

PLANAR ORTHOGONAL POLYNOMIALS AND BOUNDARY UNIVERSALITY IN THE RANDOM NORMAL MATRIX MODEL

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ABSTRACT. We show that the planar normalized orthogonal polynomials $P_{n,m}(z)$ of degree n with respect to an exponentially varying planar measure $e^{-2mQ}dA$ enjoy an asymptotic expansion

$$P_{n,m}(z) \sim m^{\frac{1}{4}} \sqrt{\phi'_\tau(z)} [\phi_\tau(z)]^n e^{mQ_\tau(z)} \left(\mathcal{B}_{0,\tau}(z) + \frac{1}{m} \mathcal{B}_{1,\tau}(z) + \frac{1}{m^2} \mathcal{B}_{2,\tau}(z) + \dots \right),$$

as $n, m \rightarrow \infty$ while the ratio $\tau = \frac{n}{m}$ is fixed. Here \mathcal{S}_τ denotes the droplet, the boundary of which is assumed to be a smooth simple closed curve, and ϕ_τ is a normalized conformal mapping from the complement \mathcal{S}_τ^c to the exterior disk \mathbb{D}_e . The functions Q_τ and $\mathcal{B}_{j,\tau}(z)$ are bounded holomorphic functions which may be expressed in terms of Q and \mathcal{S}_τ . We apply these results to obtain boundary universality in the random normal matrix model for smooth droplets, i.e., that the limiting rescaled process is the random process with correlation kernel

$$k(\xi, \eta) = e^{\xi\bar{\eta} - \frac{1}{2}(|\xi|^2 + |\eta|^2)} \operatorname{erf}(\xi + \bar{\eta}).$$

A key ingredient in the proof of the asymptotic expansion of the orthogonal polynomials is the construction of an *orthogonal foliation* – a smooth flow of closed curves near $\partial\mathcal{S}_\tau$, on each of which $P_{n,m}$ is orthogonal to lower order polynomials, with respect to an induced measure. To compute the coefficient functions, we develop an algorithm which determines the coefficients $\mathcal{B}_{j,\tau}$ successively in terms of inhomogeneous Toeplitz kernel conditions. These inhomogeneous Toeplitz kernel conditions may be understood as scalar Riemann-Hilbert problems on the Schottky double of the complement of the droplet.

1. INTRODUCTION

1.1. Historical comments on orthogonal polynomials. The early 1920s in Berlin witnessed a rapid development of the understanding of orthogonal polynomials and related kernel functions. The pioneers were Gabor Szegő, Stefan Bergman, and Salomon Bochner. One of the early results is that of Szegő [42]. He obtained the first (main) term of an asymptotic expansion of the (analytic) orthogonal polynomials in $L^2(\Gamma, ds)$, where Γ is a real-analytically smooth Jordan curve in the complex plane \mathbb{C} supplied with normalized arc length measure $ds(z) = (2\pi)^{-1}|dz|$. Let $\mathbb{C} \setminus \Gamma = \Omega \cup \Omega_e$ be the decomposition of the complement into disjoint connected components, where Ω is bounded and Ω_e is unbounded. Szegő's expansion involves the conformal mapping $\Omega_e \rightarrow \mathbb{D}_e$ which fixes the point at infinity, where \mathbb{D}_e is the exterior disk: $\mathbb{D}_e := \{z \in \mathbb{C} : |z| > 1\}$. Moreover, the expansion is valid in Ω_e , with uniform control on compact subsets. Slightly later, the Swedish mathematician Torsten Carleman [9] – inspired by the work of Szegő – considered instead the (analytic) orthogonal polynomials in $L^2(\Omega, dA)$, where $dA(z) = \pi^{-1}dx dy$ (where $z = x + iy$) is normalized area measure on the simply connected bounded domain Ω with real-analytic boundary curve Γ . He found an asymptotic formula for the orthogonal polynomials with a much smaller error (exponential decay) term than in Szegő's

case. Again, the asymptotic formula holds in Ω_e . It should be remarked that Carleman's asymptotic formula is actually valid in some fixed neighbourhood of the closure of Ω_e . Carleman's approach was to think of the $L^2(\Omega)$ -norm of holomorphic functions as the Dirichlet norm of the primitive, and then to use Green's formula to switch the integration first to the boundary and second to the exterior domain.

In contrast with the above results, the study of orthogonal polynomials on the real line has a long history. Here, we should mention the classical orthogonal polynomials associated with the names of Hermite, Laguerre, Jacobi, Gegenbauer, Chebyshev, and Legendre. The study of more general orthogonal polynomials in weighted L^2 -spaces on the line is associated with the names of Chebyshev, Markov, Stieltjes, Szegő, Bernstein, and Akhiezer, to just mention a few. The structure of general orthogonal polynomials on the line is rather rigid with the appearance of a three-term recursion relation, which has to do with the fact that multiplication by the independent variable is self-adjoint on the weighted L^2 -space. This fact is important, and is used in many approaches to the asymptotics of orthogonal polynomials. If one considers measures supported on more general curves, however, there is no analogue. In fact, if the orthogonal polynomials with respect to arc length measure on an analytic curve Γ satisfy a three-term recursion formula, then Γ is necessarily an ellipse, including the symmetric and specialized cases of a circle a line, respectively [14]. For planar orthogonal polynomials, even finite term recursion formulas occur only rarely [34].

The above-mentioned results of Szegő and Carleman for the orthogonal polynomials in $L^2(\Gamma, ds)$ and $L^2(\Omega, dA)$ do not need any three-term recursion formula and lead to a different direction of development. Given the similarity with the work presented here, we wish to highlight some results in this direction. We first describe Szegő's result. Let $\{P_0, P_1, \dots\}$ denote the sequence of orthogonal polynomials in $L^2(\Gamma, ds)$, which have

$$\int_{\Gamma} P_n(z) \bar{P}_k(z) ds = \delta_{n,k},$$

while $\deg P_n = n$ and $\lim_{z \rightarrow \infty} z^{-n} P_n(z) > 0$. Here, $\delta_{n,k}$ is the Kronecker delta symbol which equals 1 when $n = k$ and vanishes otherwise. Moreover, here and in the sequel, we use the notational convention that for a given function f , the expression \bar{f} stands for the function whose values are the complex conjugates of those of f . Then, if $\phi: \Omega_e \rightarrow \mathbb{D}_e$ denotes the exterior conformal mapping, Szegő's theorem asserts that

$$(1.1.1) \quad P_n(z) = \sqrt{\phi'(z)} [\phi(z)]^n (1 + o(1)), \quad z \in \Omega_e,$$

where the error is uniform on compact subsets. In contrast, Carleman's work concerns the normalized orthogonal polynomials $\{P_0, P_1, P_2, \dots\}$ in $L^2(\Omega)$, where Ω is a bounded simply connected domain with real-analytic boundary. Again, these are uniquely determined if we require that P_n has precise degree n and positive leading coefficient. Now let ϕ denote the conformal mapping $\phi: \Omega_e \rightarrow \mathbb{D}_e$, which fixes the point at infinity. Due to the real-analytically smooth boundary, ϕ extends to a conformal mapping of larger domain $\Omega_{\rho_0, \varepsilon} \rightarrow \mathbb{D}_e(0, \rho_0)$. Here and in the sequel, we denote by $\mathbb{D}(z_0, r)$ the open disk centered at z_0 with radius r and by $\mathbb{D}_e(z_0, r)$ the complement of the closure of this set (which may be viewed as a disk on the Riemann sphere). In the special case when $z_0 = 0$ and $r = 1$, we drop the indication of these parameters and write \mathbb{D} and \mathbb{D}_e . Then Carleman's formula states that

$$(1.1.2) \quad P_n(z) = (n+1)^{\frac{1}{2}} \phi'(z) [\phi(z)]^n (1 + O(\rho^n)), \quad z \in \Omega_{\rho_0, \varepsilon},$$

for $\rho_0 < \rho < 1$. As a consequence, the orthogonal polynomials may be thought of as push-forwards of the monomials to $A^2(\Omega)$.

One of the gems in the direction initiated by Carleman is the work of Suetin [41], which deals with domains whose boundary has a lower degree of smoothness, and the case when a weight function is present. Among many results, we recall the asymptotic formula concerning the system $\{P_n\}_n$ generated with respect to the inner product of $L^2(\Omega, \omega dA)$, where $\partial\Omega$ is

real-analytically smooth, and where ω is non-negative and Hölder continuous with exponent α . In this setting, the orthogonal polynomials are asymptotically given by

$$(1.1.3) \quad P_n(z) = (n+1)^{\frac{1}{2}} \phi'(z) [\phi(z)]^n g(z) \left(1 + O\left(\frac{\log n}{n}\right)^{\frac{\alpha}{2}} \right), \quad z \in \Omega_e,$$

where the function $g(z)$ is holomorphic in the exterior domain Ω_e and satisfies $|g(z)|^2 = \omega^{-1}$ on $\partial\Omega_e$. Let us also remark that Szegő's result (1.1.1) extends to the setting of weights as well, see [43].

We might also mention the more recent work of Miña-Díaz [33] on an integral representation of the orthogonal polynomials in terms of a kernel which is associated with conformal mapping and the Beurling transform (see, e.g., [5]).

For an exposition of some of Szegő's work on orthogonal polynomials, we refer to the books by Simon [38, 39].

1.2. Point processes on the line and plane from eigenvalues of random matrices. In connection with the study of random Hermitian matrices and one-dimensional Coulomb gas, the orthogonal polynomials with respect to exponentially varying weights are central objects. The eigenvalue process associated to a random Hermitian matrix is governed by a correlation kernel expressed in terms of the polynomial reproducing kernel

$$K_m(x, y) := \sum_{j=0}^{m-1} P_{j,m}(x) \bar{P}_{j,m}(y),$$

where the polynomials $P_{0,m}, P_{1,m}, P_{2,m}, \dots$ are normalized and orthogonal in $L^2(\mathbb{R}, \mu_{2mQ})$, where $d\mu_{2mQ}(x) = e^{-2mQ(x)} dx$, and are such that $P_{j,m}$ has precise degree j and positive leading coefficient $a_j > 0$. Here, Q is thought of as a confining potential, and the parameter m should be allowed to tend to infinity. The property characterizing the kernel $K_m(x, y)$ is the following. Let Pol_m denote the m -dimensional space of polynomials of degree $\leq m-1$. Then we have for each $f \in \text{Pol}_m$ that

$$\int_{\mathbb{R}} K_m(x, y) f(y) d\mu_{2mQ}(y) = f(x), \quad x \in \mathbb{R}.$$

For expositions on random Hermitian matrices and orthogonal polynomials, see, e.g., the books [32] and [12]. As a result of the rigid structure of the orthogonal polynomials on the line, there is a Christoffel-Darboux formula

$$K_m(x, y) = \sum_{j=0}^{m-1} P_{j,m}(x) \bar{P}_{j,m}(y) = \frac{a_{n-1}}{a_n} \frac{P_{m,m}(x) \bar{P}_{m-1,m}(y) - P_{m,m}(y) \bar{P}_{m-1,m}(x)}{x - y},$$

where it is implicit that $x, y \in \mathbb{R}$ with $x \neq y$. For an exposition of the Christoffel-Darboux formula, see [40]. The Christoffel-Darboux formula reduces the problem of analyzing the kernel to analyzing just the two highest degree polynomials, $P_{m-1,m}$ and $P_{m,m}$. By the efforts of e.g. Fokas, Its, Kitaev, and Deift and Zhou, the asymptotic behavior of the orthogonal polynomials is well understood in terms of solutions to matrix Riemann-Hilbert problems, see, e.g., [12, 13, 16, 17].

Recently, the properties of orthogonal polynomials with respect to exponentially varying planar measures of the form $e^{-2mQ} dA$ have been studied. The motivation comes from the theory of two-dimensional Coulomb gas, or Random Normal Matrix (RNM) ensembles. Indeed, the eigenvalues of a random normal matrix form a determinantal point process, and the correlation kernel is given by

$$K_m(z, w) e^{-m(Q(z)+Q(w))} = \sum_{n=0}^{m-1} P_{n,m}(z) \bar{P}_{n,m}(w) e^{-m(Q(z)+Q(w))}.$$

As a consequence of the lack of finite-term recursions, there is no Christoffel-Darboux formula, and instead we need to analyze the entire sequence $\{P_{n,m}\}_{n=0}^{m-1}$.

Macroscopically, the situation is well understood. It is known that the gas condensates to a certain compact set \mathcal{S}_1 , called the *droplet*, see the discussion in Subsection 2.6. An interesting question is how the process behaves at the microscopic level. This can be studied via the rescaled density: for a point $z_0 \in \mathbb{C}$ and $n \in \mathbb{T} = \partial\mathbb{D}$ we let

$$(1.2.1) \quad z_m(\xi) = z_0 + n \frac{\xi}{\sqrt{2m\Delta Q(z_0)}}$$

consider

$$(1.2.2) \quad \rho_m(\xi) = \frac{1}{2m\Delta Q(z_0)} K_m(z_m(\xi), z_m(\xi)) e^{-2mQ(z_m(\xi))}.$$

Near a bulk point, $z_0 \in \mathcal{S}_1^\circ$, there exists a full asymptotic expansion of the kernel, see e.g. [2, 3]. In this case $\lim_m \rho_m(\xi) = 1$, uniformly on compact subsets. Away from the bulk, i.e. for $z_0 \in \mathcal{S}_1^c$ we instead have $\lim_m \rho_m(\xi) = 0$.

Let $z_0 \in \partial\mathcal{S}_1$, such that the boundary is real-analytically smooth near z_0 , and let n be the outer normal to \mathcal{S}_1 at z_0 . In this case, it is not known what the limit of the density ρ_m is. In the case when $Q(z) = \frac{1}{2}|z|^2$, the limit is known, and this limit is expected to be universal for regular boundary points.

Conjecture 1.2.1 (boundary universality). Let $z_0 \in \partial\mathcal{S}_1$ and assume that $\partial\mathcal{S}_1$ is smooth in a neighbourhood of z_0 . Then the density ρ_m converges as $m \rightarrow \infty$ to the limit

$$\rho(\xi) = \operatorname{erf}(2 \operatorname{Re} \xi).$$

Here, we write erf for the *error function*

$$\operatorname{erf}(z) = \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-t^2/2} dt,$$

where \int is along a suitable contour from z to the origin and then from the origin to ∞ along the positive real line. This conjecture has been verified in some specific cases, and partial results have appeared recently. In connection with this we want to mention the work by Ameur, Kang and Makarov [4] who used a limiting form of Ward identities to show that if $\rho(\xi)$ is assumed to be translation invariant, then it must necessarily be as in Conjecture 1.2.1. Without simplifying assumptions, however, the full conjecture remains open. In the setting of Kähler manifolds, a similar problem appears in the context of partial Bergman kernels defined by vanishing to high order along a divisor. Under the assumption of S^1 -invariance around the divisor, Ross and Singer [36] obtain the error function asymptotics near the emergent interface around the divisor (see also the work of Zelditch and Zhou [46]). In recent work, Zelditch and Zhou [47] find that this is a universal edge phenomenon along interfaces, which appears in the context of partial Bergman kernels defined in terms of a quantized Hamiltonian.

The standard approaches to the asymptotics of Bergman kernels are local in nature, both the peak section approach of Tian (see [44]) as well as the microlocal approach by Boutet de Monvel and Sjöstrand, as explained by Berman, Berndtsson, and Sjöstrand [6]. The same applies to older work of Hörmander, Diederich, and Fefferman [15]. One reason to expect the boundary universality conjecture to be difficult is the non-locality of the correlation kernel. To illustrate this, we consider the Berezin density (associated with secondary quantization)

$$B_m^{(z_0)}(z) = K_m(z_0, z_0)^{-1} |K_m(z, z_0)|^2 e^{-2mQ(z)}$$

studied in [1] and see that for boundary points $z_0 \in \partial\mathcal{S}_1$, this density develops a ridge along the whole boundary of the spectral droplet (see Figure 1.1). It is for this reason that we focus our analysis on the orthogonal polynomials, which share the nonlocal behavior (see Figure 3.1). The Berezin density and the Palm density add up to the density of states, so that the Berezin density may be understood as the repulsive influence of a particle placed at the given point z_0 . Palm measures have been studied recently by Bufetov, Fan, and Qiu [8].

In the above context, we considered orthogonal polynomials of degree $< m$ with respect to the weight e^{-2mQ} . To be more general, we may separate the connection between the degree

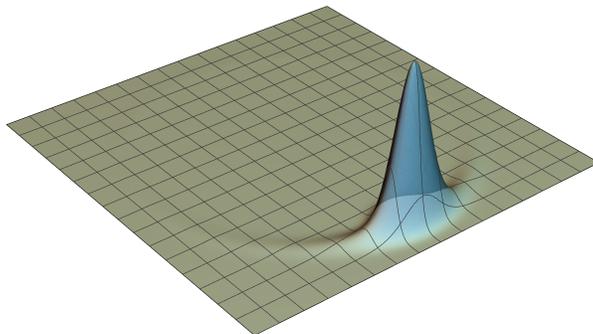


FIGURE 1.1. The Berezin density $K_m(z_0, z_0)^{-1}|K_m(z, z_0)|^2 e^{-2mQ(z)}$ with $Q(z) = \frac{1}{2}|z|^2$ for the boundary point $z_0 = 1$ and $m = 120$.

of the polynomials and the quantization parameter m . When $n = m$, the probability density $|P_{n,m}|^2 e^{-2mQ}$ turns out to condensate along the boundary $\partial\mathcal{S}_1$. More generally, when $\tau = \frac{n}{m}$ is kept fixed, the condensation occurs along the boundary $\partial\mathcal{S}_\tau$ of another spectral droplet \mathcal{S}_τ . For the special choice $Q(z) = \frac{1}{2}|z|^2 + a \operatorname{Re}(z^2)$ where $a > 0$, Lee and Riser [31] find explicitly the orthogonal polynomials, and verified Conjecture 1.2.1 in this case. Along the same lines, in [7], Balogh, Bertola, Lee and McLaughlin study the case of the potentials Q which are perturbations of the standard quadratic potential of the form

$$Q(z) = \frac{1}{2}|z|^2 - c \log|z - a|^2,$$

for some $a \in \mathbb{R}$, $c > 0$. For this choice of Q , they obtain an asymptotic expansion of the orthogonal polynomials. For choices of parameters such that the droplet \mathcal{S}_τ does not divide the plane, this is expressed in terms of the properly normalized conformal mapping of the complement \mathcal{S}_τ^c onto the exterior disk \mathbb{D}_e , denoted ϕ_τ . After some rewriting, their formula reads

$$(1.2.3) \quad P_{n,m}(z) = \left(\frac{m}{2\pi}\right)^{1/4} \sqrt{\phi'(z)} [\phi(z)]^n e^{2mQ(z)} (1 + O(m^{-1})), \quad z \in \mathcal{B}^c,$$

where $\frac{n}{m} = \tau + O(m^{-1})$, \mathcal{B} is some compact subset of the interior of \mathcal{S}_τ , and Q the bounded holomorphic function on \mathcal{B}^c with real part on $\partial\mathcal{S}_\tau$ equal to Q . Using (1.2.3), they verify Conjecture 1.2.1 for the given collection of potentials.

At the physical level, it is understood that the asymptotic formula (1.2.3) should hold for more general potentials, namely all those of the form $Q(z) = \frac{1}{2}|z|^2 + H(z)$, where H is harmonic in a neighbourhood of the droplet (so called Hele-Shaw potentials) [45]. What the higher order correction terms should look like appears not to be understood even in this situation.

In another vein, for rather general potentials Q , the mean field approximation of the random normal matrix model [2, 3] supplies information regarding the individual orthogonal polynomials. Indeed, the convergence

$$|P_{n,m}|^2 e^{-2mQ} \rightarrow \varpi(\cdot, \widehat{\mathbb{C}} \setminus \mathcal{S}_1, \infty),$$

holds as $n, m \rightarrow \infty$ with $n = m + O(1)$, in the sense of weak convergence of measures. Here, the left-hand side is interpreted as a probability measure, and the right-hand side denotes harmonic measure of the domain $\widehat{\mathbb{C}} \setminus \mathcal{S}_1$ evaluated at the point at infinity. We observe that harmonic measure is concentrated to the boundary, so that the above convergence can be interpreted as *boundary concentration*. Within the random normal matrix model, the addition of a new

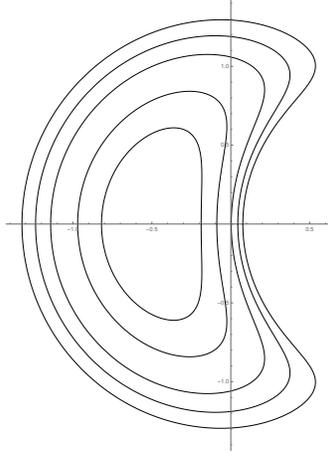


FIGURE 1.2. Laplacian growth of the compacts \mathcal{S}_τ for the potential $Q(z) = \frac{1}{2}|z|^2 - 2^{-\frac{1}{2}} \log|z-1|$ (boundary curves indicated).

particle has the net effect of adding a term $|P_{n,m}|^2 e^{-2mQ}$ of highest degree. This means that the net effect of adding a particle is felt along the boundary.

1.3. Our contribution to the asymptotics of planar orthogonal polynomials. Here, we shall study the (analytic) orthogonal polynomials in the complex plane \mathbb{C} with respect to a rather general weight e^{-2mQ} . Specifically, we will treat potentials Q , that are admissible in the sense of the definition below. For $\tau \in (0, 1]$, let

$$\mathcal{S}_\tau^* = \mathcal{S}_{Q,\tau}^* := \{z \in \mathbb{C} : \hat{Q}_\tau(z) = Q(z)\}$$

denote the coincidence set for the solution \hat{Q}_τ to the obstacle problem

$$\hat{Q}_\tau(z) = \sup \{q(z) : q \in \text{Subh}_\tau(\mathbb{C}), q \leq Q \text{ on } \mathbb{C}\},$$

where $\text{Subh}_\tau(\mathbb{C})$ is the set of subharmonic functions in the plane which grow at most like $\tau \log|z|$ at infinity. The *droplet* \mathcal{S}_τ is the support set for the measure $1_{\mathcal{S}_\tau^*} \Delta Q dA$ as a distribution, which is a subset of \mathcal{S}_τ^* , and may differ from \mathcal{S}_τ^* by a null set for the measure $|\Delta Q| dA$. The droplet \mathcal{S}_τ grows with τ according to a weighted Darcy law, and the evolution is referred to as *weighted Laplacian growth*. See Figure 1.2 for a special case.

Definition 1.3.1. The potential $Q : \mathbb{C} \rightarrow \mathbb{R}$ is said to be τ -*admissible* if the following conditions are satisfied:

- (i) Q is C^2 -smooth in the entire complex plane,
- (ii) Q is real-analytic and strictly subharmonic in a neighbourhood of \mathcal{S}_τ^* ,
- (iii) Q grows sufficiently fast at infinity:

$$(1.3.1) \quad \liminf_{|z| \rightarrow +\infty} \frac{Q(z)}{\log|z|} > \tau.$$

- (iv) The boundary $\partial \mathcal{S}_\tau^*$ is a smooth, simple and closed curve.

Note that under these conditions, $\mathcal{S}_\tau^* = \mathcal{S}_\tau$. In the sequel, we will focus on $\tau = 1$, and *assume that Q is 1-admissible*. We recall that the norm in the Hilbert space $L^2(\mathbb{C}, e^{-2mQ})$ is given by

$$(1.3.2) \quad \|f\|_{2mQ} := \left(\int_{\mathbb{C}} |f|^2 e^{-2mQ} dA \right)^{\frac{1}{2}}.$$

By applying the Gram-Schmidt procedure to the monomial sequence $1, z, z^2, \dots$ in this space, we obtain the sequence of normalized orthogonal polynomials

$$P_{0,m}, P_{1,m}, P_{2,m}, \dots,$$

where $P_{n,m}$ has degree n . Depending on the growth of potential Q , this may or may not be a finite sequence. The condition (1.3.1) with $\tau = 1$ guarantees that all polynomials of degree strictly less than $(1+\epsilon)m-1$ belong to the space $L^2(\mathbb{C}, e^{-2mQ})$, for some fixed small $\epsilon > 0$. As Q is assumed 1-admissible, the curve $\partial\mathcal{S}_1$ is smooth, simple and closed. This assumption actually has as a consequence that the same holds for the boundaries $\partial\mathcal{S}_\tau$ for $\tau \in I_{\epsilon_0} := [1 - \epsilon_0, 1 + \epsilon_0]$ for some $\epsilon_0 > 0$ (this fact is non-trivial, and has to do with local properties of the weighted Laplacian growth flow (a variant of Hele-Shaw flow), and the fact that the coincidence is set as regular as assumed, c.f. [25, 22]). By considering a smaller ϵ_0 , we can make sure this property holds on the larger interval $I_{2\epsilon_0}$ as well. Moreover, the assumptions of admissibility entail that the smooth curves $\partial\mathcal{S}_\tau$ are actually real-analytically smooth for $\tau \in I_{\epsilon_0}$. This follows from the work of Sakai [37] on boundaries with a one-sided Schwarz function, as observed in [25]. Under these assumptions, we find that the orthogonal polynomials have a full asymptotic expansion which is valid in a neighbourhood of \mathcal{S}_τ^c , with uniformly bounded error terms for $\tau \in I_{\epsilon_0}$. To set things up, we denote by ϕ_τ the conformal mapping

$$\phi_\tau : \widehat{\mathbb{C}} \setminus \mathcal{S}_\tau^c \rightarrow \mathbb{D}_e.$$

As a consequence of this, ϕ_τ extends to a conformal mapping $\mathcal{K}_{0,\tau} \rightarrow \mathbb{D}_e(0, \rho_0)$, for some compact subset $\mathcal{K}_{0,\tau}$ of \mathcal{S}_τ , and some ρ_0 with $0 < \rho_0 < 1$. We let \mathcal{Q}_τ be the bounded holomorphic function on \mathcal{S}_τ^c whose real part equals Q on $\partial\mathcal{S}_\tau$ and whose imaginary part vanishes at infinity. We need the following notion.

Definition 1.3.2. If \mathcal{K} and \mathcal{S} are compact sets in the plane with $\mathcal{K} \subset \mathcal{S}$ and

$$\text{dist}_{\mathbb{C}}(\mathcal{K}, \mathcal{S}^c) = \varepsilon,$$

we say that a compact set \mathcal{X} is *intermediate* between \mathcal{K} and \mathcal{S} if $\mathcal{K} \subset \mathcal{X} \subset \mathcal{S}$ with

$$\text{dist}_{\mathbb{C}}(\mathcal{K}, \mathcal{X}^c) \geq \frac{\varepsilon}{10} \quad \text{and} \quad \text{dist}_{\mathbb{C}}(\mathcal{X}, \mathcal{S}^c) \geq \frac{\varepsilon}{10}.$$

We recall the notation $I_{\epsilon_0} = [1 - \epsilon_0, 1 + \epsilon_0]$, where ϵ_0 is fixed and positive, with the property that the curves $\partial\mathcal{S}_\tau$ form a smooth flow of simple loops for $\tau \in I_{\epsilon_0}$.

Theorem 1.3.3. *Assume that Q is 1-admissible, and fix the precision κ which is a positive integer. Then, for each $\tau \in I_{\epsilon_0}$ there exists a compact subset $\mathcal{K}_\tau \subset \mathcal{S}_\tau$ with $\text{dist}_{\mathbb{C}}(\mathcal{K}_\tau, \partial\mathcal{S}_\tau) \geq \varepsilon$ for some positive real number ε , such that the following holds. On the complement \mathcal{K}_τ^c , there are bounded holomorphic functions $\mathcal{B}_{j,\tau}$ such that the associated function*

$$F_{n,m}^{(\kappa)} = m^{\frac{1}{4}} \sqrt{\phi'_\tau} [\phi_\tau]^n e^{m\mathcal{Q}_\tau} \sum_{j=0}^{\kappa} m^{-j} \mathcal{B}_{j,\tau},$$

approximates well the normalized orthogonal polynomials $P_{n,m}$ in the sense that we have the norm control

$$\|P_{n,m} - \chi_{0,\tau} F_{n,m}^{(\kappa)}\|_{2mQ} = O(m^{-\kappa-1})$$

as $n, m \rightarrow \infty$ while $\tau = \frac{n}{m} \in I_{\epsilon_0}$. Here, $\chi_{0,\tau}$ denotes a smooth cut-off function with $0 \leq \chi_{0,\tau} \leq 1$ and uniformly bounded gradient. In addition the function $\chi_{0,\tau}$ vanishes on \mathcal{K}_τ , and equals 1 on the set \mathcal{X}_τ^c where \mathcal{X}_τ is an intermediate set between \mathcal{K}_τ and \mathcal{S}_τ . In the above estimate, the implicit constant is uniform for $\tau \in I_{\epsilon_0}$.

In the above theorem, the products $\chi_{0,\tau} F_{n,m}^{(\kappa)}$ are thought to vanish on the set \mathcal{K}_τ , where $F_{n,m}$ may be undefined.

Remark 1.3.4. (a) We mention that the compact sets \mathcal{K}_τ as well as \mathcal{X}_τ may be obtained, e.g., as the complements of the conformal images under ϕ_τ^{-1} of the exterior disks $\mathbb{D}_e(0, \rho)$ with $\rho = \rho_0$ and $\rho = \rho'_0$, respectively, where $0 < \rho_0 < \rho'_0 < 1$. By inserting a further family of intermediate sets, we can make sure that the cut-off function $\chi_{0,\tau}$ vanishes not only on \mathcal{K}_τ , but on an intermediate set between \mathcal{K}_τ and \mathcal{X}_τ .

(b) Without loss of generality, we may assume that the cut-off function $\chi_{0,\tau}$ is uniformly smooth in the sense that for any given positive integer k the $C^k(\mathbb{C})$ -norm of $\chi_{0,\tau}$ is uniformly bounded for $\tau \in I_{\epsilon_0}$.

(c) Our approach to the proof involves Toeplitz kernel problems and the construction of an approximate orthogonal foliation flow of loops. The underlying idea is inspired by an approach to the local expansion of Bergman kernels, which involves a flow of loops emanating from the point of expansion [19].

