm{Z}_p$ (and hence, by symmetry, to $\mathrm{Z}_p\times\Omega$ as well). One way this may happen is as follows. Firstly, there is a certain point $w^0\in\mathrm{Z}_p$ for which it is easy to calculate the function $z\mapsto k^{\mathcal{H}_N(\Omega)}(z,w^0)$ explicitly. Secondly, the automorphism group of Ω is fat enough, in the sense that to each $w\in\mathrm{Z}_p$ there exists an automorphism of Ω which sends w^0 to w. Moreover, to each automorphism we need an associated unitary operator on $\mathcal{H}_N(\Omega)$ of composition type (in more detail, it should be of the type $M_F C_\phi$, where $C_\phi f=f\circ\phi$ and ϕ is the automorphism in question, while M_F denotes multiplication by a zero-free analytic function F). The automorphisms allow us to calculate $k^{\mathcal{H}_N(\Omega)}(z,w)$ for $w\in\mathrm{Z}_p$ knowing $k^{\mathcal{H}_N(\Omega)}(z,w^0)$. Note that on the set $\Omega\times\mathrm{Z}_p$, the two reproducing kernels $k^{\mathcal{H}_N(\Omega)}$ and $k^{\mathcal{G}_N(\Omega)}$ coincide.

Our goal is to express $k^{\mathcal{H}(\Omega)}$. Consider for a moment the following inner product:

$$l_N(z,w)=\big\langle \oslash_p k_w^{\mathcal{H}_N(\Omega)},\oslash_p k_z^{\mathcal{H}_N(\Omega)}\big\rangle_{\mathcal{H}_N(\mathrm{Z}_p)},\quad (z,w)\in\Omega\times\Omega. \tag{1.7}$$

Clearly, $l_N(z,w)$ is analytic in z and antianalytic in w. Since $k^{\mathcal{H}_N(\Omega)}$ and $k^{\mathcal{G}_N(\Omega)}$ coincide on the set $\Omega\times\mathrm{Z}_p$, we have

$$l_N(z,w)=\big\langle \oslash_p k_w^{\mathcal{G}_N(\Omega)},\oslash_p k_z^{\mathcal{G}_N(\Omega)}\big\rangle_{\mathcal{H}_N(\mathrm{Z}_p)},$$

and if we apply (1.4), we get

$$l_N(z,w)=\big\langle k_w^{\mathcal{G}_N(\Omega)},k_z^{\mathcal{G}_N(\Omega)}\big\rangle_{\mathcal{H}_N(\Omega)}=k^{\mathcal{G}_N(\Omega)}(z,w). \tag{1.8}$$

By (1.6), we may now write down the desired explicit formula for $k^{\mathcal{H}(\Omega)}$, valid on $\Omega\times\Omega$:

$$k^{\mathcal{H}(\Omega)}(z,w)=\sum_{N=0}^{+\infty}p(z)^N\,\bar{p}(w)^N\big\langle \oslash_p k_w^{\mathcal{H}_N(\Omega)},\oslash_p k_z^{\mathcal{H}_N(\Omega)}\big\rangle_{\mathcal{H}_N(\mathrm{Z}_p)}. \tag{1.9}$$

Applications. In Section 2, we carry out this program for classes of weighted Bergman spaces on the bidisk (with $p(z_1,z_2)=z_1-z_2$), while in Section 3, we do the same thing for the balls in $\mathbb{C}^2$ (with $p(z_1,z_2)=z_2$). Finally, in Section 4, we apply the technique to weighted Bargmann–Fock spaces on $\mathbb{C}^2$ (with $p(z_1,z_2)=z_1-z_2$).

We remark that the first norm decomposition of this type for the bidisk was obtained by Hedenmalm and Shimorin [4], who used it to substantially improve the previously known estimates of the integral means spectrum for conformal maps.

A trivial example. Let $\mathrm{d}A$ denote the normalized area element in the plane,

$$\mathrm{d}A(z) = \frac{1}{\pi}\,\mathrm{d}x\,\mathrm{d}y, \quad \text{where } z = x + iy, \tag{1.10}$$

and for α, $-1 < \alpha < +\infty$, we consider the following weighted area element in the unit disk $\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$:

$$\mathrm{d}A_\alpha(z) = (\alpha+1)\bigl(1-|z|^2\bigr)^\alpha\,\mathrm{d}A(z). \tag{1.11}$$

It is a probability measure in $\mathbb{D}$. The Hilbert space $A^2_\alpha(\mathbb{D})$ consists of all analytic functions g in $\mathbb{D}$ subject to the norm boundedness condition

$$\|g\|^2_\alpha = \int_{\mathbb{D}} \bigl|g(z)\bigr|^2\,\mathrm{d}A_\alpha(z) < +\infty. \tag{1.12}$$

Fix α, and consider the space $\mathcal{H}(\mathbb{D}) = A^2_\alpha(\mathbb{D})$ and the polynomial $p(z) = z$. Then the space $\mathcal{N}_N(\mathbb{D})$ consists of all functions that have a zero of order N at the origin, while $\mathcal{N}_N(\mathbb{D}) \ominus \mathcal{N}_{N+1}(\mathbb{D})$ is just the linear span of the function z^N. We readily find that the orthogonal expansion (1.5) condenses to the familiar

$$g(z) = \sum_{N=0}^{+\infty} c_n z^n, \qquad \|g\|^2_\alpha = \sum_{N=0}^{+\infty} \frac{N!}{(\alpha+2)_N}|c_N|^2,$$

where $(x)_n$ is the familiar Pochhammer symbol. The reproducing kernel for the space $A^2_\alpha(\mathbb{D})$ is well known:

$$k(z,w) = \sum_{N=0}^{+\infty} \frac{(\alpha+2)_N}{N!} z^N \bar{w}^N = (1-z\bar{w})^{-2-\alpha}.$$

The interesting thing is that the method outlined above applies to give the indicated representation of the reproducing kernel.

For background material on the weighted Bergman spaces $A^2_\alpha(\mathbb{D})$, see [3].

Quotient Hilbert modules. Under some natural additional assumptions, Douglas and Misra [2] characterize the quotient space (module, if we assume the existence of a natural algebra of multipliers)

$$\mathcal{H}(\Omega)/\mathcal{N}_N(\Omega) \cong \mathcal{H}(\Omega) \ominus \mathcal{N}_N(\Omega)$$

in terms of vector bundles over Z_p. In our setting, let D_p be the complex derivation ($f \in \mathcal{H}(\Omega)$)

$$D_p f(z) = \bigl\langle \nabla f(z), \nabla p(z)\bigr\rangle_{\mathbb{C}^d}, \quad z \in \Omega.$$

Here,

$$\nabla = \left(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_d}\right)$$

is the complex-analytic gradient. We should then expect that $f \in \mathcal{H}(\Omega) \ominus \mathcal{N}_N(\Omega)$ may be identified with the vector-valued function (section of the vector bundle)

$$\big(f(z), D_p f(z), D_p^2 f(z), \dots, D_p^{N-1} f(z)\big), \quad z \in \mathrm{Z}_p.$$

Indeed, it is easy to see that the above section vanishes on Z_p if and only if $f \in \mathcal{N}_N(\Omega)$. Moreover, the corresponding partial sum of the norm expansion (1.5) (with f in place of g) may then be viewed as a norm on the section. This will become clearer in the applications (see Sections 2–4).

Notation. In the rest of the paper, the notation for reproducing kernels is slightly different (with letters P and Q instead of k). Also, we should point out that in the sequel, the notation is consistent within each section, but not necessarily between sections. This mainly applies to the spaces and their reproducing kernels, as we intentionally use very similar notation to demonstrate the analogy between the three cases we study (bidisk, ball, Bargmann–Fock).

2. Weighted Bergman spaces in the bidisk

Preliminaries. The unit bidisk in $\mathbb{C}^2$ is the set

$$\mathbb{D}^2 = \big\{z = (z_1, z_2) \in \mathbb{C}^2\colon\ |z_1| < 1,\ |z_2| < 1\big\}.$$

For a survey of the function theory of the bidisk, we refer to [6]; see also [5] and [1]. Fix real parameters $\alpha, \beta, \theta, \vartheta$ with $-1 < \alpha, \beta, \theta, \vartheta < +\infty$. We consider the Hilbert space $L^2_{\alpha,\beta,\theta,\vartheta}(\mathbb{D}^2)$ of all (equivalence classes of) Borel measurable functions f in the bidisk subject to the norm boundedness condition

$$\|f\|^2_{\alpha,\beta,\theta,\vartheta} = \int_{\mathbb{D}^2} \big|f(z_1, z_2)\big|^2 |1 - \bar{z}_2 z_1|^{2\vartheta} |z_1 - z_2|^{2\theta}\, \mathrm{d}A_\alpha(z_1)\, \mathrm{d}A_\beta(z_2) < +\infty,$$

where the notation is as in (1.11); we let $\langle \cdot, \cdot \rangle_{\alpha,\beta,\theta,\vartheta}$ denote the associated sesquilinear inner product. The *weighted Bergman space* $A^2_{\alpha,\beta,\theta,\vartheta}(\mathbb{D}^2)$ is the subspace of $L^2_{\alpha,\beta,\theta,\vartheta}(\mathbb{D}^2)$ consisting of functions holomorphic in the bidisk. We need to impose a further restriction on the parameters $\alpha, \beta, \theta, \vartheta$:

$$\alpha + \beta + 2\theta + 2\vartheta + 3 > 0;$$

then the constant function 1 will belong to the space $A^2_{\alpha,\beta,\theta,\vartheta}(\mathbb{D}^2)$.

