BACKWARD SHIFT AND NEARLY INVARIANT SUBSPACES OF FOCK-TYPE SPACES

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Abstract. We study the structure of the backward shift invariant and nearly invariant subspaces in weighted Fock-type spaces $\mathcal{F}^p_W$, whose weight is not necessarily radial. We show that in the spaces $\mathcal{F}^p_W$ which contain the polynomials as a dense subspace (in particular, in the radial case) all nontrivial backward shift invariant subspaces are of the form $\mathcal{P}_n$, i.e., finite dimensional subspaces consisting of polynomials of degree at most n. In general, the structure of the nearly invariant subspaces is more complicated. In the case of spaces of slow growth (up to zero exponential type) we establish an analogue of de Branges' Ordering Theorem. We then construct examples which show that the result fails for general Fock-type spaces of larger growth.

1. Introduction

1.1. **Backward shift invariant and nearly invariant subspaces.** Shift invariant and nearly invariant subspaces form an important part of theory of spaces of analytic functions. The basic setting here is the Hardy space H^2 where the shift invariant subspaces are described by the famous Beurling theorem, while nearly invariant subspaces were studied in detail by Hayashi, Hitt and Sarason [16, 18, 25]. In the case of other classical spaces in the disc (Bergman, Dirichlet, etc.) the structure of the shift and backward shift invariant subspaces is much more complicated and a complete description seems to be out of reach (see, e.g., [23, 2, 4]).

Now we recall the necessary definitions. Let Ω be a domain in $\mathbb{C}$ with $0 \in \Omega$, and let $\mathcal{H}$ be a Banach space of functions analytic in Ω such that point evaluation functionals $f \mapsto f(w)$ are bounded for any $w \in \Omega$. In what folllows we always assume that the space $\mathcal{H}$ has the *division property*, that is, $\frac{f(z)}{z-w} \in \mathcal{H}$ whenever $f \in \mathcal{H}$, $w \in \Omega$ and $f(w) = 0$. We say that a closed linear subspace $\mathcal{M}$ of $\mathcal{H}$ is backward shift invariant if $\frac{f(z)-f(0)}{z} \in \mathcal{M}$ for any $f \in \mathcal{M}$. In other words, $\mathcal{M}$ is invariant for the backward shift $L : f \mapsto \frac{f-f(0)}{z}$.

There exists the more general notion of a nearly invariant subspace. A closed linear subspace $\mathcal{M}$ of $\mathcal{H}$ is said to be *nearly invariant* if $\frac{f(z)}{z} \in \mathcal{M}$ whenever $f \in \mathcal{M}$ and $f(0) = 0$. Clearly, if the function which is identically equal to 1 belongs to $\mathcal{M}$, then nearly invariance is equivalent to backward shift invariance. In general, $\mathcal{M}$ is backward shift invariant if and only if $\mathcal{M}$ is nearly invariant and $1 \in \mathcal{M} + z\mathcal{M}$ (i.e., $1 = f + zg$ for some $f, g \in \mathcal{M}$).

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Naturally, this condition rules out the possibility that there would exist a sequence tending to infinity such that all elements of a backward shift invariant subspace $\mathcal{M}$ would decay like $o(|z|^{-1})$ along that sequence.

The choice of the point 0 is not essential. For any $w \in \Omega$ such that there exists $f \in \mathcal{M}$ with $f(w) \neq 0$ (i.e., w is not a *common zero for* $\mathcal{M}$), one has the implications

$$\mathcal{M} \quad \text{is backward shift invariant}, \quad f \in \mathcal{M}, \quad f(w) = 0 \implies \frac{f(z) - f(w)}{z - w} \in \mathcal{M},$$

and

$$\mathcal{M} \quad \text{is nearly invariant}, \quad f \in \mathcal{M}, \quad f(w) = 0 \implies \frac{f(z)}{z - w} \in \mathcal{M}.$$

In the context of Hardy spaces in general domains the equivalence of near invariance and division invariance is shown in [2, Proposition 5.1]; a similar argument works for general Banach spaces of analytic functions [6, Proposition 7.1].

While backward shift invariant subspaces never have common zeros, this might happen for nearly invariant subspaces of a space of analytic functions with the division property. Throughout in this paper we shall consider nearly invariant subspaces *without common zeros.*

Let us make the following simple observations. If our space $\mathcal{H}$ contains the set $\mathcal{P}$ of all polynomials, then any subspace of the form $\mathcal{P}_n$ (consisting of all polynomials of degree at most n) is a backward shift invariant subspace. As we will see, for a class of weighted Fock-type spaces it is possible that all nontrivial backward shift invariant subspaces and even all nearly-invariant subspaces are of the form $\mathcal{P}_n$. Note that in this case all nearly-invariant subspaces are ordered by inclusion (recall that an operator whose invariant subspaces are ordered by inclusion is said to be *unicellular*).

It is however possible that nearly invariant subspaces have ordered structure even in the case when they are not of the form $\mathcal{P}_n$. A model situation where this property holds is given by the Ordering Theorem for de Branges spaces [12, 24] (for a generalization to the so-called Cauchy–de Branges spaces, see [1]). In the present paper we study the structure of backward shift invariant and nearly invariant subspaces for weighted Fock-type spaces of entire functions.

1.2. **Weighted Fock-type spaces.** By a *weight* we simply mean a positive function W in $\mathbb{C}$ which is measurable with respect to planar Lebesgue measure m_2 in the complex plane $\mathbb{C}$. With any weight W and $p \in [1, +\infty)$ we associate the Fock-type space of entire functions

$$\mathcal{F}^p_W = \left\{ F \in Hol(\mathbb{C}) : \|F\|^p_{\mathcal{F}^p_W} = \int_{\mathbb{C}} |F(z)|^p W(z) dm_2(z) < +\infty \right\}.$$

In what follows we shall always assume that W is bounded from above and below by positive constants on any compact. In this case, the point evaluation functionals are bounded on

$\mathcal{F}^p_W$ and $\mathcal{F}^p_W$ is a Banach space with the division property. Conversely, it is also easy to see that if point evaluation functionals are bounded on $\mathcal{F}^p_W$, we can find a weight $\tilde{W}$ which is bounded above and below by positive constants on any compact, and $\mathcal{F}^p_{\tilde{W}} = \mathcal{F}^p_W$ with equivalent norms.

In the case $p = 2$ we will omit the exponent p. Clearly, $\mathcal{F}_W$ is a reproducing kernel Hilbert space.

A typical example of a Fock-type space is a radial space $\mathcal{F}^p_W$ with $W(z) = \exp(-\varphi(|z|))$ where φ is a function on $[0, +\infty)$ such that $\log r = \mathrm{o}(\varphi(r))$, $r \to +\infty$ (to exclude the trivial finite-dimensional case). E.g., one can take $\varphi(r) = ar^\alpha$, $a, \alpha > 0$; in what follows we denote any corresponding space by $\mathcal{F}^p_\alpha$ (formally, it depends also on a, but this dependence is not essential). The case $p = 2$ and $\alpha = 2$ corresponds to the classical Bargmann–Segal–Fock space, ubiquitous in applications – from theoretical physics to time-frequency analysis. The Hilbert spaces in this scale will be denoted by $\mathcal{F}_\alpha$.

In the present paper we will also consider non-radial Fock spaces. It should be mentioned that Fock-type spaces are fairly general objects which cover even seemingly unrelated examples. It was shown in [9] that any de Branges space (whose norm is given by a weighted L^2-integral over the real axis) coincides with equivalence of norms with some Fock-type space. E.g., the Paley-Wiener space $PW_{[-a,a]}$, the image of $L^2([-a, a])$ by the Fourier transform, coincides with the space $\mathcal{F}_W$ where $W(z) = (1 + |\mathrm{Im}\, z|)^{-2} \mathrm{e}^{-2a|\mathrm{Im}\, z|}$ and, moreover, the Paley–Wiener space is the only de Branges space which can be realized as a Fock-type space with the weight depending on $\mathrm{Im}\, z$ only. The same is true for a wider class of the so-called Cauchy–de Branges spaces (for their theory see [1]): any Cauchy–de Branges can be realized as a Fock-type space.

Note also that in the radial Hilbertian Fock spaces the monomials $\{z^n\}_{n\ge 0}$ form an orthogonal basis. Therefore, one can isometrically identify such spaces with weighted spaces of sequences:

$$\mathcal{F}_W = \Big\{ F(z) = \sum_{n\ge 0} c_n z^n : \ \|F\|^2 = \sum_{n\ge 0} W_n |c_n|^2 < +\infty \Big\}, \qquad W_n = 2\pi \int_0^\infty r^{2n+1} W(r) dr.$$

1.3. **Main results.** Given a Fock-type space $\mathcal{F}^p_W$, we address the following questions:

a) When are all nontrivial backward shift invariant or nearly invariant subspaces of the form $\mathcal{P}_n$ (recall that we consider only subspaces without common zeros)?

b) When is the set of all nearly invariant subspaces totally ordered by inclusion, i.e., given two nearly invariant subspaces $\mathcal{M}_1$ and $\mathcal{M}_2$ of $\mathcal{F}^p_W$, is it true that either $\mathcal{M}_1 \subset \mathcal{M}_2$ or $\mathcal{M}_2 \subset \mathcal{M}_1$?

For backward shift invariant subspaces in radial Hilbertian Fock spaces the answer to the first question is positive and essentially known. In this case the problem is equivalent to the unicellularity of the weighted shifts studied, e.g., in [21, 26, 28, 29, 13]. One of the strongest results in this direction is due to D.V. Yakubovich [28, 29] who showed (answering a question of A.L. Shields [26]) that if $(\lambda_n)_{n\ge1}$ is a positive non-increasing sequence tending to zero, then the weighted shift

$$T:\ell^2(\mathbb{Z}_+)\to\ell^2(\mathbb{Z}_+), \qquad (c_0,c_1,\dots)\mapsto(\lambda_1c_1,\lambda_2c_2,\dots)$$

is unicellular, and so its invariant subspaces are of the form $\{(c_n)_{n\ge0}:\ c_n=0,\ n>N\}$ for some N. From this it is clear that L is unicellular on any radial Hilbertian space $\mathcal{F}_W$, hence all backward shift invariant subspaces of $\mathcal{F}_W$ are of the form $\mathcal{P}_n$.