Turning to pointwise control, it can be shown that Theorem 1.3.3 has the following consequence.

Theorem 1.3.5. *Assume that Q is 1-admissible, and let κ be an arbitrary positive integer. Then there exist bounded holomorphic functions $\mathcal{B}_{j,\tau}$ defined in a fixed neighbourhood of \mathcal{S}_τ , and compact subsets $\mathcal{K}_{m,\tau} \subset \mathcal{S}_\tau$ with $\text{dist}_{\mathbb{C}}(\partial\mathcal{K}_{m,\tau}, \partial\mathcal{S}_\tau) \geq (m^{-1} \log \log m)^{\frac{1}{2}}$, such that the asymptotic formula*

$$P_{n,m}(z) = m^{\frac{1}{4}} [\phi'_\tau(z)]^{\frac{1}{2}} [\phi_\tau(z)]^n e^{m\mathcal{Q}_\tau(z)} \left(\sum_{j=0}^{\kappa} m^{-j} \mathcal{B}_{j,\tau}(z) + O(m^{-\kappa-1}) \right),$$

holds, where the error term is uniform for $z \in \mathcal{K}_{m,\tau}^c$, as $m, n \rightarrow \infty$ and the ratio $\tau = \frac{n}{m} \in I_{\epsilon_0}$.

In other words, the orthogonal polynomials $P_{n,m}$ have an asymptotic expansion

$$P_{n,m}(z) \sim m^{\frac{1}{4}} [\phi'_\tau(z)]^{\frac{1}{2}} [\phi_\tau(z)]^n e^{m\mathcal{Q}_\tau(z)} \left(\mathcal{B}_{0,\tau}(z) + \frac{1}{m} \mathcal{B}_{1,\tau}(z) + \dots \right), \quad z \in \mathcal{K}_{m,\tau}^c$$

as $n = \tau m \rightarrow \infty$ and $\tau \in I_{\epsilon_0}$.

Remark 1.3.6. (a) It is possible to obtain the asymptotics of Theorem 1.3.5 in the complement of the smaller set $\mathcal{K}'_{m,\tau} \subset \mathcal{K}_{m,\tau} \subset \mathcal{S}_\tau$ with the property $\text{dist}_{\mathbb{C}}(\partial\mathcal{K}'_{m,\tau}, \partial\mathcal{S}_\tau) \geq A(m^{-1} \log m)^{\frac{1}{2}}$, for any given positive constant A . The difference in the proof amounts to working with $m^{A^2 D}$ instead of with $(\log m)^D$. As the positive constants A and D are fixed, this can be managed by the standard trick of expanding further, replacing the precision κ by κ' , where $\kappa' \geq \kappa + A^2 D$.

(b) Theorem 1.3.5 is a consequence of applying a certain maximum principle (Proposition 2.2.2) to the asymptotic expansion of Theorem 1.3.3. Although most of the weighted L^2 -mass of $P_{n,m}$ is concentrated near the boundary $\partial\mathcal{S}_\tau$, we obtain good pointwise asymptotics in the whole exterior of \mathcal{S}_τ , in particular at infinity. As a consequence, the κ -abschnitt $\mathfrak{f}_{n,m}^{(\kappa)}$ given by $\mathfrak{f}_{n,m}^{(\kappa)} = \sum_{j=0}^{\kappa} m^{-j} \mathcal{B}_{j,\tau}$ must satisfy

$$\text{Im } \mathfrak{f}_{n,m}^{(\kappa)}(\infty) = O(m^{-\kappa-1}).$$

This normalization is important to determine the coefficient functions in the expansion. The notion of “ κ -abschnitt” is from German and was mentioned in [21].

In order for Theorems 1.3.3 and 1.3.5 to be useful, we need to find a way to calculate the coefficient functions $\mathcal{B}_{j,\tau}$. This is explained in the following theorem. For the formulation, we need the Szegő projection $\mathbf{P}_{H_{-,0}^2}$ of $L^2(\mathbb{T})$ onto the conjugate Hardy space $H_{-,0}^2 = L^2(\mathbb{T}) \ominus H^2$ (see Section 2.5 for details). In addition, we need the *modified weight* R_τ defined in a neighbourhood of $\bar{\mathbb{D}}_e$ by

$$(1.3.3) \quad R_\tau = (Q - \check{Q}_\tau) \circ \phi_\tau^{-1},$$

where the function \check{Q}_τ is the harmonic extension of the restriction to $\hat{Q}_\tau|_{\mathcal{S}_\tau^c}$ across the boundary $\partial\mathcal{S}_\tau$. That such a harmonic extension exists is a consequence of the real-analyticity of $\partial\mathcal{S}_\tau$ which in its turn follows the 1-admissibility of Q , at least for $\tau \in I_{\epsilon_0}$.

Theorem 1.3.7. *In the asymptotic expansion of Theorem 1.3.3, we have $\mathcal{B}_{0,\tau} = \pi^{-\frac{1}{4}} e^{H_{Q,\tau}}$, where $H_{Q,\tau}$ is bounded and holomorphic in a neighbourhood of \mathcal{S}_τ^c and satisfies $\text{Im } H_{Q,\tau}(\infty) = 0$ and*

$$\text{Re } H_{Q,\tau} = \frac{1}{4} \log \Delta Q, \quad \text{on } \partial\mathcal{S}_\tau.$$

Moreover, if H_{R_τ} denotes the bounded holomorphic function on \mathbb{D}_e with

$$\operatorname{Re} H_{R_\tau} = \frac{1}{4} \log(4\Delta R_\tau) \quad \text{on } \mathbb{T},$$

and $\operatorname{Im} H_{R_\tau}(\infty) = 0$, then for $j = 1, 2, 3, \dots$, the coefficients $\mathcal{B}_{j,\tau}$ have the form

$$\mathcal{B}_{j,\tau} = [\phi'_\tau]^{\frac{1}{2}} B_{j,\tau} \circ \phi_\tau,$$

where the functions $B_{j,\tau}$ are bounded and holomorphic in a neighbourhood of $\bar{\mathbb{D}}_e$, and their restrictions to \mathbb{D}_e are given by

$$B_{j,\tau} = c_{j,\tau} e^{H_{R_\tau}} - e^{H_{R_\tau}} \mathbf{P}_{H_{0,-}^2} [e^{\bar{H}_{R_\tau}} F_{j,\tau}]$$

for some real-analytic functions $F_{j,\tau}$ on the circle \mathbb{T} and constants $c_{j,\tau} \in \mathbb{R}$. The functions $F_{j,\tau}$ as well as the constants $c_{j,\tau}$ may be computed algorithmically in terms of the potential R_τ and the functions $B_{0,\tau}, \dots, B_{j-1,\tau}$, where $B_{0,\tau} = (4\pi)^{-\frac{1}{4}} e^{H_{R_\tau}}$.

We point out that Theorem 1.3.3 and Theorem 1.3.7 together with Remark 1.3.6(a) imply that for large enough m , and for $\tau = \frac{n}{m} \in I_{e_0}$, all the zeros of the polynomial $P_{n,m}(z)$ lie in the compact subset $\mathcal{K}'_{m,\tau}$ of the interior \mathcal{S}_τ° of \mathcal{S}_τ defined in Remark 1.3.6(a).

In view of the way we defined the functions $H_{Q,\tau}$ and $H_{R,\tau}$, it is immediate that

$$H_{R_\tau} \circ \phi_\tau = \frac{1}{2} \log(2\phi'_\tau) + H_{Q,\tau}.$$

While Theorem 1.3.7 gives the asymptotic structure of the orthogonal polynomials, it remains to specify how to obtain the real-analytic functions $F_{j,\tau}$ and the constants $c_{j,\tau}$, for $j = 0, 1, 2, \dots$. For $k = 0, 1, 2, \dots$, let \mathbf{L}_k be given by

$$(1.3.4) \quad \mathbf{L}_k[f] = \sum_{\nu=k}^{3k} \frac{(-1)^{\nu-k} 2^{-\nu}}{\nu!(\nu-k)! [\partial_r^2 R_\tau(re^{i\theta})]^\nu} \partial_r^{2\nu} \left(\left[R_\tau - \frac{1}{2}(r-1)^2 \partial_r^2 R_\tau(e^{i\theta}) \right]^{\nu-k} f \right).$$

This is a differential operator of order $6k$, acting on a smooth function f defined in a neighbourhood of the unit circle. Note that the restriction $\mathbf{L}_k[f](re^{i\theta})|_{r=1}$ only involves derivatives of order at most $2k$. This operator results from the asymptotical analysis of integrals using Laplace's method, see Proposition 2.7.1 below. Later on, in Lemma 3.2.1, we show the existence of certain differential operators \mathbf{M}_k with the property that

$$\int_{\mathbb{T}} e^{i\theta} (\partial_r^2 R_\tau(re^{i\theta}))^{-\frac{1}{2}} \mathbf{L}_k[r^{1-l} f(re^{i\theta})] \Big|_{r=1} d\theta = \int_{\mathbb{T}} e^{i\theta} \mathbf{M}_k[f(re^{i\theta})] \Big|_{r=1} d\theta,$$

for $l = 1, 2, 3, \dots$. Operators similar to \mathbf{M}_k arise in the theory of pseudodifferential operators. In this context, we use them to rid the left-hand side of any unwanted dependence on the radial contribution r^{1-l} . In terms of the operators \mathbf{L}_k and \mathbf{M}_k , we may now express $F_{j,\tau}$ and $c_{j,\tau}$ as follows:

$$(1.3.5) \quad F_{j,\tau}(\theta) = \sum_{k=1}^j \mathbf{M}_k[B_{j-k,\tau}](e^{i\theta}), \quad j \geq 1,$$

and the real constants $c_{j,\tau}$ are given by $c_{0,\tau} = (4\pi)^{-1/4}$ while

$$(1.3.6) \quad c_{j,\tau} = -\frac{1}{2} (4\pi)^{\frac{1}{4}} \sum_{(i,k,l) \in \mathcal{J}_j} \int_{\mathbb{T}} (4\Delta R_\tau(e^{i\theta}))^{-\frac{1}{2}} \mathbf{L}_k[r B_{i,\tau}(re^{i\theta}) \bar{B}_{l,\tau}(re^{i\theta})] \Big|_{r=1} ds(e^{i\theta})$$

for $j = 1, 2, 3, \dots$, where $\mathcal{J}_j = \{(i, k, l) \in \mathbb{Z}^3 : i, l < j, k \geq 0, i + k + l = j\}$.

1.4. Our contribution to boundary universality for random normal matrices. As a direct consequence of Theorem 1.3.3 and Theorem 1.3.7, we manage to resolve the boundary universality conjecture (Conjecture 1.2.1) for admissible potentials.

Theorem 1.4.1. *If Q is 1-admissible and $z_0 \in \partial\mathcal{S}_1$ is a boundary point, then if ρ_m is the blow-up density given by (1.2.1) and (1.2.2), we have the convergence*

$$\lim_{m \rightarrow \infty} \rho_m(\zeta) = \operatorname{erf}(2\zeta),$$

locally uniformly on \mathbb{C} .

By polarization, it follows that a corresponding result holds for the rescaled correlation kernels

$$k_m(\xi, \eta) = \frac{1}{2m\Delta Q(z_0)} K_m(z_m(\xi), z_m(\eta)) e^{-mQ(z_m(\xi)) - mQ(z_m(\eta))}.$$

Determinantal point processes are determined by a correlation kernel, which is unique up to cocycles. Here, cocycles are understood as the multiplication by a function $c(\xi)\bar{c}(\eta)$, for some continuous unimodular function $c : \mathbb{C} \rightarrow \mathbb{T}$. Modulo such cocycles, it is known that the correlation kernels converge uniformly on compact subsets of \mathbb{C}^2 , to a function of the form

$$k(\xi, \eta) = e^{\xi\bar{\eta} - \frac{1}{2}(|\xi|^2 + |\eta|^2)} F(\xi, \eta),$$

where $F(\xi, \eta)$ is a positive definite kernel, which depends holomorphically on ξ and conjugate-holomorphically on η . Since it follows from our theorem that $F(\xi, \xi) = \operatorname{erf}(2\operatorname{Re}\xi)$, it follows that its polarization is $F(\xi, \eta) = \operatorname{erf}(\xi + \bar{\eta})$. As a consequence, we obtain the convergence

$$\lim_{m \rightarrow \infty} c_m(\xi)\bar{c}_m(\eta) k_m(\xi, \eta) = e^{\xi\bar{\eta} - \frac{1}{2}(|\xi|^2 + |\eta|^2)} \operatorname{erf}(\xi + \bar{\eta}),$$

where $c_m : \mathbb{C} \rightarrow \mathbb{T}$ is a sequence of continuous unimodular functions. We formalize this in the following corollary.

Corollary 1.4.2. *Denote by $\Phi_m = \{z_{j,m}\}_{j=1}^m$ the eigenvalues of a random normal $m \times m$ matrix associated to the weight $\exp(-2mQ)$, and define a rescaled process $\Psi_{m,z_0} = \{\zeta_{j,m}\}_{j=1}^m$ by*

$$\zeta_{j,m} = \bar{n}\sqrt{2m\Delta Q(z_0)}(z_{j,m} - z_0), \quad j = 1, \dots, m,$$

where $z_0 \in \partial\mathcal{S}_1$ and \bar{n} is the unit outer normal to \mathcal{S}_1 at z_0 , interpreted as a complex number. Then Ψ_{m,z_0} converges to a determinantal point field with correlation kernel

$$k(\xi, \eta) = e^{\xi\bar{\eta} - \frac{1}{2}(|\xi|^2 + |\eta|^2)} \operatorname{erf}(\xi + \bar{\eta}),$$

in the sense of locally uniform convergence of correlation kernels on \mathbb{C}^2 .

The necessary details are supplied in Sections 5.3, 2.6 (see also the paper [4]).

To complement the present exposition on planar orthogonal polynomials, we explain in [27] how the method of canonical positioning and orthogonal foliation flow also applies to give a full asymptotic expansion of the Bergman kernel for exponentially varying weights when one of the variables is away from the spectrum. From the physical point of view, this may be interpreted as an instance of bosonization.

In a separate work [28], we intend to explore further the implications of Theorem 1.3.3 and 1.3.7 for the theory of random normal matrices. In particular, we will study what we call *contractible potentials* Q , which are strictly subharmonic, \mathbb{C}^ω -smooth and have the additional property that $\{\partial\mathcal{S}_\tau\}_{0 < \tau \leq 1}$ is a flow of simple, closed C^ω -smooth curves which shrink down to a point. Such potentials will necessarily have a unique minimum point. For these contractible potentials we analyze the asymptotics of the free energy $\log \mathcal{Z}_{m,Q}$, where $\mathcal{Z}_{m,Q}$ denotes the partition function of the ensemble, and relate this analysis to the planar analogue of the classical Szegő limit theorem.

It has been suggested that, in analogy with the one-dimensional case, the key to the asymptotics of orthogonal polynomials ought to be Riemann-Hilbert problem techniques. For instance,

in the work of Its and Takhtajan [30] a natural *Thick Riemann-Hilbert problem*, or matrix $\bar{\partial}$ -problem, is set up. Moreover, it is shown that the 2×2 matrix built with the orthogonal polynomials with respect to the measure $e^{-2mQ}dA$ is the unique solution to this problem. However, it is not clear how to implement this approach as it is not obvious how to constructively solve these thick Riemann-Hilbert problems. Our orthogonal foliation flow approach allows reduction to one-dimensional problems, which could be understood as ordinary Riemann-Hilbert problems along the curve family in the flow. In this presentation, however, we find a more direct approach to derive the asymptotics. We supply a brief discussion of this matter in Section 6.

1.5. Comments on the exposition. In Section 2, we supply some preliminary material which will be needed later on. In particular, we discuss some aspects of weighted logarithmic potential theory and obstacle problems, and introduce weighted Laplacian growth. In addition, we collect some results on Hörmander type L^2 -estimates for the $\bar{\partial}$ -operator, and asymptotic analysis of integrals based on Laplace's method.

In Section 3, we show how the algorithm of Theorem 1.3.7 appears, assuming that Theorem 1.3.3 is valid. The proof is based on Laplace's method for asymptotic integrals, which lets us collapse planar integrals to integrals over curves. The collapsed orthogonality conditions reduces to inhomogeneous Toeplitz kernel conditions, which gives rise to the algorithm.

In Section 4, we prove the existence of the approximate orthogonal foliation flow, which then allows us to obtain Theorem 1.3.3 using Hörmander type $\bar{\partial}$ -methods. A visualization of the orthogonal foliation flow is supplied in Figure 2.1.

In Section 5, we supply proofs of Theorem 1.4.1 and Corollary 1.4.2 on boundary universality in the random normal matrix model for 1-admissible potentials.

Finally, in Section 6, we connect our orthogonal foliation flow with the Its and Takhtajan approach involving 2×2 matrix $\bar{\partial}$ -problems.

2. PRELIMINARIES

2.1. An obstacle problem and logarithmic potential theory. In this section, we follow the presentation of [22]. For a positive real parameter τ , let $\text{Subh}_\tau(\mathbb{C})$ denote the convex set of all functions $q : \mathbb{C} \rightarrow \mathbb{R} \cup \{-\infty\}$ which are subharmonic in \mathbb{C} and have the growth bound

$$q(z) \leq \tau \log |z| + O(1)$$

as $|z| \rightarrow \infty$. For lower semicontinuous potentials Q , subject to the growth condition (1.3.1), we let \hat{Q}_τ be the solution to the obstacle problem

$$(2.1.1) \quad \hat{Q}_\tau(z) := \sup \{q(z) : q \in \text{Subh}_\tau(\mathbb{C}) \text{ and } q \leq Q \text{ on } \mathbb{C}\},$$

and observe that trivially $\hat{Q}_\tau \leq Q$, and if we regularize \hat{Q}_τ on a set of logarithmic capacity 0 (and keep the same notation for the regularized function) then $\hat{Q}_\tau \in \text{Subh}_\tau(\mathbb{C})$ holds. Suppose now that Q is C^2 -smooth. Standard regularity results then give that \hat{Q}_τ is $C^{1,1}$ -smooth, so that the partial derivatives of order 2 of \hat{Q}_τ are locally bounded (in the sense of distribution theory). As a consequence of the growth condition (1.3.1) on Q , the *coincidence set* defined by

$$\mathcal{S}_\tau^* := \{z \in \mathbb{C} : \hat{Q}_\tau(z) = Q(z)\}.$$

is compact, and moreover, it follows from the smoothness that $\Delta \hat{Q}_\tau = 1_{\mathcal{S}_\tau^*} \Delta Q$ holds in the sense of distribution theory. The above obstacle problem has an important connection to potential theory. The weighted logarithmic energy, with respect to a continuous weight function $V : \mathbb{C} \rightarrow \mathbb{R}$, of a compactly supported finite real-valued Borel measure μ is defined as

$$I_V[\mu] = \int_{\mathbb{C} \times \mathbb{C}} \log \frac{1}{|z-w|} d\mu(z)d\mu(w) + 2 \int_{\mathbb{C}} V(z)d\mu(w).$$

With $V = \tau^{-1}Q$, we set out to minimize the energy $I_{\tau^{-1}Q}[\mu]$ over the all compactly supported Borel probability measures μ . There exists a unique minimizer, called the *equilibrium measure*, which we denote by μ_τ . The connection with the obstacle problem is via the following

relationship:

$$(2.1.2) \quad \frac{\tau}{2} d\mu_\tau(z) = \Delta \hat{Q}_\tau dA = 1_{\mathcal{S}_\tau^*} \Delta Q(z) dA.$$

Traditionally we call the support (as a distribution) of the equilibrium measure μ_τ the *droplet*, and denote it by \mathcal{S}_τ . In general this is a subset of the coincidence (or contact) set \mathcal{S}_τ^* . However, difference set $\mathcal{S}_\tau^* \setminus \mathcal{S}_\tau$ is small, in the sense that it is a null set with respect to the weighted area measure $|\Delta Q| dA$. In this presentation, we will assume throughout that the potential Q is 1-admissible. Under this assumption, we have the equality $\mathcal{S}_\tau = \mathcal{S}_\tau^*$ for $\tau \in I_{\epsilon_0} := [1 - \epsilon_0, 1 + \epsilon_0]$ for some small but positive ϵ_0 , and from this point onward we will use the notation \mathcal{S}_τ and think of it as the support of the equilibrium measure and the coincidence set at the same time.

2.2. Bounds on polynomials. The significance of the set \mathcal{S}_τ in relation to orthogonal polynomials is highlighted by Proposition 2.2.2 below. We begin with a useful lemma taken from [1], see Lemma 3.2.

Lemma 2.2.1. *Let u be holomorphic in a disk $\mathbb{D}(z, m^{-1/2}\delta)$. Then*

$$|u(z)|^2 e^{-2mQ(z)} \leq \frac{m e^{A\delta^2}}{\delta^2} \int_{\mathbb{D}(z, m^{-1/2})} |u|^2 e^{-2mQ} dA,$$

where A denotes the essential supremum of ΔQ on $\mathbb{D}(z, m^{-1/2}\delta)$.

This lemma is used in [1] to obtain growth bounds on the entire plane \mathbb{C} for polynomials of degree at most n . In fact, that the function is polynomial is non-essential, as long as the polynomial growth control at infinity is retained.

Proposition 2.2.2. *Let $\tau = n/m$, and suppose \mathcal{K} is a compact subset of the interior of \mathcal{S}_τ . Then there exists a constant C such that for any $u \in A_{2mQ}^2(\mathcal{K}^c)$ with $|u(z)| = O(|z|^n)$ as $|z| \rightarrow \infty$, we have that*

$$|u(z)| \leq C m^{\frac{1}{2}} \|u\|_{L^2(\mathbb{C} \setminus \mathcal{K}, e^{-2mQ})} e^{m\hat{Q}_\tau(z)}, \quad \text{dist}_{\mathbb{C}}(z, \mathcal{K}) \geq \delta m^{-1/2}.$$

Sketch of proof. Assume that $z \in \mathcal{S}_\tau \setminus \mathcal{K}$ lies at a distance of at least $m^{-1/2}\delta$ from \mathcal{K}_τ . By Proposition 2.2.1, we have the estimate

$$|u(z)|^2 \leq \frac{e^{A\delta^2}}{\delta^2} m e^{2mQ(z)} \|u\|_{L^2(\mathbb{C} \setminus \mathcal{K}, e^{-2mQ})}^2,$$

which yields the claim for $z \in \mathcal{S}_\tau \setminus \mathcal{K}$ with the constant $C = C_\delta = \delta^{-2} e^{A\delta^2}$. Now, suppose u has norm equal to 1, and let $q(z)$ be the subharmonic function

$$q(z) = \frac{1}{2m} \log \frac{|u(z)|^2}{m C_\delta}, \quad z \in \mathcal{K}^c.$$

It follows from the above estimate on $|u(z)|^2$ that $q(z) \leq Q$ for $z \in \mathcal{S}_\tau \setminus \mathcal{K}$, and the growth bound on $|u(z)|$ as $|z| \rightarrow \infty$ shows that $q(z) \leq \tau \log|z| + O(1)$ as $|z| \rightarrow \infty$. By applying the maximum principle for subharmonic functions to the function

$$q(z) - \tau \log|\phi_\tau(z)|, \quad z \in \mathcal{S}_\tau^c,$$

it follows that $q(z) \leq \hat{Q}_\tau(z)$ for $z \in \mathcal{S}_\tau^c$, which completes the proof. \square

In particular, we observe that Proposition 2.2.2 shows that $|P_{n,m}(z)|^2 e^{-2mQ}$ decays exponentially outside the set \mathcal{S}_τ if $\tau = n/m$. As alluded to in the introduction, it is possible to further localize the mass of $|P_{n,m}(z)|^2 e^{-2mQ(z)}$. The following is from [2]. Denote by $\varpi(\cdot, \hat{\mathbb{C}} \setminus \mathcal{S}_t, \infty)$ the harmonic measure of $\hat{\mathbb{C}} \setminus \mathcal{S}_t$ relative to the point at infinity.

Theorem 2.2.3. *As $m, n \rightarrow \infty$ with $\tau = \frac{n}{m} = \tau_0 + O(m^{-1})$ for some τ_0 with $0 < \tau_0 \leq 1$, we have the convergence*

$$|P_{n,m}|^2 e^{-2mQ} \rightarrow \varpi(\cdot, \widehat{\mathbb{C}} \setminus \mathcal{S}_{\tau_0}, \infty),$$

in the sense of weak convergence of measures.

We will not need this theorem per se, but merely draw intuition from it. It is clear that in the sense of $L^2(e^{-2mQ})$, the orthogonal polynomial is concentrated near $\partial\mathcal{S}_\tau$. In light of this, performing a cut-off by removing a compact subset set \mathcal{K}_τ from the interior of \mathcal{S}_τ should matter little. See Figure 3.1 for an illustration of the norm concentration.

2.3. Weighted Laplacian growth. Weighted Laplacian growth (or weighted Hele-Shaw flow) describes the movement of the boundary of a viscous fluid droplet in a porous medium, as fluid is injected into the droplet, where the weight appears as a result of the variable permeability of the medium, or, alternatively, as a result of curved geometry. For the mathematical formulation, consider a simply connected domain Ω_0 on the Riemann sphere $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ and an injection point $\alpha \in \Omega_0$. A smoothly increasing family $\{\Omega_t\}_t$ of domains is said to be a Hele-Shaw flow with weight ω , relative to the injection point α if the infinitesimal change of the measure $1_{\Omega_\tau} \omega(z) dA$ equals harmonic measure (the derivative is as usual taken in the sense of distribution theory):

$$(2.3.1) \quad \partial_t(1_{\Omega_t} \omega dA) = d\varpi(\cdot, \Omega_t, \alpha).$$

A basic reference on Hele-Shaw flow is the book [20] by Gustafsson, Teodorescu and Vasili'ev. The weighted Hele-Shaw flow problem appears to have been treated first in the paper [25] by Hedenmalm and Shimorin, where the weight was interpreted as a Riemannian metric, motivated by considerations in the potential theory of clamped plates [26]. This line of work is continued by [24], [23]. In this connection, we mention the recent work [35] by Ross and Witt-Nyström, which deals with a slightly less regular situation than the aforementioned work.

We recall that we assume that the potential Q is 1-admissible (see Definition 1.3.1). The reason why we are interested in weighted Laplacian growth is twofold. The first reason is that the compact sets \mathcal{S}_τ evolve according to weighted Laplacian growth, which is a Hele-Shaw flow for the complements \mathcal{S}_τ^c in the backward time direction as τ increases. More precisely, in [22] it is shown that in the present context the compacts \mathcal{S}_τ , $0 < \tau \leq 1$, undergo Laplacian growth with the weight $2\Delta Q$. This also carries over to the slightly larger parameter range $0 < \tau \leq 1 + \epsilon$. The second reason connects with the weak formulation of weighted Laplacian growth. In terms of weighted Hele-Shaw flow, the flow of domains $\{\Omega_t\}_t$ constitutes a weak weighted Hele-Shaw flow solution with injection point α and weight ω if

$$\int_{\Omega_t \setminus \Omega_s} h \omega dA = (t - s)h(\alpha), \quad s < t,$$

for all bounded harmonic functions h on Ω_t , for some smooth increasing real-valued function c . We shall be interested in the choices $\alpha = \infty$ and $\Omega_0 = \mathcal{S}_\tau^c$ for fixed τ , while the *orthogonal foliation flow* defined by the loops $\partial\Omega_t$ will cover a region close to the curve $\partial\mathcal{S}_\tau$ for the relevant range of parameter values t . Let p be a polynomial of degree n , whose zeros are confined to a fixed compact subset of the interior of \mathcal{S}_τ , where we keep $\tau = \frac{n}{m}$. For the moment, we consider the weighted Laplacian growth with respect to the weight $\omega = |p|^2 e^{-2mQ}$ and recall that the injection point is $\alpha = \infty$. Since the polynomial p has zeros confined to a fixed compact subset of \mathcal{S}_τ , p is automatically orthogonal to all lower order polynomials on each curve $\partial\Omega_t$ with respect to the measure $|p|^{-2} d\varpi(\cdot, \Omega_t, \alpha)$. Hence, by the weighted Laplacian growth equation (2.3.1), the integral of $|p|^{-2} d\varpi(\cdot, \Omega_t, \alpha)$ with respect to t , taken over an appropriate range of parameter $t_1 < t < t_2$, equals the measure

$$(2.3.2) \quad 1_{\Omega_{t_2} \setminus \Omega_{t_1}} \frac{\omega}{|p|^2} dA = 1_{\Omega_{t_2} \setminus \Omega_{t_1}} e^{-2mQ} dA,$$

and consequently, p is orthogonal to all lower order polynomials with respect to the measure (2.3.2). We remark that for a rather generic such polynomial p the weighted Laplacian growth will typically run into trouble with the appearance of cusps or double points and a successful

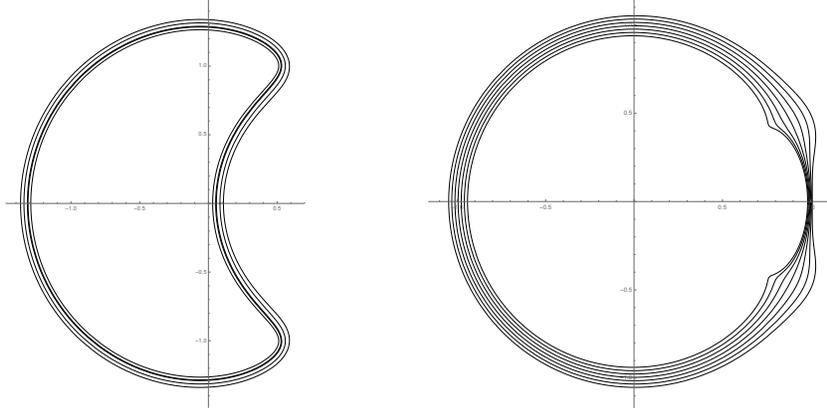


FIGURE 2.1. Approximate orthogonal foliation flow, near $\partial\mathcal{S}_1$ (left) and near \mathbb{T} (right), associated with the potential $Q(z) = \frac{1}{2}|z|^2 - 2^{-\frac{1}{2}} \log|z-1|$.

flow over a substantial range of parameters would require special properties of p . The above-mentioned orthogonal foliation flow is crucial for our proof of the main theorem (Theorem 1.3.3). An illustration of this flow is supplied in Figure 2.1. In practice, we will work with the corresponding flow around the unit circle \mathbb{T} , illustrated in the same figure.