The reproducing kernel for $A^2_{\alpha,\beta,\theta,\vartheta}(\mathbb{D}^2)$ will be denoted by

$$P_{\alpha,\beta,\theta,\vartheta} = P_{\alpha,\beta,\theta,\vartheta}(z, w),$$

where we adhere to the notational convention

$$z = (z_1, z_2), \qquad w = (w_1, w_2)$$

for points in $\mathbb{C}^2$. The kernel defines an orthogonal projection of the space $L^2_{\alpha,\beta,\theta,\vartheta}(\mathbb{D}^2)$ onto the weighted Bergman space via the formula

$$P_{\alpha,\beta,\theta,\vartheta}[f](z) = \big\langle f,\, P_{\alpha,\beta,\theta,\vartheta}(\cdot, z)\big\rangle_{\alpha,\beta,\theta,\vartheta}$$
$$= \int_{\mathbb{D}^2} f(w) P_{\alpha,\beta,\theta,\vartheta}(z, w)|1 - \bar{w}_2 w_1|^{2\vartheta} |w_1 - w_2|^{2\theta}\, \mathrm{d}A_\alpha(w_1)\, \mathrm{d}A_\beta(w_2); \quad (2.1)$$

as indicated, we shall write $P_{\alpha,\beta,\theta,\vartheta}[f]$ for the projection of a function $f \in L^2_{\alpha,\beta,\theta,\vartheta}(\mathbb{D}^2)$.

In the case $\theta = \vartheta = 0$, the reproducing kernel is readily computed:

$$P_{\alpha,\beta,0,0}(z, w) = \frac{1}{(1 - \bar{w}_1 z_1)^{\alpha+2}(1 - \bar{w}_2 z_2)^{\beta+2}}.$$

We consider the polynomial $p(z_1, z_2) = z_1 - z_2$ in the context of the introduction. In particular, for non-negative integers N, we consider the subspaces $\mathcal{N}_{\alpha,\beta,\theta,\vartheta,N}(\mathbb{D}^2)$ of functions in $A^2_{\alpha,\beta,\theta,\vartheta}(\mathbb{D}^2)$ that vanish up to order N along the diagonal

$$\mathrm{diag}(\mathbb{D}) = \big\{(z_1, z_2) \in \mathbb{D}^2\colon\ z_1 = z_2\big\}.$$

Being closed subspaces of a reproducing kernel space, the spaces $\mathcal{N}_{\alpha,\beta,\theta,\vartheta,N}(\mathbb{D}^2)$ possess reproducing kernels of their own. We shall write

$$P_{\alpha,\beta,\theta,\vartheta,N} = P_{\alpha,\beta,\theta,\vartheta,N}(z, w)$$

for these kernel functions. The operators associated with the kernels project the space $L^2_{\alpha,\beta,\theta,\vartheta}(\mathbb{D}^2)$ orthogonally onto $\mathcal{N}_{\alpha,\beta,\theta,\vartheta,N}(\mathbb{D}^2)$. As before, we write $P_{\alpha,\beta,\theta,\vartheta,N}[f]$ for the projection of a function.

Next, we define the spaces $\mathcal{M}_{\alpha,\beta,\theta,\vartheta,N}(\mathbb{D}^2)$ by setting

$$\mathcal{M}_{\alpha,\beta,\theta,\vartheta,N}\big(\mathbb{D}^2\big) = \mathcal{N}_{\alpha,\beta,\theta,\vartheta,N}\big(\mathbb{D}^2\big) \ominus \mathcal{N}_{\alpha,\beta,\theta,N+1}\big(\mathbb{C}^2\big).$$

The spaces $\mathcal{M}_{\alpha,\beta,\theta,\vartheta,N}(\mathbb{D}^2)$ also admit reproducing kernels, and their kernel functions are of the form

$$Q_{\alpha,\beta,\theta,\vartheta,N}(z, w) = P_{\alpha,\beta,\theta,\vartheta,N}(z, w) - P_{\alpha,\beta,\theta,\vartheta,N+1}(z, w).$$

We shall write $Q_{\alpha,\beta,\theta,\vartheta}$ for the kernel $Q_{\alpha,\beta,\theta,\vartheta,0}$.

We begin with the following observation.

Lemma 2.1. *We have*

$$P_{\alpha,\beta,\theta,\vartheta,N}(z, w) = (z_1 - z_2)^N (\bar{w}_1 - \bar{w}_2)^N P_{\alpha,\beta,\theta+N,\vartheta}(z, w) \quad (2.2)$$

for $z, w \in \mathbb{C}^2$.

Proof. After multiplying both sides of (2.1) by $(z_1 - z_2)^N$ and using the fact that $|w_1 - w_2|^{2N} = (w_1 - w_2)^N(\bar{w}_1 - \bar{w}_2)^N$, we see that

$$(z_1 - z_2)^N f(z) = \int_{\mathbb{D}^2} \big\{(z_1 - z_2)^N (\bar{w}_1 - \bar{w}_2)^N P_{\alpha,\beta,\theta+N,\vartheta}(z, w)\big\} \times \big\{(w_1 - w_2)^N f(w)\big\} |1 - \bar{w}_2 w_1|^{2\vartheta} |w_1 - w_2|^{2\theta} \, \mathrm{d}A_\alpha(w_1) \, \mathrm{d}A_\beta(w_2)$$

for every $f \in A^2_{\alpha,\beta,\theta+N,\vartheta}(\mathbb{D}^2)$. From this it follows that $(z_1 - z_2)^N (\bar{w}_1 - \bar{w}_2)^N P_{\alpha,\beta,\theta+N,\vartheta}(z, w)$ has the reproducing property for the space $\mathcal{N}_{\alpha,\beta,\theta,\vartheta,N}(\mathbb{D}^2)$, and the proof is complete. □

If we write, as in the introduction,

$$\mathcal{H}\big(\mathbb{D}^2\big) = A^2_{\alpha,\beta,\theta,\vartheta}\big(\mathbb{D}^2\big),$$

the argument of the proof of Lemma 2.2 actually shows that we have identified the spaces $\mathcal{H}_N(\mathbb{D}^2)$,

$$\mathcal{H}_N\big(\mathbb{D}^2\big) = A^2_{\alpha,\beta,\theta+N,\vartheta}\big(\mathbb{D}^2\big), \quad N = 0, 1, 2, \ldots.$$

At the same time, we have also identified the spaces $\mathcal{G}_N(\mathbb{D}^2)$,

$$\mathcal{G}_N\big(\mathbb{D}^2\big) = \mathcal{M}_{\alpha,\beta,\theta+N,\vartheta,0}\big(\mathbb{D}^2\big), \quad N = 0, 1, 2, \ldots.$$

As a consequence, we get that

$$Q_{\alpha,\beta,\theta,\vartheta,N}(z, w) = (z_1 - z_2)^N (\bar{w}_1 - \bar{w}_2)^N Q_{\alpha,\beta,\theta+N,\vartheta}(z, w). \tag{2.3}$$

By (1.6), we have the kernel function expansion

$$P_{\alpha,\beta,\theta,\vartheta}(z, w) = \sum_{N=0}^{+\infty} Q_{\alpha,\beta,\theta,\vartheta,N}(z, w), \tag{2.4}$$

while the orthogonal norm expansion (1.1) reads

$$\|f\|^2_{\alpha,\beta,\theta,\vartheta} = \sum_{N=0}^{+\infty} \big\|Q_{\alpha,\beta,\theta,\vartheta,N}[f]\big\|^2_{\alpha,\beta,\theta,\vartheta}, \quad f \in A^2_{\alpha,\beta,\theta,\vartheta}\big(\mathbb{D}^2\big). \tag{2.5}$$

Our next objective is to identify the Hilbert space of restrictions to the diagonal of $\mathcal{H}_N(\mathbb{D}^2) = A^2_{\alpha,\beta,\theta+N,\vartheta}(\mathbb{D}^2)$, as well as to calculate the reproducing kernel of $\mathcal{H}_N(\mathbb{D}^2)$ on the set $\mathbb{D}^2 \times \mathrm{diag}(\mathbb{D})$.

Unitary operators. The rotation operator R_ϕ (for a real parameter ϕ) defined for $f \in A^2_{\alpha,\beta,\theta,\vartheta}(\mathbb{D}^2)$ by

$$R_\phi[f](z_1, z_2) = f\big(e^{i\phi} z_1, e^{i\phi} z_2\big)$$

is clearly unitary, and we shall make use of it shortly. The following lemma supplies us with yet another family of unitary operators.

Lemma 2.2. *For every $\lambda \in \mathbb{D}$, the operator*

$$U_\lambda[f](z_1,z_2) = \frac{(1-|\lambda|^2)^{\alpha/2+\beta/2+\theta+\vartheta+2}}{(1-\bar\lambda z_1)^{\alpha+\theta+\vartheta+2}(1-\bar\lambda z_2)^{\beta+\theta+\vartheta+2}} f\left(\frac{\lambda-z_1}{1-\bar\lambda z_1}, \frac{\lambda-z_2}{1-\bar\lambda z_2}\right) \tag{2.6}$$

is unitary on the space $A^2_{\alpha,\beta,\theta,\vartheta}(\mathbb{D}^2)$, and $U^2_\lambda[f] = f$ holds for every $f \in A^2_{\alpha,\beta,\theta,\vartheta}(\mathbb{D}^2)$.

