Boundary properties of planar Green functions

Håkan Hedenmalm, Anton Baranov

November 29, 2006

The setting

- Ω – bounded simply connected domain in $\mathbb{C}$ which contains 0.

The setting

- Ω – bounded simply connected domain in $\mathbb{C}$ which contains 0.
- $\varphi : \mathbb{D} \to \Omega$ – the conformal mapping with $\varphi(0) = 0$, $\varphi'(0) > 0$.

The setting

- Ω – bounded simply connected domain in $\mathbb{C}$ which contains 0.
- $\varphi : \mathbb{D} \to \Omega$ – the conformal mapping with $\varphi(0) = 0$, $\varphi'(0) > 0$.
- $G_\Omega(z, w)$ is the Green function for Ω ($z, w \in \Omega$).

The setting

- Ω – bounded simply connected domain in $\mathbb{C}$ which contains 0.
- $\varphi : \mathbb{D} \to \Omega$ – the conformal mapping with $\varphi(0) = 0$, $\varphi'(0) > 0$.
- $G_\Omega(z, w)$ is the Green function for Ω $(z, w \in \Omega)$.
- We write $G_\Omega(z) = G_\Omega(z, 0)$.

The setting

- Ω – bounded simply connected domain in $\mathbb{C}$ which contains 0.
- $\varphi : \mathbb{D} \to \Omega$ – the conformal mapping with $\varphi(0) = 0$, $\varphi'(0) > 0$.
- $G_\Omega(z, w)$ is the Green function for Ω $(z, w \in \Omega)$.
- We write $G_\Omega(z) = G_\Omega(z, 0)$.
- Wirtinger derivatives:

$$\partial_z = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right), \qquad \bar\partial_z = \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right).$$

Multiplicative counterparts:

$$\partial_z^\times = z\partial_z, \qquad \bar\partial_z^\times = \bar z\bar\partial_z.$$

The problem

- Compare, for complex τ and real α,

$$\left|\left[\partial_z^{\times} G_\Omega(z)\right]^{\tau}\right| \quad \text{with} \quad |G_\Omega(z)|^{-\alpha}.$$

The problem

- Compare, for complex τ and real α,

$$\left|\left[\partial_z^{\times} G_\Omega(z)\right]^{\tau}\right| \quad \text{with} \quad |G_\Omega(z)|^{-\alpha}.$$

- More precisely, when do we have

$$(1) \qquad \int_\Omega \left|\left[\partial_z^{\times} G_\Omega(z)\right]^{\tau}\right| |G_\Omega(z)|^{\alpha} \, \mathrm{d}A(z) < +\infty?$$

Definition

We denote by $A_\Omega(\tau)$ the "best possible" (="smallest") α for a given τ.

Main Theorem

We have

$$A_\Omega(\tau) \le -\mathrm{Re}\,\tau + \left[\frac{9e^2}{2} + o(1)\right] |\tau|^2 \log \frac{1}{|\tau|}$$

as $|\tau| \to 0$. The $o(1)$ term is independent of the choice of the bounded simply connected domain Ω.

Remark

If $A_\Omega(\tau) + A_\Omega(-\tau) \leq 0$, our scheme of comparing the quantities in (1) in terms of L^1 integrals is very successful. It is therefore natural to view the quadratic-logarithmic remainder term in the Main Theorem as the amount by which the L^1 comparison might fail.

Remark

In terms of φ,

$$G_\Omega(\varphi(z)) = \log(|z|^2), \qquad z \in \mathbb{D},$$

and we get

$$\int_\Omega \left|\left[\partial_z^\times G(z)\right]^\tau\right| |G(z)|^\alpha \, \mathrm{d}A(z)$$
$$= \int_{\mathbb{D}} \left|\left[\frac{z\varphi'(z)}{\varphi(z)}\right]^{-\tau}\right| \left\{\log\frac{1}{|z|^2}\right\}^\alpha |\varphi'(z)|^2 \mathrm{d}A(z).$$

Integral means spectrum: definition

Let $B_\varphi(\tau)$ be "the best" β such that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\left[\frac{re^{i\theta}\varphi'(re^{i\theta})}{\varphi(re^{i\theta})}\right]^\tau\right|\mathrm{d}\theta = O\left(\frac{1}{(1-r)^\beta}\right)$$

as $r \to 1^-$. It is possible to show that

$$B_\varphi(\tau) = A_\Omega(2-\tau)+1$$

for all complex τ.