Our first main result extends the above to a large class of Fock-type spaces $\mathcal{F}^p_W$ (W not necessarily radial), containing the set of polynomials as a dense subset. To state it we introduce the following terminology. We say that $\mathcal{F}^p_W$ is a space of finite order if any function $F\in\mathcal{F}^p_W$ is of finite order. In fact, in this case there exists a uniform upper bound for the orders of elements of $\mathcal{F}^p_W$. Analogously, we say that $\mathcal{F}^p_W$ is a space of zero exponential type, if any $F\in\mathcal{F}^p_W$ is of zero type with respect to the order 1.

Theorem 1.1. *Let $\mathcal{F}^p_W$ be a space of finite order such that $\mathcal{F}^p_W$ contains the set $\mathcal{P}$ of all polynomials as a dense subset. Then any nontrivial backward shift invariant subspace is of the form $\mathcal{P}_n$ for some $n\in\mathbb{Z}_+$ and, thus, L is unicellular on $\mathcal{F}^p_W$.*

The following theorem shows that we can omit the restriction that $\mathcal{F}^p_W$ is of finite order if the weight is radial. Note that in this case, polynomials are dense in $\mathcal{F}^p_W$ whenever they are contained in it (see Proposition 3.1 below).

Theorem 1.2. *Let $\mathcal{F}^p_W$ be a radial Fock-type space, $1<p<+\infty$, containing all polynomials. Then any nontrivial backward shift invariant subspace is of the form $\mathcal{P}_n$ for some $n\in\mathbb{Z}_+$ and, thus, L is unicellular on $\mathcal{F}^p_W$.*

We now turn to the ordering property for nearly invariant subspaces. Here the threshold is given by the order 1. Spaces of smaller order have the ordering property while even in the standard radial Fock spaces of order 1, or higher, nearly invariant subspaces are not ordered by inclusion. The next theorem can be obtained by a modification of the beautiful de Branges' proof of Ordering Theorem for de Branges spaces based on a Phragmén–Lindelöf type result due to M. Heins.

Theorem 1.3. *Let $\mathcal{F}^p_W$ be a space of zero exponential type. Then the set of nearly invariant subspaces of $\mathcal{F}^p_W$ is ordered by inclusion.*

In the case when polynomials are dense in $\mathcal{F}^p_W$ this leads to a complete description of all nearly invariant subspaces.

Corollary 1.4. *Let $\mathcal{F}^p_W$ be a Fock-type space of zero exponential type which contains the polynomials as a dense subspace. Then any nontrivial nearly invariant subspace is of the form $\mathcal{P}_n$ for some $n \in \mathbb{Z}_+$. This is true, in particular, for all spaces $\mathcal{F}^p_\alpha$, $\alpha \in (0,1)$.*

Obviously, in the case $\alpha \geq 1$, the spaces $\mathcal{F}^p_\alpha$ contain finite-dimensional nearly invariant subspaces of the form $\mathrm{e}^Q \mathcal{P}_n$ where Q is a fixed polynomial of degree at most α and one cannot expect the ordered structure. However, the collection of nearly invariant subspaces is much larger. The following result is a special case of a more general construction (see Sections 5 and 6).

Theorem 1.5. *For any $\alpha \geq 1$ the space $\mathcal{F}^p_\alpha$ contains nontrivial infinite-dimensional nearly invariant subspaces.*

We will give several methods to construct nontrivial nearly invariant subspaces in Fock-type spaces. In the classical Fock space $\mathcal{F}_2$ one can give such examples using the Bargmann transform. In general spaces $\mathcal{F}_\alpha$, $\alpha \geq 1$, one can define nearly invariant subspaces imposing certain growth/decay conditions in some angles (see Section 5).

There exists a standard way to construct nearly invariant subspaces in a Banach space $\mathcal{H}$ of entire functions with bounded point evaluations and division property. Let $G \in \mathcal{H}$ have only simple zeros. Then it is easy to see that

$$\mathcal{M}_G = \overline{\mathrm{Span}}\left\{ \frac{G(z)}{z-\lambda} : \ \lambda \in \mathcal{Z}_G \right\}$$

is a nearly invariant subspace. Here $\mathcal{Z}_G$ is the zero set of G. In the case when G has multiple zeros, an obvious modification is required. Note that Corollary 1.4 admits the following reformulation in terms of approximation theory:

Corollary 1.6. *If $\mathcal{F}^p_W$ is a radial Fock-type space of zero exponential type, then the family $\left\{ \frac{G(z)}{z-\lambda} : \ \lambda \in \mathcal{Z}_G \right\}$ is complete in $\mathcal{F}^p_W$ for any transcendental entire function $G \in \mathcal{F}^p_W$ with simple zeros.*

An interesting aspect related to these examples is exploited in Section 6, where we construct a function G with an asymmetric behaviour which will be then inherited by all the non-zero elements of the corresponding subspace $\mathcal{M}_G$. In fact, all examples in Sections 5 and 6 are essentially based on finding subspaces whose non-zero elements share a certain asymmetric behaviour. This leads to the natural question whether there exist nontrivial infinite-dimensional nearly invariant subspaces satisfying additional symmetry conditions. In Section 7 we consider rotation invariance of such subspaces in radial Fock-type spaces, that is nearly invariant subspaces which are also invariant for the operator $R_\beta f(z) = f(\mathrm{e}^{i\beta} z)$.

It is easy to show that if $\beta/\pi \notin \mathbb{Q}$, then any nontrivial nearly invariant subspace $\mathcal{M}$ of a radial space $\mathcal{F}_W$ with $R_\beta \mathcal{M} \subset \mathcal{M}$ must contain polynomials up to some order and, thus, is of the form $\mathcal{P}_k$ (see Section 7).

The case when $\beta = \pi m/n$ with relatively prime m and n reduces to the case when $\beta = 2\pi/n$, and we have the following result.

Theorem 1.7. *Let $\mathcal{F}^p_W$ be a radial Fock-type space and let $n \geq 2$ be an integer such that any element of $\mathcal{F}^p_W$ has zero type with respect to order n. Then any nontrivial nearly-invariant subspace $\mathcal{M}$ invariant with respect to $R_{2\pi/n}$ is of the form $\mathcal{P}_m$.*

The restriction on the order and type is sharp. It is easy to see, e.g., that nontrivial subspaces of the Fock space from Subsection 5.1 can be invariant under $f \mapsto f(-z)$, which corresponds to $n = 2$.

We finish this Introduction with one open problem.

Problem. *Is it true that any nearly invariant subspace without common zeros in a Fock-type space $\mathcal{F}^p_\alpha$ is 1-generated, that is, of the form $\mathcal{M}_G$ for some $G \in \mathcal{F}^p_\alpha$?*

In what follows we write $U(x) \lesssim V(x)$ if there is a constant C such that $U(x) \leq CV(x)$ holds for all x in the set in question. We write $U(x) \asymp V(x)$ if both $U(x) \lesssim V(x)$ and $V(x) \lesssim U(x)$. The standard Landau notations O and o also will be used. The zero set of an entire function f will be denoted by $\mathcal{Z}_f$. We denote by $D(z,R)$ the disc with center z of radius R and by m_2 area-Lebesgue measure in $\mathbb{C}$.

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2. Preliminaries on Cauchy transforms and proof of Theorem 1.1

In what follows we will use the following two results from [7] about the asymptotic behaviour of Cauchy transforms of measures in the plane. More subtle results about Cauchy transforms on rectifiable curves were obtained in [20, 27]. We say that $\Omega \subset \mathbb{C}$ is a *set of zero planar density* if

$$\lim_{R\to+\infty} \frac{m_2(\Omega \cap D(0,R))}{R^2} = 0.$$

It is well-known that for any finite complex Borel measure ν in the plane, its Cauchy transform

$$\mathcal{C}_\nu(z) = \int_{\mathbb{C}} \frac{d\nu(\xi)}{z-\xi}$$

is well-defined in the principal value sense m_2-a.e. Note that if ν is compactly supported, we obviously have

$$\mathcal{C}_\nu(z) = \frac{\nu(\mathbb{C})}{z} + \mathrm{o}\Big(\frac{1}{z}\Big), \qquad |z| \to +\infty. \tag{2.1}$$

From the results of [20] one can easily deduce that the same asymptotics holds for an arbitrary ν outside some "small" set (see [7, Proof of Lemma 4.3]):

$$\mathcal{C}_\nu(z) = \frac{\nu(\mathbb{C})}{z} + \mathrm{o}\Big(\frac{1}{z}\Big), \qquad |z| \to +\infty, \quad z \in \mathbb{C} \setminus \Omega,$$

where Ω is a set of zero planar density.

The following result of A. Borichev (see [7, Lemma 4.2] and also [8] where an inaccuracy of the proof is corrected) can be considered as an extension of the classical Liouville theorem.

Theorem 2.1. *If an entire function f of finite order is bounded on $\mathbb{C} \setminus \Omega$ for some set Ω of zero planar density, then f is a constant.*

We will also need the following simple observation: for any finite complex measure ν we have

$$\int_{\mathbb{C}} \frac{|\mathcal{C}_\nu(z)|}{1+|z|^3} dm_2(z) \le 10|\nu|(\mathbb{C}). \tag{2.2}$$

This follows directly from Fubini's theorem and the estimate

$$\int_{\mathbb{C}} \frac{1}{|z-\zeta|(1+|z|^3)} dm_2(z) \le \int_{|z-\zeta|\ge 1} \frac{1}{1+|z|^3} dm_2(z) + \int_{|z-\zeta|<1} \frac{1}{|z-\zeta|} dm_2(z),$$

valid for any $\zeta \in \mathbb{C}$.