Proof. For real parameters p and q, we define the operator

$$\widetilde{U}_\lambda[f](z_1,z_2) = \left(\frac{1-|\lambda|^2}{(1-\bar\lambda z_1)^2}\right)^p \left(\frac{1-|\lambda|^2}{(1-\bar\lambda z_2)^2}\right)^q f\left(\frac{\lambda-z_1}{1-\bar\lambda z_1}, \frac{\lambda-z_2}{1-\bar\lambda z_2}\right).$$

We want to choose p and q so that $\widetilde{U}_\lambda$ becomes unitary. A change of variables shows that

$$\begin{aligned}
&\int_{\mathbb{D}^2} \left|\widetilde{U}_\lambda[f](z_1,z_2)\right|^2 |1-\bar z_2 z_1|^{2\vartheta} |z_1-z_2|^{2\theta} \,\mathrm{d}A_\alpha(z_1)\,\mathrm{d}A_\beta(z_2) \\
&\quad = \int_{\mathbb{D}^2} \left|f(\zeta,\xi)\right|^2 \frac{(1-|\lambda|^2)^{\alpha+\beta+2\theta+2\vartheta+4-2(p+q)}}{|1-\bar\lambda\zeta|^{2\alpha+2\theta+2\vartheta+4-4p}|1-\bar\lambda\xi|^{2\beta+2\theta+2\vartheta+4-4q}} \\
&\qquad \times |1-\bar\xi\zeta|^{2\vartheta}|\zeta-\xi|^{2\theta}\,\mathrm{d}A_\alpha(\zeta)\,\mathrm{d}A_\beta(\xi)
\end{aligned}$$

and we see that $p = 1+(\alpha+\theta+\vartheta)/2$ and $q = 1+(\beta+\theta+\vartheta)/2$ are the correct choices.

The proof of the second assertion is straightforward and therefore omitted. □

The reproducing kernel on the diagonal. We now use the operators R_ϕ and U_λ to compute the reproducing kernel on the set $\mathbb{D}^2 \times \mathrm{diag}(\mathbb{D})$. We recall the standard definition of the generalized Gauss hypergeometric function

$${}_3F_2\left(\begin{matrix} a_1 & a_2 & a_3 \\ & b_1 & b_2 \end{matrix}\middle| x\right) = 1 + \sum_{n=1}^{+\infty} \frac{(a_1)_n (a_2)_n (a_3)_n}{(b_1)_n (b_2)_n n!} x^n.$$

Theorem 2.3. *We have that*

$$\begin{aligned}
P_{\alpha,\beta,\theta,\vartheta}\big((z_1,z_2),(w_1,w_1)\big) &= Q_{\alpha,\beta,\theta,\vartheta}\big((z_1,z_2),(w_1,w_1)\big) \\
&= \frac{\sigma(\alpha,\beta,\theta,\vartheta)}{(1-\bar w_1 z_1)^{\alpha+\theta+\vartheta+2}(1-\bar w_1 z_2)^{\beta+\theta+\vartheta+2}}
\end{aligned}$$

for $z_1, z_2, w_1 \in \mathbb{D}$. Here, $\sigma(\alpha,\beta,\theta,\vartheta)$ is the positive constant given by

$$\frac{1}{\sigma(\alpha,\beta,\theta,\vartheta)} = \int_{\mathbb{D}}\int_{\mathbb{D}} |1-\bar z_2 z_1|^{2\vartheta}|z_1-z_2|^{2\theta}\,\mathrm{d}A_\alpha(z_1)\,\mathrm{d}A_\beta(z_2)$$

$$= \frac{(\beta+1)\Gamma(\alpha+2)\Gamma(\theta+1)}{(\alpha+\beta+2\theta+2\vartheta+3)\Gamma(\alpha+\theta+2)} \times {}_3F_2\left(\begin{matrix}\theta+1 & \alpha+\theta+\vartheta+2 & \alpha+\theta+\vartheta+2 \\ & \alpha+\theta+2 & \alpha+\beta+2\theta+2\vartheta+4\end{matrix}\,\middle|\,1\right).$$

Proof. By the reproducing property of $P_{\alpha,\beta,\theta,\vartheta}$, we have

$$f(0) = \big\langle f, P_{\alpha,\beta,\theta,\vartheta}(\cdot,0)\big\rangle_{\alpha,\beta,\theta,\vartheta},$$

where 0 this time denotes the origin in $\mathbb{C}^2$. The unitarity of R_ϕ gives us

$$f(0) = \big\langle R_\phi[f], R_\phi[P_{\alpha,\beta,\theta,\vartheta}](\cdot,0)\big\rangle_{\alpha,\beta,\theta,\vartheta},$$

and since $f(0,0) = R_\phi[f](0,0)$, we see from the uniqueness of the reproducing kernel that

$$P_{\alpha,\beta,\theta,\vartheta}\big(\big(e^{i\phi}z_1, e^{i\phi}z_2\big),(0,0)\big) = P_{\alpha,\beta,\theta,\vartheta}\big((z_1,z_2),(0,0)\big) = P_{\alpha,\beta,\theta,\vartheta}(z,0).$$

The function $P_{\alpha,\beta,\theta,\vartheta}(z,0)$ is holomorphic in $\mathbb{D}^2$ and can be expanded in a power series. After comparing the series expansion for the expressions on both sides of the above equality, we conclude that $P_{\alpha,\beta,\theta,\vartheta}(z,0)$ must be a (positive) constant, which we denote by $\sigma(\alpha,\beta,\theta,\vartheta)$.

Next, take $\lambda \in \mathbb{D}$ and $f \in A^2_{\alpha,\beta,\theta,\vartheta}(\mathbb{D}^2)$. Since the operators U_λ are unitary and since $U_\lambda^2[f] = f$ we obtain that

$$\begin{aligned}\big(1-|\lambda|^2\big)^{\alpha/2+\beta/2+\theta+\vartheta+2} f(\lambda,\lambda) &= U_\lambda[f](0) = \big\langle U_\lambda[f], P_{\alpha,\beta,\theta,\vartheta}(\cdot,0)\big\rangle_{\alpha,\beta,\theta,\vartheta} \\ &= \big\langle U_\lambda^2[f], U_\lambda\big[P_{\alpha,\beta,\theta,\vartheta}(\cdot,0)\big]\big\rangle_{\alpha,\beta,\theta,\vartheta} \\ &= \sigma(\alpha,\beta,\theta)\big\langle f, U_\lambda[1]\big\rangle_{\alpha,\beta,\theta,\vartheta}.\end{aligned}$$

This equality together with the uniqueness of reproducing kernels establishes that

$$P_{\alpha,\beta,\theta,\vartheta}\big((z_1,z_2),(\lambda,\lambda)\big) = \sigma(\alpha,\beta,\theta,\vartheta)\big(1-|\lambda|^2\big)^{-\alpha/2-\beta/2-\theta-\vartheta-2} U_\lambda[1](z_1,z_2),$$

which is the desired result. The explicit expression for the constant in terms of an integral over the bidisk follows if we apply the reproducing property of the kernel applied to the constant function 1; the evaluation of the integral in terms of the hypergeometric function is done by performing the change of variables

$$z_1 = \frac{z_2-\zeta}{1-\bar{z}_2\zeta}, \quad z_2 = z_2,$$

and by carrying out some tedious but straightforward calculations. □

Restrictions of reproducing kernels. From the previous subsection, we have that

$$P_{\alpha,\beta,\theta,\vartheta}\big((z_1,z_2),(w_1,w_1)\big)=\frac{\sigma(\alpha,\beta,\theta,\vartheta)}{(1-\bar{w}_1z_1)^{\alpha+\theta+\vartheta+2}(1-\bar{w}_1z_2)^{\beta+\theta+\vartheta+2}}.$$

For continuous functions $f\in L^2_{\alpha,\beta,\theta,\vartheta}(\mathbb{D}^2)$, we use the notation $\oslash f$ for the restriction to the diagonal of the function, that is,

$$(\oslash f)(z_1)=f(z_1,z_1).$$

We fix (w_1,w_1) and apply this operation to the kernel function of $A^2_{\alpha,\beta,\theta,\vartheta}(\mathbb{D}^2)$. We obtain

$$\oslash P_{\alpha,\beta,\theta,\vartheta}\big(z_1,(w_1,w_1)\big)=\frac{\sigma(\alpha,\beta,\theta,\vartheta)}{(1-\bar{w}_1z_1)^{\alpha+\beta+2\theta+2\vartheta+4}}$$

and we see that the restriction of the kernel coincides with a multiple of the kernel function for the space $A^2_{\alpha+\beta+2\theta+2\vartheta+2}(\mathbb{D})$. By the theory of reproducing kernels (see [8]), this means that the induced norm for the space $\mathcal{M}_{\alpha,\beta,\theta,\vartheta,0}(\mathbb{D}^2)$ coincides with a multiple of the norm in the aforementioned weighted Bergman space in the unit disk. An immediate consequence of this fact is the inequality

$$\frac{1}{\sigma(\alpha,\beta,\theta,\vartheta)}\|\oslash f\|^2_{\alpha+\beta+2\theta+2\vartheta+2}\leqslant\|f\|^2_{\alpha,\beta,\theta,\vartheta},\quad f\in A^2_{\alpha,\beta,\theta,\vartheta}\big(\mathbb{D}^2\big),\tag{2.7}$$

and, more importantly, the equality

$$\frac{1}{\sigma(\alpha,\beta,\theta,\vartheta)}\|\oslash f\|^2_{\alpha+\beta+2\theta+2\vartheta+2}=\big\|Q_{\alpha,\beta,\theta,\vartheta}[f]\big\|^2_{\alpha,\beta,\theta,\vartheta},\quad f\in\mathcal{M}_{\alpha,\beta,\theta,\vartheta,0}\big(\mathbb{D}^2\big).\tag{2.8}$$

The notation on the left-hand sides of (2.7) and (2.8) is in conformity with (1.12).

The next step in our program is to compute the kernel function $Q_{\alpha,\beta,\theta,\vartheta}$. In fact, we can determine the kernel in terms of an integral formula.

Lemma 2.4. *The kernel function for the space $\mathcal{M}_{\alpha,\beta,\theta,\vartheta,0}(\mathbb{D}^2)$ is given by*

$$Q_{\alpha,\beta,\theta,\vartheta}(z,w)=\int\limits_{\mathbb{D}}\frac{\sigma(\alpha,\beta,\theta,\vartheta)\,\mathrm{d}A_{\alpha+\beta+2\theta+2\vartheta+2}(\xi)}{[(1-\bar{\xi}z_1)(1-\xi\bar{w}_1)]^{\alpha+\theta+\vartheta+2}[(1-\bar{\xi}z_2)(1-\xi\bar{w}_2)]^{\beta+\theta+\vartheta+2}}$$

for $z,w\in\mathbb{D}^2$.