Consequences of Main Theorem

The *universal integral means spectrum* for the class of bounded univalent functions S_b is the function $B_b(\tau)$, obtained by taking the sup of $B_\varphi(\tau)$ over all φ. As a consequence of the Main Theorem, we get

$$B_b(2-\tau) \leq 1 - \mathrm{Re}\tau + \left[\frac{9e^2}{2} + o(1)\right] |\tau|^2 \log \frac{1}{|\tau|}$$

as $|\tau| \to 0$. For *real* τ, P. W. Jones and N. G. Makarov obtained a smaller error term:

$$B_b(2-\tau) \leq 1 - \tau + O(\tau^2), \qquad \mathbb{R} \ni \tau \to 0.$$

Cauchy and Beurling transforms

Cauchy transform:

$$\mathfrak{C}_\Omega[f](z) = \int_\Omega \frac{f(w)}{w-z}\,\mathrm{d}A(w).$$

Beurling transform:

$$\mathfrak{B}_\Omega[f](z) = \partial_z \mathfrak{C}_\Omega[f](z) = \mathrm{pv}\int_\Omega \frac{f(w)}{(w-z)^2}\,\mathrm{d}A(w).$$

It is clear that in the sense of distribution theory,

$$\bar\partial_z \mathfrak{C}_\Omega[f](z) = -f(z), \qquad \partial_z \mathfrak{C}_\Omega[f](z) = \mathfrak{B}_\Omega f(z).$$

$\mathfrak{B}_\mathbb{C}$ is *unitary* $L^2(\mathbb{C}) \to L^2(\mathbb{C})$. $\mathfrak{B}_\Omega$ is contractive $L^2(\Omega) \to L^2(\Omega)$.

Transferred Cauchy transform

We connect two functions f and g, on Ω and $\mathbb{D}$, respectively, via

$$g(z) = \bar{\varphi}'(z)\, f \circ \varphi(z),$$

and define the integral operator

$$\begin{aligned}\mathfrak{C}_\varphi[g](z) &= (\mathfrak{C}_\Omega[f]) \circ \varphi(z) \\ &= \int_{\mathbb{D}} \frac{\varphi'(w)}{\varphi(w) - \varphi(z)}\, g(w)\, \mathrm{d}A(w), \qquad z \in \mathbb{D};\end{aligned}$$

$\mathfrak{C}_\varphi$ is then a contraction $L^2(\mathbb{D}) \to W^{1,2}(\mathbb{D})/\mathbb{C}$.

Transferred (skewed) Beurling transform

It is known that $\mathfrak{B}_{\mathbb{C}}$ is bounded $L^p(\mathbb{C}) \to L^p(\mathbb{C})$, for all p with $1 < p < +\infty$. Let $K(p)$ be a positive constant such that

$$\|\mathfrak{B}_{\mathbb{C}} f\|_{L^p(\mathbb{C})} \le K(p)\, \|f\|_{L^p(\mathbb{C})}, \qquad f \in L^p(\mathbb{C}). \tag{2}$$

The optimal constant $K(p)$ in (2) is not known; however, we may choose, e. g., $K(p) = 2(p^* - 1)$, where $p^* = \max\{p, p'\}$, and $p' = p/(p-1)$ is the dual exponent (one expects $K(p) = p^* - 1$ is the optimal choice). For $0 \le \theta \le 2$, we introduce the θ-*skewed Beurling transform*, as defined by

$$\mathfrak{B}^{\theta}_{\varphi}[f] = \mathrm{pv} \int_{\mathbb{D}} \frac{\varphi'(z)^{\theta} \varphi'(w)^{2-\theta}}{(\varphi(z) - \varphi(w))^2}\, f(w)\, \mathrm{d}A(w).$$

Transferred (skewed) Beurling transform, cont

It follows from (2) that

$$\left\| \mathfrak{B}_\varphi^{2/p} f \right\|_{L^p(\mathbb{D})} \le K(p) \, \|f\|_{L^p(\mathbb{D})}, \qquad f \in L^p(\mathbb{D}),$$

for all p with $1 < p < +\infty$. In the symmetric case $\theta = 1$, we write $\mathfrak{B}_\varphi$ in place of $\mathfrak{B}_\varphi^1$, and call it the *Beurling transform*. We note that $\mathfrak{B}_\varphi$ is a contraction on $L^2(\mathbb{D})$.