We will need the following lemma, about the movement of the loops $\partial\mathcal{S}_\tau$ as τ varies.

Lemma 2.3.1. *Fix $\tau \in I_{\epsilon_0} = [1 - \epsilon_0, 1 + \epsilon_0]$. Denote by $\mathbf{n}_\tau(\zeta)$ the outer unit normal to $\partial\mathcal{S}_\tau$ at a point $\zeta \in \partial\mathcal{S}_\tau$, and let $\mathbf{n}_\tau(\zeta)\mathbb{R}$ denote the straight line which contains $\mathbf{n}_\tau(\zeta)$ and the origin. Then, if for real ε the point ζ_ε is closest to ζ in the intersection*

$$(\zeta - \mathbf{n}_\tau(\zeta)\mathbb{R}) \cap \partial\mathcal{S}_{\tau-\varepsilon},$$

we have as $\varepsilon \rightarrow 0$ that

$$\zeta_\varepsilon = \zeta - \varepsilon \mathbf{n}_\tau(\zeta) \frac{|\phi'_\tau(\zeta)|}{4\Delta Q(\zeta)} + O(\varepsilon^2)$$

and the outer normal $\mathbf{n}_{\tau-\varepsilon}(\zeta_\varepsilon)$ satisfies

$$\mathbf{n}_{\tau-\varepsilon}(\zeta_\varepsilon) = \mathbf{n}_\tau(\zeta) + O(\varepsilon).$$

Proof. We recall that the compact sets \mathcal{S}_τ evolve according to weighted Laplacian growth with respect to the weight $2\Delta Q$, see (2.1.2). If we compare with (2.3.1), we see that this means that

$$\partial_\tau(1_{\mathcal{S}_\tau} 2\Delta Q dA) = d\varpi(\cdot, \mathcal{S}_\tau^c, \infty) = |\phi'_\tau| ds,$$

where we recall that ϕ_τ is the (surjective) conformal mapping $\mathcal{S}_\tau^c \rightarrow \mathbb{D}_e$. This means that the boundary $\partial\mathcal{S}_\tau$ moves at local speed $(4\Delta Q)^{-1}|\phi'_\tau|$ in the exterior normal direction, where the number 4 appears instead of 2 as a result of the different normalizations associated with ds and dA . Since the loops $\partial\mathcal{S}_\tau$ deform smoothly, the claimed assertions follow from e.g. Taylor's formula. \square

2.4. Polynomial $\bar{\partial}$ -methods. Let ϕ be a strictly subharmonic function on \mathbb{C} . Hörmander's classical result states that the inhomogeneous $\bar{\partial}$ -equation

$$\bar{\partial}u = f$$

can be solved for any datum $f \in L^2_{\text{loc}}(\mathbb{C})$ with the estimate

$$\int_{\mathbb{C}} |u|^2 e^{-\phi} dA \leq \int_{\mathbb{C}} |f|^2 \frac{e^{-\phi}}{\Delta\phi} dA.$$

Taking this a starting point, in [1], Ameur, Hedenmalm, and Makarov investigate the case when the solution u is constrained by an additional polynomial growth condition at infinity. We now

describe this result. To this end, let $L_{\phi,n}^2(\mathbb{C})$ denote the subspace of $L_\phi^2(\mathbb{C})$ subject to the growth restraint

$$f(z) = O(|z|^{n-1})$$

near infinity. The polynomial growth Bergman space $A_{\phi,n}^2$ is analogously defined.

The following is Theorem 4.1 in [1].

Theorem 2.4.1. *Let \mathcal{T} be a compact subset of \mathbb{C} , and assume that ϕ , $\hat{\phi}$ and ϱ are real-valued $C^{1,1}$ -functions such that*

(i) $L_{\hat{\phi}}^2$ contains the function $z \mapsto (1 + |z|)^{-1}$, and there exist numbers a and b such that

$$\hat{\phi} \leq \phi + a \text{ on } \mathbb{C} \quad \text{and} \quad \phi \leq \hat{\phi} + b \text{ on } \mathcal{T}$$

(ii) $\Delta(\hat{\phi} + \varrho) > 0$ on \mathbb{C} ,

(iii) $A_{\hat{\phi}}^2 \subset A_{\phi,n}^2$,

(iv) $\nabla\varrho = 0$ on $\mathbb{C} \setminus \mathcal{T}$,

(v) there exists a number $\kappa \in (0, 1)$ such that

$$\frac{|\bar{\partial}\varrho|^2}{4\Delta(\hat{\phi} + \varrho)} \leq \frac{\kappa^2}{e^{a+b}}, \quad \text{a.e. on } \mathcal{T}.$$

Then for any $f \in L^\infty(\mathcal{T})$ the $L_{\phi,n}^2$ minimal solution to $\bar{\partial}u_{0,n} = f$ satisfies

$$\int_{\mathbb{C}} |u_{0,n}|^2 e^{\varrho - \phi} dA \leq \frac{e^{a+b}}{(1 - \kappa)^2} \int_{\mathcal{T}} |f|^2 \frac{e^{\varrho - \phi}}{\Delta(\hat{\phi} + \varrho)} dA.$$

We will require the following specialization of Theorem 2.4.1 to our needs.

Corollary 2.4.2. *Let $f \in L^\infty(\mathcal{S}_\tau)$. Then the $L_{2mQ,n}^2$ -minimal solution $u_{0,n}$ to the problem*

$$\bar{\partial}u_{0,n} = f$$

satisfies

$$(2.4.1) \quad \int_{\mathbb{C}} |u_{0,n}|^2 e^{-2mQ} dA \leq \frac{1}{2m} \int_{\mathcal{S}_\tau} |f|^2 \frac{e^{-2mQ}}{\Delta Q} dA,$$

provided that the right-hand side is finite.

Proof. We apply Theorem 2.4.1 with $\mathcal{T} = \mathcal{S}_\tau$, $\phi = 2mQ$, $\varrho = 0$, and

$$\hat{\phi} = 2m\left(1 - \frac{\varepsilon}{\tau}\right)\hat{Q}_\tau + \varepsilon m \log(1 + |z|^2).$$

Then all conditions except (iii) are trivially satisfied with $a, b = o(1)$ as $\varepsilon \rightarrow 0^+$. To see why (iii) holds, it is enough to observe that

$$\hat{\phi}(z) = 2m\tau\left(1 - \frac{\varepsilon}{\tau}\right) \log|z| + 2\varepsilon m \log|z| + O(1) = \log(|z|^{2n}) + O(1)$$

as $|z| \rightarrow \infty$. Hence the inclusion $A_{\hat{\phi}}^2 \subset \text{Pol}_n$ follows. Letting $\varepsilon \rightarrow 0^+$ for fixed (m, n) completes the proof. \square

Remark 2.4.3. We mention Theorem 2.4.1 because it helps us to build intuition on the behavior of the space of polynomials Pol_n equipped with the inner product of L_ϕ^2 . In the setting of [1], the theorem is applied (up to inessential modifications) using $\phi = 2mQ$, $\hat{\phi} = 2m\hat{Q}$ and $\mathcal{T} = \mathcal{S}_\tau$. In Theorem 2.4.1 the function ϱ , which modifies the weight, illustrates the amount of flexibility we can achieve. If \mathcal{S}_τ does not divide the plane, then ϱ is necessarily constant in the exterior, and is allowed to deviate only in the interior direction of \mathcal{S}_τ . This tells us that the exterior is rigid, while the interior is more flexible. Note that in the corollary we use the trivial modifying function $\varrho = 0$. This particular instance may be obtained more directly using Hörmander's classical $\bar{\partial}$ -estimate.

2.5. Holomorphic boundary value problems and Toeplitz operators. For the reader's convenience, we include some elementary facts from the theory of Herglotz kernels and Hardy spaces. Let f be holomorphic in the unit disk \mathbb{D} with continuous extension to the boundary. The classical Schwarz Integral Formula [18, pp. 45] states that that

$$f(z) = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \operatorname{Re}(f(\zeta)) \, ds(\zeta) + \operatorname{Im}(f(0)), \quad z \in \mathbb{D}.$$

Thus, if $F \in L^1(\mathbb{T})$ is real-valued, this allows us to solve the boundary value problem

$$\operatorname{Re} f|_{\mathbb{T}} = F$$

where f is holomorphic in the disk by the integral formula

$$f(z) = \mathbf{H}_{\mathbb{D}}[F](z) := \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} F(\zeta) \, ds(\zeta), \quad z \in \mathbb{D}.$$

Moreover, the solution is unique up to an additive imaginary constant. For us, it is more natural to work in the exterior disk. By reflection in the unit circle, we obtain the formula

$$f(z) = \mathbf{H}_{\mathbb{D}_e}[F](z) := \int_{\mathbb{T}} \frac{z + \zeta}{z - \zeta} F(\zeta) \, ds(\zeta), \quad z \in \mathbb{D}_e,$$

which we refer to as the *Herglotz transform* of F . If F is only $L^2(\mathbb{T})$ -integrable, its Herglotz transform is in the Hardy space H^2 . If we assume slightly more smoothness, e.g. that F is C^1 -smooth, then its Herglotz transform is continuous and bounded in the closed exterior disk $\bar{\mathbb{D}}_e$. Analogously, if we have a lot of smoothness, e.g. F is C^ω -smooth, then its Herglotz transform extends to a bounded analytic function on a slightly bigger exterior disk $\mathbb{D}_e(0, \rho)$ with $\rho < 1$. As the Hardy space $H^2 = H^2(\mathbb{D})$ was just mentioned, we recall the precise definition. A function f is in H^2 if it is holomorphic in \mathbb{D} with

$$\sup_{0 < r < 1} \int_{\mathbb{T}} |f(r\zeta)|^2 \, ds(\zeta) < +\infty.$$

Alternatively, in terms of the boundary values it is a closed subspace of $L^2(\mathbb{T})$ defined by the property that the Fourier coefficients with negative index vanish. The conjugate Hardy space H^2_- consists of all functions of the form \bar{f} , where $f \in H^2$, which may also be viewed as Hardy space on the exterior disk \mathbb{D}_e . In a similar fashion, the standard H^p -spaces can be defined. For $p = \infty$ the space H^∞ consists of the bounded holomorphic functions in the unit disk \mathbb{D} equipped with the supremum norm.

Associated with the Hardy and conjugate Hardy subspaces of $L^2(\mathbb{T})$ there are the orthogonal projections $\mathbf{P}_{H^2} : L^2(\mathbb{T}) \rightarrow H^2$ and $\mathbf{P}_{H^2_-} : L^2(\mathbb{T}) \rightarrow H^2_-$. These are associated with the Szegő integral kernel:

$$\mathbf{P}_{H^2} f(z) = \int_{\mathbb{T}} \frac{f(\zeta)}{1 - z\bar{\zeta}} \, ds(\zeta), \quad z \in \mathbb{D},$$

and

$$\mathbf{P}_{H^2_-} f(z) = \int_{\mathbb{T}} \frac{zf(\zeta)}{z - \zeta} \, ds(\zeta), \quad z \in \mathbb{D}_e.$$

We will also be interested in the subspace $H^2_{-,0}$ of H^2_- consisting of all functions that vanish at infinity (or equivalently, have average 0 on the unit circle). The associated projection is

$$\mathbf{P}_{H^2_{-,0}} f(z) = \int_{\mathbb{T}} \frac{\zeta f(\zeta)}{z - \zeta} \, ds(\zeta), \quad z \in \mathbb{D}_e.$$

It is clear from the above concrete formulae that the Herglotz transform $\mathbf{H}_{\mathbb{D}_e}$ can be expressed in terms of projections: $\mathbf{H}_{\mathbb{D}_e} = \mathbf{P}_{H^2_-} + \mathbf{P}_{H^2_{-,0}}$. For an $L^\infty(\mathbb{T})$ -function Θ , we define the (exterior) Toeplitz operator $\mathbf{T}_\Theta : H^2_- \rightarrow H^2_-$ by

$$\mathbf{T}_\Theta f = \mathbf{P}_{H^2_-} [\Theta f], \quad f \in H^2_-.$$

The kernel of this operator consists of all solutions in H^2_- to $\mathbf{T}_\Theta f = 0$. Assuming that Θ is non-zero almost everywhere on the circle \mathbb{T} , it follows that the condition that f belongs to the

kernel is equivalent to $f \in H_-^2 \cap \Theta^{-1}H_0^2$, where H_0^2 consists of the functions in H^2 with mean 0. If we implicitly define the function ϑ by $\Theta(z) = z\vartheta(z)$, we may rephrase this condition as

$$(2.5.1) \quad f \in H_-^2 \cap \vartheta^{-1}H^2,$$

which we refer to as a *homogeneous (exterior) Toeplitz kernel condition*. For a function F defined on the circle \mathbb{T} , we also consider the related condition

$$(2.5.2) \quad f \in H_-^2 \cap \vartheta^{-1}(-F + H^2),$$

which we refer to as an *inhomogeneous Toeplitz kernel condition*. In terms of Toeplitz operators, this condition may be written as $\mathbf{T}_{z\vartheta}f + \mathbf{P}_{H^2}[zF] = 0$. The following proposition provides the structure of solutions to the homogeneous and inhomogeneous Toeplitz kernel conditions for sufficiently regular ϑ .

Proposition 2.5.1. *Suppose that ϑ can be written in the form $\vartheta = e^{u+\bar{v}}$, where u and v are in H^∞ , and let F be a function in $L^\infty(\mathbb{T})$. Then f solves*

$$f \in H_-^2 \cap \vartheta^{-1}(-F + H^2)$$

if and only if

$$f = Ce^{-\bar{v}} - e^{-\bar{v}}\mathbf{P}_{H_{-,0}^2}[e^{-u}F],$$

for some constant C .

Proof. That $f \in H_-^2 \cap \vartheta^{-1}(-F + H^2)$ is equivalent to having

$$(2.5.3) \quad e^{\bar{v}}f \in e^{\bar{v}}H_-^2 \cap (-e^{-u}F + e^{-u}H^2) = H_-^2 \cap (-e^{-u}F + H^2).$$

Since $e^{\bar{v}}f \in H_-^2$, an application of the projection $\mathbf{P}_{H_{-,0}^2}$ gives

$$\mathbf{P}_{H_{-,0}^2}[e^{\bar{v}}f] = e^{\bar{v}}f - C$$

for some constant C . On the other hand, since $e^{\bar{v}}f \in -e^{-u}F + H^2$, it is immediate that

$$\mathbf{P}_{H_{-,0}^2}[e^{\bar{v}}f] = -\mathbf{P}_{H_{-,0}^2}[e^{-u}F],$$

since H^2 projects to $\{0\}$. It follows that

$$e^{\bar{v}}f = C + \mathbf{P}_{H_{-,0}^2}[e^{\bar{v}}f] = C - \mathbf{P}_{H_{-,0}^2}[e^{-u}F],$$

as claimed. \square

Remark 2.5.2. The Toeplitz kernel equation (2.5.3) may be viewed as a scalar Riemann-Hilbert problem with jump from the inside \mathbb{D} to the outside \mathbb{D}_e equal to $e^{-u}F$. Later, we will use the conformal mapping from the complement \mathcal{S}_τ^c to the interior, and the interpretation of the Toeplitz kernel equation in that context is as a scalar Riemann-Hilbert problem on the Schottky double of \mathcal{S}_τ^c .

2.6. The random normal matrix model. For extensive treatments of the random normal matrix ensembles, see e.g. [22, 2, 3, 4, 45]. Here we only briefly discuss the topic, in order to fix the notation and recall some basic concepts. Let M be a matrix, picked with respect to the probability measure (“tr” stands for trace)

$$d\mu_m(M) = \frac{1}{Z_{m,Q}} e^{-2m \operatorname{tr}(Q(M))} dM,$$

where dM denotes the measure induced from Lebesgue measure on the manifold of normal $m \times m$ matrices embedded canonically into \mathbb{C}^{m^2} , and where $Z_{m,Q}$ is a normalizing constant. Such a matrix M has a set of m random eigenvalues, which we denote by $\Phi_m = \{z_{1,m}, \dots, z_{m,m}\}$. It is known that the eigenvalues follow the law

$$(2.6.1) \quad d\mathbb{P}_m(z_1, \dots, z_m) = \frac{1}{Z_{m,Q}} \left[\prod_{j < k} |z_j - z_k|^2 \right] e^{-2m \sum_{j=1}^m Q(z_j)} dA^{\otimes n}(z_1, \dots, z_m),$$

where $\mathcal{Z}_{m,Q}$ is a normalizing constant, known as the *partition function* of the ensemble. Here, $dA^{\otimes n}$ stands for volume measure in \mathbb{C}^n normalized by the factor π^{-n} . We recognize this as the law for Coulomb gas with m particles at the inverse temperature $\beta = 2$ in the external field Q . Courtesy of the fact that the product expression in (2.6.1) may be written as the square modulus of a Vandermonde determinant, these ensembles are determinantal. That is, if the k -point intensities $R_{k,m}(z_1, \dots, z_k)$ are defined as the intensities associated to finding points simultaneously at the locations z_1, \dots, z_k , then we may compute $R_{k,m}$ by

$$(2.6.2) \quad R_{k,m}(z_1, \dots, z_k) = \det (K_m(z_j, z_l))_{1 \leq j, l \leq k}.$$

Here K_m is the *correlation kernel*

$$K_m(z, w) = K_m(z, w) e^{-m(Q(z)+Q(w))}, \quad z, w \in \mathbb{C}$$

where K_m is the reproducing kernel for the space Pol_m , supplied with the inner product of the space $L^2_{2mQ}(\mathbb{C})$. We remark that the correlation kernel K_m is not uniquely determined by the above-mentioned intensities, since any kernel modified by a cocycle

$$K_m^c(z, w) = c(z)\bar{c}(w)K_m(z, w),$$

will generate the same point process by the determinantal formula (2.6.2). Here, the cocycle is associated with a continuous unimodular function $c : \mathbb{C} \rightarrow \mathbb{T}$. This means that in terms of convergence of point processes, we need only correlation kernel convergence modulo cocycles. It is known (see [22, 45]) that Φ_m condensates to the droplet \mathcal{S}_1 as $m \rightarrow \infty$. Indeed, if ν_m denotes the empirical measure

$$\nu_m = \frac{1}{m} \sum_{z \in \Phi_m} \delta_z,$$

then almost surely, ν_m converges weakly to μ_{2Q} , the equilibrium measure, the support of which equals \mathcal{S}_1 . We will be interested in rescaling the point process near a boundary point z_0 in the outer normal direction \mathbf{n} , in order to understand the microscopic behavior of the point process. To rescale around the point z_0 we use the linear transformation

$$z_m(\zeta) := z_0 + \mathbf{n} \frac{\zeta}{\sqrt{2m\Delta Q(z_0)}}.$$

We recall that $\Phi_m = \{z_{j,m}\}_j$ denotes the original point process, and introduce the rescaled local process around the point z_0 by $\Psi_m = \{\zeta_{j,m}\}_j$, where

$$z_{j,m} = z_m(\zeta_{j,m}), \quad j = 1, \dots, m.$$

Similarly, we denote by k_m the rescaled correlation kernel

$$k_m(\xi, \eta) = \frac{1}{2m\Delta Q(z_0)} K_m(z_m(\xi), z_m(\eta)).$$

The following is from [4].

Theorem 2.6.1. *To a given sequence of positive integers \mathcal{N} , there exist a subsequence $\mathcal{N}^* \subset \mathcal{N}$ and an Hermitian entire function $F(\xi, \eta)$ such that*

$$\lim_{\mathcal{N}^* \ni m \rightarrow \infty} c_m(\xi)\bar{c}_m(\eta) k_m(z_m(\xi), z_m(\eta)) = e^{\xi\bar{\eta} - \frac{1}{2}(|\xi|^2 + |\eta|^2)} F(\xi, \eta),$$

for some sequence of continuous unimodular functions $c_m : \mathbb{C} \rightarrow \mathbb{T}$.

2.7. Steepest descent analysis. When describing our computational algorithm in Section 3.3, we will find need for the following result ([29], p. 220, Theorem 7.7.5). For the formulation, we need some notation. For an open subset Ω of \mathbb{R} , we let $C^k(\Omega)$ denote the space of k times differentiable functions on Ω , and for a compact subset K of \mathbb{R} , we let $C_0^k(K)$ denote the space

k times differentiable, compactly supported functions on \mathbb{R} whose support is contained in K . The norm in the space $C^k(\Omega)$ is defined as

$$\|u\|_{C^k(\Omega)} = \sum_{j=0}^k \|u^{(j)}\|_{L^\infty(\Omega)},$$

and the norm in $C_0^k(K)$ is analogously defined.

Proposition 2.7.1. *Let $K \subset \mathbb{R}$ be a compact interval, Ω an open neighbourhood of K , and k a positive integer. If $u \in C_0^{2k}(K)$, $V \in C^{3k+1}(\Omega)$ and $V \geq 0$ in X , $V'(x_0) = 0$, $V''(x_0) > 0$, and $V' \neq 0$ in $K \setminus \{x_0\}$, then, for $\omega > 0$, we have*

$$(2.7.1) \quad \left| e^{\omega V(x_0)} \int_K u(x) e^{-\omega V(x)} dx - \left(\frac{2\pi}{\omega V''(x_0)} \right)^{\frac{1}{2}} \sum_{j=0}^{k-1} \omega^{-j} \mathbf{L}_j u(x_0) \right| \leq C \omega^{-k} \|u\|_{C^{2k}(K)}.$$

Here, C is bounded when V stays in a bounded set in $C^{3k+1}(\Omega)$, and $|x - x_0|/|V'(x)|$ has a uniform bound. With

$$W_{x_0}(x) := V(x) - V(x_0) - \frac{1}{2}(x - x_0)^2 V''(x_0),$$

we have

$$\mathbf{L}_j u(x) := \sum_{(k,l): l-k=j, 2l \geq 3k} \frac{(-1)^k 2^{-l}}{k! l! [V''(x_0)]^l} \partial_x^{2l} (W_{x_0}^k u)(x).$$

In the definition of the above differential operator \mathbf{L}_j , it is implicit that the summation takes place over non-negative integers k and l .

The following proposition is tailored to our needs, based on Proposition 2.7.1.

Proposition 2.7.2. *Let three reals ρ_0, ρ_1, ρ_2 be given, with $0 < \rho_0 < 1 < \rho_1 < \rho_2$. Assume that $V : [\rho_0, \infty) \rightarrow \mathbb{R}$ is C^{3k+1} -smooth, and that V has a unique minimum at 1, with $V(1) = V'(1) = 0$. Suppose furthermore that we have*

- (a) *the convexity bound $V'' \geq \alpha$ on (ρ_0, ρ_2) for some real $\alpha > 0$,*
- (b) *and that V has a bound from below of the form $V(x) \geq \vartheta \log x$ on the interval $[\rho_1, \infty)$, for some real constant $\vartheta > 0$.*

If the function $u : (\rho_0, \infty) \rightarrow \mathbb{C}$ is bounded and continuous throughout, and in addition u is C^{2k} -smooth on the interval $[0, \rho_2]$ and vanishes on $[0, \rho_0]$, then we have

$$\int_{\rho_0}^{\infty} u(x) e^{-\omega V(x)} dx = \left(\frac{2\pi}{\omega V''(1)} \right)^{\frac{1}{2}} \sum_{j=0}^{k-1} \omega^{-j} \mathbf{L}_j [u](1) + E,$$

where the error term $E = E(\omega, k, u, \vartheta, \rho_0, \rho_1, \rho_2)$ enjoys the bound

$$|E| \leq C_1 \omega^{-k} \|u\|_{C^{2k}([\rho_0, \rho_2])} + \|u\|_{L^\infty([\rho_1, \infty))} \rho_1^{-\omega \vartheta + 1},$$

provided that $\omega > \frac{2}{\vartheta}$, where C_1 remains uniformly bounded when V stays in a bounded set of $C^{3k+1}([\rho_0, \rho_2])$.

Sketch of proof. Let χ be a smooth cut-off function with $0 \leq \chi \leq 1$ throughout, which equals 1 on the interval $[\rho_0, \rho_1]$, and vanishes on $[\rho_2, \infty)$. We use the cut-off function to split the integral

$$\int_{\rho_0}^{\infty} u(x) e^{-\omega V(x)} dx = \int_{\rho_0}^{\rho_2} \chi(x) u(x) e^{-\omega V(x)} dx + \int_{\rho_1}^{\infty} (1 - \chi(x)) u(x) e^{-\omega V(x)} dx.$$

The first integral gives the main contribution, which is estimated using Proposition 2.7.1. The other two integrals are estimated using the given bounds from below on V . The details are omitted. \square

2.8. Notation. For the convenience of the reader, we supply a list of commonly used notation.

$\mathbb{C}, \mathbb{D}, \mathbb{T}$	Complex plane, open unit disk and unit circle, respectively.
$\partial_z, \bar{\partial}_z$	Wirtinger derivatives, given by $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$, $\bar{\partial}_z = \frac{1}{2}(\partial_x + i\partial_y)$, where $z = x + iy$.
X^c	Complement of the set X , understood as $\mathbb{C} \setminus X$ unless specified otherwise.
Δ	Laplacian, which factorizes as $\Delta = \partial\bar{\partial}$. N.B.: this equals one-quarter of the usual laplacian.
Q	Subharmonic, smooth real-valued function with superlogarithmic growth.
\hat{Q}_τ	Solution to obstacle problem – maximal subharmonic function with $\hat{Q}_\tau \leq Q$ and $\hat{Q}_\tau(z) = \tau \log z + O(1)$ as $z \rightarrow \infty$.
$\mathcal{S}_\tau, \mathcal{S}_\tau^*$	The droplet and the coincidence set, which coincide under our assumptions.
ϕ_τ	Conformal mapping $\phi_\tau : \mathcal{S}_\tau^c \rightarrow \mathbb{D}_e$ with $\phi'_\tau(\infty) > 0$.
\check{Q}_τ	Harmonic extension of $\hat{Q}_\tau _{\mathcal{S}_\tau^c}$ across $\partial\mathcal{S}_\tau$.
Q_τ^\otimes	Bounded harmonic extension of $Q _{\partial\mathcal{S}_\tau}$ to \mathcal{S}_τ^c .
\mathcal{Q}_τ	Holomorphic function on \mathcal{S}_τ^c with $\text{Re } \mathcal{Q}_\tau = Q_\tau^\otimes$.
χ_0, χ_1	Various smooth cut-off functions.
$\varpi(E, \Omega, z_0)$	Harmonic measure of E relative to (Ω, z_0) .
\mathbf{H}_Ω	The Herglotz operator for a domain Ω containing the point at infinity.
$\mathbf{P}_{H^2}, \mathbf{P}_{\underline{H^2}}$	Orthogonal projection onto Hardy spaces.
\mathcal{I}_n	Indexing set $\{(j, l) \in \mathbb{N}^2 : 2j + l \leq n\}$.

3. HEURISTIC ALGORITHM

3.1. Introduction of quasipolynomials and a renormalizing ansatz. In the present section, we discuss the *approximate orthogonal quasipolynomials* $F_{n,m}$, by which we mean certain functions which behave like orthogonal polynomials in $L^2(\mathbb{C}, e^{-2mQ})$, in a sense specified below. Let \mathcal{K}_τ be an appropriately chosen compact subset of the droplet \mathcal{S}_τ , which lies at a fixed positive distance from $\partial\mathcal{S}_\tau$. Moreover, we require that the conformal mapping $\phi_\tau : \mathcal{S}_\tau \rightarrow \mathbb{D}_e$ extends to a (surjective) conformal mapping

$$\phi_\tau : \mathcal{K}_\tau^c \rightarrow \mathbb{D}_e(0, \rho_0), \quad \tau \in I_{\epsilon_0},$$

for some ρ_0 with $0 < \rho_0 < 1$. In what follows, we will disregard the behavior on the compact set \mathcal{K}_τ . We expect this to have little effect, and will be justified a posteriori, since the orthogonal polynomials are highly concentrated near $\partial\mathcal{S}_\tau$ (see Figure 3.1 for an illustration of the behavior of $|P_{n,m}|^2 e^{-2mQ(z)}$ in a special case).

In the context of the following definition, we use the standard notation that $A \asymp B$ provided that $A = O(B)$ and $B = O(A)$ in some limiting procedure involving positive quantities A and B .

Definition 3.1.1. We say that a function F is a *quasipolynomial* on \mathcal{K}_τ^c of degree n if it is defined and holomorphic on $\mathbb{C} \setminus \mathcal{K}_\tau$, with polynomial growth near infinity: $|F(z)| \asymp |z|^n$ as $|z| \rightarrow \infty$.

In the context of this definition, a quasipolynomial F of degree n has $F(z) = az^n + O(|z|^{n-1})$ near infinity, for some complex number $a \neq 0$. We refer to the number a as the *leading coefficient* of the quasipolynomial F .

We now fix a positive integer κ , which we think of as an accuracy parameter. Moreover, we denote by $\chi_{0,\tau}$ a smooth cut-off function that vanishes on \mathcal{K}_τ and equals 1 on \mathcal{X}_τ^c , where \mathcal{X}_τ denotes an intermediate set between \mathcal{K}_τ and \mathcal{S}_τ . In addition, we shall require that the $C^{2(\kappa+1)}$ -norm of $\chi_{0,\tau}$ remains uniformly bounded for $\tau \in I_{\epsilon_0}$.