Our proofs are often based on duality arguments. We cannot identify the dual space to $\mathcal{F}^p_W$ with the space $\mathcal{F}^q_W$, $1/p + 1/q = 1$, unless $p = 2$. However, by the Hahn–Banach theorem, any continuous linear functional on $\mathcal{F}^p_W$, $p \in [1, +\infty)$, may be represented in the form

$$\Psi_g(f) = \int_{\mathbb{C}} fgW\, dm_2, \qquad g \in L^q(W), \tag{2.3}$$

where $1/p+1/q = 1$, $L^q(W) = \{f : fW^{1/q} \in L^q(\mathbb{C}, dm_2)\}$. In the proof of Theorem 1.2 we will need a more detailed information about the structure of the functionals and will show that the function g can be chosen to have a certain "antianalyticity" (see Proposition 3.2).

Proof of Theorem 1.1. Let $\mathcal{M}$ be a nontrivial backward shift invariant subspace of $\mathcal{F}^p_W$ and let Ψ_g, $g \in L^q(W)$, be a functional annihilating $\mathcal{M}$. To simplify notation we write $d\mu$ in place of Wm_2. Let $f \in \mathcal{M}$. Then we have, for any $z \in \mathbb{C}$,

$$\int_{\mathbb{C}} \frac{f(\zeta) - f(z)}{\zeta - z} g(\zeta) d\mu(\zeta) = 0,$$

whence

$$f(z)\mathcal{C}_{g\mu}(z) = \mathcal{C}_{fg\mu}(z).$$

Since $\mathcal{P}$ is dense in $\mathcal{F}^p_W$, we can choose the smallest $n \in \mathbb{Z}_+$ such that $\Psi_g(z^n) = \int_{\mathbb{C}} \zeta^n g(\zeta) d\mu(\zeta) \neq 0$. Then

$$\int_{\mathbb{C}} \frac{z^n - \zeta^n}{z - \zeta} g(\zeta) d\mu(\zeta) = 0$$

whence, by (2.1), there exists a set Ω of zero planar density such that

$$\mathcal{C}_{g\mu}(z) = \frac{1}{z^n}\mathcal{C}_{\zeta^n g\mu}(z) = \frac{\alpha}{z^{n+1}} + \mathrm{o}\Big(\frac{1}{z^{n+1}}\Big), \qquad \mathcal{C}_{fg\mu}(z) = \mathrm{O}\Big(\frac{1}{z}\Big),$$

when $|z| \to +\infty$, $z \in \mathbb{C} \setminus \Omega$, and $\alpha \neq 0$. We conclude that $|f(z)| = \mathrm{O}(|z|^n)$ outside a set of zero planar measure. Since $\mathcal{F}^p_W$ is of finite order, f is a polynomial of degree at most n by Theorem 2.1. $\square$

Remark 2.2. 1. The above argument applies to a somewhat more general class of spaces than Fock-type spaces. The only property of the space we used is that there exists a measure μ in $\mathbb{C}$ such that $\|f\|_{\mathcal{F}^p_W} = \|f\|_{L^p(\mu)}$ for any $f \in \mathcal{F}^p_W$.

2. In general, if the conditions of Theorem 1.1 are not satisfied, even the ordering property for backward shift invariant spaces may fail. Simple examples of this type are provided by Paley–Wiener spaces. Let us denote, as usual, by PW_I the image of $L^2(I)$ by the the Fourier transform. As mentioned in the Introduction, by the result in [9], $PW_{[-a,a]}$ is a Fock-type space, and $PW_{[0,a]}$, $PW_{[-a,0]}$ are backward shift invariant subspaces which are orthogonal to each other.

3. Proof of Theorem 1.2

The density of polynomials in radial spaces $\mathcal{F}^p_W$ is apparently well known. We include a very short proof of this fact for the sake of completeness.

Proposition 3.1. *If W is a radial weight with the property $\int_{\mathbb{C}} |z|^n W(z) dm_2(z) < +\infty$ for any n, then polynomials are dense in $\mathcal{F}^p_W$ for all $p \in [1, +\infty)$.*

Proof. Given $f \in \mathcal{F}^p_W$ and $\theta \in [0, 2\pi]$, we have that $f_\theta(z) = f(\mathrm{e}^{\mathrm{i}\theta} z)$ belongs to $\mathcal{F}^p_W$ and $\|f_\theta\|_{\mathcal{F}^p_W} = \|f\|_{\mathcal{F}^p_W}$. In particular, if $\theta_n \to \theta$, from the equality of the norms together with the pointwise convergence $f_{\theta_n}(z) \to f_\theta(z)$, $z \in \mathbb{C}$, we deduce that $\|f_{\theta_n} - f_\theta\|_{\mathcal{F}^p_W} \to 0$, Now let $g \in L^q(W)$ or $(L^\infty(\mathbb{C})$ if $p = 1)$ annihilate the polynomials, let $f \in \mathcal{F}^p_W$ and let

$$u(\mathrm{e}^{\mathrm{i}\theta}) = \int_{\mathbb{C}} f_\theta g W dm_2, \qquad \theta \in [0, 2\pi].$$

By the above argument u is continuous on the unit circle. By Fubini's theorem, its Fourier coefficients satisfy

$$\int_0^{2\pi} u(\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{-\mathrm{i}n\theta}\frac{d\theta}{2\pi} = \int_{\mathbb{C}} g(z)\Big(\int_0^{2\pi} f_\theta(z)\mathrm{e}^{-\mathrm{i}n\theta}\frac{d\theta}{2\pi}\Big)W(z)dm_2(z) = 0, \quad n \in \mathbb{Z},$$

since the inner integral either vanishes, or it is a constant multiple of z^n. Thus $u = 0$ on the unit circle, whence g annihilates any $f \in \mathcal{F}^p_W$. □

A key step in the proof of Theorem 1.2 is a simple identity for continuous linear functionals Ψ_g on $\mathcal{F}^p_W$. With the representation (2.3) we have that if $f \in \mathcal{F}^p_W$, $\zeta \in \mathbb{C}$, and $T_\zeta f(z) = L(1-\zeta L)^{-1}f(z) = \frac{f(z)-f(\zeta)}{z-\zeta}$, then

$$\Psi_g(T_\zeta f) = f(\zeta)\mathcal{C}_{g\mu}(\zeta) - \mathcal{C}_{fg\mu}(\zeta), \tag{3.1}$$

m_2-a.e., where $\mu = Wm_2$. In order to prove Theorem 1.2, we shall assume that the left hand side vanishes and estimate $|\mathcal{C}_{fg\mu}|$ from above and $|\mathcal{C}_{g\mu}|$ from below, in order to obtain an estimate for $|f|$. While (2.2) provides a sufficiently good upper estimate for $\mathcal{C}_{fg\mu}$, the lower estimate for $\mathcal{C}_{g\mu}$ is more subtle and is addressed in the proposition below.

We shall denote by H^p the usual Hardy spaces on the unit disc $\mathbb{D}$, by A the disc algebra and by P_+ the usual Riesz projection

$$P_+u(\rho\mathrm{e}^{\mathrm{i}\theta}) = \frac{1}{2\pi}\int_0^{2\pi}\frac{u(\mathrm{e}^{\mathrm{i}t})}{1-\rho\mathrm{e}^{\mathrm{i}(\theta-t)}}dt, \qquad \rho < 1,\ \theta \in [0,2\pi],$$

where u is a function in L^1 on the unit circle. Recall that functions in H^p (or A) can be identified with Fourier series in $L^p([0,2\pi])$ (or $C([0,2\pi])$) whose negative coefficients vanihsh, and this identification defines an isometry between the spaces in question. Moreover, by the famous M. Riesz theorem P_+ is a bounded operator from $L^p([0,2\pi])$ onto H^p, whenever $1 < p < +\infty$.

Proposition 3.2. *Let $1 < p < +\infty$, $\frac{1}{p}+\frac{1}{q} = 1$, and let $\mathcal{F}^p_W$ contain all polynomials. Then there exists $K > 0$ depending only on p such that every continuous linear functional Ψ on $\mathcal{F}^p_W$ can be represented in the form (2.3) with $g \in L^q(W)$ such that:*

- (i) *$\|g\|_{L^q(W)} \le K\|\Psi\|$, $g \in C(\mathbb{C})$, and for all $r > 0$, there exists a function $u_r \in A$ such that $u_r(\mathrm{e}^{\mathrm{i}t}) = \overline{g}(r\mathrm{e}^{\mathrm{i}t})$;*
- (ii) *The Cauchy transform $\mathcal{C}_{g\mu}$ is continuous on $\mathbb{C}$, satisfies for $r > 0$*

$$\mathcal{C}_{g\mu}(r\mathrm{e}^{\mathrm{i}t}) = \frac{2\pi}{r\mathrm{e}^{\mathrm{i}t}}\int_0^r \overline{u_\rho}\Big(\frac{\rho\mathrm{e}^{\mathrm{i}t}}{r}\Big)\rho W(\rho)d\rho; \tag{3.2}$$

- (iii) *There exists a function $U_r \in A$ such that $U_r(\mathrm{e}^{\mathrm{i}t}) = \overline{\mathcal{C}_{g\mu}}(r\mathrm{e}^{\mathrm{i}t})$ and*

$$\|U_r\|_{H^q} \le \frac{K}{r}\|\Psi\|\Big(\int_0^r \rho W(\rho)d\rho\Big)^{\frac{1}{p}}. \tag{3.3}$$