The basic identity

We have the identity

$$\log \frac{z(\varphi(z)-\varphi(\zeta))}{(z-\zeta)\varphi(z)} + \log(1-\bar{z}\zeta) \\ = \int_{\mathbb{D}} \frac{\varphi'(w)}{\varphi(w)-\varphi(z)} \, \frac{\zeta}{1-\bar{w}\zeta} \, \mathrm{d}A(w) = \zeta \, \mathfrak{C}_{\varphi}[g_{\zeta}](z),$$

where

$$g_{\zeta}(w) = \frac{1}{1-\bar{w}\zeta}.$$

Grunsky identity (integral form)

We have

$$\frac{\varphi'(z)\varphi'(\zeta)}{(\varphi(z)-\varphi(\zeta))^2}-\frac{1}{(z-\zeta)^2}$$
$$=\int_{\mathbb{D}}\frac{\varphi'(z)\varphi'(w)}{(\varphi(w)-\varphi(z))^2}\,\frac{1}{(1-\bar{w}\zeta)^2}\,\mathrm{d}A(w)=\mathfrak{B}_\varphi[k_\zeta](z),$$

where

$$k_\zeta(w)=\frac{1}{(1-\bar{w}\zeta)^2}.$$

Grunsky identity (operator form)

Let

$$\mathfrak{P}[f](z) = \int_{\mathbb{D}} \frac{f(w)}{(1-\bar{w}z)^2}\,\mathrm{d}A(w).$$

Then

(3) $$\mathfrak{B}_\varphi - \mathfrak{B} = \mathfrak{P}\mathfrak{B}_\varphi = \mathfrak{B}_\varphi\bar{\mathfrak{P}} = \mathfrak{P}\mathfrak{B}_\varphi\bar{\mathfrak{P}}.$$

The strong Grunsky inequality is equivalent to the statement that $\mathfrak{B}_\varphi - \mathfrak{B}$ is a contraction on $L^2(\mathbb{D})$, which immediately follows from (3).

Skewed Grunsky identity (operator form)

Let $\mathfrak{D}$ denote the operator

$$\mathfrak{D}[f](z) = \int_{\mathbb{D}} \frac{f(w)}{(w-z)(1-\bar{w}z)} \,\mathrm{d}A(w),$$

and $\mathfrak{M}_F$ the operator of multiplication by F. For $0 < \theta < 2$, we have the operator identity

$$\mathfrak{B}_\varphi^\theta - \mathfrak{B} + (\theta - 1)\mathfrak{D}\mathfrak{M}_{1-|z|^2}\mathfrak{M}_{\varphi''/\varphi'} = \mathfrak{P}\mathfrak{B}_\varphi^\theta.$$

Variant of basic identity

$$\log \frac{z(\varphi(z)-\varphi(\zeta))}{(z-\zeta)\varphi(z)} - \zeta(1-|\zeta|^2)\left[\frac{\varphi'(\zeta)}{\varphi(\zeta)-\varphi(z)} - \frac{1}{\zeta - z}\right] + \log\left(1-\bar{z}\zeta\right) + \bar{z}\zeta\,\frac{1-|\zeta|^2}{1-\bar{z}\zeta} = \zeta^2 \int_{\mathbb{D}} \frac{\varphi'(w)}{\varphi(w)-\varphi(z)}\,\frac{\bar{\zeta}-\bar{w}}{(1-\bar{w}\zeta)^2}\,\mathrm{d}A(w).$$

Variant of basic identity, cont

Put $z = \zeta$:

$$\log \frac{z\varphi'(z)}{\varphi(z)} + \log\left(1 - |z|^2\right) = z^2 \int_{\mathbb{D}} \frac{\varphi'(w)}{\varphi(w) - \varphi(z)} \frac{\bar{z} - \bar{w}}{(1 - \bar{w}z)^2} \, \mathrm{d}A(w) + O(1).$$

Marcinkiewicz-Zygmund integrals

Suppose $0 < \kappa, \gamma < 1$. Let $\delta(w)$ be the Euclidean distance from w to $\mathbb{C} \setminus \Omega$. Marcinkiewicz-Zygmund integral:

$$I_\kappa(z) = \int_\Omega \min\left\{ \frac{\delta(w)^\kappa}{|z-w|^{2+\kappa}}, \frac{\gamma^{-2-\kappa}}{\delta(w)^2} \right\} \mathrm{d}A(w).$$