Proof. In the notation of the introduction, this is the identity (with $N=0$)

$$k^{\mathcal{G}_N(\Omega)}(z,w)=\big\langle\oslash_p k_w^{\mathcal{H}_N(\Omega)},\oslash_p k_z^{\mathcal{H}_N(\Omega)}\big\rangle_{\mathcal{H}_N(\mathbb{Z}_p)},$$

which follows from (1.7) and (1.8). □

We may now replace the terms on the right-hand side in (2.5) by norms taken in weighted spaces in the unit disk.

Lemma 2.5. *For each $N = 0, 1, 2, \ldots$, we have the equality of norms*

$$\big\| Q_{\alpha,\beta,\theta,\vartheta,N}[f] \big\|^2_{\alpha,\beta,\theta,\vartheta} = \frac{1}{\sigma(\alpha,\beta,\theta+N,\vartheta)} \left\| \oslash \left[\frac{P_{\alpha,\beta,\theta,\vartheta,N}[f]}{(z_1-z_2)^N} \right] \right\|^2_{\alpha+\beta+2\theta+2\vartheta+2N+2},$$

for all $f \in A^2_{\alpha,\beta,\theta,\vartheta}(\mathbb{D}^2)$.

Proof. The statement follows from a combination of (2.2) and (2.8). □

We need one more result in order to complete the norm expansion for the bidisk. In what follows, we use the notation $\partial_{z_1} f$ for the partial derivative of f with respect to the variable z_1.

Lemma 2.6. *For $k = 0, 1, 2, \ldots$, we have, for each $f \in A^2_{\alpha,\beta,\theta,\vartheta}(\mathbb{D}^2)$,*

$$\oslash\big[\partial^k_{z_1} f\big] = \sum_{n=0}^{k} n! \binom{k}{n} \frac{(\alpha+\theta+\vartheta+n+2)_{k-n}}{(\alpha+\beta+2\theta+2\vartheta+2n+4)_{k-n}} \partial^{k-n}_{z_1} \oslash \left[\frac{P_{\alpha,\beta,\theta,\vartheta,n}[f]}{(z_1-z_2)^n} \right].$$

Proof. We recall that

$$P_{\alpha,\beta,\theta,\vartheta}(z,w) = \sum_{N=0}^{+\infty} (z_1-z_2)^N (\bar{w}_1-\bar{w}_2)^N Q_{\alpha,\beta,\theta+N,\vartheta}(z,w),$$

whence it follows that

$$\oslash \partial^k_{z_1} P_{\alpha,\beta,\theta,\vartheta}\big(z_1,(w_1,w_2)\big) = \sum_{n=0}^{k} n! \binom{k}{n} (\bar{w}_1-\bar{w}_2)^N \oslash \partial^{k-n}_{z_1} Q_{\alpha,\beta,\theta+n,\vartheta}\big(z_1,(w_1,w_2)\big). \quad (2.9)$$

Differentiation of the integral formula of Lemma 2.4 and taking the diagonal restriction leads to the equality

$$\begin{aligned} &\oslash \partial^{k-n}_{z_1} Q_{\alpha,\beta,\theta+n,\vartheta}\big(z_1,(w_1,w_2)\big) \\ &\quad = (\alpha+\theta+\vartheta+n+2)_{k-n} \sigma(\alpha,\beta,\theta+n,\vartheta) \\ &\qquad \times \int_{\mathbb{D}} \frac{\bar{\xi}^{k-n} \mathrm{d}A_{\alpha+\beta+2\theta+2\vartheta+2n+2}(\xi)}{(1-\bar{\xi}z_1)^{\alpha+\beta+2\theta+2\vartheta+2n+4-(k-n)} (1-\xi\bar{w}_1)^{\alpha+\theta+\vartheta+n+2} (1-\xi\bar{w}_2)^{\beta+\theta+\vartheta+n+2}}. \end{aligned}$$

We now note that the expression

$$\frac{\bar{\xi}^{k-n}}{(1-\bar{\xi}z_1)^{\alpha+\beta+2\theta+2\vartheta+2n+4+(k-n)}}$$

is a multiple of the reproducing kernel of the space $A^2_{\alpha+\beta+2\theta+2\vartheta+2}(\mathbb{D})$, differentiated $k-n$ times. Invoking the reproducing property of this kernel, we obtain that

$$\oslash\partial_{z_1}^{k-n} Q_{\alpha,\beta,\theta+n,\vartheta}\big(z_1,(w_1,w_2)\big) = \frac{(\alpha+\theta+\vartheta+n+2)_{k-n}}{(\alpha+\beta+2\theta+2\vartheta+2n+4)_{k-n}}\partial_{z_1}^{k-n}\oslash[P_{\alpha,\beta,\theta+n,\vartheta}]\big(z_1,(w_1,w_2)\big).$$

This result, together with the identities (2.9) and (2.2), yields the desired equality, and the proof is complete. □

We remark that Lemma 2.6 is rather the opposite to what we need; it expresses the known quantity $\oslash[\partial_{z_1}^k f]$ in terms of the quantities we should like to understand. Nevertheless, it is possible to invert the assertion of Lemma 2.6 and express the unknown quantities in terms of known quantities.

The diagonal norm expansion for the bidisk. The above lemma finally allows us to express each term in the right-hand side of (2.5) in terms of diagonal restrictions of derivatives of the original function.

Lemma 2.7. *Put*

$$a_{k,N} = \frac{(-1)^{N-k}}{k!(N-k)!}\frac{(\alpha+\theta+\vartheta+k+2)_{N-k}}{(\alpha+\beta+2\theta+2\vartheta+N+k+3)_{N-k}}.$$

Then, for all $N = 0, 1, 2, \ldots$, the equality

$$\oslash\left[\frac{P_{\alpha,\beta,\theta,\vartheta}[f]}{(z_1-z_2)^N}\right] = \sum_{k=0}^{N} a_{k,N}\partial_{z_1}^{N-k}\oslash\big[\partial_{z_1}^k f\big],$$

holds for each $f \in A^2_{\alpha,\beta,\theta,\vartheta}(\mathbb{D}^2)$.

Proof. In view of the previous lemma it is enough to check that

$$\sum_{k=n}^{N} a_{k,N} n!\binom{k}{n}\frac{(\alpha+\theta+\vartheta+n+2)_{k-n}}{(\alpha+\beta+2\theta+2\vartheta+2n+4)_{k-n}} = \delta_{n,N},$$

where $\delta_{n,N}$ is the Kronecker delta. This amounts to performing some rather straight-forward calculations. First we note that

$$(\alpha+\theta+\vartheta+k+2)_{N-k}(\alpha+\theta+\vartheta+n+2)_{k-n} = (\alpha+\theta+\vartheta+2+n)_{N-n}$$

and since this last expression does not depend on k, we can factor it out from the above sum. This reduces our task to showing that

$$\sum_{k=n}^{N}\frac{(-1)^{N-k}}{(N-k)!(k-n)!(\alpha+\beta+2\theta+2\vartheta+N+k+3)_{N-n}(\alpha+\beta+2\theta+2\vartheta+2n+4)_{k-n}} = \delta_{n,N}.$$

We note that this is true when $n = N$. It remains to show that the left-hand side vanishes whenever $n < N$. Next, the fact that

$$\begin{aligned}&(\alpha+\beta+2\theta+2\vartheta+N+k+3)_{N-k}(\alpha+\beta+2\theta+2\vartheta+2n+4)_{k-n}\\&\quad=\frac{(\alpha+\beta+2\theta+2\vartheta+2n+4)_{2N-2n-1}}{(\alpha+\beta+2\theta+2\vartheta+k+n+4)_{N-n-1}}\end{aligned}$$

implies that we need only study the sum

$$\sum_{k=n}^{N}\frac{(-1)^{N-k}}{(N-k)!(k-n)!}(\alpha+\beta+2\theta+2\vartheta+k+n+4)_{N-n-1}.$$

Shifting the sum by setting $j = k - n$ and $M = N - n$, introducing the variable

$$\lambda=\alpha+\beta+2\theta+2\vartheta+2n+4,$$

and performing some manipulations, we find that the above sum transforms to

$$\frac{(-1)^M}{M!}\sum_{j=0}^{M}(-1)^j\binom{M}{j}(\lambda+j)_{M-1}.$$

This is an iterated difference of order M, and as $(\lambda)_{M-1}$ is a polynomial of degree $M-1$, the iterated difference vanishes whenever $M > 0$. The proof is complete. □

We now combine our results and obtain the norm expansion for the unit bidisk.

Theorem 2.8. *For any $f \in A^2_{\alpha,\beta,\theta,\vartheta}(\mathbb{D}^2)$, we have*

$$\|f\|^2_{\alpha,\beta,\theta,\vartheta}=\sum_{N=0}^{+\infty}\frac{1}{\sigma(\alpha,\beta,\theta+N,\vartheta)}\left\|\sum_{k=0}^{N}a_{k,N}\partial_{z_1}^{N-k}\oslash\left[\partial_{z_1}^{k}f\right]\right\|^2_{\alpha+\beta+2\theta+2\vartheta+2N+2},$$

where

$$\begin{aligned}\frac{1}{\sigma(\alpha,\beta,\theta,\vartheta)}&=\frac{(\beta+1)\Gamma(\alpha+2)\Gamma(\theta+1)}{(\alpha+\beta+2\theta+2\vartheta+3)\Gamma(\alpha+\theta+2)}\\&\quad\times{}_3F_2\left(\begin{matrix}\theta+1 & \alpha+\theta+\vartheta+2 & \alpha+\theta+\vartheta+2\\ & \alpha+\theta+2 & \alpha+\beta+2\theta+2\vartheta+4\end{matrix}\middle|1\right)\end{aligned}$$

and

$$a_{k,N}=\frac{(-1)^{N-k}}{k!(N-k)!}\frac{(\alpha+\theta+\vartheta+k+2)_{N-k}}{(\alpha+\beta+2\theta+2\vartheta+N+k+3)_{N-k}}.$$

The expression for the reproducing kernel of $A^2_{\alpha,\beta,\theta,\vartheta}(\mathbb{D}^2)$. We now combine (2.3), (2.4) and the integral expression for $Q_{\alpha,\beta,\theta,\vartheta}$ given in Lemma 2.4 and supply an explicit series and integral expression for the full reproducing kernel function of $A^2_{\alpha,\beta,\theta,\vartheta}(\mathbb{D}^2)$.