Definition 3.1.2. We say that a sequence $\{F_{n,m}\}_{n,m}$ of quasipolynomials of degree n on \mathcal{K}_τ^c is *normalized and approximately orthogonal* (of accuracy κ) if the following asymptotic conditions (i)-(iii) are met as $m \rightarrow \infty$ while $\tau = \frac{n}{m} \in I_{\epsilon_0}$:

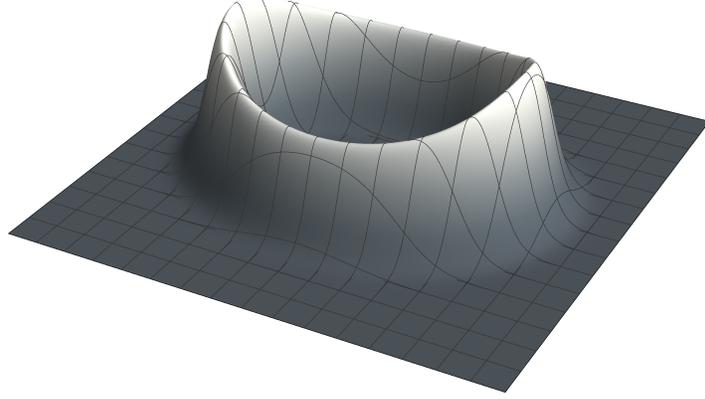


FIGURE 3.1. The orthogonal polynomial density $|P_{n,m}(z)|^2 e^{-2mQ(z)}$ for $n = 6$, $m = 20$ and $Q(z) = \frac{1}{2}|z|^2 - \operatorname{Re}(tz^2)$, where $t = 0.4$.

(i) we have the approximate orthogonality

$$\forall p \in \operatorname{Pol}_n : \int_{\mathbb{C}} \chi_{0,\tau} F_{n,m}(z) \bar{p}(z) e^{-2mQ(z)} dA(z) = O\left(m^{-\kappa - \frac{1}{3}} \|p\|_{2mQ}\right),$$

(ii) the quasipolynomials $F_{n,m}$ have approximately unit norm,

$$\int_{\mathbb{C}} \chi_{0,\tau}^2 |F_{n,m}(z)|^2 e^{-2mQ(z)} dA(z) = 1 + O(m^{-\kappa - \frac{1}{3}}),$$

(iii) and the quasipolynomial $F_{n,m}$ has leading coefficient $a_{n,m}$ at infinity which is approximately real and positive, in the sense that

$$\frac{\operatorname{Im} a_{n,m}}{\operatorname{Re} a_{n,m}} = O(m^{-\kappa - \frac{1}{12}})$$

where all the implied constants are uniform.

In terms of the above definition, Theorem 1.3.3 implies in particular that $F_{n,m}^{(\kappa)}$ is a sequence of approximately orthogonal quasipolynomials with accuracy κ . The fraction $\frac{1}{3}$ which appears in the definition is convenient in our calculations. The concept would be meaningful even if this number were replaced by e.g. $\frac{1}{5}$.

We proceed to list some functions related to Q .

- (i) \hat{Q}_τ is the solution to the obstacle problem (2.1.1). Recall that \hat{Q}_τ is harmonic function on \mathcal{S}_τ^c which equals Q on $\partial\mathcal{S}_\tau$, and grows like $\tau \log|z|$ as $|z| \rightarrow \infty$.
- (ii) \check{Q}_τ is defined as the harmonic extension of the restriction of \hat{Q}_τ to $\mathbb{C} \setminus \mathcal{S}_\tau$ across $\partial\mathcal{S}_\tau$.
- (iii) Q_τ^\circledast is the bounded harmonic function on \mathcal{S}_τ^c which equals Q on $\partial\mathcal{S}_\tau$.
- (iv) \mathcal{Q}_τ is a bounded holomorphic function in \mathcal{S}_τ^c such that $\operatorname{Re} \mathcal{Q}_\tau = Q_\tau^\circledast$ on \mathcal{S}_τ^c .

Since Q is assumed 1-admissible, the curves $\Gamma := \partial\mathcal{S}_\tau$ remain real-analytically smooth and simple for $\tau \in I_{\epsilon_0} = [1 - \epsilon_0, 1 + \epsilon_0]$. By possibly making \mathcal{K}_τ a little bigger, we may assume that the functions Q_τ^\circledast and \check{Q}_τ are harmonic, and that \mathcal{Q}_τ is holomorphic, in the domain $\mathbb{C} \setminus \mathcal{K}_\tau$. We define the operator $\mathbf{\Lambda}_{n,m}$ by

$$(3.1.1) \quad \mathbf{\Lambda}_{n,m}[f](z) := \phi'_\tau(z) [\phi_\tau(z)]^n e^{m\mathcal{Q}_\tau(z)} (f \circ \phi_\tau)(z), \quad \tau = \frac{n}{m}.$$

If f is well-defined in a neighbourhood of $\bar{\mathbb{D}}_e(0, \rho_0)$, then $\mathbf{\Lambda}_{n,m}[f]$ is well-defined in a neighbourhood of $\mathbb{C} \setminus \mathcal{K}_\tau$. Observe that

$$(3.1.2) \quad \int_{\mathcal{K}_\tau^c} \mathbf{\Lambda}_{n,m}[f] \overline{\mathbf{\Lambda}_{n,m}[g]} e^{-2mQ} dA = \int_{\mathcal{K}_\tau^c} f \circ \phi_\tau \bar{g} \circ \phi_\tau e^{-2m(Q - \log|\phi_\tau| - \operatorname{Re} \mathcal{Q}_\tau)} |\phi'_\tau|^2 dA \\ = \int_{\mathbb{D}_e(0, \rho_0)} f \bar{g} e^{-2mR_\tau} dA,$$

where we write

$$R_\tau := (Q - \check{Q}_\tau) \circ \phi_\tau^{-1},$$

and the first equality holds since the Green function for \mathcal{S}_τ^c with pole at infinity is given by $G_{\infty, \tau} = \log|\phi_\tau|$, so that we may decompose

$$\check{Q}_\tau = Q_\tau^\circledast + \tau \log|\phi_\tau|.$$

The function R_τ is a central object, and we turn to some of its basic properties.

Proposition 3.1.3. *The function R_τ is defined on $\mathbb{D}_e(0, \rho_0)$, and is real-analytic in a neighbourhood of \mathbb{T} . Moreover, near the unit circle R_τ satisfies*

$$R_\tau(re^{i\theta}) = 2\Delta R_\tau(e^{i\theta})(1-r)^2 + O((1-r)^3), \quad r \rightarrow 1,$$

where the implied constant is uniform for $e^{i\theta} \in \mathbb{T}$ and $\tau \in I_{\epsilon_0}$. Moreover, it has the growth bound from below

$$R_\tau(z) \geq \vartheta \log|z|, \quad z \in \mathbb{D}_e(0, \rho_1),$$

for some real parameters $\vartheta > 0$ and $\rho_1 > 1$, which do not depend on $\tau \in I_{\epsilon_0}$.

Sketch of proof. The assertion on the local behaviour near the circle \mathbb{T} results from an application of Taylor's formula, using that along the boundary $\partial\mathcal{S}_\tau$ we have $Q = \check{Q}_\tau$, $\nabla Q = \nabla \check{Q}_\tau$ while

$$\partial_n^2(Q - \check{Q}_\tau) = (\partial_n^2 + \partial_t^2)(Q - \check{Q}_\tau) = 4\Delta Q.$$

Here we recall that ∂_n and ∂_t denote the normal and tangential derivatives, respectively. We turn to the global estimate from below on R_τ . By the assumption (1.3.1) with $\tau = 1$ on the growth of Q near infinity, and the growth control

$$\check{Q}_\tau(z) = \hat{Q}_\tau(z) = \tau \log|z| + O(1), \quad \text{as } |z| \rightarrow \infty,$$

it follows that

$$\liminf_{z \rightarrow \infty} \frac{(Q - \check{Q}_\tau)(z)}{\log|z|} > 0$$

for $\tau \in I_{\epsilon_0}$, provided that ϵ_0 is small enough. Since $|\phi_\tau^{-1}(z)| \asymp |z|$ near infinity, we see that

$$\lim_{|z| \rightarrow \infty} \frac{R_\tau(z)}{\log|z|} > 0.$$

There is no point in \mathbb{D}_e where R_τ vanishes, since the coincidence set (where \hat{Q}_τ and Q coincide) equals \mathcal{S}_τ (see Definition 1.3.1). We may conclude that the ratio $\frac{R_\tau(z)}{\log|z|}$ is bounded below by a positive constant ϑ on the exterior disk $\mathbb{D}_e(0, \rho_1)$. A careful analysis of this argument shows that we may assume that ϑ does not depend on τ , as long as $\tau \in I_{\epsilon_0}$. \square

Remark 3.1.4. Since \check{Q}_τ is harmonic on \mathcal{K}_τ^c , we find that

$$\Delta R_\tau = \Delta(Q - \check{Q}_\tau) \circ \phi_\tau^{-1} = |(\phi_\tau^{-1})'|^2 (\Delta Q) \circ \phi_\tau^{-1},$$

which shows that near the circle \mathbb{T} , we have uniform bound of ΔR_τ from below by a positive constant. As a consequence the same holds for $\partial_r^2 R_\tau(re^{i\theta})$ for r close to 1 (cf. Proposition 2.7.2).

Proposition 3.1.3 explains why near the unit circle, the function e^{-2mR_τ} may be thought of as a Gaussian wave around the unit circle \mathbb{T} .

We return to the operator $\mathbf{\Lambda}_{n,m}$, defined in (3.1.1). It renormalizes the weight, and transports holomorphic functions in \mathcal{K}_τ^c to the exterior disk $\mathbb{D}_e(0, \rho_0)$. In the sequel, we will refer to $\mathbf{\Lambda}_{n,m}$ as *the canonical positioning operator*. Its basic properties are summarized in the following proposition.

Proposition 3.1.5. *The canonical positioning operator $\mathbf{\Lambda}_{n,m}$ is an isometric isomorphism $L_{2mR_\tau}^2(\mathbb{D}_e(0, \rho_0)) \rightarrow L_{2mQ}^2(\mathcal{K}_\tau^c)$, and the inverse operator is given by*

$$\mathbf{\Lambda}_{n,m}^{-1}[g](z) = z^{-n}[\phi_\tau^{-1}]'(z) e^{-m(\mathcal{Q}_\tau \circ \phi_\tau^{-1})(z)} (g \circ \phi_\tau^{-1})(z), \quad g \in L_{2mQ}^2(\mathcal{K}_\tau^c).$$

Moreover, the operator $\mathbf{\Lambda}_{n,m}$ preserves holomorphicity, and in addition, it maps the subspace $A_{2mR_\tau,0}^2(\mathbb{D}_e(0, \rho_0))$ onto $A_{2mQ,n}^2(\mathcal{K}_\tau^c)$.

Proof. As direct consequence of the (3.1.2), we see that $L_{2mR_\tau}^2(\mathbb{D}_e(0, \rho_0))$ is mapped isometrically into $L_{2mQ}^2(\mathcal{K}_\tau^c)$, and moreover if $\mathbf{\Lambda}_{n,m}^{-1}$ is given by the above formula, we see that it is actually the inverse to $\mathbf{\Lambda}_{n,m}$. By definition, $\mathbf{\Lambda}_{n,m}[f]$ is holomorphic in \mathcal{K}_τ^c , if f is holomorphic in $\mathbb{D}_e(0, \rho_0)$. It follows that $\mathbf{\Lambda}_{n,m}$ is actually an isometric isomorphism $A_{2mR_\tau}^2(\mathbb{D}_e(0, \rho_0)) \rightarrow A_{2mQ}^2(\mathcal{K}_\tau^c)$. It remains to note that $\mathbf{\Lambda}_{m,n}$ maps bijectively

$$A_{2mR_\tau,0}^2(\mathbb{D}_e(0, \rho_0)) \rightarrow A_{2mQ,n}^2(\mathcal{K}_\tau^c),$$

which is a direct consequence of the fact that $|\phi_\tau(z)| \asymp |z|$ as $|z| \rightarrow \infty$. \square

The canonical positioning operator is useful to our finding the asymptotic expansion formula for the orthogonal polynomials.

3.2. Implementation of the radial Laplace method. We turn to the algorithm of Theorem 1.3.7. To proceed, we need two families of differential operators. We recall the differential operators \mathbf{L}_k appearing in the application of Laplace's method in Proposition 2.7.1. We need to apply these operators to functions defined in a neighbourhood of the unit circle, and we apply them in the radial direction. So, for functions $f(re^{i\theta})$, we put

$$\mathbf{L}_k[f](re^{i\theta}) = \sum_{\nu=k}^{3k} \frac{(-1)^{\nu-k} 2^{-\nu}}{\nu!(\nu-k)![\partial_r^2 R_\tau(re^{i\theta})]^\nu} \partial_r^{2\nu} \left([W_\tau(re^{i\theta})]^{\nu-k} f(re^{i\theta}) \right),$$

where

$$W_\tau(re^{i\theta}) = R_\tau(re^{i\theta}) - \frac{1}{2}(r-1)^2 \partial_x^2 R_\tau(xe^{i\theta}) \Big|_{x=1}.$$

The second family of operators is defined implicitly in the following Lemma, which is inspired by the theory of pseudodifferential operators.

Lemma 3.2.1. *Let k be a non-negative integer. Then there exist partial differential operators \mathbf{M}_k of order $2k$ with real-analytic coefficients, such that for any integer $l \geq 0$ and any function smooth function f defined in a neighbourhood of \mathbb{T} , we have that*

$$\int_{\mathbb{T}} e^{il\theta} (\partial_r^2 R_\tau(re^{i\theta}))^{-\frac{1}{2}} \mathbf{L}_k[r^{1-l} f(re^{i\theta})] \Big|_{r=1} d\theta = \int_{\mathbb{T}} e^{il\theta} \mathbf{M}_k[f](e^{i\theta}) d\theta.$$

Proof. We first observe that by integration by parts, multiplication by l corresponds to applying the differential operator $i\partial_\theta$ inside the integral:

$$l \int_{\mathbb{T}} f(\theta) e^{il\theta} d\theta = \int_{\mathbb{T}} i\partial_\theta f(\theta) e^{il\theta} d\theta.$$

From this it is immediate that the formula

$$(3.2.1) \quad p(l) \int_{\mathbb{T}} f(\theta) e^{il\theta} d\theta = \int_{\mathbb{T}} p(i\partial_\theta) f(\theta) e^{il\theta} d\theta$$

holds for polynomials p . Structurally, $\mathbf{L}_k[r^{1-l}f(re^{i\theta})]$ can be written as

$$(3.2.2) \quad \mathbf{L}_k[r^{1-l}f(re^{i\theta})] = \sum_{\nu=k}^{3k} b_\nu(re^{i\theta}) \partial_r^{2\nu} [[W_\tau(re^{i\theta})]^{\nu-k} r^{1-l} f(re^{i\theta})],$$

where b_ν is the real-analytic function given by

$$b_\nu(re^{i\theta}) = \frac{(-1)^{\nu-k} 2^{-\nu}}{\nu! (\nu-k)! [\partial_r^2 R_\tau(re^{i\theta})]^\nu}.$$

We observe that by the Leibniz rule

$$(3.2.3) \quad \begin{aligned} \partial_r^j (r^{1-l} f(re^{i\theta})) \Big|_{r=1} &= \sum_{i=0}^j \binom{j}{i} (-1)^{j-i} (l-1)_{j-i} r^{1-l-j+i} \partial_r^i f(re^{i\theta}) \Big|_{r=1} \\ &= \sum_{i=0}^j \binom{j}{i} (-1)^{j-i} (l-1)_{j-i} \partial_r^i f(re^{i\theta}) \Big|_{r=1}, \end{aligned}$$

where $(x)_i = x(x+1)\cdots(x+i-1)$ denotes the standard Pochhammer symbol. We return to the formula (3.2.2) for \mathbf{L}_k . Again by the Leibniz formula we have that

$$\begin{aligned} \partial_r^{2\nu} [W_\tau^{\nu-k}(e^{i\theta}) r^{1-l} f(re^{i\theta})] \Big|_{r=1} &= \sum_{j=0}^{2\nu} \binom{2\nu}{j} \partial_r^{2\nu-j} ([W_\tau(re^{i\theta})]^{\nu-k}) \partial_r^j (r^{1-l} f(re^{i\theta})) \Big|_{r=1} \\ &= \sum_{j=0}^{3k-\nu} \binom{2\nu}{j} \partial_r^{2\nu-j} ([W_\tau(re^{i\theta})]^{\nu-k}) \partial_r^j (r^{1-l} f(re^{i\theta})) \Big|_{r=1} \\ &= \sum_{j=0}^{3k-\nu} \sum_{i=0}^j (-1)^{j-i} \binom{2\nu}{j} \binom{j}{i} (l-1)_{j-i} \partial_r^{2\nu-j} ([W_\tau(re^{i\theta})]^{\nu-k}) \partial_r^i f(re^{i\theta}) \Big|_{r=1}, \end{aligned}$$

where the truncation of the sum follows from an application of the flatness of W_τ near the unit circle \mathbb{T} , and the last equality is due to (3.2.3). We write the expression for $\mathbf{L}_k[r^{1-l}f(re^{i\theta})]$ as

$$\mathbf{L}_k[r^{1-l}f(re^{i\theta})] \Big|_{r=1} = \sum_{\nu=k}^{3k} \sum_{j=0}^{3k-\nu} \sum_{i=0}^j (l-1)_{j-i} c_{i,j,\nu}(e^{i\theta}) \partial_r^i f(re^{i\theta}) \Big|_{r=1},$$

where

$$c_{i,j,\nu}(e^{i\theta}) = (-1)^{j-i} \binom{2\nu}{j} \binom{j}{i} (l-1)_{j-i} b_\nu(e^{i\theta}) \partial_r^{2\nu-j} ([W_\tau(re^{i\theta})]^{\nu-k}) \Big|_{r=1}.$$

Changing the order of summation, we arrive at

$$\begin{aligned} \left(\partial_r^2 R_\tau(re^{i\theta}) \right)^{-\frac{1}{2}} \mathbf{L}_k[r^{1-l}f(re^{i\theta})] \Big|_{r=1} &= \sum_{i=0}^{2k} \sum_{j=i}^{2k} (-1)^{j-i} \binom{j}{i} (l-1)_{i-j} \left(\partial_r^2 R_\tau(re^{i\theta}) \right)^{-\frac{1}{2}} d_j(e^{i\theta}) \partial_r^i f(re^{i\theta}) \Big|_{r=1}, \end{aligned}$$

where

$$d_j(e^{i\theta}) = \sum_{\nu=k}^{3k-j} \binom{2\nu}{j} b_\nu(e^{i\theta}) \partial_r^{2\nu-j} ([W_\tau(re^{i\theta})]^{\nu-k}) \Big|_{r=1}.$$

It follows from (3.2.1) that the asserted identity holds with \mathbf{M}_k given by

$$\mathbf{M}_k[f](e^{i\theta}) = \sum_{i=0}^{2k} \sum_{j=i}^{2k} (-1)^{j-i} \binom{j}{i} (i\partial_\theta - 1)_{i-j} \left[\left(\partial_r^2 R_\tau(re^{i\theta}) \right)^{-\frac{1}{2}} d_j(e^{i\theta}) \partial_r^i f(re^{i\theta}) \right] \Big|_{r=1}.$$

The proof of the lemma is complete. \square

3.3. Algorithmic computation of the coefficients in the asymptotic expansion. In this section we supply the proof of Theorem 1.3.7, and explain the underlying computational algorithm. The main point is that we show how to iteratively obtain the coefficients, given that an asymptotic expansion exists, as formulated in Theorem 1.3.3. The proof of Theorem 1.3.3 is then supplied later on in Section 4.

Proof of Theorem 1.3.7. Fix the precision κ to be a positive integer. Let $F_{n,m}^{(\kappa)}$ be the approximate orthogonal quasipolynomials from Theorem 1.3.3 with the expansion

$$F_{n,m}^{(\kappa)}(z) = \left(\frac{m}{2\pi}\right)^{\frac{1}{4}} \sqrt{\phi'_\tau(z)} [\phi_\tau(z)]^n e^{m\mathcal{Q}_\tau(z)} \sum_{j=0}^{\kappa} m^{-j} \mathcal{B}_{j,\tau}(z),$$

where the functions $\mathcal{B}_{j,\tau}$ are bounded and holomorphic on \mathcal{K}_τ^c for some compact subset \mathcal{K}_τ of \mathcal{S}_τ° , which we may assume to be the conformal image of the exterior disk $\mathbb{D}_e(0, \rho_0)$ under the mapping ϕ_τ^{-1} . If we make the ansatz

$$\mathcal{B}_{j,\tau}(z) = (2\pi)^{1/4} \sqrt{\phi'_\tau(z)} (B_{j,\tau} \circ \phi_\tau)(z),$$

we may express $F_{n,m}$ using the canonical positioning operator as $F_{n,m}^{(\kappa)} = m^{\frac{1}{4}} \mathbf{\Lambda}_{n,m} [f_{n,m}^{(\kappa)}]$, where

$$(3.3.1) \quad f_{n,m}^{(\kappa)}(z) = \sum_{j=0}^{\kappa} m^{-j} B_{j,\tau}(z), \quad z \in \mathbb{D}_e(0, \rho_0).$$

According to Theorem 1.3.3, the functions $F_{n,m}^{(\kappa)}$ have the approximate orthogonality property

$$(3.3.2) \quad \int_{\mathbb{C}} \chi_{0,\tau} F_{n,m}^{(\kappa)} \bar{p} e^{-2mQ} dA = O(m^{-\kappa-1} \|p\|_{2mQ}), \quad p \in \text{Pol}_n.$$

The function $\chi_{0,\tau}$ is a cut-off function with $0 \leq \chi \leq 1$ throughout \mathbb{C} , such that $\chi_{0,\tau}$ vanishes on \mathcal{K}_τ and equals 1 on \mathcal{X}_τ^c , where \mathcal{K}_τ lies at a fixed positive distance from $\partial\mathcal{S}_\tau$, and \mathcal{X}_τ is an intermediate set between them (cf. Definiton 1.3.2). We consider the associated cut-off function $\chi_{1,\tau} = \chi_{0,\tau} \circ \phi_\tau^{-1}$, tacitly extended to vanish where it is undefined. Without loss of generality, we may assume that $\chi_{1,\tau}$ is radial. By Remark 1.3.4, we may assume that $\chi_{1,\tau}$ vanishes on $\mathbb{D}(0, \rho'_0)$ for some number ρ'_0 with $\rho_0 < \rho'_0 < 1$. In order to compute the functions $B_{j,\tau}$, we would like to apply equation (3.3.2) to

$$q(z) = \mathbf{\Lambda}_{n,m} [z^{-l}] = \phi'_\tau(z) [\phi_\tau(z)]^{n-l} e^{m\mathcal{Q}_\tau(z)}$$

for a positive integer l , but this function is not a polynomial! To fix this, we consider the $L^2_{2mQ,n}$ -minimal solution v to the $\bar{\partial}$ -problem

$$\bar{\partial}v = \bar{\partial}(\chi_{0,\tau}q) = q \bar{\partial}\chi_{0,\tau}.$$

If v is the solution, then the difference $\chi_{0,\tau}q - v$ will be an entire function with the polynomial growth bound $O(|z|^{n-1})$ at infinity, and hence a polynomial of degree less than or equal to $n - 1$. By the estimate of Corollary 2.4.2, we have the norm control

$$\int_{\mathbb{C}} |v|^2 e^{-2mQ} dA \leq \frac{1}{2m} \int_{\mathbb{C}} |q|^2 |\bar{\partial}\chi_{0,\tau}|^2 \frac{e^{-2mQ}}{\Delta Q} dA \leq \frac{A^2}{2m\alpha_1} \int_{\mathcal{X}_\tau \setminus \mathcal{K}_\tau} |q|^2 e^{-2mQ} dA,$$

where we have used that there exists a positive real α_1 such that $\Delta Q \geq \alpha_1$ holds on \mathcal{S}_τ , which contains the support of $\bar{\partial}\chi_{0,\tau}$, and that we have the bound $|\bar{\partial}\chi_{0,\tau}| \leq A$. Since the support of $\bar{\partial}\chi_{0,\tau}$ lies in \mathcal{K}_τ^c , we may use the structure of q as $q = \mathbf{\Lambda}_{n,m} [z^{-l}]$ and Proposition 3.1.5

$$\int_{\mathcal{X}_\tau \setminus \mathcal{K}_\tau} |q|^2 e^{-2mQ} dA = \int_{\rho_0 \leq |z| \leq \rho''_0} |z|^{-2l} e^{-2mR_\tau(z)} dA(z),$$

where ρ''_0 is associated with a natural choice of the intermediate set \mathcal{X}_τ as the image of an exterior disk under ϕ_τ^{-1} , and satisfies $\rho_0 < \rho'_0 < \rho''_0 < 1$. Due to Proposition 3.1.3, this immediately gives that for any fixed positive integer l

$$\int_{\mathbb{C}} |v|^2 e^{-2mQ} dA = O(e^{-\epsilon_1 m})$$

as m, n tend to infinity while $\tau = \frac{n}{m} \in I_{\epsilon_0}$, for some positive real ϵ_1 . This means that for a fixed positive integer l , we have for $q = \mathbf{\Lambda}[z^{-l}]$ the approximate orthogonality

$$(3.3.3) \quad \int_{\mathbb{C}} \chi_{0,\tau}^2 F_{n,m}^{(\kappa)} \bar{q} e^{-2mQ} dA = O(m^{-\kappa-1}),$$

where we have used that $\chi_{0,\tau} q - v$ is a polynomial of degree at most $n-1$, and the above smallness of v . If we use the canonical positioning operator as in Proposition 3.1.5 in polarized form, (3.3.3) reads in polar coordinates

$$(3.3.4) \quad m^{\frac{1}{4}} \int_{\mathbb{T}} e^{il\theta} \int_{\rho_0}^{\infty} r^{1-l} \chi_{1,\tau}^2(r) f_{n,m}^{(\kappa)}(re^{i\theta}) e^{-2mR_{\tau}(re^{i\theta})} dr ds(e^{i\theta}) = O(m^{-\kappa-1}),$$

for fixed l . We now apply proposition 2.7.2 to the radial integral, with $V(r) = 2R_{\tau}(re^{i\theta})$. Note that $\partial_r^2 R_{\tau}(re^{i\theta})|_{r=1} = 4\Delta R_{\tau}(e^{i\theta})$. As a consequence, the inner integral in (3.3.4) has an expansion

$$\begin{aligned} \int_{\rho_0}^{\infty} r^{1-l} \chi_{1,\tau}^2(r) f_{n,m}^{(\kappa)}(re^{i\theta}) e^{-2mR_{\tau}(re^{i\theta})} dr &= \left(\frac{\pi}{4m\Delta R_{\tau}(e^{i\theta})} \right)^{\frac{1}{2}} \sum_{j=0}^{\kappa} m^{-j} \mathbf{L}_j[r^{1-l} f_{m,n}^{(\kappa)}(re^{i\theta})] \Big|_{r=1} \\ &+ O\left(m^{-\kappa-1} \|r^{1-l} \chi_{1,\tau}^2 f_{n,m,\theta}^{(\kappa)}\|_{C^2(\kappa+1)([\rho_0, \rho_2])} + \|r^{1-l} \chi_{1,\tau}^2 f_{n,m,\theta}^{(\kappa)}\|_{L^{\infty}([\rho_1, \infty))} \rho_1^{-m\vartheta+1} \right), \end{aligned}$$

where we to simplify the notation we use the subscript θ to denote the radial restriction $f_{\theta}(r) = f(re^{i\theta})$. Here, ϑ, α and ρ_1 are some real numbers with $\vartheta > 0$, $\alpha > 0$ and $1 < \rho_1 < \rho_2$, which are independent of $\tau \in I_{\epsilon_0}$. By applying the standard Cauchy estimates to the functions $f_{m,n}$, and by Remark 1.3.4 (both part (a) and (b) are needed) we have uniform control on the norms

$$\|r^{1-l} \chi_{1,\tau}^2 f_{n,m,\theta}^{(\kappa)}\|_{C^2(\kappa+1)([\rho_0, \rho_2])} \quad \text{and} \quad \|r^{1-l} \chi_{1,\tau}^2 f_{n,m,\theta}^{(\kappa)}\|_{L^{\infty}([\rho_1, \infty))}$$

provided that l is fixed, and that $f_{m,n}^{(\kappa)}$ are uniformly bounded. For fixed l , it follows that

$$(3.3.5) \quad \int_{\rho_0}^{\infty} r^{1-l} \chi_{1,\tau}^2(r) f_{n,m}^{(\kappa)}(re^{i\theta}) e^{-2mR_{\tau}(re^{i\theta})} dr = \left(\frac{\pi}{4m\Delta R_{\tau}(e^{i\theta})} \right)^{\frac{1}{2}} \sum_{j=0}^{\kappa} m^{-j} \mathbf{L}_j[r^{1-l} f_{m,n}^{(\kappa)}(re^{i\theta})] \Big|_{r=1} + O(m^{-\kappa-1}),$$

where the implied constant is uniformly bounded as long as $f_{n,m}^{(\kappa)}$ is uniformly bounded on $\mathbb{D}_e(0, \rho_0)$. By expanding the expression (3.3.1) for $f_{n,m}^{(\kappa)}$, it follows from (3.3.5) that

$$(3.3.6) \quad \int_{\rho_0}^{\infty} r^{1-l} \chi_{1,\tau}^2(r) f_{n,m}^{(\kappa)}(re^{i\theta}) e^{-2mR_{\tau}(re^{i\theta})} dr = \left(\frac{\pi}{4m\Delta R_{\tau}(e^{i\theta})} \right)^{\frac{1}{2}} \sum_{k=0}^{\kappa} m^{-k} \mathbf{L}_k[r^{1-l} f_{n,m}^{(\kappa)}(re^{i\theta})] \Big|_{r=1} + O(m^{-\kappa-1}) \\ = \left(\frac{\pi}{4m\Delta R_{\tau}(e^{i\theta})} \right)^{\frac{1}{2}} \sum_{j=0}^{\kappa} m^{-j} \sum_{k=0}^j \mathbf{L}_k[r^{1-l} B_{j-k,\tau}(re^{i\theta})] \Big|_{r=1} + O(m^{-\kappa-1}),$$

as $m \rightarrow \infty$. We multiply the expression (3.3.6) by $e^{il\theta}$ and integrate with respect to θ to get

$$\begin{aligned} m^{\frac{1}{4}} \int_{\mathbb{T}} e^{il\theta} \int_{\rho_0}^{\infty} r^{1-l} \chi_{1,\tau}^2(r) f_{n,m}^{(\kappa)}(re^{i\theta}) e^{-2mR_{\tau}(re^{i\theta})} dr ds(e^{i\theta}) \\ = \sum_{j=0}^{\kappa} m^{-j-\frac{1}{4}} \int_{\mathbb{T}} e^{il\theta} \left(\frac{\pi}{4\Delta R_{\tau}(e^{i\theta})} \right)^{\frac{1}{2}} \sum_{k=0}^j \mathbf{L}_k[r^{1-l} B_{j-k,\tau}(re^{i\theta})] \Big|_{r=1} ds(e^{i\theta}) + O(m^{-\kappa-\frac{3}{4}}), \end{aligned}$$

as $m \rightarrow \infty$. This is an asymptotic series, and so is (3.3.4), only that all the coefficients vanish in the latter, and only the error term remains. Since two asymptotic series coincide only if they coincide term by term, we find that for integers $j = 0, \dots, \kappa$,

$$\int_{\mathbb{T}} e^{il\theta} (4\Delta R_\tau(e^{i\theta}))^{-\frac{1}{2}} \sum_{k=0}^j \mathbf{L}_k[r^{1-l} B_{j-k,\tau}(re^{i\theta})] \Big|_{r=1} ds(e^{i\theta}) = 0, \quad l = 1, 2, 3, \dots$$

This condition looks like the standard condition membership in the Hardy space H^2 . The problem with this is that the functions unfortunately depend on the parameter l , so the criterion does not apply. To remedy this, we apply Lemma 3.2.1, which gives

$$(3.3.7) \quad \int_{\mathbb{T}} e^{il\theta} \sum_{k=0}^j \mathbf{M}_k[B_{j-k,\tau}](e^{i\theta}) ds(e^{i\theta}) = 0, \quad l = 1, 2, 3, \dots,$$

which is now of the desired form. So, by the standard Fourier analytic characterization of the Hardy space, the equation (3.3.7) is equivalent to having

$$(3.3.8) \quad \sum_{k=0}^j \mathbf{M}_k[B_{j-k,\tau}] \Big|_{\mathbb{T}} \in H^2, \quad j = 0, \dots, \kappa.$$

We look at the case $j = 0$ first. Then (3.3.8) says that $\mathbf{M}_0[B_{0,\tau}] \Big|_{\mathbb{T}} \in H^2$. The operator \mathbf{M}_0 , with the defining property given by Lemma 3.2.1, has the form

$$(3.3.9) \quad \mathbf{M}_0[f](e^{i\theta}) = (4\Delta R_\tau(e^{i\theta}))^{-\frac{1}{2}} f(e^{i\theta}).$$

We recall that it is given that $B_{0,\tau}$ is bounded and holomorphic in a neighbourhood of the closed exterior disk \mathbb{D}_e , so that in particular $B_{0,\tau} \Big|_{\mathbb{T}} \in H_-^2$. If we combine this with the observation that $\mathbf{M}_0[B_{0,\tau}] \Big|_{\mathbb{T}} \in H^2$ together with the explicit expression (3.3.9) for \mathbf{M}_0 , we arrive at

$$(3.3.10) \quad B_{0,\tau} \Big|_{\mathbb{T}} \in (4\Delta R_\tau)^{\frac{1}{2}} H^2 \cap H_-^2.$$

Let H_{R_τ} be a bounded holomorphic function in \mathbb{D}_e such that

$$(3.3.11) \quad \operatorname{Re} H_{R_\tau} = \frac{1}{2} \log(4\Delta R_\tau)^{\frac{1}{2}} = \frac{1}{4} \log(4\Delta R_\tau), \quad \text{on } \mathbb{T}.$$

It follows from the given regularity of R_τ that H_{R_τ} is a bounded holomorphic function in the exterior disk, which extends holomorphically to a neighbourhood of \mathbb{D}_e . We may rewrite (3.3.10) in the form

$$B_{0,\tau} \Big|_{\mathbb{T}} \in e^{2\operatorname{Re} H_{R_\tau}} H^2 \cap H_-^2.$$

By Proposition 2.5.1 applied with $u = v = -\bar{H}_{R_\tau}$ and $F = 0$, it follows that $B_{0,\tau}$ is of the form

$$(3.3.12) \quad B_{0,\tau} = c_{0,\tau} e^{H_{R_\tau}}$$

for some constant $c_{0,\tau}$.