Zygmund showed (essentially) in 1969 that

$$\left\| e^{\lambda I_\kappa} - 1 \right\|_{L^1(\mathbb{C})} \leq \frac{\kappa\,|\Omega|_A}{\kappa - 9e|\lambda|\gamma^{-\kappa}(2+\kappa)} - |\Omega|_A$$

for complex λ with

$$|\lambda| < \frac{\kappa\,\gamma^\kappa}{9e(2+\kappa)}.$$

Uniform Sobolev imbedding

We work with

$$\widetilde{\mathfrak{C}}_\varphi[f](z) = \int_{\mathbb{D}} \frac{\varphi'(w)}{\varphi(w)-\varphi(z)} \frac{\bar z - \bar w}{1-\bar w z} f(w)\,\mathrm{d}A(w).$$

For $0<\kappa<1$, we consider the Lebesgue space

$$X_\kappa(\mathbb{D}) = L^p(\mathbb{D},\mu),$$

where

$$p = \frac{2+\kappa}{1+\kappa}, \qquad \mathrm{d}\mu(z) = (1-|z|^2)^{-\kappa/(1+\kappa)}\,\mathrm{d}A(z).$$

By Hölder's inequality, we get

$$\left|\widetilde{\mathfrak{C}}_\varphi[f](z)\right| \le \left\{ \int_{\mathbb{D}} \left| \frac{(w-z)\varphi'(w)}{(1-\bar w z)(\varphi(w)-\varphi(z))} \right|^{2+\kappa} \times (1-|w|^2)^\kappa\,\mathrm{d}A(w) \right\}^{1/(2+\kappa)} \times \|f\|_{X_\kappa(\mathbb{D})}.$$

Uniform Sobolev imbedding, cont

The function

$$J_\kappa[\varphi](z) = \int_{\mathbb{D}} \left| \frac{(w-z)\varphi'(w)}{(1-\bar{w}z)(\varphi(w)-\varphi(z))} \right|^{2+\kappa} (1-|w|^2)^\kappa \mathrm{d}A(w)$$

is essentially the Marcinkiewicz-Zygmund integral:

$$J_k[\varphi](z) \le 4^\kappa\, I_\kappa(\varphi(z)) + O(1), \qquad z \in \mathbb{D}.$$

Uniform Sobolev imbedding, statement

For complex λ with

$$|\lambda| < \frac{\kappa 4^{-\kappa}}{9e(2+\kappa)},$$

we have

$$\int_{\mathbb{D}} \exp\left\{|\lambda| \sup_{f\in \mathrm{ball}(X_\kappa(\mathbb{D}))} \left| \widetilde{\mathfrak{C}}_\varphi[f](z) \right|^{2+\kappa}\right\} |\varphi'(z)|^2 \, \mathrm{d}A(z) < +\infty.$$

Proof of the main theorem

By the variant of the basic identity,

$$\log \frac{z\varphi'(z)}{\varphi(z)} + \log(1-|z|^2) = z^2 \widetilde{\mathfrak{C}}_\varphi[g_z](z) + O(1),$$

where

$$g_z(w) = \frac{1}{1-\bar{w}z}.$$

We plan to apply the Uniform Sobolev Imbedding to the function $f = f_z = g_z/\|g_z\|_{X_\kappa(\mathbb{D})}$. We get

$$\|g_z\|_{X_\kappa(\mathbb{D})}^{2+\kappa} \sim \left[\frac{\Gamma\left(\frac{1-\kappa}{1+\kappa}\right)}{\Gamma\left(\frac{1}{1+\kappa}\right)^2} \log \frac{1}{1-|z|^2} \right]^{1+\kappa}.$$

Let Λ be such that

$$\Lambda > \left[\frac{\Gamma\left(\frac{1-\kappa}{1+\kappa}\right)}{\Gamma\left(\frac{1}{1+\kappa}\right)^2} \right]^{1+\kappa}.$$

Proof of the main theorem, cont(1)