Theorem 2.9. *The reproducing kernel function of the space $A^2_{\alpha,\beta,\theta,\vartheta}(\mathbb{D}^2)$ is*

$$P_{\alpha,\beta,\theta,\vartheta}(z,w) = \sum_{N=0}^{+\infty} \sigma(\alpha,\beta,\theta+N,\vartheta)(z_1-z_2)^N(\bar{w}_1-\bar{w}_2)^N \times \int_{\mathbb{D}} \frac{\mathrm{d}A_{\alpha+\beta+2\theta+2\vartheta+2N+2}(\xi)}{[(1-\bar{\xi}z_1)(1-\bar{w}_1\xi)]^{\alpha+\theta+\vartheta+N+2}[(1-\bar{\xi}z_2)(1-\bar{w}_2\xi)]^{\beta+\theta+\vartheta+N+2}}.$$

The weighted Hardy space case. We look at a special case of the identity of Theorem 2.9. First, we set $\vartheta = 0$ and note that in this case, the expression for the constant $\sigma(\alpha,\beta,\theta,\vartheta)$ reduces to

$$\frac{1}{\sigma(\alpha,\beta,\theta,0)} = \frac{\Gamma(\alpha+2)\Gamma(\beta+2)\Gamma(\theta+1)\Gamma(\alpha+\beta+2\theta+3)}{\Gamma(\alpha+\theta+2)\Gamma(\beta+\theta+2)\Gamma(\alpha+\beta+\theta+3)},$$

and if we also put $\alpha = \beta = -1$, we get

$$\frac{1}{\sigma(-1,-1,\theta,0)} = \frac{\Gamma(2\theta+2)}{(2\theta+1)[\Gamma(\theta+1)]^2}.$$

Next, we recall that in the limit $\alpha \to -1$, the weighted measure $\mathrm{d}A_\alpha(z_1)$ degenerates to arc-length measure on the unit circle. This means that letting α and β tend to -1 corresponds to considering the weighted Hardy space $H^2_\theta(\mathbb{D}^2)$, with norm defined by

$$\|f\|^2_{H^2_\theta(\mathbb{D}^2)} = \int_{\mathbb{T}^2} \big|f(z)\big|^2 |z_1-z_2|^{2\theta}\,\mathrm{d}s_1(z_1)\,\mathrm{d}s_1(z_2), \tag{2.10}$$

where $\mathrm{d}s_1$ is the normalized Lebesgue measure on the unit circle. Hence, Theorem 2.9 leads to a norm expansion for the weighted Hardy space. We state this as a corollary.

Corollary 2.10. *For any $f \in H^2_\theta(\mathbb{D}^2)$, we have*

$$\|f\|^2_{H^2_\theta(\mathbb{D}^2)} = \sum_{N=0}^{+\infty} \frac{\Gamma(2\theta+2N+2)}{(2\theta+2N+1)[\Gamma(\theta+N+1)]^2} \left\| \sum_{k=0}^{N} b_{k,N}\,\partial_{z_1}^{N-k} \oslash \big[\partial_{z_1}^k f\big] \right\|^2_{2\theta+2N}$$

where

$$b_{k,N} = \frac{(-1)^{N-k}}{k!(N-k)!}\,\frac{(\theta+k+1)_{N-k}}{(2\theta+N+k+1)_{N-k}}.$$

3. Weighted Bergman spaces in the unit ball

Preliminaries. The unit ball in $\mathbb{C}^2$ is the set

$$\mathbb{B}^2 = \big\{z = (z_1, z_2) \in \mathbb{C}^2\colon\ |z_1|^2 + |z_2|^2 < 1\big\}.$$

For a survey of the function theory of the ball, we refer to [7]; see also [5] and [1]. We consider weighted spaces $L^2_{\alpha,\beta,\theta}(\mathbb{B}^2)$ consisting of (equivalence classes of) Borel measurable functions f on $\mathbb{B}^2$ with

$$\|f\|^2_{\alpha,\beta,\theta} = \int\limits_{\mathbb{B}^2} \big|f(z)\big|^2 |z_2|^{2\theta} \big(1 - |z_1|^2 - |z_2|^2\big)^{\alpha} \big(1 - |z_1|^2\big)^{\beta}\, \mathrm{d}A(z_1, z_2) < +\infty,$$

where α, β, θ have $-1 < \alpha, \beta, \theta < +\infty$, and

$$\mathrm{d}A(z_1, z_2) = \mathrm{d}A(z_1)\, \mathrm{d}A(z_2).$$

The Bergman space $A^2_{\alpha,\beta,\theta}(\mathbb{B}^2)$ is the subspace of $L^2_{\alpha,\beta,\theta}(\mathbb{B}^2)$ consisting of functions f that are holomorphic in $\mathbb{B}^2$. In this section, we find the orthogonal decomposition of functions in $A^2_{\alpha,\beta,\theta}(\mathbb{B}^2)$ along the zero variety

$$\big\{(z_1, z_2) \in \mathbb{B}^2\colon\ z_2 = 0\big\},$$

corresponding to the polynomial $p(z_1, z_2) = z_2$. In the special case $\beta = \theta = 0$, the Bergman kernel is well known:

$$P_{\alpha,0,0}(z, w) = \frac{1}{(1 - \bar{w}_1 z_1 - \bar{w}_2 z_2)^{\alpha+3}}.$$

The norm expansion and the kernel function for $A^2_{\alpha,\beta,\theta}(\mathbb{B}^2)$ can be found using the techniques of the introduction and the previous section. The corresponding unitary operator U_λ on $A^2_{\alpha,\beta,\theta}(\mathbb{B}^2)$ one should use in this case is given by

$$U_\lambda[f](z_1, z_2) = \frac{(1 - |\lambda|^2)^{\frac{\alpha+\beta+\theta+3}{2}}}{(1 - \bar{\lambda} z_1)^{\alpha+\beta+\theta+3}} f\left(\frac{\lambda - z_1}{1 - \bar{\lambda} z_1}, -\frac{\sqrt{1 - |\lambda|^2}}{1 - \bar{\lambda} z_1} z_2\right), \quad \lambda \in \mathbb{D},$$

and we may again identify the restricted kernel $\oslash P_{\alpha,\beta,\theta}$ (with $z, w \in \mathbb{D} \times \{0\}$) with a multiple of the kernel of a weighted Bergman space in the unit disk. However, it turns out that there is an easier way to obtain the norm expansion and the explicit expression for the kernel function for $A^2_{\alpha,\beta,\theta}(\mathbb{B}^2)$.

By Taylor's formula, any function $f \in A^2_{\alpha,\beta,\theta}(\mathbb{B}^2)$ has a decomposition

$$f(z) = \sum_{N=0}^{+\infty} g_N(z_1) z_2^N, \quad \text{where } g_N(z_1) = \frac{1}{N!} \partial^N_{z_2} f(z_1, 0).$$

It is easy to see that the summands in this decomposition are orthogonal in the space $A^2_{\alpha,\beta,\theta}(\mathbb{B}^2)$ for different N, and hence

$$\|f\|^2_{\alpha,\beta,\theta} = \sum_{N=0}^{+\infty} \|g_N(z_1)z_2^N\|^2_{\alpha,\beta,\theta}. \tag{3.1}$$

The norm expansion for the ball. All we need is the following lemma.

Lemma 3.1. *We have that*

$$\|g_N(z_1)z_2^N\|^2_{\alpha,\beta,\theta} = \frac{\Gamma(\alpha+1)\Gamma(\theta+N+1)}{(\alpha+\beta+\theta+N+2)\Gamma(\alpha+\theta+N+2)}\|g_N\|^2_{\alpha+\beta+\theta+N+1}.$$

Proof. We make the change of variables

$$z_1 = z_1, \qquad z_2 = \bigl(1-|z_1|\bigr)^{1/2}u,$$

and get

$$\begin{aligned}
&\|g_N(z_1)z_2^N\|^2_{\alpha,\beta,\theta} \\
&\quad= \int_{\mathbb{B}^2} \bigl|g_N(z_1)\bigr|^2 |z_2|^{2\theta+2N}\bigl(1-|z_1|^2-|z_2|^2\bigr)^{\alpha}\bigl(1-|z_1|^2\bigr)^{\beta}\,\mathrm{d}A(z_1,z_2) \\
&\quad= \int_{\mathbb{D}} \bigl|g_N(z_1)\bigr|^2\bigl(1-|z_1|^2\bigr)^{\alpha+\beta+\theta+N+1}\,\mathrm{d}A(z_1) \int_{\mathbb{D}} |u|^{2(\theta+N)}\bigl(1-|u|^2\bigr)^{\alpha}\,\mathrm{d}A(u),
\end{aligned} \tag{3.2}$$

whence the assertion follows. □

We obtain the norm expansion (1.5) for the case of the ball.