Proof. (i) Let Ψ be a continuous linear functional on $\mathcal{F}^p_W$ and choose, by the Hahn-Banach theorem, $h \in L^q(W)$ such that

$$\Psi(f) = \int_{\mathbb{C}} fhW dm_2, \qquad \|h\|_{L^q(W)} = \|\Psi\|.$$

Since $1 < p < +\infty$, $\mathcal{F}^p_W$ is reflexive, since it is a closed subspace of the reflexive space $L^p(W)$. Therefore, there exists $f_0 \in \mathcal{F}^p_W$, $\|f_0\|_{\mathcal{F}^p_W} = 1$, such that $\Psi(f_0) = \|\Psi\|$. Then for the functions $f_0 W^{\frac{1}{p}}$, $hW^{\frac{1}{q}}$, we have equality in the Hölder inequality, which implies that $h = \overline{f}_0 |f_0|^{p-2}$, m_2-a.e. on $\mathbb{C}$. Put $F_r(\mathrm{e}^{\mathrm{i}t}) = \overline{f}_0 |f_0|^{p-2}(r\mathrm{e}^{\mathrm{i}t})$, $r \ge 0$, $t \in [0, 2\pi]$. It is easy to verify that $\overline{f}_0 |f_0|^{p-2} \in \mathrm{Lip}^\beta(\Omega)$ for any compact subset Ω of $\mathbb{C}$ with $\beta = \min\{p, 2\} - 1$. Therefore, the function $u_r = P_+ F_r$ belongs to A and, moreover, is in $\mathrm{Lip}^\beta(\overline{\mathbb{D}})$ (see, e.g., [14, Theorem 5.1]). Let

$$g(r\mathrm{e}^{\mathrm{i}t}) = \overline{u_r}(\mathrm{e}^{\mathrm{i}t}).$$

Then the function g is well-defined everywhere in $\mathbb{C}$. Let us show that g is continuous. Let

$$g_n(r\mathrm{e}^{\mathrm{i}t}) = \overline{u_r}\Big(\big(1 - \frac{1}{n}\big)\mathrm{e}^{\mathrm{i}t}\Big).$$

Since the functions u_r are in $\mathrm{Lip}^\beta(\overline{\mathbb{D}})$ uniformly with respect to r in any bounded set, we conclude that $g_n(r\mathrm{e}^{\mathrm{i}t})$ converge to $g(r\mathrm{e}^{\mathrm{i}t})$ locally uniformly in r and uniformly in t. Finally, it is obvious that $g_n(r'\mathrm{e}^{\mathrm{i}t})$ converges uniformly in t to $g(r\mathrm{e}^{\mathrm{i}t})$ as $r' \to r$ for any fixed n. We conclude that $g \in C(\mathbb{C})$.

Using polar coordinates the estimate $\|g\|_{L^q(W)} \le K\|h\|_{L^q(W)} = K\|\Psi\|$ follows by the M. Riesz theorem. Finally, since $\mathcal{F}^p_W$ consists of functions which are continuous on every closed disc and analytic inside, we have by definition

$$\int_0^{2\pi} f(r\mathrm{e}^{\mathrm{i}t})(u_r(\mathrm{e}^{\mathrm{i}t}) - F_r(\mathrm{e}^{\mathrm{i}t}))dt = 0,$$

hence using polar coordinates,

$$\begin{aligned}\Psi(f) &= \int_0^{+\infty} rW(r) \int_0^{2\pi} f(r\mathrm{e}^{\mathrm{i}t}) F_r(\mathrm{e}^{\mathrm{i}t}) dt dr \\ &= \int_0^{+\infty} rW(r) \int_0^{2\pi} f(r\mathrm{e}^{\mathrm{i}t}) u_r(\mathrm{e}^{\mathrm{i}t}) dt dr = \int_{\mathbb{C}} fgW dm_2.\end{aligned}$$

(ii) The function $\mathcal{C}_{g\mu}$ is well-defined everywhere in $\mathbb{C}$ and, for a fixed $\zeta \ne 0$, we have

$$\mathcal{C}_{g\mu}(\zeta) = \lim_{\delta \to 0} \int_{\big||z|-|\zeta|\big|>\delta} \frac{g(z)}{\zeta - z} W(z) dm_2(z).$$

Using (i) and a straightforward calculation we obtain

$$\int_0^{2\pi} \frac{g(\rho\mathrm{e}^{\mathrm{i}t})}{\zeta - \rho\mathrm{e}^{\mathrm{i}t}} dt = 0, \text{ if } \rho > |\zeta|, \qquad \int_0^{2\pi} \frac{g(\rho\mathrm{e}^{\mathrm{i}t})}{\zeta - \rho\mathrm{e}^{\mathrm{i}t}} dt = \frac{2\pi}{\zeta} \overline{u}_\rho\big(\rho/\overline{\zeta}\big), \text{ if } \rho < |\zeta|.$$

Hence,

$$\mathcal{C}_{g\mu}(\zeta) = \frac{2\pi}{\zeta} \lim_{\delta\to 0} \int_0^{|\zeta|-\delta} \overline{u}_\rho\Big(\frac{\rho}{\overline{\zeta}}\Big)\rho W(\rho)d\rho = \frac{2\pi}{\zeta}\int_0^{|\zeta|} \overline{u}_\rho\big(\rho/\overline{\zeta}\big)\rho W(\rho)d\rho,$$

since the functions u_ρ are uniformly bounded for $\rho \le |\zeta|$. This proves formula (3.2). In particular, $\mathcal{C}_{g\mu}$ is continuous at ζ.

(iii) It follows from (3.2) that the nonnegative Fourier coefficients of the function $\mathcal{C}_{g\mu}(re^{it})$ vanish, therefore there exists a function $U_r \in A$ such that $U_r(e^{it}) = \overline{\mathcal{C}_{g\mu}}(re^{it})$, $U_r(0) = 0$. Finally, by (i), $\|u_r\|^q_{H^q} = \frac{1}{2\pi}\int_0^{2\pi} |g(re^{it})|^q dt$, and the inequality (3.3) follows by the Hölder inequality together with the fact that

$$\int_0^{2\pi} |u_\rho(se^{it})|^q dt \le 2\pi\|u_\rho\|^q_{H^q}, \qquad 0 < s \le 1.$$

□

Proof of Theorem 1.2. The density of polynomials in $\mathcal{F}^p_W$ has already been proved in Proposition 3.1. We are going to show that if $f \in \mathcal{F}^p_W$ and Ψ is a non-zero continuous linear functional on $\mathcal{F}^p_W$ with $\Psi(T_\zeta f) = 0$, $\zeta \in \mathbb{C}$, then f is a polynomial of degree at most $n+1$, where n is the smallest nonnegative integer such that $\Psi(z^n) \ne 0$. Since polynomials are dense in $\mathcal{F}^p_W$ and $\Psi \ne 0$, such an integer exists.

We shall consider first the case when $n = 0$. Consider the representation of Ψ given by Proposition 3.2. The equality (3.1) gives

$$f(\zeta)\mathcal{C}_{g\mu}(\zeta) = \mathcal{C}_{fg\mu}(\zeta), \tag{3.4}$$

m_2-a.e. For any fixed $r > 0$, the function $e^{it} \mapsto f(re^{it})U_r(e^{it})$ belongs to the disc algebra A. Using also (3.4) we obtain for all $\delta \in (0,1)$

$$\begin{aligned}\int_0^{2\pi} |f(\delta re^{it})U_r(\delta e^{it})|dt &\le \int_0^{2\pi} |f(re^{it})U_r(e^{it})|dt \\ &= \int_0^{2\pi} |f(re^{it})\mathcal{C}_{g\mu}(re^{it})|dt = \int_0^{2\pi} |\mathcal{C}_{fg\mu}(re^{it})|dt.\end{aligned}$$

Given an H^q-function v with $v(0) = 0$, we can write for $|z| < 1$, $v(z) = zv'(0) + z^2 v_1(z)$, where $\|v_1\|_{H^q} \le 2\|v\|_{H^q}$, and a standard estimate gives for all $\delta \in (0, 1/2)$,

$$|v(\delta e^{it})| \ge \delta|v'(0)| - \frac{2^{1+\frac{1}{q}}\delta^2}{(1-\delta)^{\frac{1}{q}}}\|v\|_{H^q} \ge \delta|v'(0)| - K'\delta^2\|v\|_{H^q}$$

with K' depending on q only.

In order to apply this estimate to U_r we note first that

$$\begin{aligned}U_r'(0) &= \frac{2\pi}{r}\int_0^r u_\rho(0)\rho W(\rho)d\rho \\ &= \frac{1}{r}\int_0^r \Big(\int_0^{2\pi} u_\rho(e^{it}))dt\Big)\rho W(\rho)d\rho = \frac{1}{r}\int_{|z|<r} \overline{g}(z)W(z)dm_2(z),\end{aligned}$$

whence $rU_r'(0) \to \overline{\Psi(1)}$ as $r \to +\infty$. Together with the estimate (3.3) this gives for $r > 0$, $\delta \in (0, 1/2)$,

$$|U_r(\delta \mathrm{e}^{\mathrm{i}t})| \geq \frac{\delta}{r} \left| \int_{|z|<r} \overline{g}(z) W(z) dm_2(z) \right| - K'' \delta^2 \|\Psi\|,$$

with $K'' = K'K\big(\int_{\mathbb{C}} W(z) dm_2(z)\big)^{\frac{1}{p}}$. Then there exists $r_0 > 0$, $\delta_0 \in (0, 1)$, such that for $r > r_0$, $\delta \in (0, \delta_0)$ we have

$$|U_r(\delta \mathrm{e}^{\mathrm{i}t})| \geq \frac{\delta |\Psi(1)|}{2r}.$$

Using this in (3.5) we obtain for r, δ as above,

$$\int_0^{2\pi} |f(\delta r \mathrm{e}^{\mathrm{i}t})| dt \leq \frac{2r}{\delta |\Psi(1)|} \int_0^{2\pi} |\mathcal{C}_{fg\mu}(r \mathrm{e}^{\mathrm{i}t})| dt,$$

and from (2.2) it follows that

$$\int_{\mathbb{C}} \frac{|f(\delta z)|}{1 + |z|^4} dm_2(z) < +\infty.$$

This implies that f is a polynomial of degree at most one and finishes the proof in the case when $\Psi(1) \neq 0$.