We now proceed to consider more generally $j = 1, 2, 3, \dots$. If we separate out the term corresponding to $k = 0$ from equation (3.3.8), we find that

$$(3.3.13) \quad \frac{B_{j,\tau}}{(4\Delta R_\tau)^{\frac{1}{2}}} + \sum_{k=1}^j \mathbf{M}_k[B_{j-k,\tau}] \Big|_{\mathbb{T}} \in H^2, \quad j = 1, \dots, \kappa.$$

This equation allows us to compute $B_{j,\tau}$, given that we have already obtained the functions $B_{0,\tau}, \dots, B_{j-1,\tau}$. Indeed, if we put

$$F_{j,\tau} = \sum_{k=1}^j \mathbf{M}_k[B_{j-k,\tau}],$$

which involves only the functions $B_{0,\tau}, \dots, B_{j-1,\tau}$, we may write (3.3.13) in the form

$$B_{j,\tau} \Big|_{\mathbb{T}} \in H_-^2 \cap (4\Delta R_\tau)^{\frac{1}{2}} (-F_{j,\tau} + H^2) = H_-^2 \cap e^{2\operatorname{Re} H_{R_\tau}} (-F_{j,\tau} + H^2),$$

which by Proposition 2.5.1 has the solution

$$(3.3.14) \quad B_{j,\tau} = c_{j,\tau} e^{H_{R\tau}} - e^{H_{R\tau}} \mathbf{P}_{H_{0,-}^2} [e^{\bar{H}_{R\tau}} F_{j,\tau}],$$

for some constant $c_{j,\tau}$. Since $B_{0,\tau}$ is known up to a constant multiple, this allows us to iteratively derive $B_{j,\tau}$ for $j = 1, \dots, \kappa$. The only remaining freedom is the choice of the constants $c_{j,\tau}$ for $j = 0, \dots, \kappa$. We proceed to determine them. Since the orthogonal polynomials $P_{n,m}$ are normalized, it follows from Theorem 1.3.3 together with the triangle inequality that

$$\|\chi_{0,\tau} F_{n,m}^{(\kappa)}\|_{2mQ} = 1 + O(m^{-\kappa-1})$$

as $m \rightarrow \infty$. Since $\chi_{0,\tau} F_{n,m}^{(\kappa)} = m^{\frac{1}{4}} \mathbf{A}_{n,m} [\chi_{1,\tau} f_{n,m}^{(\kappa)}]$, it follows from the isometric property described in Proposition 3.1.5 that

$$(3.3.15) \quad m^{\frac{1}{2}} \int_{\mathbb{C}} \chi_{1,\tau}^2 |f_{n,m}^{(\kappa)}|^2 e^{-2mR\tau} dA = \int_{\mathbb{C}} \chi_{0,\tau}^2 |F_{n,m}^{(\kappa)}|^2 e^{-2mQ} dA = 1 + O(m^{-\kappa-1}).$$

Here, the integrals are over the whole plane, although the isometry is only over the complements of certain compact subsets. However, since we interpret the products with the cut-off functions as vanishing where the cut-off function vanishes itself, this is of no concern to us. We now expand $f_{n,m}^{(\kappa)}$ according to (3.3.1), so that by equation (3.3.15),

$$(3.3.16) \quad 2m^{\frac{1}{2}} \sum_{j,k=0}^{\kappa} m^{-(j+k)} \int_{\mathbb{T}} \int_{\rho_0}^{\infty} \chi_{1,\tau}^2(r) B_{j,\tau}(re^{i\theta}) \bar{B}_{k,\tau}(re^{i\theta}) e^{-2mR\tau(re^{i\theta})} r dr ds(e^{i\theta}) \\ = 1 + O(m^{-\kappa-1}),$$

where the factor 2 appears as a result of our normalizations. This equation is what will give us the values of the constants $c_{j,\tau}$. We turn first to the case $j = 0$. By a trivial version of Proposition 2.7.2, for any integers j, k with $0 \leq j, k \leq \kappa$ we have the rough estimate

$$\int_{\rho_0}^{\infty} \chi_{1,\tau}^2(r) B_{j,\tau}(re^{i\theta}) \bar{B}_{k,\tau}(re^{i\theta}) e^{-2mR\tau(re^{i\theta})} r dr ds(re^{i\theta}) = O(m^{-\frac{1}{2}}),$$

where the implicit constant is uniform for $\tau \in I_{\epsilon_0}$. If we disregard all the contributions in (3.3.16) which are of order $O(m^{-\frac{1}{2}})$, we see that only $j = k = 0$ gives a nontrivial contribution. The term corresponding to $j = k = 0$ in (3.3.16) can be expanded using the Laplace method of Proposition 2.7.2 (recall the formula (3.3.12) for $B_{0,\tau}$), to give

$$2m^{\frac{1}{2}} \int_{\mathbb{T}} \int_{\rho_0}^{\infty} \chi_{1,\tau}^2(r) |B_{0,\tau}(re^{i\theta})|^2 e^{-2mR\tau(re^{i\theta})} r dr ds(e^{i\theta}) \\ = 2m^{\frac{1}{2}} |c_{0,\tau}|^2 \int_{\mathbb{T}} \left(\frac{\pi}{4m\Delta R_{\tau}(e^{i\theta})} \right)^{\frac{1}{2}} \mathbf{L}_0 [re^{2\operatorname{Re} H_{R\tau}(re^{i\theta})}] \Big|_{r=1} ds + O(m^{-\frac{1}{2}}).$$

Since in general, for a smooth function f we have that $\mathbf{L}_0[f(r)]|_{r=1} = f(1)$, the leading contribution simplifies to (recall the definition (3.3.11) of $H_{R\tau}$),

$$2m^{\frac{1}{2}} |c_{0,\tau}|^2 \int_{\mathbb{T}} \left(\frac{\pi}{4m\Delta R_{\tau}(e^{i\theta})} \right)^{\frac{1}{2}} \mathbf{L}_0 [re^{2\operatorname{Re} H_{R\tau}(re^{i\theta})}] \Big|_{r=1} ds \\ = 2\pi^{\frac{1}{2}} |c_{0,\tau}|^2 \int_{\mathbb{T}} (4\Delta R_{\tau}(e^{i\theta}))^{-\frac{1}{2}} e^{2\operatorname{Re} H_{R\tau}(e^{i\theta})} ds(e^{i\theta}) \\ = 2\pi^{\frac{1}{2}} |c_{0,\tau}|^2 \int_{\mathbb{T}} ds(e^{i\theta}) = 2\pi^{\frac{1}{2}} |c_{0,\tau}|^2.$$

As this is the leading contribution to (3.3.16), we must have $2\pi^{\frac{1}{2}} |c_{0,\tau}|^2 = 1$. This determines the constant $c_{0,\tau}$ up to a unimodular factor, and we choose $c_{0,\tau} = (4\pi)^{-\frac{1}{4}}$.

We turn to the remaining coefficients $c_{j,\tau}$, for $j = 1, \dots, \kappa$. By applying the Laplace method of Proposition 2.7.1 to the radial integral in the formula (3.3.16), we arrive at

$$2\pi^{\frac{1}{2}} \sum_{j=0}^{\kappa} m^{-j} \sum_{(i,k,l) \in \mathcal{J}_j^*} \int_{\mathbb{T}} (4\Delta R_\tau(e^{i\theta}))^{-\frac{1}{2}} \mathbf{L}_k[rB_{i,\tau}(re^{i\theta})\bar{B}_{l,\tau}(re^{i\theta})] \Big|_{r=1} ds(e^{i\theta}) = 1 + O(m^{-\kappa-\frac{1}{2}}),$$

where \mathcal{J}_j^* denotes the index set $\mathcal{J}_j^* = \{(i, k, l) \in \mathbb{N}^3 : i + k + l = j\}$, and \mathbb{N} stands for the natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$. As this represents an equality of asymptotic series, we may identify term by term. The term with $j = 0$ was already analyzed, and it follows that for $j = 1, \dots, \kappa$ we have

$$(3.3.17) \quad \sum_{(i,k,l) \in \mathcal{J}_j^*} \int_{\mathbb{T}} (4\Delta R_\tau(e^{i\theta}))^{-\frac{1}{2}} \mathbf{L}_k[rB_{i,\tau}(re^{i\theta})\bar{B}_{l,\tau}(re^{i\theta})] \Big|_{r=1} ds(e^{i\theta}) \\ = 2 \operatorname{Re} \int_{\mathbb{T}} (4\Delta R_\tau(e^{i\theta}))^{-\frac{1}{2}} \mathbf{L}_0[rB_{j,\tau}(re^{i\theta})\bar{B}_{0,\tau}(re^{i\theta})] \Big|_{r=1} ds(e^{i\theta}) \\ + \sum_{(i,k,l) \in \mathcal{J}_j} \int_{\mathbb{T}} (4\Delta R_\tau(e^{i\theta}))^{-\frac{1}{2}} \mathbf{L}_k[rB_{i,\tau}(re^{i\theta})\bar{B}_{l,\tau}(re^{i\theta})] \Big|_{r=1} ds(e^{i\theta}) = 0,$$

where \mathcal{J}_j denotes the restricted index set $\mathcal{J}_j = \mathcal{J}_j^* \cap \{(i, k, l) \in \mathcal{J}_j^* : i, l < j\}$, and where we separate out the terms involving the leading term $B_{j,\tau}$. We successfully resolve the first term on the right-hand side of (3.3.17), while the second term is much more complicated. However, we may observe that it only depends on the functions $B_{\nu,\tau}$ with $\nu = 0, \dots, j-1$, and hence only on the constants $c_{\nu,\tau}$ with $\nu = 0, \dots, j-1$. This allows us to algorithmically determine these constants, albeit with increasing degree of complexity. As for the first term on the right-hand side, we observe that the operator $\mathbf{L}_0|_{r=1}$ only evaluates at $r = 1$. Using the structure of $B_{j,\tau}$ as given by (3.3.14), we find that

$$\int_{\mathbb{T}} (4\Delta R_\tau(e^{i\theta}))^{-\frac{1}{2}} \mathbf{L}_0[rB_{j,\tau}(re^{i\theta})\bar{B}_{0,\tau}(re^{i\theta})] \Big|_{r=1} ds(e^{i\theta}) \\ = \int_{\mathbb{T}} (4\Delta R_\tau(e^{i\theta}))^{-\frac{1}{2}} B_{j,\tau}(e^{i\theta})\bar{B}_{0,\tau}(e^{i\theta}) ds(e^{i\theta}) \\ = c_{0,\tau} \int_{\mathbb{T}} (4\Delta R_\tau(e^{i\theta}))^{-\frac{1}{2}} e^{2\operatorname{Re} H_{R_\tau}(re^{i\theta})} (c_{j,\tau} - \mathbf{P}_{H_{-0}^2} [e^{\bar{H}_{R_\tau}} F_{j,\tau}](e^{i\theta})) ds(e^{i\theta}) \\ = c_{0,\tau} \int_{\mathbb{T}} (c_{j,\tau} - \mathbf{P}_{H_{-0}^2} [e^{\bar{H}_{R_\tau}} F_{j,\tau}](e^{i\theta})) ds(e^{i\theta}) = c_{0,\tau} c_{j,\tau}.$$

Here we use the definition (3.3.11) of H_{R_τ} and the fact that the projection $\mathbf{P}_{H_{-0}^2}$ maps into a subspace of functions with mean 0. Assume now that j is given, and that we have determined $c_{k,\tau}$ for $k = 0, \dots, j-1$. The above equality together with (3.3.17) then gives that

$$2 \operatorname{Re} c_{j,\tau} c_{0,\tau} = - \sum_{(i,k,l) \in \mathcal{J}_j} \int_{\mathbb{T}} (4\Delta R_\tau(e^{i\theta}))^{-\frac{1}{2}} \mathbf{L}_k[rB_{i,\tau}(re^{i\theta})\bar{B}_{l,\tau}(re^{i\theta})] \Big|_{r=1} ds(e^{i\theta}).$$

Since $c_{0,\tau} = (4\pi)^{-\frac{1}{4}}$, we obtain that

$$\operatorname{Re} c_{j,\tau} = -\frac{1}{2} (4\pi)^{\frac{1}{4}} \sum_{(i,k,l) \in \mathcal{J}_j} \int_{\mathbb{T}} (4\Delta R_\tau(e^{i\theta}))^{-\frac{1}{2}} \mathbf{L}_k[rB_{i,\tau}(re^{i\theta})\bar{B}_{l,\tau}(re^{i\theta})] \Big|_{r=1} ds(e^{i\theta}).$$

This completes the proof. \square

4. THE EXISTENCE OF ASYMPTOTIC EXPANSIONS

4.1. Approximately orthogonal quasipolynomials and orthogonal foliations. We recall from Section 3.1 the canonical positioning operator $\mathbf{A}_{n,m}$

$$\mathbf{A}_{n,m}[f](z) = \phi'_\tau(z) [\phi_\tau(z)]^n e^{m\mathcal{Q}_\tau(z)} (f \circ \phi_\tau)(z), \quad z \in \mathcal{K}_\tau^c,$$

defined for f holomorphic in $\mathbb{D}_e(0, \rho_0)$. We intend to show the existence of a sequence of *approximately orthogonal quasipolynomials* with an asymptotic expansion. Specifically, in Section 4.8, we will prove the following result.

Proposition 4.1.1. *Let $\kappa \in \mathbb{N}$ be given. There exists a sequence of normalized approximately orthogonal quasipolynomials $F_{n,m}^{(\kappa)}$ to accuracy κ in the sense of Definition 3.1.2, of the form*

$$F_{n,m}^{(\kappa)}(z) = m^{\frac{1}{4}} \phi'_\tau(z) [\phi_\tau(z)]^n e^{m\mathcal{Q}_\tau(z)} (f_{n,m}^{(\kappa)} \circ \phi_\tau)(z)$$

where

$$f_{n,m}^{(\kappa)}(z) = \sum_{j=0}^{\kappa} m^{-j} B_{j,\tau}(z),$$

and $B_{j,\tau}$ are holomorphic functions in $\mathbb{D}_e(0, \rho_0)$, which are uniformly bounded for $\tau \in I_{\epsilon_0}$.

We will obtain Proposition 4.1.1 as a consequence of the existence of what we call the *approximate orthogonal foliation flow of simple loops* $\Gamma_{n,m,t}$, parameterized by the parameter t . For convenience of notation, let δ_m be the number

$$\delta_m = m^{-1/2} \log m.$$

A conformal mapping ψ of the exterior disk \mathbb{D}_e onto a domain containing the point at infinity is said to be *normalized* if it maps ∞ to ∞ , and has $\psi'(\infty) > 0$. Given a smooth family ψ_t of normalized conformal mappings on the exterior disk, indexed by a real parameter t close to 0, such that the image domains $\Omega_t := \psi_t(\mathbb{D}_e)$ increase with t , we may form the *foliation mapping* Ψ by the formula

$$\Psi(z) = \psi_{1-|z|} \left(\frac{z}{|z|} \right),$$

for z in some annulus \mathbb{A} containing the unit circle. The foliation mapping Ψ maps \mathbb{A} onto the domain \mathcal{D} covered by the boundaries

$$\mathcal{D} = \bigcup_t \psi_t(\mathbb{T}).$$

Moreover, the Jacobian J_Ψ is given by

$$(4.1.1) \quad J_\Psi(z) = -\operatorname{Re} \left\{ \frac{\bar{z}}{|z|^2} \partial_t \psi_t \left(\frac{z}{|z|} \right) \overline{\psi'_t \left(\frac{z}{|z|} \right)} \right\} \Big|_{t=1-|z|}.$$

Since the loops $\Gamma_{n,m,t}$ of the orthogonal foliation flow are Jordan curves, they each divide the plane into two components. We denote the normalized conformal mapping of the exterior disk \mathbb{D}_e onto the unbounded component of $\mathbb{C} \setminus \Gamma_{n,m,t}$ by $\psi_{n,m,t}$. Denote the corresponding foliation mapping by $\Psi_{n,m}$. We may integrate over a flow, encoded by a foliation mapping Ψ as follows: If we denote by \mathbb{A}_ϵ the annulus $\mathbb{A}_\epsilon = \mathbb{D}(0, 1 + \epsilon) \setminus \overline{\mathbb{D}(0, 1 - \epsilon)}$, we have for integrable f ,

$$(4.1.2) \quad \begin{aligned} \int_{\Psi(\mathbb{A}_\epsilon)} f dA &= \int_{\mathbb{A}_\epsilon} f \circ \Psi J_\Psi dA \\ &= 2 \int_{-\epsilon}^{\epsilon} \int_{\mathbb{T}} f \circ \psi_t(\zeta) (1-t) J_\Psi((1-t)\zeta) ds(\zeta) dt. \end{aligned}$$

The existence of the foliation flow may be phrased as follows.

Lemma 4.1.2. *Fix the precision parameter κ to be a positive integer. For $\tau = \frac{n}{m} \in I_{\epsilon_0}$, there exist compact subsets $\mathcal{K}_\tau \subset \mathcal{S}_\tau$ with*

$$\inf_{\tau \in I_{\epsilon_0}} \operatorname{dist}_{\mathbb{C}}(\mathcal{K}_\tau, \partial \mathcal{S}_\tau) > 0,$$

and bounded holomorphic functions $B_{j,\tau}$ on \mathcal{K}_τ^c for $j = 1 \dots, \kappa$ and a smooth family of normalized conformal mappings $\{\psi_{n,m,t}\}_{n,m,t}$ on \mathbb{D}_e , such that if $f_{n,m}^{(\kappa)}$ is given as a κ -abschnitt of an

asymptotic expansion (3.3.1), then for $\zeta \in \mathbb{T}$,

$$(4.1.3) \quad m^{\frac{1}{2}} |f_{n,m} \circ \psi_{n,m,t}(\zeta)|^2 e^{-2m(R_\tau \circ \psi_{n,m,t})(\zeta)} (1-t) J_{\Psi_{n,m}}((1-t)\zeta) \\ = \frac{m^{\frac{1}{2}}}{(4\pi)^{\frac{1}{2}}} e^{-mt^2} + O(m^{-\kappa} e^{-2mR \circ \psi_{n,m,t}}),$$

where the implicit constant is uniform for $|t| \leq \delta_m$ while $\tau \in I_{\epsilon_0}$. Moreover, if $\mathcal{D}_{n,m} = \bigcup_{|t| \leq \delta_m} \psi_{n,m,t}(\mathbb{T})$, then $\text{dist}_{\mathbb{C}}(\mathcal{D}_{n,m}^c, \partial \mathcal{S}_\tau) \geq c_0 \delta_m$ for some constant $c_0 > 0$ so it holds that

$$\int_{\mathcal{D}_{n,m}} m^{\frac{1}{2}} |f_{n,m}^{(\kappa)}|^2 e^{-2mR_\tau} dA = 1 + O(\delta_m^{2\kappa+1}) = 1 + O(m^{-\kappa - \frac{1}{3}}).$$

Remark 4.1.3. The equation (4.1.3) may be understood as an approximate *weighted Polubarina-Galin equation* with weight $|f_{n,m}^{(\kappa)}|^2 e^{-2mR_\tau}$, and variable speed of expansion. Indeed, we should compare with equation (6.11) in [25], which states in a similar context that along concentric circles,

$$J_\Psi = \omega^{-1} \circ \Psi,$$

where Ψ is a foliation mapping, and ω denotes a weight. In comparison, our factor $(4\pi)^{-\frac{1}{2}} e^{-mt^2}$ appears as consequence of the variable speed.

4.2. The orthogonal foliation flow I. Renormalization. In this section we present the outlines of the construction of the approximate orthogonal foliation flow, and the first step in the algorithm that produces it. In order to proceed with less obscuring notation, we consider a smooth family of bounded holomorphic functions $f_s(z)$, a smooth family of normalized conformal mappings $\psi_{s,t}$. Moreover, we denote by R a weight whose properties mirror those of R_τ , captured in the following definition. We denote by $\mathbb{A}(\rho_1, \rho_2)$ the annulus

$$\mathbb{A}(\rho_1, \rho_2) := \mathbb{D}(0, \rho_2) \setminus \bar{\mathbb{D}}(0, \rho_1),$$

for positive real numbers ρ_1 and ρ_2 with $\rho_1 < \rho_2$.

For a real-analytic function R there exists a *polarization* $R(z, w)$, which is holomorphic in (z, \bar{w}) and has $R(z, z) = R(z)$. This is easy to see using convergent local Taylor series expansions of $R(z)$ in the coordinates which are the real and imaginary parts, $\text{Re } z$ and $\text{Im } z$. By replacing $\text{Re } z$ by $\frac{1}{2}(z + w)$ and $\text{Im } z$ by $\frac{1}{2i}(z - w)$ in this expansion, we obtain the polarization $R(z, w)$. We observe that if $R(z, w)$ is such a polarization of a function $R(z)$ which is real-analytically smooth near the circle \mathbb{T} and in addition is quadratically flat there, then $R(z, w)$ factors as $R(z, w) = (1 - z\bar{w})^2 R_0(z, w)$, where $R_0(z, w)$ is holomorphic in (z, \bar{w}) in a neighborhood of the diagonal where both variables are near \mathbb{T} .

Definition 4.2.1. For positive real numbers ρ_0, σ_0 where $\rho_0 < 1$, we denote by $\mathcal{W}(\rho_0, \sigma_0)$ the class of C^2 -smooth functions $R : \mathbb{D}_e(0, \rho_0) \rightarrow \mathbb{R}_{+,0} := \mathbb{R}_+ \cup \{0\}$, such that R and ∇R both vanish on \mathbb{T} , while $\Delta R > 0$ holds on \mathbb{T} , and in addition, R is real-analytic in the annulus $\mathbb{A}(\rho, \rho_0^{-1})$ with polarization $R(z, w)$ which is holomorphic in (z, \bar{w}) on the $2\sigma_0$ -fattened diagonal annulus

$$\hat{\mathbb{A}}(\rho_0, \sigma_0) := \{(z, w) \in \mathbb{A}(\rho_0, \rho_0^{-1}) \times \mathbb{A}(\rho_0, \rho_0^{-1}) : |z - w| \leq 2\sigma_0\},$$

and factors as $R(z, w) = (1 - z\bar{w})^2 R_0(z, w)$, where $R_0(z, w)$ is holomorphic in (z, \bar{w}) and bounded and bounded away from 0 on the set $\hat{\mathbb{A}}(\rho, \sigma)$, while, in addition,

$$R_0(z, z) \geq \alpha(R) > 0, \quad z \in \mathbb{A}(\rho_0, \rho_0^{-1}),$$

and further away,

$$\inf_{z \in \mathbb{D}_e(0, \rho_0^{-1})} \frac{R(z)}{\log|z|} = \theta(R) > 0.$$

We say that a subset $\mathfrak{S} \subset \mathcal{W}(\rho_0, \sigma_0)$ is a *uniform family*, provided that for each $R \in \mathfrak{S}$, the corresponding $R_0(z, w)$ is uniformly bounded and bounded away from 0 on $\hat{\mathbb{A}}(\rho_0, \sigma_0)$ while the controlling constants such as $\alpha(R)$ and $\theta(R)$ are uniformly bounded away from 0.

If a function $f(z, w)$ is holomorphic in (z, \bar{w}) , we may consider the associated function

$$(4.2.1) \quad f_{\mathbb{T}}(z) = f\left(z, \frac{1}{\bar{z}}\right)$$

which is then holomorphic in z , wherever it is well-defined. We note that $f_{\mathbb{T}}(z) = f(z, z)$ on the circle \mathbb{T} . We recall the notation of Definition 4.2.1.

Proposition 4.2.2. *Suppose that $f(z, w)$ is holomorphic in (z, \bar{w}) on the domain $\hat{\mathbb{A}}(\rho, \sigma)$, where $0 < \rho < 1$ and $\sigma > 0$. Then the function $f_{\mathbb{T}}(z)$, which extends the restriction of the diagonal function $f(z, z)$, where $z \in \mathbb{T}$, is holomorphic on the annulus*

$$\rho' < |z| < (\rho')^{-1},$$

where

$$\rho' = \max \left\{ \rho, \sqrt{1 + \sigma^2} - \sigma \right\}.$$

Proof. The function $f_{\mathbb{T}}(z) = f(z, \bar{z}^{-1})$ is automatically holomorphic in the variable z in the domain

$$\left| z - \frac{1}{\bar{z}} \right| \leq 2\sigma,$$

provided that $z \in \mathbb{A}(\rho, \rho^{-1})$. By checking these requirements carefully, the assertion follows. \square

The setting which will prove useful to us is when we may control certain related quantities and their polarizations, which is possible on smaller thickened annuli. The polarization of $\log \Delta R$ appears later in the induction algorithm, while $\log(z\partial\hat{R})$ is important for the control associated with the implicit function theorem.