Theorem 3.2. *For any* $f \in A^2_{\alpha,\beta,\theta}(\mathbb{B}^2)$,

$$\|f\|^2_{\alpha,\beta,\theta} = \sum_{N=0}^{+\infty} \frac{\Gamma(\alpha+1)\Gamma(\theta+N+1)}{(\alpha+\beta+\theta+N+2)\Gamma(\alpha+\theta+N+2)(N!)^2}\bigl\|\partial^N_{z_2} f(z_1,0)\bigr\|^2_{\alpha+\beta+\theta+N+1}.$$

Weighted Hardy spaces. As in the case of the bidisk, we derive a corollary concerning weighted Hardy spaces also for the ball. We have

$$\lim_{\alpha\to -1+0}(\alpha+1)(\alpha+2)\|f\|^2_{\alpha,\beta,\theta} = \int_{\partial\mathbb{B}^2} \bigl|f(z)\bigr|^2|z_2|^{2\theta}\bigl(1-|z_1|^2\bigr)^{\beta}\,\mathrm{d}s_3(z),$$

where $\mathrm{d}s_3$ is the normalized Lebesgue measure on $\partial\mathbb{B}^2$. The right-hand side of the last formula represents the norm of a function in the weighted Hardy space denoted by $H^2_{\beta,\theta}(\mathbb{B}^2)$. As a corollary, we obtain the following decomposition of the norm of functions from this weighted Hardy space:

Corollary 3.3. *For any* $f \in H^2_{\beta,\theta}(\mathbb{B}^2)$,

$$\|f\|^2_{H^2_{\beta,\theta}(\mathbb{B}^2)} = \sum_{N=0}^{+\infty} \frac{1}{(\beta+\theta+N+1)(N!)^2} \left\|\partial^N_{z_2} f(z_1,0)\right\|^2_{\beta+\theta+N}.$$

An expression for the reproducing kernel of $A^2_{\alpha,\beta,\theta}(\mathbb{B}^2)$. Now, we derive an explicit formula for the reproducing kernel for the space $A^2_{\alpha,\beta,\theta}(\mathbb{B}^2)$. In conformity with the notation in the introduction, we denote by $\mathcal{M}_{\alpha,\beta,\theta,N}(\mathbb{B}^2)$ the subspace of $A^2_{\alpha,\beta,\theta}(\mathbb{B}^2)$ consisting of functions of the form $f(z) = z_2^N g(z_1)$. An easy calculation based on Lemma 3.1 establishes the following result.

Lemma 3.4. *The reproducing kernel for* $\mathcal{M}_{\alpha,\beta,\theta,N}(\mathbb{B}^2)$ *is given by the formula*

$$Q_{\alpha,\beta,\theta,N}(z,w) = \frac{(\alpha+\beta+\theta+N+2)\Gamma(\alpha+\theta+N+2)}{\Gamma(\alpha+1)\Gamma(\theta+N+1)} \times \frac{(z_2\bar{w}_2)^N}{(1-z_1\bar{w}_1)^{\alpha+\beta+\theta+N+3}}.$$

Since $A^2_{\alpha,\beta,\theta}(\mathbb{B}^2)$ is the orthogonal sum of the subspaces $\mathcal{M}_{\alpha,\beta,\theta,N}(\mathbb{B}^2)$, its reproducing kernel $P_{\alpha,\beta,\theta}$ is given by the sum

$$\begin{aligned}
P_{\alpha,\beta,\theta}(z,w) &= \sum_{N=0}^{+\infty} Q_{\alpha,\beta,\theta,N}(z,w) \\
&= \frac{\Gamma(\alpha+\theta+2)}{\Gamma(\alpha+1)\Gamma(\theta+1)} \frac{1}{(1-z_1\bar{w}_1)^{\alpha+\beta+\theta+3}} \\
&\quad \times \sum_{N=0}^{+\infty} \frac{(\alpha+\beta+\theta+N+2)(\alpha+\theta+2)_N}{(\theta+1)_N} \left(\frac{z_2\bar{w}_2}{1-z_1\bar{w}_1}\right)^N \\
&= \frac{\Gamma(\alpha+\theta+2)}{\Gamma(\alpha+1)\Gamma(\theta+1)} \frac{1}{(1-z_1\bar{w}_1)^{\theta+\alpha+\beta+3}} \\
&\quad \times \left[(\alpha+\theta+2)\,{}_2F_1\left(\alpha+\theta+3,1;\theta+1;\frac{z_2\bar{w}_2}{1-z_1\bar{w}_1}\right)\right. \\
&\quad \left. + \beta\,{}_2F_1\left(\alpha+\theta+2,1;\theta+1;\frac{z_2\bar{w}_2}{1-z_1\bar{w}_1}\right)\right].
\end{aligned}$$

Here, ${}_2F_1$ stands for the classical Gauss hypergeometric function. We formulate the result as a theorem.

Theorem 3.5. *The kernel function for the space* $A^2_{\alpha,\beta,\theta}(\mathbb{B}^2)$ *is*

$$\begin{aligned}
P_{\alpha,\beta,\theta}(z,w) &= \frac{\Gamma(\alpha+\theta+2)}{\Gamma(\alpha+1)\Gamma(\theta+1)} \frac{1}{(1-\bar{w}_1z_1)^{\alpha+\beta+\theta+3}} \\
&\quad \times \left[(\alpha+\theta+2)\,{}_2F_1\left(\alpha+\theta+3,1;\theta+1;\frac{z_2\bar{w}_2}{1-\bar{w}_1z_1}\right)\right.
\end{aligned}$$

$$+\beta\,{}_2F_1\left(\alpha+\theta+2,\,1;\,\theta+1;\,\frac{z_2\bar{w}_2}{1-\bar{w}_1z_1}\right)\Bigg]. \tag{3.3}$$

Remark 3.6. It would be natural to consider more general Hilbert space norms of the type

$$\|f\|^2_{\alpha,\beta,\theta,\gamma}=\int\limits_{\mathbb{B}^2}\big|f(z)\big|^2|z_2|^{2\theta}\big(1-|z_1|^2-|z_2|^2\big)^{\alpha}\big(1-|z_1|^2\big)^{\beta}\big(1-|z_2|^2\big)^{\gamma}\,\mathrm{d}A(z_1,z_2),$$

which are symmetric with respect to an interchange of the variables z_1 and z_2 (if simultaneously β and γ are interchanged). Here, we must suppose that $-1<\alpha,\beta,\gamma,\theta<+\infty$. The already treated case corresponds to $\gamma=0$. The above analysis applies here as well, but, unfortunately, the formulas become rather complicated; this is why we work things out for $\gamma=0$ only.

4. Weighted Bargmann–Fock spaces in $\mathbb{C}^2$

Preliminaries. Fix a real parameter γ with $0<\gamma<+\infty$. The classical one-variable Bargmann–Fock space—denoted by $A^2_\gamma(\mathbb{C})$—consists of all entire functions of one complex variable with

$$\|f\|^2_\gamma=\int\limits_{\mathbb{C}}\big|f(z)\big|^2e^{-\gamma|z|^2}\,\mathrm{d}A(z)<+\infty, \tag{4.1}$$

the associated sesquilinear inner product is denoted by $\langle\cdot,\cdot\rangle_\gamma$. The reproducing kernel of this Hilbert space is well known:

$$P_\gamma(z,w)=\gamma e^{\gamma\bar{w}z},\quad z,w\in\mathbb{C}.$$

Next, fix real parameters α,β,θ with $0<\alpha,\beta<+\infty$ and $-1<\theta<+\infty$. We consider the Hilbert space $L^2_{\alpha,\beta,\theta}(\mathbb{C}^2)$ of all (equivalence classes of) Borel measurable functions f on $\mathbb{C}^2$ subject to the norm boundedness condition

$$\|f\|^2_{\alpha,\beta,\theta}=\int\limits_{\mathbb{C}}\int\limits_{\mathbb{C}}\big|f(z_1,z_2)\big|^2|z_1-z_2|^{2\theta}e^{-\alpha|z_1|^2-\beta|z_2|^2}\,\mathrm{d}A(z_1)\,\mathrm{d}A(z_2)<+\infty;$$

we let $\langle\cdot,\cdot\rangle_{\alpha,\beta,\theta}$ denote the associated sesquilinear inner product. The *weighted Bargmann–Fock space* $A^2_{\alpha,\beta,\theta}(\mathbb{C}^2)$ is the subspace of $L^2_{\alpha,\beta,\theta}(\mathbb{C}^2)$ consisting of the entire functions.

The reproducing kernel for $A^2_{\alpha,\beta,\theta}(\mathbb{C}^2)$ will be denoted by

$$P_{\alpha,\beta,\theta}=P_{\alpha,\beta,\theta}(z,w),$$

where $z=(z_1,z_2)$ and $w=(w_1,w_2)$ are two points in $\mathbb{C}^2$. The kernel defines an orthogonal projection of the space $L^2_{\alpha,\beta,\theta}(\mathbb{C}^2)$ onto the weighted Bargmann–Fock space via the formula

$$\begin{aligned}P_{\alpha,\beta,\theta}[f](z)&=\big\langle f,\,P_{\alpha,\beta,\theta}(\cdot,z)\big\rangle_{\alpha,\beta,\theta}\\&=\int\limits_{\mathbb{C}}\int\limits_{\mathbb{C}}f(w)\,P_{\alpha,\beta,\theta}(z,w)|w_1-w_2|^{2\theta}e^{-\alpha|z_1|^2-\beta|z_2|^2}\,\mathrm{d}A(w_1)\,\mathrm{d}A(w_2);\end{aligned}$$

as indicated, we shall write $P_{\alpha,\beta,\theta}[f]$ for the projection of a function $f\in L^2_{\alpha,\beta,\theta}(\mathbb{C}^2)$.