If the smallest integer n with $\Psi(z^n) \neq 0$ is strictly positive, we observe that $\Psi_n(f) = \Psi(z^n f)$ defines a continuous linear functional on the space $\mathcal{F}^p_{W_n}$ with radial weight $W_n(z) = |z|^{-n} W(z)$. The space $\mathcal{F}^p_{W_n}$ contains polynomials and all functions of the form $L^n h$, $h \in \mathcal{F}^p_W$. Moreover, since $h - z^n L^n h$ is a polynomial of degree strictly less than n, and $L^n T_\zeta = T_\zeta L^n$, we have

$$\Psi_n(T_\zeta L^n f) = \Psi_n(L^n T_\zeta f) = \Psi(z^n L^n T_\zeta f) = \Psi(T_\zeta f) = 0, \qquad \zeta \in \mathbb{C}.$$

Since $\Psi_n(1) \neq 0$, the previous argument shows that $L^n f$ is a polynomial of degree at most one, which completes the proof. $\square$

4. The Ordering Theorem for Fock-type spaces of small growth

The key idea of the proof of Theorem 1.3 is due to L. de Branges [12, Theorem 35]. It is based on an application of a deep result by M. Heins [17] (see also [12, Lemma 8] or [24, Proposition 26]): *if f and g are entire function of zero exponential type and*

$$\min\big(|f(z)|, |g(z)|\big) \lesssim 1, \qquad z \in \mathbb{C},$$

then either f or g is a constant function. However, we will need a slightly refined version of Heins' theorem where the estimate holds everywhere up to a small exceptional set. The proof of the following statement is identical to the proof of Heins' theorem (see proof of Proposition 26 in [24]) and we omit it.

Theorem 4.1. *Let f and g be entire function of zero exponential type such that*

$$\min\big(|f(z)|, |g(z)|\big) \lesssim 1, \qquad z \in \mathbb{C} \setminus \Omega,$$

where the set Ω satisfies

$$\int_\Omega \frac{dm_2(z)}{|z|+1} < +\infty.$$

Then either f or g is a constant function.

Proof of Theorem 1.3. Let $\mathcal{M}_1$, $\mathcal{M}_2$ be two nearly invariant subspaces of $\mathcal{F}^p_W$. Assume that neither $\mathcal{M}_1 \subset \mathcal{M}_2$ nor $\mathcal{M}_2 \subset \mathcal{M}_1$. Then we can choose two nonzero functionals Ψ_{F_1} and Ψ_{F_2}, $F_1, F_2 \in L^q(W)$ such that Ψ_{F_1} annihilates $\mathcal{M}_2$ but not $\mathcal{M}_1$, while Ψ_{F_2} annihilates $\mathcal{M}_1$ but not $\mathcal{M}_2$.

Let $F \in \mathcal{M}_1$ and $G \in \mathcal{M}_2$. Define two functions

$$f(z) = \Psi_{F_1}\left(\frac{F - \frac{F(z)}{G(z)}G}{\zeta - z}\right) = \int_{\mathbb{C}} \frac{F(\zeta) - \frac{F(z)}{G(z)}G(\zeta)}{\zeta - z} F_1(\zeta) d\mu(\zeta), \tag{4.1}$$

$$g(z) = \Psi_{F_2}\left(\frac{G - \frac{G(z)}{F(z)}F}{\zeta - z}\right) = \int_{\mathbb{C}} \frac{G(\zeta) - \frac{G(z)}{F(z)}F(\zeta)}{\zeta - z} F_2(\zeta) d\mu(\zeta), \tag{4.2}$$

where $\mu = W m_2$. The functions f and g are well-defined and analytic on the sets $\{z : G(z) \neq 0\}$ and $\{z : F(z) \neq 0\}$, respectively.

Step 1: *f and g are entire functions of zero exponential type, f does not depend on the choice of G and g does not depend on the choice of F.*

Let f_1 be a function associated in a similar way to $G_1 \in \mathcal{M}_2$,

$$f_1(z) = \int \frac{F(\zeta) - \frac{F(z)}{G_1(z)}G_1(\zeta)}{\zeta - z} F_1(\zeta) d\mu(\zeta).$$

Then, for $G(z) \neq 0$ and $G_1(z) \neq 0$, we have

$$f_1(z) - f(z) = \frac{F(z)}{G(z)G_1(z)} \int \frac{G_1(z)G(\zeta) - G(z)G_1(\zeta)}{\zeta - z} F_1(\zeta) d\mu(\zeta) = 0,$$

since $\frac{G_1(z)G - G(z)G_1}{\zeta - z} \in \mathcal{M}_2$.

Now choosing G such that $G(z) \neq 0$ we can extend f analytically to a neighborhood of the point z. Thus, f and g are entire functions.

Since $fG = G\mathcal{C}_{FF_1\mu} - F\mathcal{C}_{GF_1\mu}$, it follows from (2.2) that

$$\int_{D(z,1)} |f(\zeta)G(\zeta)| dm_2(\zeta) \lesssim (|z|^3 + 1) \sup_{\zeta \in D(z,1)} \big(|F(\zeta)| + |G(\zeta)|\big),$$

whence fG is an entire function of zero type. Since G itself is of zero exponential type, f is of zero type by the standard properties of entire functions (see, e.g., [19, Chapter 1, §9]).

Step 2: *Either f or g is identically zero.*

Given z such that $F(z)\neq 0$, $G(z)\neq 0$, we have

$$\begin{aligned} |f(z)| &\le \Big|\int \frac{F(\zeta)F_1(\zeta)}{\zeta-z}d\mu(\zeta)\Big| + \Big|\frac{F(z)}{G(z)}\Big|\cdot\Big|\int \frac{G(\zeta)F_1(\zeta)}{\zeta-z}d\mu(\zeta)\Big|,\\ |g(z)| &\le \Big|\int \frac{G(\zeta)F_2(\zeta)}{\zeta-z}d\mu(\zeta)\Big| + \Big|\frac{G(z)}{F(z)}\Big|\cdot\Big|\int \frac{F(\zeta)F_2(\zeta)}{\zeta-z}d\mu(\zeta)\Big|. \end{aligned} \tag{4.3}$$

Note that, by (2.2), for any finite measure ν, there exists a set of finite measure such that $|C_\nu(z)|\le |z|^3$ outside this set. Then we can find a set Ω of finite measure such that

$$|f(z)|\lesssim |z|^3\Big(1+\Big|\frac{F(z)}{G(z)}\Big|\Big), \qquad |g(z)|\lesssim |z|^3\Big(1+\Big|\frac{G(z)}{F(z)}\Big|\Big), \qquad z\notin\Omega.$$

We conclude that

$$\min\big(|f(z)|,|g(z)|\big)\lesssim |z|^3, \qquad z\notin\Omega.$$

By Theorem 4.1 either f or g is a polynomial.

Assume that f is a nonzero polynomial. By (2.1), there exists a set Ω of zero area density such that

$$\Big|\int \frac{F(\zeta)F_1(\zeta)}{\zeta-z}d\mu(\zeta)\Big| + \Big|\int \frac{G(\zeta)F_1(\zeta)}{\zeta-z}d\mu(\zeta)\Big| = \mathrm{O}\Big(\frac{1}{|z|}\Big), \qquad z\notin\Omega.$$

Hence, $|F(z)/G(z)|\to+\infty$ as $|z|\to+\infty$, $z\notin\Omega$, and so

$$|g(z)| \le \Big|\int \frac{G(\zeta)F_2(\zeta)}{\zeta-z}d\mu(\zeta)\Big| + \Big|\frac{G(z)}{F(z)}\Big|\cdot\Big|\int \frac{F(\zeta)F_2(\zeta)}{\zeta-z}d\mu(\zeta)\Big| = \mathrm{O}\Big(\frac{1}{|z|}\Big), \qquad w\notin\Omega\cup\widetilde{\Omega},$$

where $\widetilde{\Omega}$ is another set of zero area density. Thus, g tends to zero outside a set of zero density and so $g\equiv 0$ by Theorem 2.1.

Step 3: *End of the proof.*

Without loss of generality, let $f\equiv 0$. Then

$$\frac{F(z)}{G(z)}\int \frac{G(\zeta)F_1(\zeta)}{\zeta-z}d\mu(\zeta) = \int \frac{F(\zeta)F_1(\zeta)}{\zeta-z}d\mu(\zeta)$$

for any $F\in\mathcal{M}_1$, $G\in\mathcal{M}_2$.

Recall that Ψ_{F_1} does not annihilate $\mathcal{M}_1$ and so we can choose $F\in\mathcal{M}_1$ such that $\Psi_{F_1}(F)=\int FF_1 d\mu\neq 0$. Then, by (2.1),

$$\Big|\int \frac{F(\zeta)F_1(\zeta)}{\zeta-z}d\mu(\zeta)\Big| \gtrsim \frac{1}{|z|}, \qquad z\notin\Omega,$$

for some set Ω of zero density. Since $\Psi_{F_1}(G)=0$ for any $G\in\mathcal{M}_2$, we have (again by (2.1))

$$\Big|\int \frac{G(\zeta)F_1(\zeta)}{\zeta-z}d\mu(\zeta)\Big| = \mathrm{o}\Big(\frac{1}{|z|}\Big), \qquad |z|\to+\infty,\ z\notin\widetilde{\Omega},$$

where $\widetilde{\Omega}$ is another set of zero density. We conclude that $|F(z)/G(z)|\to+\infty$ when $|z|\to+\infty$ outside the set of zero density $\Omega\cup\widetilde{\Omega}$ (for any $G\in\mathcal{M}_2$). Applying this fact and (2.1) to g we conclude that $|g(z)| \to 0$ outside a set of zero density and so $g \equiv 0$ by Theorem 2.1.