Proposition 4.2.3. *If R belongs to a uniform family $\mathfrak{S} \subset \mathcal{W}(\rho_0, \sigma_0)$ for some positive reals ρ_0, σ_0 with $\rho_0 < 1$, and if $\hat{R} = \sqrt{R}$ is chosen so that $\hat{R}(z)$ is positive for $|z| > 1$ and negative for $|z| < 1$, then there exist positive ρ_1, σ_1 with $\rho_0 \leq \rho_1 < 1$ and $\sigma_1 \leq \sigma_0$, such that the polarizations of the functions $\log \Delta R$, \hat{R} , $\log(z\partial\hat{R})$ are all holomorphic in (z, \bar{w}) and uniformly bounded on the $2\sigma_1$ -fattened diagonal annulus $\hat{\mathbb{A}}(\rho_1, \sigma_1)$.*

Proof. This follows from the assumptions on the uniform family, if we use the standard Cauchy estimates plus the fact that $\log \Delta R = \log(2R_0)$ and $\log(z\partial\hat{R}) = \frac{1}{2} \log R_0$ hold on the unit circle \mathbb{T} . \square

Remark 4.2.4. Suppose a real-analytic function $F(z)$ admits a polarization $F(z, w)$ which is holomorphic in (z, \bar{w}) for $(z, w) \in \hat{\mathbb{A}}(\rho, \sigma)$, and let f be given in terms of the Herglotz kernel by $f = \mathbf{H}_{\mathbb{D}_e}[F|_{\mathbb{T}}]$. We note that by the properties of the Herglotz kernel, f may be obtained by the formula $f = 2P_{H^2_{-,0}}[F|_{\mathbb{T}}] + \langle F \rangle_{\mathbb{T}}$, where $\langle F \rangle_{\mathbb{T}}$ denotes the average of F on the unit circle. Let $F_{\mathbb{T}}$ be as in (4.2.1), and express it in terms of its Laurent series, which by Proposition 4.2.2 converges in the annulus $\mathbb{A}(\rho', (\rho')^{-1})$:

$$F_{\mathbb{T}}(z) = \sum_{n \in \mathbb{Z}} a_n z^n.$$

In terms of the Laurent series, $P_{H^2_{-,0}}[F|_{\mathbb{T}}]$ equals $\sum_{n < 0} a_n z^n$ and $\langle F \rangle_{\mathbb{T}} = a_0$. As a consequence, $\mathbf{P}_{H^2_{-,0}}[F|_{\mathbb{T}}]$ defines a holomorphic function on the exterior disk $\mathbb{D}_e(0, \rho')$ and hence, f is holomorphic on $\mathbb{D}_e(0, \rho')$ as well.

For an integer n , we denote by \mathcal{I}_n the triangular index set

$$\mathcal{I}_n = \{(j, l) \in \mathbb{N}^2 : 2j + l \leq n\},$$

and supply it with the inherited lexicographic ordering \prec :

$$(i, k) \prec (j, l) \text{ if } i < j \text{ or } i = j \text{ and } k < l.$$

Recall the number ρ' from Proposition 4.2.2, which is defined in terms of the pair ρ_1, σ_1 supplied in Proposition 4.2.3.

The following is an analogue of Lemma 4.1.2.

Proposition 4.2.5. *Let κ be a given positive integer and let $R \in \mathcal{W}(\rho_0, \sigma_0)$, for some ρ_0, σ_0 with $0 < \rho_0 < 1$ and $\sigma_0 > 0$. Then there exist a radius ρ'' with $\rho' < \rho'' < 1$, bounded holomorphic functions $f_s = \sum_{j=0}^{\kappa} s^j B_j$ on $\mathbb{D}_e(0, \rho')$ and normalized conformal mappings*

$$\psi_{s,t} = \psi_{0,t} + \sum_{\substack{(j,l) \in \mathcal{I}_{2\kappa+1} \\ j \geq 1}} s^j t^l \hat{\psi}_{j,l}$$

defined on $\mathbb{D}_e(0, \rho'')$, such that $\psi_{s,t}(\mathbb{D}_e(0, \rho'')) \subset \mathbb{D}_e(0, \rho')$ holds for s and t close to 0, while the domains $\psi_{s,t}(\mathbb{D}_e)$ increase with t . Moreover, on the circle \mathbb{T} the functions f_s and $\psi_{s,t}$ have the property that

$$(4.2.2) \quad |f_s \circ \psi_{s,t}(\zeta)|^2 e^{-2s^{-1}R \circ \psi_{s,t}(\zeta)} \operatorname{Re} \left(-\bar{\zeta} \partial_t \psi_{s,t}(\zeta) \overline{\psi'_{s,t}(\zeta)} \right) \\ = e^{-t^2/s} \left\{ (4\pi)^{-\frac{1}{2}} + O(|s|^{\kappa+\frac{1}{2}} + |t|^{2\kappa+1}) \right\}.$$

Here, the implicit constant remains uniformly bounded as long as R is confined to a uniform family in $\mathcal{W}(\rho_0, \sigma_0)$, for fixed ρ_0, σ_0 .

The first step towards finding the conformal mappings $\psi_{s,t}$ is to note the following: if we set $h_s = \log|f_s|^2$, we find by taking logarithms that

$$(4.2.3) \quad h_s \circ \psi_{s,t}(\zeta) - 2s^{-1}(R \circ \psi_{s,t})(\zeta) + \log \operatorname{Re} \left(-\bar{\zeta} \partial_t \psi_{s,t} \overline{\psi'_{s,t}(\zeta)} \right) = -s^{-1}t^2 + O(1),$$

as $s, t \rightarrow 0$. Next, we multiply both sides by s , to obtain

$$(4.2.4) \quad sh_s \circ \psi_{s,t}(\zeta) - 2R \circ \psi_{s,t}(\zeta) + s \log \operatorname{Re} \left(-\bar{\zeta} \partial_t \psi_{s,t} \overline{\psi'_{s,t}(\zeta)} \right) = -t^2 + O(s).$$

Finally, we take the limit as $s \rightarrow 0$ expecting that $h_s \circ \psi_{s,t}$ and $\log \operatorname{Re}(-\bar{\zeta} \partial_t \psi_{s,t} \overline{\psi'_{s,t}})$ remain bounded, and arrive at the equation

$$R \circ \psi_{0,t}(\zeta) = \frac{t^2}{2}.$$

As a consequence, $\psi_{0,t}$ should be a conformal mapping onto the exterior of the appropriate level curve of the weight R .

Proposition 4.2.6. *There exists a positive number t_0 , and a real-analytically smooth family $\{\psi_{0,t}\}_{t \in (-t_0, t_0)}$ of normalized conformal mappings $\mathbb{D}_e \rightarrow \Omega_t$, where Ω_t is the unbounded component of $\mathbb{C} \setminus \Gamma_t$, and where Γ_t are real-analytically smooth, simple closed level curves of R :*

$$R|_{\Gamma_t} = \frac{t^2}{2}.$$

Moreover, $\Omega_0 = \mathbb{D}_e$ and Ω_t increases with t .

Proof. The assumed strict subharmonicity of R gives that there exists a neighbourhood U of \mathbb{T} such that $\nabla R|_{U \setminus \mathbb{T}} \neq 0$. This shows that the level sets must be simple and closed curves, for $|t|$ sufficiently small. Indeed, if a curve would possess a loop, then R would have to have a local extremal point inside the loop, which is impossible. Since ∇R vanishes on \mathbb{T} , we cannot apply the implicit function theorem directly to R to obtain the result. However, the function

$$\tilde{R}(re^{i\theta}) := \frac{R(re^{i\theta})}{(r-1)^2}$$

is, in view of Proposition 3.1.3, strictly positive and real-analytic in a neighbourhood of the unit circle \mathbb{T} . We form the square root $\hat{R} = \sqrt{\tilde{R}}$ by

$$\hat{R}(re^{i\theta}) = (r-1) \sqrt{\tilde{R}(re^{i\theta})},$$

where the square root on the right-hand side is the standard square root of a positive number. We may now apply the implicit function theorem to the function \hat{R} . The result follows immediately by applying the Riemann mapping theorem to the exterior of the resulting analytic level curves of \hat{R} . \square

Proposition 4.2.6 tells us that the conformal mappings $\psi_{0,t}$ extend to some domain containing $\bar{\mathbb{D}}_e$, but supplies little information on how much bigger such a domain is allowed to be. We will discuss this issue in Subsection 4.3 below. Along the way, we also obtain an alternative proof of Proposition 4.2.6, which may be viewed as a quantitative version of the implicit function theorem in the given context.

The Taylor coefficients $\hat{\psi}_{0,l}$ (in the flow variable t) of the conformal mappings $\psi_{0,t}$ can be explicitly computed in terms of R , using a higher order version of Nehari's formula for conformal mappings to nearly circular domains. We will return to this in Section 4.6. Before we continue, we recall the following elementary lemma, which allows us to draw the conclusion that the mappings $\psi_{s,t}$ are actually conformal.

Lemma 4.2.7. *Let \mathcal{D} denote a simply connected domain in $\hat{\mathbb{C}}$ containing the point at infinity, whose boundary $\partial\mathcal{D}$ is a simple closed curve. Denote by f a holomorphic function $f : \mathbb{D}_e \rightarrow \mathcal{D}$ such that $f'(z) \neq 0$ in \mathbb{D}_e and $f(z) = cz + O(1)$ as $|z| \rightarrow \infty$, which maps \mathbb{T} bijectively onto $\partial\mathcal{D}$. Then f is a conformal mapping.*

This result is well-known.

From this it immediately follows that the mappings $\psi_{s,t}$ are conformal when s is small enough. That the derivatives are non-vanishing is immediate, and a small computation allows one to show that for small enough $|s|$ and $|t|$, $\psi_{s,t}$ is injective on \mathbb{T} . Let us expand on the latter claim: we write

$$\psi_{s,t}(\zeta) = \psi_{0,t}(\zeta) + s\psi_{s,t}(\zeta),$$

where $\psi_{s,t}$ is some holomorphic function, which is uniformly bounded in the relevant parameter range. But then, if $\psi_{s,t}$ maps two distinct points $\zeta_1, \zeta_2 \in \mathbb{T}$ to the same point, it follows that

$$(4.2.5) \quad s^{-1} = \frac{\psi_{s,t}(\zeta_1) - \psi_{s,t}(\zeta_2)}{\psi_{0,t}(\zeta_1) - \psi_{0,t}(\zeta_2)}.$$

Since the functions $\psi_{0,t}$ and $\psi_{s,t}$ are smooth up to the boundary, and since $\psi'_{0,t} \neq 0$, the quotients

$$D_{s,t}(\zeta, \eta) := \frac{\psi_{s,t}(\zeta) - \psi_{s,t}(\eta)}{\psi_{0,t}(\zeta) - \psi_{0,t}(\eta)}$$

are uniformly bounded. It follows that if

$$s^{-1} > D := \sup_{s,t} D_{s,t}(\zeta, \eta),$$

where the supremum is taken over all s and t sufficiently close to zero, we arrive at a contradiction in (4.2.5), so $\psi_{s,t}$ must be univalent, and thus conformal.

4.3. The smoothness of level curves, the implicit function theorem, and Toeplitz kernel equations. We consider a function R , which is assumed to belong to the class $\mathcal{W}(\rho_0, \sigma_0)$ of Definition 4.2.1, which is a quantitative way to say that R is real-analytic near the unit circle \mathbb{T} , and vanishes along with its normal derivative on \mathbb{T} , while ΔR is positive on \mathbb{T} . Next, we recall the definition of the square root \hat{R} of R from the proof of Proposition 4.2.6. This function is also real-analytic near the circle, vanishes on \mathbb{T} but its gradient there is non-zero and points in the direction of the outward normal. As such, there exists a quantitative way to say this, which we now state. To this end, in analogy with Definition 4.2.1, we let $\hat{\rho}$ and $\hat{\sigma}$ denote numbers with $0 < \hat{\rho} < 1$ and $\hat{\sigma} > 0$ such that \hat{R} has a polarization $\hat{R}(z, w)$ which is holomorphic in (z, \bar{w}) for (z, w) in the $2\hat{\sigma}$ -fattened diagonal annulus $\hat{A}(\hat{\rho}, \hat{\sigma})$, such that

$$\sup_{(z,w) \in \hat{A}(\hat{\rho}, \hat{\sigma})} |\log(z\partial_z \hat{R}(z, w))| < +\infty.$$

In this subsection, we will focus on the conformal mappings $\psi_{0,t}$, and specifically the region to which they extend holomorphically. We recall that these mappings satisfy

$$\hat{R} \circ \psi_{0,t}(\zeta) = -\frac{t}{\sqrt{2}}, \quad \zeta \in \mathbb{T}.$$

Upon differentiating in t , we obtain

$$\partial_r \hat{R} \circ \psi_{0,t} \partial_t |\psi_{0,t}| + \partial_\theta \hat{R} \circ \psi_{0,t} \partial_t \arg \psi_{0,t} = -\frac{1}{\sqrt{2}},$$

which we may rewrite as

$$\begin{aligned} r \partial_r \hat{R} \circ \psi_{0,t} \partial \log |\psi_{0,t}| + \partial_\theta \hat{R} \circ \psi_{0,t} \partial_t \arg \psi_{0,t} \\ = \operatorname{Re} \left\{ (r \partial_r \hat{R} - i \partial_\theta \hat{R}) \circ \psi_{0,t} \partial_t \log \frac{\psi_{0,t}}{\zeta} \right\} = -\frac{1}{\sqrt{2}}, \end{aligned}$$

where we have divided by the coordinate function ζ in order to avoid issues with branch cuts of the logarithm. The differential operator acting on \hat{R} may be rewritten as $2z\partial_z$, so we may once again simplify

$$\operatorname{Re} \left\{ (2z\partial_z \hat{R}) \circ \psi_{0,t} \partial_t \log \frac{\psi_{0,t}}{\zeta} \right\} = -\frac{1}{\sqrt{2}}.$$

If we introduce the notation $\nu_t = \log(2z\partial_z \hat{R}) \circ \psi_{0,t}$ and $f_t = \partial_t \log \frac{\psi_{0,t}}{\zeta}$, this may be rewritten

$$e^{\nu_t} f_t + e^{\bar{\nu}_t} \bar{f}_t = -\sqrt{2}.$$

Here, the function $(2z\partial_z \hat{R}) \circ \psi_{0,t}$ evaluated at $t = 0$ is just $\sqrt{2\Delta R}$ on the circle \mathbb{T} , so there are no problems with taking the logarithm in the definition of ν_t for small t . Next, we make the decomposition $\nu_t = \nu_t^+ + \nu_t^-$, where $\nu_t^+ \in H^2$ and $\nu_t^- \in H_{-,0}^2$, and write $\tilde{f}_t = e^{\nu_t^-} f_t$. It is clear that $\tilde{f}_t \in H_-^2$. If we multiply the above equation by $e^{-2\operatorname{Re} \nu_t^+}$, we arrive at

$$e^{-\nu_t^+} \tilde{f}_t + e^{-\nu_t^+} \bar{\tilde{f}}_t = 2 \operatorname{Re} \{ e^{-\nu_t^+} \tilde{f}_t \} = -\sqrt{2} e^{-2\operatorname{Re} \nu_t^+},$$

where we point out that $e^{-\nu_t^+} \tilde{f}_t \in H_-^2$. This may be recognized as a Toeplitz kernel condition, and it has the solution

$$\tilde{f}_t = -\frac{1}{\sqrt{2}} e^{\nu_t^+} \mathbf{H}_{\mathbb{D}_e} [e^{-2\operatorname{Re} \nu_t^+}],$$

that is,

$$(4.3.1) \quad f_t = -\frac{1}{\sqrt{2}} e^{\bar{\nu}_t^+ - \nu_t^-} \mathbf{H}_{\mathbb{D}_e} [e^{-2\operatorname{Re} \nu_t^+}].$$

Let us recall that in the equation (4.3.1), the functions f_t and ν_t may be expressed in terms of \hat{R} and $\psi_{0,t}$. Let us write

$$(4.3.2) \quad g_t(\zeta) = \log \frac{\psi_{0,t}(\zeta)}{\zeta} \quad \text{and} \quad \mu(z) = \log(2z\partial_z \hat{R}(z)),$$

where both logarithms may be understood in terms of the principal branch of the logarithm. In terms of these functions, the equation (4.3.1) becomes the following non-linear differential equation in t :

$$(4.3.3) \quad \partial_t g_t = -\frac{1}{\sqrt{2}} \exp \left\{ \overline{\mathbf{P}_{H^2}[\mu \circ \psi_{0,t}]} - \mathbf{P}_{H_{-,0}^2}[\mu \circ \psi_{0,t}] \right\} \mathbf{H}_{\mathbb{D}_e} \left[\exp \left\{ -2 \operatorname{Re} \mathbf{P}_{H^2}[\mu \circ \psi_{0,t}] \right\} \right].$$

It is not difficult to see that the equation (4.3.3) may be solved by an iterative procedure, if we rewrite it in integral form

$$(4.3.4) \quad g_t = -\int_0^t \frac{1}{\sqrt{2}} \exp \left\{ \overline{\mathbf{P}_{H^2}[\mu \circ \psi_{0,\theta}]} - \mathbf{P}_{H_{-,0}^2}[\mu \circ \psi_{0,\theta}] \right\} \mathbf{H}_{\mathbb{D}_e} \left[\exp \left\{ -2 \operatorname{Re} \mathbf{P}_{H^2}[\mu \circ \psi_{0,\theta}] \right\} \right] d\theta.$$

As a first order approximation, we start with $\psi_{0,t}^{[0]}(\zeta) = \zeta$, and use the formula (4.3.4) to define $g_t^{[j+1]}$ in terms of $\psi_{0,t}^{[j]}$, for $j = 0, 1, 2, \dots$ by integration. The process is interlaced with computing $\psi_{0,t}^{[j+1]} := \zeta \exp(g_t^{[j+1]})$, and results in convergent sequences $g_t^{[j]}$ and $\psi_{0,t}^{[j]}$.

Next, we are interested in analyzing where the function $\psi_{0,t}$ extends to as a holomorphic mapping. To this end, we recall that the function μ given by (4.3.2) has a well-defined polarization to $\hat{\mathbb{A}}(\hat{\rho}, \hat{\sigma})$. It is clear that if $\psi_{0,t}$ maps $\mathbb{A}(\rho_t, \rho_t^{-1})$ into $\mathbb{A}(\hat{\rho}', (\hat{\rho}')^{-1})$, we obtain the estimate

$$\|\partial_t g_t\|_{H^\infty(\mathbb{A}(\rho_t, \rho_t^{-1}))} \leq \frac{\sqrt{2}}{1 - \rho_t^2} \exp \left\{ 5 \frac{\|\mu\|_{H^\infty(\mathbb{A}(\hat{\rho}', (\hat{\rho}')^{-1}))}}{1 - \rho_t^2} \right\},$$

where we use the estimate

$$\|\mathbf{P}_{H^2}[f]\|_{H^\infty(\mathbb{D}(0, \rho_t^{-1}))} \leq \frac{\|f\|_{H^\infty(\mathbb{A}(\rho_t, \rho_t^{-1}))}}{1 - \rho_t^2},$$

and the analogous estimate for $\mathbf{P}_{H^2_-,0}[f]$. Assume for the moment that $\rho_t < 1$ is monotonically increasing in $|t|$, and recall that $\psi_{0,t}(\zeta) = \zeta \exp(g_t)$. In light of the above estimate of $\partial_t g_t$, we obtain

$$\|g_t\|_{H^\infty(\mathbb{A}(\rho_t, \rho_t^{-1}))} \leq \frac{\sqrt{2}|t|}{1 - \rho_t^2} \exp \left\{ 5 \frac{\|\mu\|_{H^\infty(\mathbb{A}(\hat{\rho}', (\hat{\rho}')^{-1}))}}{1 - \rho_t^2} \right\} =: C_t |t|,$$

where C_t is defined implicitly by the last relation. This leads to the control

$$e^{-C_t |t|} \rho_t \leq |\psi_{0,t}(\zeta)| \leq e^{C_t |t|} (\rho_t)^{-1}, \quad \zeta \in \mathbb{A}(\rho_t, \rho_t^{-1}),$$

which means that $\psi_{0,t}$ maps the annulus $\mathbb{A}(\rho_t, \rho_t^{-1})$ into $\mathbb{A}(\hat{\rho}', (\hat{\rho}')^{-1})$, provided that

$$e^{-C_t |t|} \rho_t \geq \hat{\rho}'.$$

Let us make the ansatz $\rho_t = \hat{\rho}' e^{A|t|}$, for some constant A . The above requirement is then satisfied provided that $A \geq C_t$. If we restrict t to have

$$(4.3.5) \quad |t| \leq \frac{\log \frac{1}{\hat{\rho}'}}{2A},$$

it is immediate that

$$\frac{1}{1 - \rho_t^2} \leq \frac{1}{1 - \hat{\rho}'^2}.$$

This then gives the estimate for C_t

$$C_t \leq \frac{\sqrt{2}}{1 - \hat{\rho}'^2} \exp \left\{ 5 \frac{\|\mu\|_{H^\infty(\mathbb{A}(\hat{\rho}', (\hat{\rho}')^{-1}))}}{1 - \hat{\rho}'^2} \right\},$$

where the right-hand side does not depend on t . We may finally choose A to be this constant,

$$A = \frac{\sqrt{2}}{1 - \hat{\rho}'^2} \exp \left\{ 5 \frac{\|\mu\|_{H^\infty(\mathbb{A}(\hat{\rho}', (\hat{\rho}')^{-1}))}}{1 - \hat{\rho}'^2} \right\}$$

and obtain that $\psi_{0,t}$ is holomorphic in the exterior disk $\mathbb{D}_e(0, \rho_t)$, where

$$\rho_t = \hat{\rho}' e^{A|t|},$$

provided that t satisfies (4.3.5).

4.4. The orthogonal foliation flow II. Overview of the algorithm. We now proceed to describe the outlines of the algorithm. With the notation

$$(4.4.1) \quad \omega_{s,t}(\zeta) = |(f_s \circ \psi_{s,t})(\zeta)|^2 e^{-2s^{-1} \{(R \circ \psi_{s,t})(\zeta) - \frac{t^2}{2}\}} \operatorname{Re} \left(-\bar{\zeta} \partial_t \psi_{s,t}(\zeta) \overline{\psi'_{s,t}(\zeta)} \right)$$

we may rewrite the flow condition (4.2.2) as

$$(4.4.2) \quad \partial_s^j \partial_k^l \omega_{s,t}(\zeta) \Big|_{s=t=0} = \begin{cases} (4\pi)^{-\frac{1}{2}} & \text{for } \zeta \in \mathbb{T} \text{ and } (j, l) = (0, 0), \\ 0 & \text{for } \zeta \in \mathbb{T} \text{ and } (j, l) \in \mathcal{I}_{2\kappa} \setminus \{(0, 0)\}. \end{cases}$$

provided that the functions f_s and $\psi_{s,t}$ obtained by solving these equations do not degenerate, as long as R remains in a bounded set of $\mathcal{W}(\rho_0, \sigma_0)$ for some ρ_0 which is bounded away from 1 and $\sigma_0 > 0$. It turns out (see Proposition 4.6.4) that for $j, l \geq 1$, it holds that

$$(4.4.3) \quad \frac{1}{(j-1)!} \partial_s^{j-1} \partial_t^l \omega_{s,t}(\zeta) \Big|_{s=t=0} = 4(4\pi)^{-\frac{1}{2}} \Delta R(\zeta) \operatorname{Re}(\bar{\zeta} \hat{\psi}_{j,l-1}(\zeta)) \operatorname{Re}(\bar{\zeta} \hat{\psi}_{0,1}(\zeta)) + \mathfrak{F}_{j-1,l}(\zeta),$$

where $\mathfrak{F}_{j-1,l}$ is real-valued and real-analytic, and depends only on B_0, \dots, B_{j-1} and $\hat{\psi}_{p,q}$ where $(p, q) \prec (j, l-1)$. Moreover, when $l = 0$ we have

$$(4.4.4) \quad \frac{1}{j!} \partial_s^j \omega_{s,t}(\zeta) \Big|_{s=t=0} = 2 \operatorname{Re} (\bar{B}_0(\zeta) B_j(\zeta)) + \mathfrak{F}_{j,0}(\zeta)$$

where $\mathfrak{F}_{j,0}$ depends only on B_0, \dots, B_{j-1} and $\hat{\psi}_{p,q}$ for $(p, q) \prec (j+1, 0)$.

STEP 1. Let $\psi_{0,t}$ be the normalized conformal mappings to the exterior of level curves of R , as given by Proposition 4.2.6. In particular, this determines uniquely the coefficient functions $\hat{\psi}_{0,l}$, for $l = 0, \dots, 2\kappa + 1$ (see Proposition 4.6.1 below).

STEP 2. By evaluating $\omega_{s,t} \Big|_{s=t=0}$, we obtain from (4.4.2) that

$$|B_0(\zeta)|^2 \operatorname{Re}(-\bar{\zeta} \hat{\psi}_{0,1}(\zeta)) = (4\pi)^{-\frac{1}{2}}.$$

As $\hat{\psi}_{0,1}$ is already known and the above real part is strictly positive on \mathbb{T} (see Proposition 4.6.1 below), this gives $|B_0|^2$ on the unit circle \mathbb{T} . We choose B_0 to be the outer function in \mathbb{D}_e with these boundary values, with the additional normalization $B_0(\infty) > 0$.

We proceed from Step 2 to Step 3 with $j = 1$.

STEP 3. We have determined B_0, \dots, B_{j-1} and $\hat{\psi}_{a,b}$ for all $(a, b) \prec (j, 0)$. For $l = 1$, we may then compute $\mathfrak{F}_{j-1,1}$ (see Proposition 4.6.4 below), which by the equations (4.4.2) and (4.4.3) gives an equation for $\hat{\psi}_{j,0}$. The equation takes the form $\operatorname{Re}(\bar{\zeta} \hat{\psi}_{j,0}) = g_{j,0}$ on \mathbb{T} , for some known real-valued real-analytic expression $g_{j,0}$, and we solve it by the formula

$$\hat{\psi}_{j,0}(\zeta) = \zeta \mathbf{H}_{\mathbb{D}_e}[g_{j,0}](\zeta).$$

This means that we extend the background data to all $\hat{\psi}_{p,q}$ with $(p, q) \prec (j, 1)$, and we may apply the same procedure with $l = 2$ to determine $\mathfrak{F}_{j-1,2}$ followed by $\hat{\psi}_{j,1}$. We continue iterating the same procedure to cover $l = 3, 4, \dots, 2(\kappa - j) + 2$. Ultimately, in this step we obtain the functions $\hat{\psi}_{j,l}$ for $l = 0, \dots, 2(\kappa - j) + 1$.

STEP 4. At this stage, using Step 3, we have at our disposal the functions B_0, \dots, B_{j-1} , and $\hat{\psi}_{p,q}$ for all $(p, q) \prec (j+1, 0)$. This allows us to compute $\mathfrak{F}_{j,0}$ (see Proposition 4.6.4 below), and from (4.4.2) and (4.4.4), we derive an equation of the form $\operatorname{Re}(\bar{B}_0 B_j) = h_j$, for some known real-analytic real-valued expression h_j . We solve this explicitly by

$$B_j(\zeta) = B_0(\zeta) \mathbf{H}_{\mathbb{D}_e} \left[\frac{h_j}{|B_0|^2} \right](\zeta), \quad \zeta \in \mathbb{D}_e.$$

Due to the smoothness of the function $h_j/|B_0|^2$, the function B_j extends holomorphically across the boundary \mathbb{T} .

STEP 5. Finally, we iterate Steps 3 and 4 with j replaced by $j+1$, until all coefficients B_j for $j = 0, \dots, \kappa$ and $\hat{\psi}_{j,l}$ for all $(j, l) \in \mathcal{I}_{2\kappa+1}$ have been determined.

Remark 4.4.1. (a) If we apply the above algorithm to the function $R = R_\tau$, the functions B_j obtained here are (up to a constant multiple) the same as those appearing in Theorem 1.3.7. This algorithm is in principle an alternative route towards finding them explicitly. However, since this algorithm involves the additional functions $\hat{\psi}_{j,l}$, this is not feasible even for $\kappa = 2$.

(b) The functions B_j and $\hat{\psi}_{j,l}$ determined iteratively by the above algorithmic procedure all have the required properties: each function B_j is bounded and holomorphic in the exterior disk, and each function $\hat{\psi}_{j,l}$ is holomorphic in the exterior disk and meets the normalization

$\hat{\psi}'_{j,l}(\infty) > 0$. Moreover, all the above-mentioned functions extend holomorphically across the boundary \mathbb{T} , and we tacitly extend them to the larger region $\mathbb{D}_e(0, \rho)$ for some positive $\rho < 1$.

4.5. The multivariate Faà di Bruno formula. We recall Faà di Bruno's formula in several variables, and study some of the properties. Using standard multi-index notation, we may introduce the *lexicographic ordering*: if α and β are two multi-indices

$$\alpha = (\alpha_1, \dots, \alpha_n) \quad \text{and} \quad \beta = (\beta_1, \dots, \beta_n),$$

we say that $\alpha \prec \beta$, if either $\alpha_1 < \beta_1$ or $\alpha_1 = \beta_1, \dots, \alpha_k = \beta_k$ while $\alpha_{k+1} < \beta_{k+1}$ holds for some $1 \leq k \leq n$. If $\alpha \prec \beta$ or $\alpha = \beta$, we agree that $\alpha \preceq \beta$. For multi-indices α and β in \mathbb{N}^n we write

$$\begin{aligned} |\alpha| &= \sum_i |\alpha_i| \\ \alpha! &= \prod_i (\alpha_i!) \\ \xi^\beta &= \prod_i \xi_i^{\beta_i}, \quad \xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n, \\ \partial^\alpha f(\mathbf{x}) &= \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} f(\mathbf{x}), \quad \mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n. \end{aligned}$$

We will need the index set

$$\begin{aligned} \mathcal{T}_{m;d,n} &= \{(\alpha_1, \dots, \alpha_m; \beta_1, \dots, \beta_m) \in (\mathbb{N}^d)^m \times (\mathbb{N}^n)^m : \\ &0 \prec \alpha_1 \prec \alpha_2 \prec \dots \prec \alpha_m \text{ and } \forall i = 1, \dots, m : |\beta_i| > 0\}. \end{aligned}$$

Proposition 4.5.1 (Faà di Bruno's formula, [11]). *Let f be a real-valued function defined in a domain $\mathcal{D} \subset \mathbb{R}^n$, of class C^k , and let g be defined and C^k in a domain $\mathcal{D}' \subset \mathbb{R}^d$ such that g takes values in \mathcal{D} . Then, for any multi-index ν with $|\nu| = k$, we have on \mathcal{D}'*

$$\partial^\nu (f \circ g) = \sum_{1 \leq |\mu| \leq k} (\partial^\mu f) \circ g \mathcal{G}_{\mu,\nu}(g),$$

where the function $\mathcal{G}_{\mu,\nu}(g)$ is given by

$$\mathcal{G}_{\mu,\nu}(g) = \nu! \sum_{m=1}^k \sum_{(\alpha;\beta) \in S_m(\mu,\nu)} \prod_{j=1}^m \frac{[\partial^{\alpha_j} g]^{\beta_j}}{\beta_j! [\alpha_j!]^{|\beta_j|}},$$

and where the index set is defined as

$$S_m(\mu, \nu) = \left\{ (\alpha; \beta) = (\alpha_1, \dots, \alpha_m; \beta_1, \dots, \beta_m) \in \mathcal{T}_{m;d,n} : \sum \beta_i = \mu, \sum |\beta_i| \alpha_i = \nu \right\}.$$

Here, since g is assumed vector-valued, the derivative $\partial^{\alpha_j} g$ is also vector-valued, and the power $[\partial^{\alpha_j} g]^{\beta_j}$ is taken with respect to the multi-index notation and produces a real-valued function.