In the case $\theta = 0$, the reproducing kernel is readily computed:

$$P_{\alpha,\beta,0,0}(z, w) = \alpha\beta e^{\alpha z_1 \bar{w}_1 + \beta z_2 \bar{w}_2}.$$

We consider the polynomial $p(z_1, z_2) = z_1 - z_2$ in the context of the introduction. In particular, for non-negative integers N, we consider the subspaces $\mathcal{N}_{\alpha,\beta,\theta,N}(\mathbb{C}^2)$ of functions in $A^2_{\alpha,\beta,\theta}(\mathbb{C}^2)$ that vanish up to order N along the diagonal

$$\operatorname{diag}(\mathbb{C}) = \big\{(z_1, z_2) \in \mathbb{C}^2\colon\ z_1 = z_2\big\}.$$

Being closed subspaces of a reproducing kernel space, the spaces $\mathcal{N}_{\alpha,\beta,\theta,N}(\mathbb{C}^2)$ possess reproducing kernels of their own. We shall write

$$P_{\alpha,\beta,\theta,N} = P_{\alpha,\beta,\theta,N}(z, w)$$

for these kernel functions. The operators associated with the kernels project the space $L^2_{\alpha,\beta,\theta}(\mathbb{C}^2)$ orthogonally onto $\mathcal{N}_{\alpha,\beta,\theta,\vartheta,N}(\mathbb{D}^2)$. As before, we write $P_{\alpha,\beta,\theta,N}[f]$ for the projection of a function.

Next, we define the spaces $\mathcal{M}_{\alpha,\beta,\theta,N}(\mathbb{C}^2)$ by setting

$$\mathcal{M}_{\alpha,\beta,\theta,N}\big(\mathbb{C}^2\big) = \mathcal{N}_{\alpha,\beta,\theta,N}\big(\mathbb{C}^2\big) \ominus \mathcal{N}_{\alpha,\beta,\theta,N+1}\big(\mathbb{C}^2\big).$$

The spaces $\mathcal{M}_{\alpha,\beta,\theta,N}(\mathbb{C}^2)$ also admit reproducing kernels, and their kernel functions are of the form

$$Q_{\alpha,\beta,\theta,\vartheta,N}(z, w) = P_{\alpha,\beta,\theta,N}(z, w) - P_{\alpha,\beta,\theta,N+1}(z, w).$$

We shall write $Q_{\alpha,\beta,\theta}$ for the kernel $Q_{\alpha,\beta,\theta,0}$.

As in the case of the weighted Bergman spaces on the bidisk, we make the following observation. We suppress the proof, as it is virtually identical to that of Lemma 2.2.

Lemma 4.1. *We have*

$$P_{\alpha,\beta,\theta,N}(z, w) = (z_1 - z_2)^N (\bar{w}_1 - \bar{w}_2)^N P_{\alpha,\beta,\theta+N}(z, w)$$

for $z, w \in \mathbb{C}^2$.

If we write, as in the introduction,

$$\mathcal{H}\big(\mathbb{C}^2\big) = A^2_{\alpha,\beta,\theta}\big(\mathbb{C}^2\big),$$

we may identify the spaces $\mathcal{H}_N(\mathbb{C}^2)$,

$$\mathcal{H}_N\big(\mathbb{C}^2\big) = A^2_{\alpha,\beta,\theta+N}\big(\mathbb{C}^2\big), \quad N = 0, 1, 2, \ldots,$$

and the spaces $\mathcal{G}_N(\mathbb{C}^2)$ as well:

$$\mathcal{G}_N\big(\mathbb{C}^2\big) = \mathcal{M}_{\alpha,\beta,\theta+N,0}\big(\mathbb{C}^2\big), \quad N = 0, 1, 2, \ldots.$$

As a consequence, we get that

$$Q_{\alpha,\beta,\theta,N}(z,w) = (z_1 - z_2)^N (\bar{w}_1 - \bar{w}_2)^N Q_{\alpha,\beta,\theta+N}(z,w). \tag{4.2}$$

By (1.6), we have the kernel function expansion

$$P_{\alpha,\beta,\theta}(z,w) = \sum_{N=0}^{+\infty} Q_{\alpha,\beta,\theta,N}(z,w), \tag{4.3}$$

while the orthogonal norm expansion (1.1) reads

$$\|f\|^2_{\alpha,\beta,\theta} = \sum_{N=0}^{+\infty} \big\| Q_{\alpha,\beta,\theta,N}[f] \big\|^2_{\alpha,\beta,\theta}, \quad f \in A^2_{\alpha,\beta,\theta}\big(\mathbb{C}^2\big). \tag{4.4}$$

Our next objective is to identify the Hilbert space of restrictions to the diagonal of $\mathcal{H}_N(\mathbb{C}^2) = A^2_{\alpha,\beta,\theta+N}(\mathbb{C}^2)$, as well as to calculate the reproducing kernel of $\mathcal{H}_N(\mathbb{C}^2)$ on the set $\mathbb{C}^2 \times \mathrm{diag}(\mathbb{C})$.

Unitary operators. The rotation operator R_ϕ (for a real parameter ϕ) defined for $f \in A^2_{\alpha,\beta,\theta}(\mathbb{C}^2)$ by

$$R_\phi[f](z_1,z_2) = f\big(e^{i\phi}z_1, e^{i\phi}z_2\big)$$

is clearly unitary, and we shall make use of it shortly. The following lemma supplies us with yet another family of unitary operators.

Proposition 4.2. *For every $\lambda \in \mathbb{C}$, the operator*

$$U_\lambda[f](z_1,z_2) = e^{-(\alpha+\beta)|\lambda|^2/2} e^{-\alpha\bar{\lambda}z_1 - \beta\bar{\lambda}z_2} f(z_1+\lambda, z_2+\lambda)$$

is unitary on the space $A^2_{\alpha,\beta,\theta}(\mathbb{C}^2)$, and its adjoint is $U^*_\lambda = U_{-\lambda}$.

The proof amounts to making a couple of elementary changes of variables in integrals, and is therefore left out.

The reproducing kernel on the diagonal. We now use the operators R_ϕ and U_λ to compute the reproducing kernel on the set $\mathbb{C}^2 \times \mathrm{diag}(\mathbb{C})$.

Theorem 4.3. *We have*

$$P_{\alpha,\beta,\theta}\big((z_1,z_2),(w_1,w_1)\big) = Q_{\alpha,\beta,\theta}\big((z_1,z_2),(w_1,w_1)\big) = \sigma(\alpha,\beta,\theta) e^{\alpha\bar{w}_1 z_1 + \beta\bar{w}_1 z_2}$$

for $z_1, z_2, w_1 \in \mathbb{C}$. Here, $\sigma(\alpha,\beta,\theta)$ is the positive constant given by

$$\frac{1}{\sigma(\alpha,\beta,\theta)} = \int_{\mathbb{C}}\int_{\mathbb{C}} |z_1 - z_2|^{2\theta} e^{-\alpha|z_1|^2 - \beta|z_2|^2} \,\mathrm{d}A(z_1)\,\mathrm{d}A(z_2) = \frac{(\alpha+\beta)^\theta}{(\alpha\beta)^{\theta+1}}\Gamma(\theta+1).$$

Proof. By using the unitarity of the rotation operator R_ϕ, we get as in the proof of Theorem 2.3 that the function $P_{\alpha,\beta,\theta}(\cdot,0)$ is positive constant, which we denote by $\sigma(\alpha,\beta,\theta)$.

Next, take $\lambda\in\mathbb{C}$ and $f\in A^2_{\alpha,\beta,\theta}(\mathbb{C}^2)$. As the operators U_λ are unitary, and as $U_\lambda^*=U_{-\lambda}$, we find that

$$\begin{aligned} e^{-(\alpha+\beta)|\lambda|^2/2}f(\lambda,\lambda)=U_\lambda[f](0)&=\big\langle U_\lambda[f],P_{\alpha,\beta,\theta}(\cdot,0)\big\rangle_{\alpha,\beta,\theta}\\ &=\big\langle f,U_{-\lambda}\big[P_{\alpha,\beta,\theta}(\cdot,0)\big]\big\rangle_{\alpha,\beta,\theta}=\sigma(\alpha,\beta,\theta)\big\langle f,U_{-\lambda}[1]\big\rangle_{\alpha,\beta,\theta}. \end{aligned}$$

This equality together with the uniqueness of reproducing kernels establishes that

$$P_{\alpha,\beta,\theta}\big((z_1,z_2),(\lambda,\lambda)\big)=\sigma(\alpha,\beta,\theta)e^{(\alpha+\beta)|\lambda|^2/2}U_{-\lambda}[1](z_1,z_2),$$

which is the desired result. The explicit expression for the constant in terms of an integral over the bidisk follows if we apply the reproducing property of the kernel applied to the constant function 1. The evaluation of the integral in terms of the Gamma function is done by performing a suitable change of variables. □

Restrictions of reproducing kernels. From the previous subsection, we have that

$$P_{\alpha,\beta,\theta}\big((z_1,z_2),(w_1,w_1)\big)=\sigma(\alpha,\beta,\theta)e^{\alpha\bar w_1z_1+\beta\bar w_1z_2}.$$

For continuous functions $f\in L^2_{\alpha,\beta,\theta}(\mathbb{C}^2)$, we use the notation $\oslash f$ for the restriction to the diagonal of the function, that is,

$$(\oslash f)(z_1)=f(z_1,z_1),\quad z_1\in\mathbb{C},$$

just like in Section 2. We fix w_1 and apply this operation to the reproducing kernel function of $A^2_{\alpha,\beta,\theta}(\mathbb{C}^2)$. We obtain

$$\oslash P_{\alpha,\beta,\theta}\big(z_1,(w_1,w_1)\big)=\sigma(\alpha,\beta,\theta)e^{(\alpha+\beta)\bar w_1z_1}$$

and we see that the restriction of the kernel coincides with a multiple of the reproducing kernel function for the space $A^2_{\alpha+\beta}(\mathbb{C})$. By the theory of reproducing kernels (see [8]), this means that the induced norm for the space $\mathcal{M}_{\alpha,\beta,\theta,0}(\mathbb{C}^2)$ coincides with a multiple of the norm in the aforementioned Bargmann–Fock space of one variable. An immediate consequence of this fact is the inequality

$$\frac{\alpha+\beta}{\sigma(\alpha,\beta,\theta)}\|\oslash f\|^2_{\alpha+\beta}\leqslant\|f\|^2_{\alpha,\beta,\theta},\quad f\in A^2_{\alpha,\beta,\theta}\big(\mathbb{C}^2\big),\tag{4.5}$$

and, more importantly, the equality

$$\frac{\alpha+\beta}{\sigma(\alpha,\beta,\theta)}\|\oslash f\|^2_{\alpha+\beta}=\big\|Q_{\alpha,\beta,\theta}[f]\big\|^2_{\alpha,\beta,\theta},\quad f\in\mathcal{M}_{\alpha,\beta,\theta,0}\big(\mathbb{C}^2\big).\tag{4.6}$$

The notation on the left-hand sides of (4.5) and (4.6) is in conformity with (4.1).