Thus, for any $G \in \mathcal{M}_2$, we have

$$\frac{G(z)}{F(z)} \int \frac{F(\zeta)F_2(\zeta)}{\zeta - z} d\mu(\zeta) = \int \frac{G(\zeta)F_2(\zeta)}{\zeta - z} d\mu(\zeta) \tag{4.4}$$

and we may repeat the above argument. Choose $G \in \mathcal{M}_2$ such that $\Psi_{F_2}(G) = \int GF_2 d\mu \neq 0$. Then, by (2.1), the modulus of the right-hand side in (4.4) is $\gtrsim |z|^{-1}$, while the left-hand side is $o(|z|^{-1})$ when $|z| \to +\infty$ outside a set of zero density. This contradiction proves Theorem 1.3. $\square$

5. Failure of the Ordering Theorem in spaces of large growth

5.1. **Examples via the Bargmann transform.** We start with the case of the classical space $\mathcal{F}_2$ with the weight $W(z) = \exp(-|z|^2)$. The Bargmann (Segal–Bargmann) transform is the isometry $\mathcal{B}$ from $L^2(\mathbb{R})$ onto $\mathcal{F}_2$ given by

$$\mathcal{B}u(z) = \int_{\mathbb{R}} \mathrm{e}^{-\frac{t^2+z^2}{2} + \sqrt{2}tz} u(t)dt, \qquad u \in L^2(\mathbb{R}).$$

Let $I \subsetneq \mathbb{R}$ be a nonempty open interval and let

$$\mathcal{M}_I = \mathcal{B}L^2_I, \qquad L^2_I = \big\{u: \ u = 0 \text{ a.e. off } I\big\}.$$

Obviously, $\mathcal{M}_I$ is nontrivial and closed in $\mathcal{F}_2$. We want to show that these subspaces are nearly invariant. This will follow directly from an alternative description in terms of the Paley–Wiener type spaces PW_I. Recall that PW_I is by definition the image of L^2_I under the Fourier transform. Also, if $I = [a, b]$, then $f \in PW_I$ if and only if f is the restriction to the real axis of an entire function F which is in $L^2(\mathbb{R})$ and satisfies

$$|F(z)| \leq \begin{cases} C\mathrm{e}^{b \operatorname{Im} z}, & \operatorname{Im} z > 0, \\ C\mathrm{e}^{a \operatorname{Im} z}, & \operatorname{Im} z < 0. \end{cases}$$

Proposition 5.1. *For a nonempty bounded open interval $I \subset \mathbb{R}$ we have*

$$\mathcal{M}_I = \big\{f \in \mathcal{F}_2: \ \mathrm{e}^{-\frac{z^2}{2}} f(-\mathrm{i}z) \in PW_{\sqrt{2}I}\big\}.$$

In particular, $\mathcal{M}_I$ is nearly invariant with no common zeros.

Proof. From the definition of $\mathcal{B}$ we have for $f = \mathcal{B}u \in \mathcal{F}_2$, $u \in L^2_I$,

$$\mathrm{e}^{-\frac{z^2}{2}} f(-\mathrm{i}z) = \int_{\mathbb{R}} \mathrm{e}^{-\frac{t^2}{2} - \sqrt{2}\mathrm{i}zt} u(t)dt \in PW_{\sqrt{2}I}.$$

Conversely, any function v in L^2_I with bounded I can be written as $v(t) = \mathrm{e}^{-t^2/2}u(t)$ and so

$$\int_I \mathrm{e}^{-\sqrt{2}\mathrm{i}zt} v(t)dt = e^{-\frac{z^2}{2}} f(-\mathrm{i}z)$$

for $f = \mathcal{B}u \in \mathcal{F}_2$.

It follows immediately from the structure of the Paley–Wiener spaces that $\mathcal{M}_I$ is nearly invariant. $\square$

Clearly, subspaces $\mathcal{M}_I$ are not ordered by inclusion. For example, if $I \cap J = \emptyset$, we have $\mathcal{M}_I \cap \mathcal{M}_J = \{0\}$, and both are nontrivial.

5.2. **Phragmén–Lindelöf approach.** Next we consider a more general construction of nontrivial nearly invariant subspaces in the radial Fock-type spaces $\mathcal{F}^p_\alpha$, $\alpha \geq 1$, with the weight $W_\alpha(z) = \exp(-|z|^\alpha)$. The idea behind our construction is to consider subspaces consisting of functions satisfying a growth restriction in a fixed angle. The remarkable fact is that the growth restriction and the angle can be chosen such that the Phragmén–Lindelöf principle guarantees that the corresponding subspace is closed in $\mathcal{F}^p_\alpha$.

Let us record first the standard pointwise estimate in these spaces. To this end, note that for any $\alpha \geq 1$ we have $W_\alpha(z) \asymp W_\alpha(\zeta)$, $|\zeta - z| \leq |z|^{2-2\alpha}$, $|z| \geq 1$. Thus, using the fact that $|f(z)|^p \leq \frac{1}{\pi r^2}\int_{D(z,r)} |f(\zeta)|^p dm_2(\zeta)$, we get

$$|f(z)|^p \mathrm{e}^{-|z|^\alpha} \leq A|z|^{2\alpha-2}\|f\|^p_{\mathcal{F}^p_\alpha}, \qquad |z| > 1, \tag{5.1}$$

for some constant $A = A(\alpha, p)$.

Now let $\Omega = \{z : 0 < \arg z < 2\pi/\alpha\}$ and let $\Gamma = \partial\Omega$. Fix some polynomial P with zeros outside $\overline{\Omega}$ and $\deg P \geq (2\alpha - 2)/p$. For any $a \geq 0$, put

$$\mathcal{M}_a = \left\{ f \in \mathcal{F}^p_\alpha : \ \left|\frac{f(z)}{P(z)}\mathrm{e}^{-\frac{1}{p}z^\alpha}\right| \leq C\mathrm{e}^{a|z|^{\alpha/2}}, \ z \in \Omega, \text{ for some } C > 0 \right\}.$$

Proposition 5.2. *$\mathcal{M}_a$ is a closed nearly invariant subspace of $\mathcal{F}^p_\alpha$, with no common zeros in $\mathbb{C}$.*

Proof. It is sufficient to show that $\mathcal{M}_a$ is closed, since the remaining assertions are immediate. Let $f_n \in \mathcal{M}_a$ be a sequence convergent in $\mathcal{F}^p_\alpha$ to some function f_0. Then we have

$$\left|\frac{f_n(z)}{P(z)}\mathrm{e}^{-\frac{1}{p}z^\alpha}\right| \leq C_n \mathrm{e}^{a|z|^{\alpha/2}}, \qquad z \in \Omega,$$

for some constants C_n. We need to show that the constants C_n can be replaced by a uniform bound. Once this is done, by taking the pointwise limit it follows that f_0 satisfies the same estimate growth estimate in the angle Ω, i.e. $f_0 \in \mathcal{M}_a$.

Since $\sup_n \|f_n\|_{\mathcal{F}^p_\alpha} < +\infty$, it follows from (5.1) that there is a constant A_0 (independent of n) such that

$$\Big|\frac{f_n(z)}{P(z)}\mathrm{e}^{-\frac{1}{p}z^\alpha}\Big| \le A_0, \qquad z \in \Gamma.$$

For $z \in \mathbb{C}^+$, put

$$g_n(z) = \frac{\mathrm{e}^{z^2/p} f_n(z^{2/\alpha})}{P(z^{2/\alpha})}.$$

Then g_n is analytic in $\mathbb{C}^+$ and continuous up to the boundary, $|g_n(x)| \le A_0$, $x \in \mathbb{R}$, and

$$|g_n(z)| \le C_n e^{a|z|}, \qquad z \in \mathbb{C}^+.$$

By the classical Phragmén–Lindelöf principle, $|g_n(z)| \le A_0 e^{a \operatorname{Im} z}$, $z \in \mathbb{C}^+$, whence for all n we can replace C_n by A_0 and the result follows. $\square$

One can also consider subspaces with $a < 0$ if we replace the estimate in the definition of $\mathcal{M}_a$ by two conditions:

$$\Big|\frac{f(z)}{P(z)}\mathrm{e}^{-\frac{1}{p}z^\alpha}\Big| \le C\mathrm{e}^{a|z|^{\alpha/2}}, \quad \arg z = \pi/\alpha, \qquad \Big|\frac{f(z)}{P(z)}\mathrm{e}^{-\frac{1}{p}z^\alpha}\Big| \le C\mathrm{e}^{d|z|^{\alpha/2}}, \quad z \in \Omega,$$

for some $d > 0$. Also, in general, if α is sufficiently large, one can impose conditions in several angles of size $2\pi/\alpha$ in order to obtain further examples of closed nontrivial nearly invariant subspaces.

Example 5.3. In the case of the Fock space $\mathcal{F}_2$ one can impose conditions in both half-planes:

$$\mathcal{M}_{a,b} = \{f \in \mathcal{F}_2 : |f(z)\mathrm{e}^{-z^2/2}| \le C\,\mathrm{e}^{b \operatorname{Im} z}, \operatorname{Im} z > 0, \text{ and } |f(z)\mathrm{e}^{-z^2/2}| \le C\mathrm{e}^{a \operatorname{Im} z}, \operatorname{Im} z < 0\}.$$

These subspaces actually coincide with those constructed in Subsection 5.1.

Example 5.4. Consider the space $\mathcal{F}_1$ with the weight $W(z) = \exp(-|z|)$. Then $f \in \mathcal{M}_a$ if and only if $f \in \mathcal{F}_1$ and

$$|f(z)\mathrm{e}^{-z/2}| \le C\mathrm{e}^{a|z|^{1/2}}, \qquad z \in \mathbb{C}.$$

Similar classes of functions appeared in the paper of Gurarii [15] in connection to the study of primary ideals in weighted L^1 spaces (Beurling algebras).

Let us show that for any $\alpha > 1$ the subspaces $\mathcal{M}_a$ in $\mathcal{F}^p_\alpha$ are nontrivial.

Lemma 5.5. *For any $\alpha \ge 1$ and $a > 0$ there exists entire function $f \in \mathcal{F}^p_\alpha$ such that*

$$\Big|f(z)\mathrm{e}^{-\frac{z^\alpha}{p}}\Big| \le \mathrm{e}^{a|z|^{\alpha/2}}, \qquad \arg z \in [0, 2\pi/\alpha]. \tag{5.2}$$

Proof. If $\alpha = m \in \mathbb{N}$ and $a > 0$, we can put

$$f(z) = \mathrm{e}^{\frac{1}{p}z^m} \frac{\sin(az^{m/2})}{z^{m/2}} (P(z))^{-1}$$

for a polynomial P whose zeros are a subset of the zero set of $\sin(az^{m/2})$. Then $f \in \mathcal{F}^p_m$ and

$$\left| f(z)\mathrm{e}^{-\frac{1}{2}z^m} \right| \le C\mathrm{e}^{a|z|^{m/2}}, \qquad z \in \mathbb{C}.$$

In the case when α is noninteger, the construction is not so explicit. It is based on a subtle atomization theorem of R. S. Yulmukhametov [30]. Let $\Omega = \{z : \ 0 < \arg z < 2\pi/\alpha\}$. First we consider the case $a = 0$ and construct a nonzero function $f_1 \in \mathcal{F}^p_\alpha$ such that

$$|f_1(z)\mathrm{e}^{-\frac{z^\alpha}{p}}| \le 1, \qquad z \in \Omega. \tag{5.3}$$

Fix a small number $\varepsilon \in \big(0, \pi\big(1 - \frac{1}{\alpha}\big)\big)$ and put

$$h(\theta) = \begin{cases} \cos\alpha\theta, & \theta \in [0, 2\pi/\alpha + \varepsilon] \cup [2\pi - \varepsilon, 2\pi], \\ \cos\alpha\varepsilon, & \theta \in [2\pi/\alpha + \varepsilon, 2\pi - \varepsilon], \end{cases}$$

and

$$u(z) = \frac{1}{p} r^\alpha h(\theta).$$

It is easy to see that u is a subharmonic function in $\mathbb{C}$ (it is harmonic in Ω and subharmonic in $\mathbb{C} \setminus \overline{\Omega}$). Next, we are going to use the following approximation result of [30].