Let us now specialize Proposition 4.5.1 to our situation. We will be interested in the case of $n = d = 2$. By setting $F(r, \theta) = R(re^{i\theta})$, we may write

$$R \circ \psi_{s,t} = F \circ \Phi_{s,t}, \quad \Phi_{s,t} = (|\psi_{s,t}|, \arg \psi_{s,t}).$$

If we denote by $D_{r,\theta}^\mu$ the differential operator

$$D_{r,\theta}^\mu = \partial_r^{\mu_1} \partial_\theta^{\mu_2}, \quad \mu = (\mu_1, \mu_2),$$

we obtain by applying Proposition 4.5.1 to $F \circ \Phi_{s,t}$ with $\nu = (j, l)$ that on the circle \mathbb{T} ,

$$(4.5.1) \quad \partial_s^j \partial_t^l (R \circ \psi_{s,t}) \Big|_{s=t=0} = \sum_{2 \leq |\mu| \leq j+l} D_{r,\theta}^\mu R(\psi_{s,t}) \mathcal{G}_{\mu,(j,l)}(\Phi_{s,t}) \Big|_{s=t=0},$$

where the terms corresponding to indices μ with $|\mu| = 1$ are dropped, and the reason is the following. First, we have that $\psi_{0,0}(\zeta) = \zeta$, and second, the function R together with its gradient vanish along the unit circle \mathbb{T} . In the context of (4.5.1), we should point out that the multi-index derivatives that appear in the expression $\mathcal{G}_{\mu,(j,l)}(\Phi_{s,t})$, as defined in Proposition 4.5.1,

are taken with respect to the variables (s, t) . As for the (suppressed) variable $\zeta \in \mathbb{T}$, it is considered fixed.

We will be interested in identifying the *maximal* index (a, b) with respect to the lexicographical ordering, such that the partial derivative $\partial_s^a \partial_t^b \Phi_{s,t}$ appears non-trivially in the right-hand side expression of (4.5.1).

Proposition 4.5.2. *Let ν and μ be double-indices with $2 \leq |\mu| \leq |\nu|$ and $\mu \notin \{(1, 1), (0, 2)\}$. Let $(\alpha; \beta) \in S_m(\mu, \nu)$. Then*

(i) *If $\nu = (j, l)$, where $j, l \geq 1$, then for all $i = 1, \dots, m$, we have that $\alpha_i \preceq (j, l - 1)$, where equality holds if and only if $i = m = 2$, $\mu = (2, 0)$, and*

$$(\alpha; \beta) = ((0, 1), (j, l - 1); (1, 0), (1, 0)).$$

(ii) *If $\nu = (j, 0)$ with $j \geq 3$, then α_i is of the form $(a, 0)$ with $a \leq j - 1$. Moreover, equality holds if and only if $i = m = 2$, $\mu = (2, 0)$, and*

$$(\alpha; \beta) = ((1, 0), (j - 1, 0); (1, 0), (1, 0)).$$

(iii) *If $\nu = (0, l)$ with $l \geq 3$, then α_i is of the form $(0, b)$ with $b \leq l - 1$. Moreover, equality holds if and only if $\mu = (2, 0)$ and*

$$(\alpha; \beta) = ((0, 1), (0, l - 1); (1, 0), (1, 0)).$$

(iv) *If $\nu = (2, 0)$, then necessarily $\mu = (2, 0)$ and the only non-trivial index $(\alpha; \beta)$ is*

$$(\alpha; \beta) = ((1, 0); (0, 2)).$$

(v) *If $\nu = (0, 2)$, then necessarily $\mu = (2, 0)$ and the only non-trivial index $(\alpha; \beta)$ is*

$$(\alpha; \beta) = ((0, 1); (0, 2)).$$

Note that since $|\nu| \geq 2$, the above list covers all the possibilities.

Proof of Proposition 4.5.2. We will show how to obtain (i), (ii) and (iv). The remaining cases (iii) and (v) are analogous and omitted. We recall the compatibility conditions of the index set $S_m(\mu, \nu)$: the assertion $(\alpha; \beta) \in S_m(\mu, \nu)$ means that

$$(4.5.2) \quad \sum_{i=1}^m |\beta_i| \alpha_i = \nu, \quad \sum_{i=1}^m \beta_i = \mu,$$

where each β_i meets $|\beta_i| \geq 1$, and the multi-indices α_i are strictly increasing with i in the lexicographical ordering. From these assumptions it follows that each α_i satisfies $\alpha \preceq \nu$.

Turning to (i), we see that equality $\alpha_i = (j, l)$ could hold only if $m = 1$, $\alpha_1 = (j, l)$ and $\beta_1 = (1, 0)$. But then $|\mu| = 1$, which contradicts the assumption that $|\mu| \geq 2$. Hence, for any index i , we have $\alpha_i \preceq (j, l - 1)$. However, if equality holds here, that is, if for some i_0 we have $\alpha_{i_0} = (j, l - 1)$, we find from (4.5.2) that $|\beta_{i_0}| = 1$, the sum on the left-hand side, taken over all other indices $i \neq i_0$, equals $(0, 1)$. As a consequence, we find that $m = 2$ and that $\alpha = ((0, 1), (j, l - 1))$. Since the only admissible index μ with $|\mu| = 2$ is $\mu = (2, 0)$, the rest of the claim is immediate.

We next consider (ii). In a similar manner as above, since the weighted sum of the multi-indices α_i equals $(j, 0)$, we see that α_i is of the form $\alpha_i = (a_i, 0)$ for any i . It is moreover clear that $a_{i_0} = j$ could only occur for some i_0 only if $i_0 = m = 1$, $|\beta_1| = 1$ and $|\mu| = 1$, which again is contrary to our assumption that $|\mu| \geq 2$. Next, the only way we could have $\alpha_{i_0} = (j - 1, 0)$ for some i_0 is if $i_0 = m = 2$ and correspondingly $\alpha = ((1, 0), (j - 1, 0))$. The remaining properties are immediate.

Finally, to see why (iv) holds, we again find that each α_i is of the form $\alpha_i = (a_i, 0)$. Since the multi-indices α_i are assumed strictly increasing with i , the combined index $\alpha = ((1, 0), (1, 0))$ is not admissible. Recall that $\{|\beta_i| \alpha_i : i = 1, \dots, m\}$ sums up to ν . The only remaining way to obtain $\nu = (2, 0)$ is if $m = 1$ so that the sum has only one element, which is automatically $(2, 0)$ itself. If $\alpha_1 = (2, 0)$, then $|\beta_1| = 1$, so $|\mu| = 1$, which is contrary to our assumption

that $|\boldsymbol{\mu}| \geq 2$. The only remaining alternative is that $\boldsymbol{\alpha}_1 = (1, 0)$, and $|\boldsymbol{\beta}_1| = 2$. Since $\boldsymbol{\beta}_1 = \boldsymbol{\mu}$, and the only admissible $\boldsymbol{\mu}$ of length 2 is $\boldsymbol{\mu} = (2, 0)$, it follows that $\boldsymbol{\beta}_1 = (2, 0)$, and the claim follows. \square

We note that in each of the cases (i)-(v), the maximal $\boldsymbol{\alpha}_i$ occurs as the index $\boldsymbol{\alpha}_m$, where $(\boldsymbol{\alpha}; \boldsymbol{\beta}) \in S_m(\boldsymbol{\mu}, \boldsymbol{\nu})$ and $\boldsymbol{\mu} = (2, 0)$. We will find the notation

$$(4.5.3) \quad \mathcal{G}_{(2,0),\boldsymbol{\nu}}^*(\Phi_{s,t}) = \boldsymbol{\nu}! \sum_{m=1}^{|\boldsymbol{\nu}|} \sum_{(\boldsymbol{\alpha}; \boldsymbol{\beta}) \in S_m^*(\boldsymbol{\nu})} \prod_{j=1}^m \frac{[\partial^{\boldsymbol{\alpha}_j} \Phi_{s,t}]^{\boldsymbol{\beta}_j}}{\boldsymbol{\beta}_j! [\boldsymbol{\alpha}_j!]^{|\boldsymbol{\beta}_j|}}$$

useful. Here, the sum is taken over the index set $S_m^*(\boldsymbol{\nu})$, defined as follows. Let

$$A(\boldsymbol{\nu}) = \max_m \max_{(\boldsymbol{\alpha}; \boldsymbol{\beta}) \in S_m((2,0), \boldsymbol{\nu})} \boldsymbol{\alpha}_m$$

where the maximum is taken lexicographically over the range $m = 1, \dots, |\boldsymbol{\nu}|$. Moreover, denote by $(\boldsymbol{\alpha}^*; \boldsymbol{\beta}^*)$ the unique pair such that $\boldsymbol{\alpha}_i^* = A(\boldsymbol{\nu})$ for some i (see Proposition 4.5.2 for details). We then put

$$S_m^*(\boldsymbol{\nu}) = \begin{cases} S_m((2,0), \boldsymbol{\nu}), & \text{if } (\boldsymbol{\alpha}^*; \boldsymbol{\beta}^*) \notin S_m((2,0), \boldsymbol{\nu}), \\ S_m((2,0), \boldsymbol{\nu}) \setminus \{(\boldsymbol{\alpha}^*; \boldsymbol{\beta}^*)\}, & \text{if } (\boldsymbol{\alpha}^*; \boldsymbol{\beta}^*) \in S_m((2,0), \boldsymbol{\nu}). \end{cases}$$

4.6. The orthogonal foliation flow III. The construction. In this section we obtain Proposition 4.2.5. To this end, we need to fill in the blanks of the algorithmic procedure outlined in Subsection 4.4.

We now explore STEP 1 of the algorithmic procedure outlined in Subsection 4.4. We recall the notation $\Phi_{0,t} = (|\psi_{0,t}|, \arg \psi_{0,t})$.

Proposition 4.6.1. *The coefficients $\hat{\psi}_{0,l}$ of the conformal mapping $\psi_{0,t}$ with Taylor expansion near the origin*

$$\psi_{0,t}(\zeta) = \sum_{l=0}^{2\kappa+1} t^l \hat{\psi}_{0,l}(\zeta) + O(t^{2\kappa+2}),$$

are uniquely determined by the level-curve requirement

$$R \circ \psi_{0,t}(\zeta) = \frac{t^2}{2}, \quad \zeta \in \mathbb{T},$$

the monotonicity condition that the images $\psi_{0,t}(\mathbb{D}_e)$ grow with t , and the normalization $\psi'_{0,t}(\infty) > 0$. Moreover, as such they are given by

$$\begin{aligned} \hat{\psi}_{0,0}(\zeta) &= \zeta, \\ \hat{\psi}_{0,1}(\zeta) &= -\zeta \mathbf{H}_{\mathbb{D}_e} [(4\Delta R)^{-\frac{1}{2}}](\zeta), \end{aligned}$$

and, more generally, by

$$\hat{\psi}_{0,l}(\zeta) = \zeta \mathbf{H}_{\mathbb{D}_e} [(4\Delta R)^{-\frac{1}{2}} \mathfrak{G}_l](\zeta), \quad l = 2, \dots, 2\kappa + 3,$$

where $\mathfrak{G}_l(\zeta)$ is a real-analytic function on the circle \mathbb{T} , which may be expressed in terms of the functions $\hat{\psi}_{0,0}, \dots, \hat{\psi}_{0,l-1}$ by the formula

$$\begin{aligned} \mathfrak{G}_l(\zeta) := \frac{1}{(l+1)!} & \left\{ 4\Delta R(\zeta) \mathcal{G}_{(2,0),(0,l+1)}^*(\Phi_{0,t}) \Big|_{t=0} + \mathfrak{H}_l \right. \\ & \left. + \sum_{3 \leq |\boldsymbol{\mu}| \leq l+1} \partial_r^{\mu_1} \partial_\theta^{\mu_2} R(\zeta) \mathcal{G}_{\boldsymbol{\mu},(0,l+1)}(\Phi_{0,t}) \Big|_{t=0} \right\}, \end{aligned}$$

where

$$\mathfrak{H}_l := \partial_t^l |\psi_{0,t}| \Big|_{t=0} - l! \operatorname{Re}(\bar{\zeta} \hat{\psi}_{0,l}).$$

The coefficient functions $\hat{\psi}_{j,l}$ extend holomorphically to the domain $\mathbb{D}_e(0, \rho')$.

Proof. That $\hat{\psi}_{0,0}(\zeta) = \zeta$ follows since R vanishes only on \mathbb{T} . By Taylor's formula, we have that

$$(4.6.1) \quad (R \circ \psi_{0,t})(\zeta) = \sum_{j=0}^{2\kappa+1} \frac{t^j}{j!} \partial_t^j (R \circ \psi_{0,t})(\zeta) \Big|_{t=0} + O(|t|^{2\kappa+2}).$$

Since by assumption $R \circ \psi_{0,t}(\zeta) = \frac{t^2}{2}$ holds on \mathbb{T} , we find that

$$(4.6.2) \quad \partial_t^l (R \circ \psi_{0,t})(\zeta) \Big|_{t=0} = \begin{cases} 1, & \text{for } l = 2 \\ 0, & \text{otherwise.} \end{cases}$$

The above formula for $\hat{\psi}_{0,0}$ is readily verified. Indeed, the level curve $R = 0$ corresponding to $t = 0$ is the unit circle, and the normalization at infinity forces $\hat{\psi}_{0,0}(\zeta) = \zeta$. Moreover, expression

$$\partial_t (R \circ \psi_{0,t}) \Big|_{t=0} = 0$$

as the function R and its gradient vanish on the circle \mathbb{T} . Thus the equation (4.6.2) for $l = 1$ gives us no additional information. We now consider the second derivative separately. A short computation using the chain rule and the flatness of R near the unit circle \mathbb{T} shows that

$$\partial_t^2 (R \circ \psi_{0,t}) \Big|_{t=0} = 4\Delta R [\operatorname{Re}(\bar{\zeta} \hat{\psi}_{0,1})]^2.$$

If we solve for $\operatorname{Re}(\bar{\zeta} \hat{\psi}_{0,1})$ in (4.6.2) for $l = 2$ using the negative square root, we find that

$$\operatorname{Re}(\bar{\zeta} \hat{\psi}_{0,1}) = -(4\Delta R)^{-\frac{1}{2}}, \quad \text{on } \mathbb{T},$$

which is compatible with the growth of the domains $\psi_{0,t}(\mathbb{D}_e)$ as t increases. We finally solve this equation by the formula

$$(4.6.3) \quad \hat{\psi}_{0,1}(\zeta) = -\zeta \mathbf{H}_{\mathbb{D}_e} [(4\Delta R)^{-\frac{1}{2}}](\zeta),$$

as in STEP 3 of the algorithmic procedure in Subsection 4.4. As $(4\Delta R)^{-\frac{1}{2}}$ has a polarization which is holomorphic in (z, \bar{w}) for $(z, w) \in \hat{\mathbb{A}}(\rho, \sigma)$, the function $\hat{\psi}_{0,1}$ extends holomorphically to $\mathbb{D}_e(0, \rho')$, by Proposition 4.2.2 and Remark 4.2.4.

We find the higher order Taylor coefficients by applying Faà di Bruno's formula to the composition $F \circ \Phi_t$, where $F(r, \theta) := R(re^{i\theta})$. First, recall that by the properties of R , we have that

$$D_{r,\theta}^\mu R|_{\mathbb{T}} = 0, \quad \mu \in \{(0,0), (1,0), (0,1), (1,1), (0,2)\}.$$

Consequently, we cannot have non-zero contributions from μ with $|\mu| \leq 1$. As a result, on the circle $\zeta \in \mathbb{T}$ we have for $l = 2, 3, 4, \dots$

$$(4.6.4) \quad \begin{aligned} \partial_t^{l+1} (R \circ \psi_{0,t}) \Big|_{t=0} &= \sum_{2 \leq |\mu| \leq l+1} (\partial_r^{\mu_1} \partial_\theta^{\mu_2} R) \mathcal{G}_{\mu,(0,l+1)}(\Phi_{0,t}) \Big|_{t=0} \\ &= 4(l+1)\Delta R \partial_t^l |\psi_{0,t}| \partial_t |\psi_{0,t}| \Big|_{t=0} + \mathcal{G}_{(2,0),(l+1)}^*(\Phi_{0,t}) \Big|_{t=0} \\ &\quad + \sum_{3 \leq |\mu| \leq l+1} (\partial_r^{\mu_1} \partial_\theta^{\mu_2} R) \mathcal{G}_{\mu,(0,l+1)}(\Phi_{0,t}) \Big|_{t=0} \\ &= 4(l+1)! \Delta R \operatorname{Re}(\bar{\zeta} \hat{\psi}_{0,l}) \operatorname{Re}(\bar{\zeta} \hat{\psi}_{0,1}) + \mathfrak{H}_l + \mathcal{G}_{(2,0),(0,l+1)}^*(\Phi_{0,t}) \Big|_{t=0} \\ &\quad + \sum_{3 \leq |\mu| \leq l+1} (\partial_r^{\mu_1} \partial_\theta^{\mu_2} R) \mathcal{G}_{\mu,(0,l+1)}(\Phi_{0,t}) \Big|_{t=0}, \end{aligned}$$

where we recall that

$$\mathfrak{H}_l = \partial_t^l |\psi_{0,t}| \Big|_{t=0} - l! \operatorname{Re}(\bar{\zeta} \hat{\psi}_{0,l}).$$

A short computation shows that the highest order derivatives cancel out, and that \mathfrak{H}_l may be expressed in terms of the lower order Taylor coefficients $\hat{\psi}_{0,b}$ for $b \leq l-1$. We recall that the expression $\mathcal{G}_{(2,0),(0,l+1)}^*(\Phi_{0,t})$ appearing in the above formula is defined in (4.5.3). If we write

$$\mathfrak{E}_l(\zeta) = \frac{1}{(l+1)!} \left\{ \mathfrak{H}_l(\zeta) + \mathcal{G}_{(2,0),(0,l+1)}^*(\Phi_{0,t}) \Big|_{t=0} + \sum_{3 \leq |\mu| \leq l+1} (\partial_r^{\mu_1} \partial_\theta^{\mu_2} R) \mathcal{G}_{\mu,(0,l+1)}(\Phi_t) \Big|_{t=0} \right\},$$

we claim that \mathfrak{G}_l may be expressed in terms of the functions $\hat{\psi}_{0,b}$ for $b \leq l-1$. Indeed, we saw previously that \mathfrak{H}_l has this property. That the same holds for the remaining two terms of the above formula is a consequence of Proposition 4.5.2.

By (4.6.3), we have on the unit circle \mathbb{T} that

$$4\Delta R \operatorname{Re}(\bar{\zeta}\hat{\psi}_{0,1}) = (4\Delta R)^{\frac{1}{2}}(4\Delta R)^{\frac{1}{2}} \operatorname{Re}(\bar{\zeta}\hat{\psi}_{0,1}) = -(4\Delta R)^{\frac{1}{2}}.$$

In view of this and the calculation (4.6.4), we may express the condition (4.6.2) in the form

$$-(4\Delta R)^{\frac{1}{2}} \operatorname{Re}(\bar{\zeta}\hat{\psi}_{0,l}) + \mathfrak{G}_l = 0.$$

This type of equation we have encountered previously, and we know that a solution $\hat{\psi}_{0,l}$ is given by the formula

$$\hat{\psi}_{0,l}(\zeta) = \zeta \mathbf{H}_{\mathbb{D}_e} \left[\frac{\mathfrak{G}_l}{(4\Delta R)^{\frac{1}{2}}} \right](\zeta).$$

In this manner, we iteratively determine the functions $\hat{\psi}_{0,l}$ for $l = 2, \dots, 2\kappa+1$. As the function $\mathfrak{G}_l(4\Delta R)^{-\frac{1}{2}}$ has a polarization which is holomorphic in (z, \bar{w}) for $(z, w) \in \hat{\mathbb{A}}(\rho, \sigma)$, it follows that $\hat{\psi}_{0,l}$ is holomorphic in $\mathbb{D}_e(0, \rho')$. \square

We need three further propositions of a rather technical character. The following proposition gives us structural information regarding the expansion of the composition $R \circ \psi_{s,t}$ in terms of powers of s and t . As for the formulation, we retain the notation introduced in connection with Faà di Bruno's formula in Subsection 4.5.

Proposition 4.6.2. *On the unit circle \mathbb{T} , the function $R \circ \psi_{s,t}$ enjoys the expansion*

$$(4.6.5) \quad R \circ \psi_{s,t} = R \circ \psi_{0,t} + \sum_{(j,l) \in \mathcal{I}_{2\kappa}} s^{j+1} t^l \mathfrak{A}_{j,l} + \mathcal{O}(|s|(|s|^{\kappa+\frac{1}{2}} + |t|^{2\kappa+1})),$$

where $\mathfrak{A}_{0,0} = 0$, while for the remaining indices $(j,l) \neq (0,0)$, we have

$$\mathfrak{A}_{j,l} = \begin{cases} \frac{1}{(j+1)!(l-1)!} 4\Delta R (\partial_s^{j+1} \partial_t^{l-1} |\psi_{s,t}|)(\partial_t |\psi_{0,t}|) \Big|_{s=t=0} + \frac{1}{(j+1)!} \mathfrak{R}_{j,l}, & \text{for } j \geq 0 \text{ and } l \geq 1, \\ \frac{1}{(j+1)!} \sum_{2 \leq |\mu| \leq j+1} D_{r,\theta}^\mu R \mathcal{G}_{\mu,(j+1,0)}(\Phi_{s,t}) \Big|_{s=t=0}, & \text{for } j \geq 2 \text{ and } l = 0, \\ 4\Delta R [\operatorname{Re}(\bar{\zeta}\hat{\psi}_{1,0})]^2, & \text{for } j = 1 \text{ and } l = 0. \end{cases}$$

Here, the functions $\mathfrak{R}_{j,l}$ are given by

$$\mathfrak{R}_{j,l} = 4\Delta R \mathcal{G}_{(2,0),(j+1,l)}^*(\Phi_{s,t}) \Big|_{s=t=0} + \sum_{3 \leq |\mu| \leq j+l+1} D_{r,\theta}^\mu R \mathcal{G}_{\mu,(j+1,l)}(\Phi_{s,t}) \Big|_{s=t=0}.$$

In particular, the terms $\mathfrak{R}_{j,l}$ may be expressed in terms of the partial derivatives $\partial_s^a \partial_t^b \psi_{s,t}$ with $(a,b) \prec (j+1, l-1)$, while the coefficients $\mathfrak{A}_{j,0}$ may be expressed in terms of the partial derivatives $\partial_s^a \partial_t^b \psi_{s,t}$ with $b = 0$ and $a \leq j$. Moreover, the implied constant in (4.6.5) remains bounded if the weight R is confined to a uniform family in $\mathcal{W}(\rho_0, \sigma_0)$ for some fixed $\rho_0 < 1$ and $\sigma_0 > 0$, while the functions $\psi_{s,t}$ are assumed smooth with bounded norms in $C^{2\kappa+4}$ with respect to (s,t) in a neighbourhood of $(0,0)$, uniformly on the circle \mathbb{T} .

Proof. The fact that $R \circ \psi_{s,t}$ enjoys an expansion of the form (4.6.5) for some coefficients $\mathfrak{A}_{j,l}$ with the given error term is an immediate consequence of the multivariate Taylor's formula. The coefficients $\mathfrak{A}_{j,l}$ are then obtained from an application of Faà di Bruno's formula (4.5.1). It remains to identify the coefficients $\mathfrak{A}_{j,l}$. By the flatness of R near the circle \mathbb{T} , we have

$$D_{r,\theta}^\mu R|_{\mathbb{T}} = 0, \quad \mu \in \{(0,0), (1,0), (0,1), (1,1), (0,2)\},$$

so no index μ with $|\mu| \leq 1$ gives non-trivial contributions.

We first consider the case $l \geq 1$, and study the derivative $\partial^\nu (R \circ \psi_{s,t}) \Big|_{s=t=0}$, where $\nu = (j+1, l)$. If $|\mu| = 2$, and $(\alpha; \beta) \in S_m(\mu, \nu)$, then Proposition 4.5.2 gives that α_m , which is maximal among the indices α_i in our lexicographical ordering, is bounded above by $(j+1, l-1)$. Moreover, the index $(j+1, l-1)$ appears only once across all summation indices m and μ with

$|\boldsymbol{\mu}| = 2$, and the only way that we can achieve $\boldsymbol{\alpha}_m = (j+1, l-1)$ is if $m = 2$, $\boldsymbol{\alpha}_1 = (0, 1)$, and $\boldsymbol{\beta}_1 = \boldsymbol{\beta}_2 = (1, 0)$.

In a similar fashion, Proposition 4.5.2 shows that if $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in S_m(\boldsymbol{\mu}, \boldsymbol{\nu})$ with $|\boldsymbol{\mu}| \geq 3$, then $\boldsymbol{\alpha}_m \prec (j+1, l-1)$.

A short computation using the fact that $\partial_r^2 R = 4\Delta R$ on \mathbb{T} (see Proposition 3.1.3) shows that the coefficient $\mathfrak{A}_{j,l}$ has the required form, and since the remainder term $\mathfrak{R}_{j,l}$ is not summed over the index $(\boldsymbol{\alpha}^*; \boldsymbol{\beta}^*)$, which is strictly maximal according to the above discussion, it follows that $\mathfrak{R}_{j,l}$ has the asserted properties.

The assertion $\mathfrak{A}_{0,0} = 0$ is a consequence of the fact that both the function R and its gradient ∇R vanish along \mathbb{T} . The formula supplied for $\mathfrak{A}_{j,0}$ with $j \geq 1$ is a direct consequence of the Faà di Bruno formula (4.5.1). With regards to the claimed property of $\mathfrak{A}_{j,0}$, it suffices to note that any index $\boldsymbol{\alpha}_i$, where $(\boldsymbol{\alpha}; \boldsymbol{\beta}) \in S_m(\boldsymbol{\mu}, \boldsymbol{\nu})$ with $\boldsymbol{\nu} = (j+1, 0)$ is of the form $\boldsymbol{\alpha}_i = (a, 0)$ for some a with $1 \leq a \leq j$, by Proposition 4.5.2. This finishes the proof. \square

For a positive integer $r \in \mathbb{Z}_+$ and $(j, l) \in \mathcal{I}_{2\kappa}$, we use the notation $X_r(j, l)$ for the set

$$X_r(j, l) = \left\{ (j_i, l_i)_{i=1}^r : \sum_{i=1}^r j_i = j, \sum_{i=1}^r l_i = l \right\}.$$

In preparation for the next proposition, we observe that by Proposition 4.6.2

$$(4.6.6) \quad \frac{1}{s} \left\{ R \circ \psi_{s,t} - \frac{t^2}{2} \right\} = \sum_{(i,l) \in \mathcal{I}_{2\kappa}} s^j t^l \mathfrak{A}_{j,l} + O(|s|^{\kappa+\frac{1}{2}} + |t|^{2\kappa+1}).$$

holds on the unit circle \mathbb{T} , since there, $R \circ \psi_{0,t} = \frac{t^2}{2}$ holds as a matter of definition. We recall that f_s and $\psi_{s,t}$ denote functions of the form

$$f_s(\zeta) = \sum_{j=0}^{\kappa} s^j B_j(\zeta) \quad \text{and} \quad \psi_{s,t}(\zeta) = \psi_{0,t}(\zeta) + \sum_{\substack{(j,l) \in \mathcal{I}_{\kappa+1} \\ j \geq 1}} s^j t^l \hat{\psi}_{j,l}(\zeta),$$

where the B_j are some bounded holomorphic functions in a neighbourhood of the exterior disk \mathbb{D}_e , and where $\psi_{0,t}$ is a conformal mapping onto the exterior of the level curves Γ_t of R as above, and where $\hat{\psi}_{j,l}$ are holomorphic functions on $\mathbb{D}_e(0, \rho')$ with bounded derivative normalized at infinity by $\hat{\psi}'_{j,l}(\infty) \in \mathbb{R}$. Let us denote by $\mathfrak{U}_{j,l}$, $\mathfrak{V}_{j,l}$ and $\mathfrak{W}_{j,l}$ the coefficients from the following three expansions (for $\zeta \in \mathbb{T}$):

$$(4.6.7) \quad |f_s \circ \psi_{s,t}(\zeta)|^2 = \sum_{(j,l) \in \mathcal{I}_{2\kappa}} s^j t^l \mathfrak{U}_{j,l}(\zeta) + O(|s|^{\kappa+\frac{1}{2}} + |t|^{2\kappa+1}),$$

$$(4.6.8) \quad e^{-2s^{-1}\{R \circ \psi_{s,t}(\zeta) - \frac{t^2}{2}\}} = \sum_{(j,l) \in \mathcal{I}_{2\kappa}} s^j t^l \mathfrak{V}_{j,l}(\zeta) + O(|s|^{\kappa+\frac{1}{2}} + |t|^{2\kappa+1}),$$

$$(4.6.9) \quad \operatorname{Re}(-\bar{\zeta} \partial_t \psi_{s,t}(\zeta) \overline{\psi'_{s,t}(\zeta)}) = \sum_{(j,l) \in \mathcal{I}_{2\kappa}} s^j t^l \mathfrak{W}_{j,l}(\zeta) + O(|s|^{\kappa+\frac{1}{2}} + |t|^{2\kappa+1}).$$

The following proposition tells us how to obtain these coefficients in terms of the successive partial derivatives of the functions B_j and $\psi_{s,t}$.