The next step in our program is to compute the kernel function $Q_{\alpha,\beta,\theta}$.

Proposition 4.4. *The kernel function for the space $\mathcal{M}_{\alpha,\beta,\theta,0}(\mathbb{C}^2)$ is given by*

$$Q_{\alpha,\beta,\theta}(z,w) = \frac{(\alpha\beta)^{\theta+1}}{(\alpha+\beta)^{\theta}\Gamma(\theta+1)} e^{(\alpha\bar{w}_1+\beta\bar{w}_2)(\alpha z_1+\beta z_2)/(\alpha+\beta)}, \quad z, w \in \mathbb{C}^2.$$

Proof. In the notation of the introduction, we have the identity (with $N = 0$)

$$k^{\mathcal{G}_N(\Omega)}(z,w) = \big\langle \oslash_p k_w^{\mathcal{H}_N(\Omega)}, \oslash_p k_z^{\mathcal{H}_N(\Omega)} \big\rangle_{\mathcal{H}_N(\mathbb{Z}_p)},$$

by a combination of (1.7) and (1.8). In the notation of this section, this means that

$$Q_{\alpha,\beta,\theta}(z,w) = \frac{\alpha+\beta}{\sigma(\alpha,\beta,\theta)} \big\langle \oslash P_{\alpha,\beta,\theta}(\cdot,w), \oslash P_{\alpha,\beta,\theta}(\cdot,z) \big\rangle_{\alpha+\beta}, \quad z, w \in \mathbb{C}^2,$$

so that by applying Theorem 4.3, we get

$$Q_{\alpha,\beta,\theta}(z,w) = (\alpha+\beta)\sigma(\alpha,\beta,\theta) \int_{\mathbb{C}} e^{(\alpha z_1+\beta z_2)\bar{\xi}} e^{(\alpha\bar{w}_1+\beta\bar{w}_2)\xi} e^{-(\alpha+\beta)|\xi|^2} \, \mathrm{d}A(\xi).$$

It just remains to evaluate the integral. □

The expression for the reproducing kernel of $A^2_{\alpha,\beta,\theta}(\mathbb{C}^2)$. In view of Proposition 4.4, (4.2), and (4.3), we may now derive an explicit expression for the reproducing kernel of $A^2_{\alpha,\beta,\theta}(\mathbb{C}^2)$.

Corollary 4.5. *The reproducing kernel for $A^2_{\alpha,\beta,\theta}(\mathbb{C}^2)$ is given by*

$$P_{\alpha,\beta,\theta}(z,w) = \frac{(\alpha\beta)^{\theta+1}}{(\alpha+\beta)^{\theta}} e^{(\alpha\bar{w}_1+\beta\bar{w}_2)(\alpha z_1+\beta z_2)/(\alpha+\beta)} E_\theta\left(\frac{\alpha\beta}{\alpha+\beta}(z_1-z_2)(\bar{w}_1-\bar{w}_2)\right),$$

where

$$E_\theta(x) = \sum_{N=0}^{+\infty} \frac{x^N}{\Gamma(\theta+N+1)}, \quad x \in \mathbb{C}.$$

The diagonal norm expansion for the Bargmann–Fock space. Having obtained the reproducing kernel in explicit form, we only need to write the norm decomposition (4.4) in desired form.

Lemma 4.6. *For each $N = 0, 1, 2, \ldots$, we have the equality of norms*

$$\big\| Q_{\alpha,\beta,\theta,N}[f] \big\|^2_{\alpha,\beta,\theta} = \frac{(\alpha+\beta)^{\theta+N+1}\Gamma(\theta+N+1)}{(\alpha\beta)^{\theta+N+1}} \left\| \oslash \left[\frac{P_{\alpha,\beta,\theta,N}[f]}{(z_1-z_2)^N} \right] \right\|^2_{\alpha+\beta},$$

for all $f \in A^2_{\alpha,\beta,\theta}(\mathbb{C}^2)$.

Proof. The statement follows from a combination of Lemma 4.1 and (4.6), plus the evaluation of $\sigma(\alpha, \beta, \theta + N)$. □

All that remains for us to do is to make the right-hand side of the expression in Lemma 4.6 sufficiently explicit.

Lemma 4.7. *For all $k = 0, 1, 2, \ldots$, and each $f \in A^2_{\alpha,\beta,\theta}(\mathbb{C}^2)$, we have*

$$\oslash \partial_{z_1}^k [f] = \sum_{n=0}^{k} n! \binom{k}{n} \left(\frac{\alpha}{\alpha+\beta} \right)^{k-n} \partial_{z_1}^{k-n} \oslash \left[\frac{P_{\alpha,\beta,\theta,n}[f]}{(z_1 - z_2)^n} \right]. \tag{4.7}$$

Proof. We observe that

$$\oslash \big[\partial_{z_1}^j Q_{\alpha,\beta,\theta,N} \big](z_1, (w_1, w_2)) = \left(\frac{\alpha}{\alpha+\beta} \right)^j \partial_{z_1}^j \oslash [Q_{\alpha,\beta,\theta,N}](z_1, (w_1, w_2)).$$

The rest of the proof is obtained by mimicking the arguments of Lemma 2.6. □

It is quite easy to invert Lemma 4.7:

Lemma 4.8. *For all $N = 0, 1, 2, \ldots$ and each $f \in A^2_{\alpha,\beta,\theta}(\mathbb{C}^2)$, the equality*

$$\oslash \left[\frac{P_{\alpha,\beta,\theta}[f]}{(z_1 - z_2)^N} \right](z_1) = \sum_{k=0}^{N} \frac{(-1)^{N-k}}{k!(N-k)!} \left(\frac{\alpha}{\alpha+\beta} \right)^{N-k} \partial_{z_1}^{N-k} \oslash \big[\partial_{z_1}^k f \big](z_1), \quad z_1 \in \mathbb{C}, \tag{4.8}$$

holds for each $f \in A^2_{\alpha,\beta,\theta}(\mathbb{C}^2)$.

Proof. In view of Lemma 4.7, it is enough to check that

$$\sum_{k=n}^{N} \frac{(-1)^{N-k}}{k!(N-k)!} n! \binom{k}{n} \left(\frac{\alpha}{\alpha+\beta} \right)^{N-n} = \delta_{n,N},$$

where $\delta_{n,N}$ is the Kronecker delta. Firstly, we observe that the equality holds for $n = N$. Secondly, we observe that it is equivalent to show that

$$\sum_{k=n}^{N} \frac{(-1)^{N-k}}{k!(N-k)!} n! \binom{k}{n} = \sum_{k=n}^{N} \frac{(-1)^{N-k}}{(N-k)!(k-n)!} = 0$$

whenever $n < N$. The expression in the middle is an $(N-n)$th difference of a constant function, which of course is 0 for $n < N$. We are done. □

We now combine our results and obtain the norm expansion for the Bargmann–Fock space.

Theorem 4.9. *Let $c_{k,N}(\alpha,\beta)$ be given by*

$$c_{k,N}(\alpha,\beta) = (-1)^{N-k}\binom{N}{k}\left[\frac{\alpha}{\alpha+\beta}\right]^{N-k}.$$

Then, for each $f \in A^2_{\alpha,\beta,\theta}(\mathbb{C}^2)$, we have

$$\|f\|^2_{\alpha,\beta,\theta} = \sum_{N=0}^{+\infty} \frac{(\alpha+\beta)^{\theta+N+1}\Gamma(\theta+N+1)}{(\alpha\beta)^{\theta+N+1}[N!]^2} \left\| \sum_{k=0}^{N} c_{k,N}(\alpha,\beta)\partial_{z_1}^{N-k} \oslash \left[\partial_{z_1}^k f\right] \right\|^2_{\alpha+\beta}.$$

Remark 4.10. There is an alternative way to obtain the norm expansion and the explicit expression for the reproducing kernel in the Bargmann–Fock space $A^2_{\alpha,\beta,\theta}(\mathbb{C}^2)$. The change of variables

$$\begin{cases} z_1 = w_1 + \beta w_2, \\ z_2 = w_1 - \alpha w_2 \end{cases}$$

transforms the norm in $A^2_{\alpha,\beta,\theta}(\mathbb{C}^2)$ into the expression

$$\|f\|_{\alpha,\beta,\theta} = (\alpha+\beta)^{2\theta+2} \int_{\mathbb{C}}\int_{\mathbb{C}} \left|g(w_1,w_2)\right|^2 |w_2|^{2\theta} e^{-(\alpha+\beta)|w_1|^2-\alpha\beta(\alpha+\beta)|w_2|^2}\, \mathrm{d}A(w_1)\,\mathrm{d}A(w_2),$$

where $g(w_1,w_2) = f(w_1+\beta w_2, w_1-\alpha w_2)$. The reproducing kernel and the norm expansion about the hyperplane $w_2 = 0$ with respect to the latter norm can be calculated by separation of variables. Shifting back to the original variables (z_1,z_2), then, we obtain the reproducing kernel and norm expansion for $A^2_{\alpha,\beta,\theta}(\mathbb{C}^2)$.

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