Theorem 5.6. *Let u be a subharmonic function in the complex plane of finite order ρ. Then there exists an entire function F such that for every $\beta \ge \rho$*

$$|\log|F(z)| - u(z)| \le C_\beta \log|z|, \qquad z \in \mathbb{C} \setminus E_\beta,$$

for some $C_\beta > 0$, and the set E_β can be covered by a family of discs $D(z_j, r_j)$ such that

$$\sum_{|z_j|>R} r_j = \mathrm{o}(R^{\rho-\beta}), \qquad R \to +\infty.$$

Let F be an entire function constructed from u via this theorem (here $\rho = \alpha$ and we can find a sufficiently large $\beta > \alpha$ to ensure that e^u is almost constant inside the exceptional discs $D(z_j, r_j)$). Then it is easy to see that $f_1 = F/P$ is in $\mathcal{F}^p_\alpha$ and satisfies inequality (5.3) for some polynomial P. The estimate remains valid inside the discs $D(z_j, r_j)$ by the maximum principle.

To construct a function in $\mathcal{M}_a$, $a > 0$, but not in $\mathcal{M}_{a'}$ for $a' < a$, we construct an entire function f_2 such that, for $z = r\mathrm{e}^{i\theta}$,

$$|f_2(z)| \le \begin{cases} \mathrm{e}^{r^{\alpha/2}\sin(\alpha\theta/2)}, & \theta \in [0, 2\pi/\alpha], \\ 1, & \theta \notin [0, 2\pi/\alpha], \end{cases} \tag{5.4}$$

Then the function $f(z) := f_1(z)f_2(z)$ satisfies (5.2), but not a similar estimate with $a' < a$.

The function

$$g(z) = \frac{\sin(z^{\alpha/2})}{z^{\alpha/2}}$$

is analytic in the sector $\Omega = \{z : \ 0 < \arg z < 2\pi/\alpha\}$ and bounded in $\partial\Omega$. Put

$$g_1(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{g(w)}{z-w}dw, \qquad z \in \mathbb{C}\setminus\overline{\Omega}.$$

It is well known that g_1 extends to an entire function. Indeed, g_1 is analytic in $\mathbb{C}\setminus\overline{\Omega}$. Let $\Omega_R = \Omega\cap\{z : |z| > R\}$ and $g_R(z) = \frac{1}{2\pi i}\int_{\partial\Omega_R}\frac{g(w)}{z-w}dw$, $z \in \mathbb{C}\setminus\overline{\Omega}_R$. Then the function g_R is an analytic continuation of g_1 to $\mathbb{C}\setminus\overline{\Omega}_R$ and, thus, g_1 extends to an entire function. By the Sokhotski–Plemelj theorem we get

$$|g_1(z)| \le \mathrm{e}^{r^{\alpha/2}\sin(\alpha\theta/2)} + \mathrm{O}(r), \quad \theta \in [0, 2\pi].$$

Using standard estimates of Cauchy transform we get $|g_1(z)| \le 10(1+|z|)$, $z \notin \Omega$. Hence, function $f_2(z) = \frac{g_1(z)}{z-z_0}$ (where z_0 is some zero of g_1) satisfies (5.4). □

6. Counterexamples constructed via generating functions

In this section we discuss an additional, quite general method to construct nontrivial nearly invariant subspaces.

Let $\mathcal{F}_W$ be a Fock-type space (not necessarily radial). In this section we consider only Hilbertian case, though all arguments easily carry over to L^p-setting. We denote by k_z the reproducing kernel of $\mathcal{F}_W$ at the point z. In what follows we will assume that our weight has some mild regularity, more precisely, there exists $N \ge 0$ such that

$$C_1|z|^{-N} \le \frac{W(\zeta)}{W(z)} \le C_2|z|^N, \qquad \zeta \in D(z, |z|^{-N}),\ |z| > 1. \tag{6.1}$$

Now assume that there exists an entire function $G \in \mathcal{F}_W$ such that the zeros $\{t_n\}$ of G are simple and power separated, i.e.,

$$\mathrm{dist}(t_n, \{t_m\}_{m\neq n}) \gtrsim (|t_n|+1)^{-N_1}, \tag{6.2}$$

and also its derivative at the zeros has almost maximal growth (again up to a power factor):

$$\|k_{t_n}\| \le (|t_n|+1)^{N_2}|G'(t_n)|. \tag{6.3}$$

Here N_1, N_2 are some positive constants.

All these assumptions are standard and natural. For many Fock-type spaces (including standard spaces $\mathcal{F}_\alpha$, $\alpha > 0$) one can choose the function G which approximates nicely the weight: there exists $N > 0$ such that

$$(|z|+1)^{-N}W^{-1/2}(z)\mathrm{dist}(z, \mathcal{Z}_G) \lesssim |G(z)| \lesssim (|z|+1)^N W^{-1/2}(z)\mathrm{dist}(z, \mathcal{Z}_G) \tag{6.4}$$

for all $z \in \mathbb{C}$. Since, under condition (6.1) one has $\|k_z\| \lesssim (|z|+1)^N W(z)$ (for some, probably different, $N>0$), (6.3) follows from (6.4).

In the classical Fock space (with $\varphi(z) = \pi|z|^2$) the corresponding function G is the the Weierstrass function

$$\sigma(z) = z \prod_{\lambda \in (\mathbb{Z}+i\mathbb{Z})\setminus\{0\}} \Big(1 - \frac{z}{\lambda}\Big) e^{\frac{z}{\lambda} + \frac{z^2}{2\lambda^2}}.$$

In general, for a subharmonic weight W the function G can be obtained by atomization of ΔW (see, e.g., [30] or [11, Proposition 8.1]).

Theorem 6.1. *Let $\mathcal{F}_W$ and G satisfy conditions (6.1), (6.2) and (6.3). Assume that there exists an entire function T such that $\mathcal{Z}_T \subset \mathcal{Z}_G$, both $\mathcal{Z}_T$ and $\mathcal{Z}_G \setminus \mathcal{Z}_T$ are infinite, and the following holds:*

$$|T(z)| \gtrsim (|z|+1)^{-N}, \qquad z \notin \cup_n D_n, \tag{6.5}$$

where $D_n = D(t_n, (|t_n|+1)^{-N}/10)$, and

$$|T(t_n)| \lesssim 1, \qquad t_n \in \mathcal{Z}_G \setminus \mathcal{Z}_T. \tag{6.6}$$

Then $\mathcal{F}_W$ contains a nontrivial infinite-dimensional subspace of the form $\mathcal{M}_g$ for some $g \in \mathcal{F}_W$.

Proof. We may assume that the constants N, N_1, N_2 in (6.1)–(6.6) are the same. Dividing T by a polynomial of sufficiently large degree with zeros in $\mathcal{Z}_T$ we may assume, without loss of generality, that

$$\sum_n \frac{|T(t_n)|}{|G'(t_n)|} \|k_{t_n}\| < +\infty. \tag{6.7}$$

Next we construct an entire function G_0 whose zeros $\lambda_n = t_n + \delta_n$, $t_n \in \mathcal{Z}_G \setminus \mathcal{Z}_T$, are very small but *nonzero* perturbations of zeros of G/T so that

$$|G_0(z)T(z)| \asymp |G(z)|, \qquad z \notin \cup_n D_n. \tag{6.8}$$

and

$$\sum_n \Big| \frac{G_0(t_n)T(t_n)}{G'(t_n)} \Big| < +\infty \tag{6.9}$$

Now let P be a polynomial with $\mathcal{Z}_P \subset \mathcal{Z}_{G_0}$. Put $g = \frac{G_0}{P}$. We claim that if the degree of P is sufficiently large, then $g \in \mathcal{F}_W$ and $\mathcal{M}_g = \overline{\mathrm{Span}}\big\{\frac{g(z)}{z-\lambda} : \ \lambda \in \mathcal{Z}_g\big\}$ is a nontrivial nearly invariant subspace. We have

$$|g(z)| \asymp \frac{|G(z)|}{|P(z)T(z)|} \lesssim \frac{|z|^N G(z)}{|P(z)|}, \qquad z \notin \cup_n D_n.$$

Thus, if $\deg P \geq N$, then $W^{-1/2} g \in L^2(\mathbb{C} \setminus \cup_n D_n)$. Since the weight W satisfies (6.1) and the discs D_n are disjoint, we conclude that $g \in \mathcal{F}_W$ if $\deg P$ is sufficiently large.

Now we construct a function which is orthogonal to $\mathcal{M}_g$. Let $b_n = T(t_n)\|k_{t_n}\|/G'(t_n)$ and let

$$f = \sum_n \overline{b_n} k_{t_n}.$$

In view of (6.7) $f \in \mathcal{F}_W$. Clearly, for $\lambda \in \mathcal{Z}_g$,

$$\left(\frac{g(z)}{z-\lambda}, f\right)_{\mathcal{F}_W} = \sum_n b_n \frac{g(t_n)}{t_n - \lambda} = \sum_n \frac{g(t_n)T(t_n)}{G'(t_n)(t_n-\lambda)}$$

(note that $\lambda \neq t_n$ by the construction of G_0). We claim that

$$\sum_n \frac{g(t_n)T(t_n)}{G'(t_n)(z-t_n)} = \frac{g(z)T(z)}{G(z)}.$$

Then, putting $z=\lambda$ yields that $\left(\frac{g(z)}{z-\lambda}, f\right)_{\mathcal{F}_W} = 0$.