We now find that for

$$|\lambda| < \frac{\kappa 4^{-\kappa}}{9e(2+\kappa)},$$

$$\int_{\mathbb{D}} \exp\left\{ \frac{|\lambda|}{\Lambda} \left| 1 - \frac{\log \frac{z\,\varphi'(z)}{\varphi(z)}}{\log \frac{1}{1-|z|^2}} \right|^{2+\kappa} \log \frac{1}{1-|z|^2} \right\} \times |\varphi'(z)|^2 \mathrm{d}A(z) < +\infty.$$

It remains to apply a *linear approximation argument*. We consider the convexity estimate (a, b complex)

$$|a|^{2+\kappa} = |\bar{a}|^{2+\kappa} \geq |b|^{2+\kappa} - (2+\kappa)|b|^{\kappa}\mathrm{Re}\big[b(\bar{b} - a)\big]$$
$$= |b|^{2+\kappa} + (2+\kappa)|b|^{\kappa}\big[\mathrm{Re}\, b - |b|^2\big] - (2+\kappa)|b|^{\kappa}\mathrm{Re}\big[b(1-a)\big].$$

Proof of the main theorem, cont(2)

We apply this to

$$a = 1 - \log \frac{z\varphi'(z)}{\varphi(z)} \log \frac{1}{1-|z|^2},$$

and obtain

$$\begin{aligned}\left|1 - \frac{\log \frac{z\,\varphi'(z)}{\varphi(z)}}{\log \frac{1}{1-|z|^2}}\right|^{2+\kappa} & \log \frac{1}{1-|z|^2} \\ &\geq \left[|b|^{2+\kappa} + (2+\kappa)|b|^{\kappa}\left[\mathrm{Re} b - |b|^2\right]\right] \log \frac{1}{1-|z|^2} \\ &\qquad - (2+\kappa)|b|^{\kappa}\mathrm{Re}\left[b \log \frac{z\varphi'(z)}{\varphi(z)}\right]\end{aligned}$$

for any $b \in \mathbb{C}$.

Proof of the main theorem, cont(3)

We now insert this estimate into the estimate we got from the Uniform Sobolev Imbedding, and find that

$$\int_{\mathbb{D}} \exp\left\{ \frac{|\lambda|}{\Lambda}\left[|b|^{2+\kappa} + (2+\kappa)|b|^{\kappa}\left[\mathrm{Re}\, b - |b|^2\right]\right] \log\frac{1}{1-|z|^2} \right.$$
$$\left. - \frac{|\lambda|}{\Lambda}(2+\kappa)|b|^{\kappa}\mathrm{Re}\left[b \log \frac{z\varphi'(z)}{\varphi(z)}\right]\right\} |\varphi'(z)|^2 \mathrm{d}A(z) < +\infty.$$

Next, we assume $b \neq 0$, and put $\tau = \Lambda^{-1}|\lambda|(2+\kappa)\,|b|^{\kappa} b$. Note also that

Proof of the main theorem, cont(4)

$$\exp\left\{-\frac{|\lambda|}{\Lambda}(2+\kappa)\,|b|^{\kappa}\,\mathrm{Re}\left[b\log\frac{z\varphi'(z)}{\varphi(z)}\right]\right\} = \left|\left[\frac{z\varphi'(z)}{\varphi(z)}\right]^{-\tau}\right|.$$

We now get that (in view of the restrictions on λ, Λ)

$$\int_{\mathbb{D}}\left|\left[\frac{z\varphi'(z)}{\varphi(z)}\right]^{-\tau}\right|\left(1 - |z|^2\right)^{-\mathrm{Re}\,\tau+R(\tau)}|\varphi'(z)|^2\mathrm{d}A(z) < +\infty$$

holds so long as $R(\tau)$ satisfies

Proof of the main theorem, cont(5)

$$R(\tau) > R_0(\tau) :=$$

$$\inf_{0<\kappa<1} \left(\frac{9e4^{\kappa}}{\kappa}\right)^{1/(1+\kappa)} \frac{(1+\kappa)\Gamma\left(\frac{1-\kappa}{1+\kappa}\right)}{(2+\kappa)\Gamma\left(\frac{1}{1+\kappa}\right)^2} |\tau|^{(2+\kappa)/(1+\kappa)}.$$

The choice (for small $|\tau|$)

$$\kappa = \frac{1}{\log \frac{1}{|\tau|}},$$

yields the asserted asymptotics.