Let

$$H(z) = \sum_n \frac{g(t_n)T(t_n)}{G'(t_n)(z-t_n)} - \frac{g(z)T(z)}{G(z)}.$$

Since the residues coincide, H is an entire function. Note that

$$\left|\frac{g(z)T(z)}{G(z)}\right| = \left|\frac{G_0(z)T(z)}{G(z)}\right| \cdot \frac{1}{|P(z)|} \lesssim \frac{1}{|P(z)|}, \qquad z \notin \cup_n D_n.$$

Combining this with (6.9) we conclude that $|H(z)| \lesssim 1 + |z|^N$ and also $\liminf_{|z|\to+\infty} |H(z)| = 0$. Thus, $H \equiv 0$. $\square$

Here are some concrete examples where Theorem 6.1 applies.

Example 6.2. All examples of Subsection 5.1 can be obtained by the construction of Theorem 6.1. E.g., let $W(z) = \exp(-\pi|z|^2)$ and $\mathcal{M} = \mathrm{e}^{\pi z^2/2} PW_{[-\pi,\pi]}$. Let $G = \sigma$ be the Weierstrass function and $T(z) = \frac{\sigma(z)e^{-\pi z^2/2}}{\sin \pi z}$. Then the subspace constructed in Theorem 6.1 starting from these G and T coincides with $\mathcal{M}$. In the case of $\mathcal{M} = \mathrm{e}^{\pi z^2/2} PW_{[-a,a]}$ one should replace $\sin \pi z$ by an appropriate product of sine-type functions with zeros in the lattice $\mathbb{Z} + i\mathbb{Z}$ intersected with a strip around $\mathbb{R}$.

Example 6.3. A different example in the classical Fock space $\mathcal{F}_2$ with $W(z) = \exp(-\pi|z|^2)$ can be obtained if we consider

$$T(z) = \prod_{k=1}^{+\infty} \left(1 - \frac{\mathrm{e}^{2\pi i z}}{\mathrm{e}^{2\pi k}}\right).$$

Then T is an entire function with zeros $z_{m,k} = m - ik$, $m \in \mathbb{Z}$, $k \in \mathbb{N}$, and it is easy to see that T satisfies (6.5)–(6.6) with $G = \sigma$. This function was used in [1] to provide counterexamples to the Ordering Theorem for nearly invariant subspaces in Cauchy–de Branges spaces. The corresponding nearly invariant subspace coincides with the subspace $\mathcal{M}_0$ from Proposition 5.2.

Example 6.4. Consider $\mathcal{F}_1$, i.e., let $W(z) = \exp(-|z|)$. Then, by an atomization procedure one can find an entire function G satisfying (6.4) (see, e.g., [30] or [11, Proposition 8.1]). Moreover, it is easy to choose its zeros t_n so that $\{t_n\} \supset \{\beta^{-1}n^2\}_{n\ge1}$ for some $\beta > 0$. Then we set

$$T(z) = G(z)\mathrm{e}^{-z}\left(\frac{\sin \pi\beta\sqrt{z}}{\sqrt{z}}\right)^{-1}.$$

Conditions (6.5)–(6.6) are clearly satisfied.

7. Rotation-invariant nearly invariant subspaces

Recall that these are nearly invariant subspaces of $\mathcal{F}^p_W$ (W radial), which are also invariant under the isometry R_β, defined by $R_\beta f(z) = f(\mathrm{e}^{\mathrm{i}\beta}z)$, $f \in \mathcal{F}^p_W$. As we mentioned in the introduction, in the case when $\beta/\pi \notin \mathbb{Q}$, such a subspace must have the form $\mathcal{P}_k$, for some integer $k \ge 0$.

To verify this, note that the symmetrizations defined for integers $k \ge 0$, $N \ge 1$, by

$$R_{k,N} = \frac{1}{N}\sum_{j=0}^{N-1} \mathrm{e}^{-\mathrm{i}jk\beta}R^j_\beta, \tag{7.1}$$

are uniformly bounded on $\mathcal{F}^p_W$, and if f is a polynomial, then $R_{k,N}f$ converges in the norm of $\mathcal{F}^p_W$ to the monomial $\frac{f^{(k)}(0)}{k!}z^k$ as $N \to +\infty$. By Proposition 3.1, the polynomials are dense in $\mathcal{F}^p_W$, hence the above assertion holds true for every $f \in \mathcal{F}^p_W$. Thus, if $\mathcal{M} \subset \mathcal{F}^p_W$ is a closed nearly invariant subspace without common zeros, then it contains the monomial z^k whenever there exists $f \in \mathcal{M}$, $f^{(k)}(0) \ne 0$. In particular, it contains the constants which makes $\mathcal{M}$ backward shift invariant. But then, if $\mathcal{M}$ contains the monomial z^k, it will also contain z^l, $0 \le l \le k$. If $\mathcal{M}$ contains all the monomials, then $\mathcal{M} = \mathcal{F}^p_W$, given that the polynomials are dense. We conclude that if $\mathcal{M}$ is a nontrivial nealy invariant subspace, then there must exist k_0 such that $f^{(k)}(0) = 0$ whenever $f \in \mathcal{M}$, $k \ge k_0$. From this, the claim is immediate.

Proof of Theorem 1.7. Throughout the proof we assume that $\beta = 2\pi/n$. Given an R_β-invariant subspace $\mathcal{M}$ of $\mathcal{F}^p_W$, we denote by $\mathcal{M}^s$ the set of all functions F in $\mathcal{M}$ such that $R_\beta F = F$. In other words, with the notation in (7.1), $\mathcal{M}^s = R_{0,n}\mathcal{M}$.

Now let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two nearly invariant subspaces invariant also with respect to R_β. We will show that either $\mathcal{M}^s_1 \subset \mathcal{M}^s_2$ or $\mathcal{M}^s_2 \subset \mathcal{M}^s_1$. As in the proof of Theorem 1.3, we fix nonzero functions $F_1, F_2 \in L^q(W)$ such that Ψ_{F_1} annihilates $\mathcal{M}^s_2$ but not $\mathcal{M}^s_1$, while

Ψ_{F_2} annihilates $\mathcal{M}_1^s$ but not $\mathcal{M}_2^s$. Note that, if $R_\beta F = F$, then

$$\Psi_{F_1}(F) = \frac{1}{n}\sum_{j=0}^{n-1}\int_{\mathbb{C}} F(\mathrm{e}^{\mathrm{i}j\beta}z)F_1(z)W(z)dm_2(z)$$
$$= \frac{1}{N}\int_{\mathbb{C}} F(z)\sum_{j=0}^{N-1} F_1(\mathrm{e}^{-\mathrm{i}j\beta}z)W(z)dm_2(z) = \Psi_{F_1^s}(F).$$

Thus, we may assume without loss of generality that $R_\beta F_1 = F_1$, $R_\beta F_2 = F_2$.

Define entire functions f and g, by the formulas (4.1) and (4.2), respectively. Then, for $\mu = Wm_2$,

$$f(\mathrm{e}^{\mathrm{i}\beta}z) = \int \frac{F(\zeta) - \frac{F(\mathrm{e}^{\mathrm{i}\beta}z)}{G(\mathrm{e}^{\mathrm{i}\beta}z)}G(\zeta)}{\zeta - \mathrm{e}^{\mathrm{i}\beta}z}F_1(\zeta)d\mu(\zeta)$$
$$= \int \frac{F(\mathrm{e}^{\mathrm{i}\beta}u) - \frac{F(\mathrm{e}^{\mathrm{i}\beta}z)}{G(\mathrm{e}^{\mathrm{i}\beta}z)}G(\mathrm{e}^{\mathrm{i}\beta}u)}{\mathrm{e}^{\mathrm{i}\beta}u - \mathrm{e}^{\mathrm{i}\beta}z}F_1(\mathrm{e}^{\mathrm{i}\beta}u)d\mu(u) = \mathrm{e}^{-\mathrm{i}\beta}f(z).$$

Thus, $\tilde{f}(z) = zf(z)$ satisfies $\tilde{f}(\mathrm{e}^{\mathrm{i}\beta}z) = \tilde{f}(z)$, whence $\tilde{f}(z) = f_1(z^n)$. Since all functions in $\mathcal{F}^p_W$ are of zero type with respect to the order n, the same is true for the function f (see [19, Chapter 1, §9]). We conclude that f_1 is a function of zero exponential type. Analogously, $zg(z) = g_1(z^n)$, where g_1 is of zero exponential type.

By (2.2), there is a set Ω of finite measure such that

$$|f_1(z^n)| \lesssim |z|^3\left(1 + \left|\frac{F(z)}{G(z)}\right|\right), \qquad |g_1(z^n)| \lesssim |z|^3\left(1 + \left|\frac{G(z)}{F(z)}\right|\right), \qquad |z| > 1, z \notin \Omega.$$

Hence, $\min(|f_1(z)|, |g_1(z)|) \lesssim |z|^3$ outside Ω and, by the Heins Theorem 4.1, either f_1 or g_1 is a polynomial. Repeating the arguments from the proof of Theorem 1.3 we arrive at a contradiction.

We have shown that the symmetrized parts of nearly invariant rotation invariant subspaces are ordered by inclusion. In particular, for any infinite-dimensional rotation invariant subspace $\mathcal{M}$, its symmetrized part $\mathcal{M}^s$ must contain the collection of polynomials of the form $\sum_{k=0}^m p_k z^{kn}$, the symmetrized part of $\mathcal{P}_{mn}$. Since $\mathcal{M}$ is also nearly invariant, we conclude that $\mathcal{M}$ contains all the polynomials. If $\mathcal{M}$ is instead finite-dimensional, there exists m such that $\mathcal{P}_m^s = \mathcal{M}^s$ and so $\mathcal{M} = \mathcal{P}_k$ for some k. ∎

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