Proposition 4.6.3. *In the above context, the coefficients $\mathfrak{U}_{j,l}$, $\mathfrak{V}_{j,l}$ and $\mathfrak{W}_{j,l}$ are obtained on the unit circle \mathbb{T} as follows. We have that $\mathfrak{V}_{0,0} = 1$, while for $(j, l) \in \mathcal{I}_{2\kappa} \setminus \{(0, 0)\}$*

$$\mathfrak{V}_{j,l} = \sum_{r=1}^{j+l} \sum_{X_r(j,l)} \frac{r!}{(j'_1)! \cdots (j'_r)! (l'_1)! \cdots (l'_r)!} \mathfrak{A}_{j'_1, l'_1} \cdots \mathfrak{A}_{j'_r, l'_r},$$

where $\mathfrak{A}_{j,l}$ denote the coefficients obtained in Proposition 4.6.2. Moreover, the remaining coefficients are obtained by

$$\mathfrak{U}_{j,l} = \sum_{\substack{p_1+p_2+i+k=j \\ q_1+q_2=l}} \frac{\partial_s^{p_1} \partial_t^{q_1} (B_i \circ \psi_{s,t}) \partial_s^{p_2} \partial_t^{q_2} (\bar{B}_k \circ \psi_{s,t})}{p_1! p_2! q_1! q_2!} \Big|_{s=t=0}, \quad (j,l) \in \mathcal{I}_{2\kappa},$$

$$\mathfrak{W}_{j,l} = \sum_{\substack{p+i=j \\ q+k=l}} (k+1) \operatorname{Re} \left(-\bar{\zeta} \hat{\psi}_{i,k+1} \overline{\hat{\psi}'_{p,q}} \right), \quad (j,l) \in \mathcal{I}_{2\kappa}.$$

Here, the summation takes place over all the relevant tuples $(p_1, p_2, q_1, q_2, i, k)$ and (p, q, i, k) , respectively, where the entries are all non-negative integers.

Proof. This follows from an application of the multivariate Taylor's formula, together with Faà di Bruno's formula (Proposition 4.5.1), and the equation (4.6.6) above. \square

Next, let $\omega_{s,t}$ denote the function

$$\omega_{s,t} = |f_s \circ \psi_{s,t}|^2 e^{-2s^{-1} \{R \circ \psi_{s,t} - \frac{t^2}{2}\}} \operatorname{Re} \left(-\bar{\zeta} \partial_t \psi_{s,t} \overline{\psi'_{s,t}} \right),$$

and define implicitly the coefficients $\mathfrak{C}_{j,l}$ of the expansion of $\omega_{s,t}$ on the circle \mathbb{T} :

$$\omega_{s,t} = \sum_{(j,l) \in \mathcal{I}_{2\kappa}} s^j t^l \mathfrak{C}_{j,l} + O(|s|^{\kappa+\frac{1}{2}} + |t|^{2\kappa+1}).$$

Proposition 4.6.4. *The coefficients $\mathfrak{C}_{j,l}$ in the above expansion are given by*

$$\mathfrak{C}_{0,0} = |B_0|^2 \operatorname{Re} \left(-\bar{\zeta} \hat{\psi}_{0,1} \right) = |B_0|^2 (4\Delta R)^{-\frac{1}{2}},$$

and for $l = 0$ and $j = 1, 2, 3, \dots$ by

$$\mathfrak{C}_{j,0} = 2 \operatorname{Re} (\bar{B}_0 B_j) (4\Delta R)^{-\frac{1}{2}} + \mathfrak{F}_{j,0},$$

where

$$\mathfrak{F}_{j,0} = (4\Delta R)^{-\frac{1}{2}} \sum_{\substack{0 \leq p, q \leq j-1 \\ p+q+r=j}} 2 \operatorname{Re} (\bar{B}_p B_q) \mathfrak{A}_{r,0},$$

while for $j = 0, 1, 2, \dots$ and $l = 1, 2, 3, \dots$ the coefficient function $\mathfrak{C}_{j,l}$ meets

$$\mathfrak{C}_{j,l} = (4\pi)^{\frac{1}{2}} \operatorname{Re} (\bar{\zeta} \hat{\psi}_{j+1, l-1}) (4\Delta R)^{\frac{1}{2}} + \mathfrak{F}_{j,l},$$

where

$$\mathfrak{F}_{j,l} = (4\pi)^{-\frac{1}{2}} (4\Delta R)^{\frac{1}{2}} \{ \mathfrak{S}_{j,l} + \mathfrak{R}_{j,l} \} + \sum_{\substack{(j_1, l_1, j_2, l_2, j_3, l_3) \in X_3(j,l) \\ (j_2, l_2) \prec (j,l)}} \mathfrak{U}_{j_1, l_1} \mathfrak{W}_{j_2, l_2} \mathfrak{W}_{j_3, l_3}.$$

Here, we recall that the functions $\mathfrak{R}_{j,l}$ were defined in Proposition 4.6.2, and on the unit circle \mathbb{T} , the function $\mathfrak{S}_{j,l}$ is given by

$$\begin{aligned} \mathfrak{S}_{j,l} &= \partial_s^{j+1} \partial_t^{l-1} |\psi_{s,t}| \Big|_{s=t=0} - (j+1)!(l-1)! \operatorname{Re} (\bar{\zeta} \hat{\psi}_{j+1, l-1}) = \sum_{k=2}^{j+l} (-1)^k \left(-\frac{1}{2}\right)_k (j+1)!(l-1)! \\ &\times \sum_{m=1}^{j+l} \sum_{S_m(k, (j+1, l-1))} \prod_{i=1}^m \frac{1}{\beta_i! [\alpha_i!]^{\beta_i}} \left(\sum_{\mathbf{0} \leq \gamma \leq \alpha_i} \frac{\alpha_i!}{\gamma! (\alpha_i - \gamma)!} \partial_{s,t}^{\gamma} \psi_{s,t} \partial_{s,t}^{\alpha_i - \gamma} \bar{\psi}_{s,t} \right)^{\beta_i} \Big|_{s=t=0} \\ &+ \frac{1}{2} \sum_{\mathbf{0} \prec \gamma \prec (j+1, l-1)} \frac{(j+1)!(l-1)!}{\gamma! ((j+1, l-1) - \gamma)!} \partial_{s,t}^{\gamma} \psi_{s,t} \partial_{s,t}^{(j+1, l-1) - \gamma} \bar{\psi}_{s,t} \Big|_{s=t=0}, \end{aligned}$$

where for a multi-index $\gamma \in \mathbb{N}^2$, $\partial_{s,t}^{\gamma} = \partial_s^{\gamma_1} \partial_t^{\gamma_2}$. In particular, the expression for the function $\mathfrak{F}_{j,0}$ involves only the coefficients B_0, \dots, B_{j-1} and $\hat{\psi}_{a,b}$ for $(a,b) \prec (j+1, 0)$, whereas for $l \geq 1$, the expression for the function $\mathfrak{F}_{j,l}$ involves only B_0, \dots, B_j and $\hat{\psi}_{a,b}$ for $(a,b) \prec (j+1, l-1)$.

Proof. Again, this follows from an application of Taylor’s formula, together with Faà di Bruno’s formula. Note that the endpoints of the ordering in the final sum defining $\mathfrak{S}_{j,l}$ are not included in the summation, as a consequence of cancellation in the defining expression

$$\mathfrak{S}_{j,l} = \partial_s^{j+1} \partial_t^{l-1} |\psi_{s,t}|_{s=t=0} - (j+1)!(l-1)! \operatorname{Re}(\bar{\zeta} \hat{\psi}_{j+1,l-1}).$$

The fact that the expression for $\mathfrak{S}_{j,l}$ does not include the maximal partial derivative of $\psi_{s,t}$, corresponding to the index $(j+1, l-1)$, is a consequence of an analogue of Proposition 4.5.2 (adapted to include the simpler instance of compositions $f \circ g$ when g is a scalar-valued function, and the gradient of f does not vanish). The omitted details are left to the reader. \square

4.7. The orthogonal foliation flow IV. Putting the pieces together. We are now ready to formalize the algorithm of Subsection 4.4. We restate the flow equation (4.4.2) in terms of the coefficient functions $\mathfrak{C}_{j,l}$ for the expansion of $\omega_{s,t}$ on the circle:

$$(4.7.1) \quad \mathfrak{C}_{j,l} = \begin{cases} (4\pi)^{-\frac{1}{2}} & \text{for } \zeta \in \mathbb{T} \text{ and } (j,l) = (0,0), \\ 0 & \text{for } \zeta \in \mathbb{T} \text{ and } (j,l) \in \mathcal{I}_{2\kappa} \setminus \{(0,0)\}. \end{cases}$$

This property is key to our completing the proof of Proposition 4.2.5. We solve for the coefficients of f_s and $\psi_{s,t}$ iteratively, according to the algorithm outlined in Subsection 4.4.

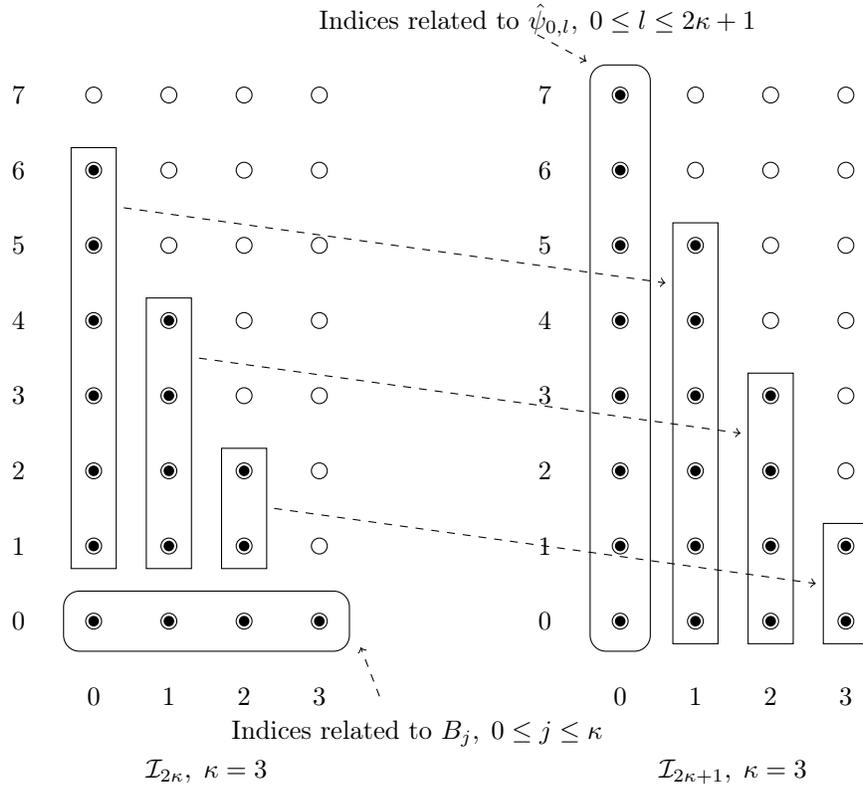


FIGURE 4.1. Illustrations of the index sets $\mathcal{I}_{2\kappa}$ and $\mathcal{I}_{2\kappa+1}$ (marked in black), related to the coefficients $\mathfrak{C}_{j,l}$ (left) and the conformal maps $\hat{\psi}_{j+1,l-1}$ (right), respectively. Indices in sharp boxes are involved in STEP 3, in the determination of the coefficients $\hat{\psi}_{j,l}$ with $j \geq 1$.

Proof of Proposition 4.2.5. In view of Propositions 4.2.6 and 4.6.1, the conformal mapping $\psi_{0,t}$ and its Taylor coefficients $\hat{\psi}_{0,l}$ for $l = 0, 1, \dots, 2\kappa + 1$ with respect to the time parameter t of the flow are well-defined, and they satisfy the required smoothness properties: for t near the origin, $\psi_{0,t}$ extends conformally across the boundary \mathbb{T} to an exterior disk $\mathbb{D}_e(0, \rho'')$ for some

$0 < \rho'' < 1$. Moreover, the derivative $\psi'_{0,t}$ remains uniformly bounded as long as the weight R is confined to a uniform family in $\mathcal{W}(\rho_0, \sigma_0)$ for fixed ρ_0 and σ_0 . In addition, the coefficient functions extend holomorphically to $\mathbb{D}_e(0, \rho')$, by the token of Remark 4.2.4. This completes STEP 1 of the algorithmic procedure.

Turning our attention to STEP 2, we recall from Proposition 4.6.1 that on the circle \mathbb{T} , we have $\operatorname{Re}(-\bar{\zeta}\hat{\psi}_{0,1}) = (4\Delta R(\zeta))^{-\frac{1}{2}}$. Hence, the equation

$$\mathfrak{C}_{0,0} = |B_0(\zeta)|^2 \operatorname{Re}(-\bar{\zeta}\hat{\psi}_{0,1}(\zeta)) = (4\pi)^{-\frac{1}{2}},$$

which is (4.7.1) for $(j, l) = (0, 0)$, together with our requirements that the function B_0 be outer on \mathbb{D}_e with $B_0(\infty) > 0$, tell us that

$$B_0(\zeta) = (4\pi)^{-\frac{1}{4}} \exp \left\{ \frac{1}{2} \mathbf{H}_{\mathbb{D}_e} [\log(4\Delta R)^{\frac{1}{2}}](\zeta) \right\} = (4\pi)^{-\frac{1}{4}} \exp \left\{ \frac{1}{4} \mathbf{H}_{\mathbb{D}_e} [\log 4\Delta R](\zeta) \right\}.$$

This formula initially defines B_0 in the exterior disk \mathbb{D}_e . In view of the given smoothness of the weight R , and the fact that the polarization of ΔR remains zero-free on $\hat{\mathbb{A}}(\rho, \sigma)$, the function $\log(4\Delta R)$ has a holomorphic polarization on $\hat{\mathbb{A}}(\rho, \sigma)$. It follows from Remark 4.2.4 that B_0 extends holomorphically across the unit circle \mathbb{T} to the domain $\mathbb{D}_e(0, \rho')$. Moreover, B_0 remains uniformly bounded provided that R is confined to a uniform family in $\mathcal{W}(\rho_0, \sigma_0)$. This completes STEP 2.

We proceed to STEP 3 of the algorithmic procedure. We are now in the following situation. For some $j_0 \geq 1$, the functions B_0, \dots, B_{j_0-1} and $\hat{\psi}_{j,l}$ for all $(j, l) \in \mathcal{I}_{2\kappa+1}$ with $(j, l) \prec (j_0, 0)$ are already known, and in addition, the relations (4.7.1) are met for all $(j, l) \in \mathcal{I}_{2\kappa}$ with $(j, l) \preceq (j_0 - 1, 0)$. Moreover, all the above-mentioned functions are holomorphic on $\mathbb{D}_e(0, \rho')$. We will now show how this allows us to obtain the relations (4.7.1) for all indices $(j, l) \in \mathcal{I}_{2\kappa}$ with $(j, l) \prec (j_0, 0)$, by making appropriate choices of the functions $\hat{\psi}_{j_0, l-1}$ for $l \geq 1$ with $(j_0, l-1) \in \mathcal{I}_{2\kappa+1}$. The only additional tuples in (4.7.1) are those $(j, l) \in \mathcal{I}_{2\kappa}$ of the form $(j, l) = (j_0 - 1, l)$, where $l \geq 1$.

To achieve the above, we assume that we have completed this procedure up to some $l = l_0 \geq 0$, and turn to the next equation, which reads $\mathfrak{C}_{j_0-1, l_0+1} = 0$, as long as $(j_0 - 1, l_0 + 1) \in \mathcal{I}_{2\kappa}$. If l_0 is too large for this to happen, we are in fact done. In view of Proposition 4.6.4, the equation $\mathfrak{C}_{j_0-1, l_0+1} = 0$ may be written in the form

$$(4\pi)^{-\frac{1}{4}} (4\Delta R)^{\frac{1}{2}} \operatorname{Re}(\bar{\zeta}\hat{\psi}_{j_0, l_0}) + \mathfrak{F}_{j_0-1, l_0+1} = 0,$$

where $\mathfrak{F}_{j_0-1, l_0+1}$ are real-analytic functions on the circle \mathbb{T} that may be expressed in terms of the known functions B_0, \dots, B_{j_0-1} and $\hat{\psi}_{j,l}$ for $(j, l) \in \mathcal{I}_{2\kappa+1}$ with $(j, l) \preceq (j_0, l_0 - 1)$. We provide a solution to this equation by the formula

$$\hat{\psi}_{j_0, l_0} = -(4\pi)^{\frac{1}{4}} \zeta \mathbf{H}_{\mathbb{D}_e} \left[\frac{\mathfrak{F}_{j_0-1, l_0+1}}{(4\Delta R)^{\frac{1}{2}}} \right].$$

The function $\mathfrak{F}_{j_0-1, l_0+1}$ has a polarization which is holomorphic in (z, \bar{w}) for $(z, w) \in \hat{\mathbb{A}}(\rho, \sigma)$, and the same holds for the weight R . As a consequence, it follows that $\hat{\psi}_{j_0, l_0}$ extends holomorphically to exterior the disk $\mathbb{D}_e(0, \rho')$, and that $\hat{\psi}_{j_0, l_0}(\zeta) = O(|\zeta|)$ with an implicit constant which is uniformly bounded, provided that R is confined to a uniform family in $\mathcal{W}(\rho_0, \sigma_0)$.

We now turn to STEP 4. After the completion of STEP 3, the situation is as follows: The functions B_0, \dots, B_{j_0-1} and $\hat{\psi}_{j,l}$ for $(j, l) \in \mathcal{I}_{2\kappa+1}$ with $(j, l) \prec (j_0 + 1, 0)$ are known, and the relations (4.7.1) are met for all $(j, l) \in \mathcal{I}_{2\kappa}$ with $(j, l) \prec (j_0, 0)$. In this step, we need to find the function B_{j_0} , and verify that the relation (4.7.1) is then met with $(j, l) = (j_0, 0)$. To this end, we apply Proposition 4.6.4, and observe that the equation (4.7.1) with $(j, l) = (j_0, 0)$ is equivalent to having

$$\mathfrak{C}_{j_0,0} = 2 \operatorname{Re}(\bar{B}_0 B_{j_0})(4\Delta R)^{-\frac{1}{2}} + \mathfrak{F}_{j_0,0} = 0,$$

where $\mathfrak{F}_{j_0,0}$ is a real-valued real-analytic function expressed in terms of the known functions B_0, \dots, B_{j_0-1} and $\hat{\psi}_{j,l}$ for $(j, l) \in \mathcal{I}_{2\kappa+1}$ with $(j, l) \prec (j_0 + 1, 0)$. The above equation $\mathfrak{C}_{j_0,0} = 0$

may be interpreted as an equation for the unknown function B_{j_0} , with solution

$$B_{j_0} = -\frac{1}{2}B_0\mathbf{H}_{\mathbb{D}_e} \left[\frac{(4\Delta R)^{\frac{1}{2}}\mathfrak{F}_{j_0,0}}{|B_0|^2} \right] = -\pi^{\frac{1}{2}}B_0\mathbf{H}_{\mathbb{D}_e} [\mathfrak{F}_{j_0,0}].$$

This function B_{j_0} extends holomorphically to the exterior disk $\mathbb{D}_e(0, \rho')$, and remains uniformly bounded if the weight R is confined to a uniform family in $\mathcal{W}(\rho_0, \sigma_0)$. Moreover, we observe that B_{j_0} has the required normalization at infinity: $\text{Im } B_{j_0}(\infty) = 0$.

We finally turn to STEP 5. The key observation is that we are now in a position to return to STEP 3 followed by STEP 4 with j_0 replaced by $j_0 + 1$. Since STEP 1 and STEP 2 combine to form the initial data for STEPS 3 AND 4 with $j_0 = 1$, the algorithm produces iteratively the entire set of coefficient functions, and solves in the process all the equations (4.7.1) for $(j, l) \in \mathcal{I}_{2\kappa}$.

Equipped with the functions B_j for $j = 0, \dots, \kappa$, the conformal mappings $\psi_{0,t}$ and the coefficients $\hat{\psi}_{j,l}$ for $(j, l) \in \mathcal{I}_{2\kappa+1} \cap \{(j, l) : j \geq 1\}$, we observe that the functions f_s and $\psi_{s,t}$ given by

$$f_s(\zeta) = \sum_{j=0}^{\kappa} s^j B_j(\zeta) \quad \psi_{s,t}(\zeta) = \psi_{0,t} + \sum_{\substack{(j,l) \in \mathcal{I}_{2\kappa+1} \\ j \geq 1}} s^j t^l \hat{\psi}_{j,l}(\zeta)$$

are well-defined, and have the desired smoothness and mapping properties. As $\psi_{s,t}$ is a perturbation of the identity, and as the finite collection of coefficient functions all extend holomorphically past the boundary \mathbb{T} of the exterior disk, it follows that the radii ρ'' may be chosen with the stated properties. The conclusion of Proposition 4.2.5 is now an immediate consequence of the relations (4.7.1) for the Taylor coefficients of the function

$$\omega_{s,t}(\zeta) = |f_s \circ \psi_{s,t}(\zeta)|^2 e^{-2s^{-1}\{(R \circ \psi_{s,t})(\zeta) - \frac{t}{2}\}} \text{Re} \left(-\bar{\zeta} \partial_t \psi_{s,t}(\zeta) \overline{\psi'_{s,t}(\zeta)} \right), \quad \zeta \in \mathbb{T}$$

in the variables (s, t) near $(0, 0)$, verified in the above algorithm. These Taylor coefficients were calculated in (4.6.7), (4.6.8) and (4.6.9) and later combined in Proposition 4.6.4. \square

4.8. The orthogonal foliation flow IV. The implementation scheme. The hard work was completed in the previous subsection. The existence of the orthogonal foliation flow now follows succinctly, if we put $s = m^{-1}$. In terms of notation, we change to more convenient subscripts. So, we write $\psi_{n,m,t}$ and $f_{n,m}^{(\kappa)}$ when we mean the mapping $\psi_{s,t}$ and the function f_s , respectively, associated with the choices $s = m^{-1}$ and $R = R_\tau$ with $\tau = n/m$. In the previous section the parameter κ was needed but it was suppressed in the notation, here we sometimes prefer to express the dependence explicitly.

Proof of Lemma 4.1.2. In view of Proposition 3.1.3, the collection R_τ of weights with $\tau \in \mathcal{I}_{\epsilon_0}$, is a uniform family in $\mathcal{W}(\rho_0, \sigma_0)$ for some numbers ρ_0, σ_0 with $0 < \rho_0 < 1$ and $\sigma_0 > 0$. The flow equation (4.1.3) of Lemma 4.1.2 now follows immediately from the assertion of Proposition 4.2.5, together with the observation that

$$\begin{aligned} & m^{\frac{1}{2}} \int_{\mathcal{D}_{n,m}} |f_{n,m}^{(\kappa)}|^2 e^{-2mR_\tau} dA \\ &= 2m^{\frac{1}{2}} \int_{-\delta_m}^{\delta_m} \int_{\mathbb{T}} |f_{n,m}^{(\kappa)} \circ \psi_{n,m,t}(\zeta)|^2 e^{-2mR_\tau \circ \psi_{n,m,t}(\zeta)} \text{Re} \left(-\bar{\zeta} \partial_t \psi_{n,m,t}(\zeta) \overline{\psi'_{n,m,t}(\zeta)} \right) ds(\zeta) dt \\ &= 2m^{\frac{1}{2}} \int_{-\delta_m}^{\delta_m} \left((4\pi)^{-\frac{1}{2}} + O(\delta_m^{2\kappa+1}) \right) e^{-mt^2} dt = 1 + O(\delta_m^{2\kappa+1}) = 1 + O(m^{-\kappa-\frac{1}{3}}), \end{aligned}$$

where we use (4.1.2) to integrate over the flow in the coordinates $(t, \zeta) \in [-\delta_m, \delta_m] \times \mathbb{T}$. Moreover, we observe that $f_{n,m}^{(\kappa)}$ is zero-free in $\mathbb{D}_e(0, \rho')$ for large enough m , as the leading term $B_{0,\tau}$ has this property. Here, we recall that ρ' is defined in terms of ρ and σ' in Proposition 4.2.2. \square

We next turn to the result on the structure of the normalized approximately orthogonal quasipolynomials, Proposition 4.1.1.

Proof of Proposition 4.1.1. We recall the definition of the canonical positioning operator $\Lambda_{n,m}$ from Subsection 3.1, and write

$$(4.8.1) \quad F_{n,m}^{(\kappa)}(z) = m^{\frac{1}{4}} \Lambda_{n,m}[f_{n,m}^{(\kappa)}](z),$$

where $f_{n,m}^{(\kappa)}$ is the function from Lemma 4.1.2, which is defined and holomorphic on \mathcal{K}_τ^c , for some compact subset \mathcal{K}_τ of the interior \mathcal{S}_τ° of \mathcal{S}_τ , which stays away from the boundary $\partial\mathcal{S}_\tau$ in the sense that

$$\inf_{\tau \in \mathcal{I}_{\epsilon_0}} \text{dist}_{\mathbb{C}}(\mathcal{K}_\tau, \partial\mathcal{S}_\tau) > 0.$$

We recall that $\chi_{0,\tau}$ denotes a smooth cut-off function which vanishes on \mathcal{K}_τ and equals 1 on \mathcal{X}^τ , where \mathcal{X}^τ is an intermediate set between \mathcal{K}_τ and \mathcal{S}_τ^c . In line with Remark 1.3.4(a), we may insert a further intermediate set \mathcal{X}'_τ between \mathcal{K}_τ and \mathcal{X}_τ , such that $\chi_{0,\tau}$ vanishes on \mathcal{X}'_τ as well (and not just on \mathcal{K}_τ).

The functions $f_{n,m}^{(\kappa)}$ are bounded and holomorphic on \mathcal{K}_τ^c . As the leading term $B_{0,\tau}$ in the expansion of $f_{n,m}^{(\kappa)}$ does not vanish at infinity, it follows that for large enough m , the same can be said for $f_{n,m}^{(\kappa)}$. In view of this, the functions $F_{n,m}^{(\kappa)}$ are quasipolynomials of order n on \mathcal{K}_τ^c in the sense of Definition 3.1.1. Moreover, the definition (4.8.1) of the functions $F_{n,m}^{(\kappa)}$ together with the definition (3.3.1) of the functions $f_{n,m}^{(\kappa)}$ shows that $F_{n,m}^{(\kappa)}$ has the indicated form. What remains for us to do is to verify the properties (i), (ii), and (iii) of Definition 3.1.2.

To this end, we recall the definition of the domain $\mathcal{D}_{n,m}$ from Lemma 4.1.2, which is a certain closed neighbourhood of the unit circle which arises from our orthogonal foliation flow. We recall that

$$\text{dist}_{\mathbb{C}}(\mathcal{D}_{n,m}^c, \mathbb{T}) \geq c_0 \delta_m$$

holds for some fixed constant $c_0 > 0$. We first check property (ii) of Definition 3.1.2. As a step in this direction, we claim that most of the weighted L^2 -mass of the function $\chi_{1,\tau} f_{n,m}^{(\kappa)}$ lies in the domain $\mathcal{D}_{n,m}$. We know that the functions $f_{n,m}^{(\kappa)}$ are bounded uniformly in \mathcal{K}_τ^c independently of m and n while $\tau \in I_{\epsilon_0}$, so that

$$(4.8.2) \quad \chi_{1,\tau} |f_{n,m}^{(\kappa)}| \leq C_0$$

holds in the whole plane \mathbb{C} , for some constant C_0 . Let \mathcal{D}_τ denote a bounded domain which only depends only on τ and contains $\mathbb{D} \cup \mathcal{D}_{n,m}$, such that the bound from below $R_\tau(z) \geq \theta_0 \log|z|$ holds outside \mathcal{D}_τ , for some $\theta_0 > 0$. That such a domain exists for sufficiently large m is shown in Proposition 3.1.3. On the other hand, as $R_\tau \in \mathcal{W}(\rho_0, \sigma_0)$ in the sense of Definition 4.2.1, on the set $\mathcal{D}_\tau \cap \mathbb{D}_e(0, \rho_0) \setminus \mathcal{D}_{n,m}$ we have the estimate

$$e^{-2mR_\tau} \leq e^{-\alpha_0(\log m)^2}$$

for some constant $\alpha_0 > 0$, at least if \mathcal{D}_τ is chosen small enough. Here, we recall that ρ_0 is a radius with $0 < \rho_0 < 1$. It now follows that we have

$$(4.8.3) \quad m^{\frac{1}{2}} \int_{\mathbb{C} \setminus \mathcal{D}_{n,m}} \chi_{1,\tau}^2 |f_{n,m}^{(\kappa)}|^2 e^{-2mR_\tau} \leq C_0^2 m^{\frac{1}{2}} \int_{\mathbb{C} \setminus \mathcal{D}_\tau} e^{-2m\theta_0 \log|z|} dA \\ + C_0^2 m^{\frac{1}{2}} \int_{\mathcal{D}_\tau \cap \mathbb{D}(0, \rho_0) \setminus \mathcal{D}_{n,m}} e^{-\alpha_0(\log m)^2} dA = O(m^{\frac{1}{2}} e^{-\alpha_0(\log m)^2}) = O(m^{-\alpha_0 \log m + \frac{1}{2}}).$$

In the above calculation, we observe that the integral over $\mathbb{C} \setminus \mathcal{D}_\tau$ gives an exponentially small contribution, dominated by the indicated error term. It follows that

$$\begin{aligned} m^{\frac{1}{2}} \int_{\mathbb{C}} \chi_{1,\tau}^2 |f_{n,m}^{(\kappa)}|^2 e^{-2mR_\tau} dA &= m^{\frac{1}{2}} \int_{\mathcal{D}_{n,m}} |f_{n,m}^{(\kappa)}|^2 e^{-2mR_\tau} dA \\ &\quad + m^{\frac{1}{2}} \int_{\mathbb{C} \setminus \mathcal{D}_{n,m}} \chi_{1,\tau}^2 |f_{n,m}^{(\kappa)}|^2 e^{-2mR_\tau} dA = 1 + O(m^{-\kappa - \frac{1}{3}}), \end{aligned}$$

where we use that $\chi_{1,\tau} = 1$ holds on $\mathcal{D}_{n,m}$ together with our foliation flow Lemma 4.1.2 and the estimate (4.8.3). Hence, by the isometric property of $\mathbf{\Lambda}_{n,m}$ from Proposition 3.1.5, it follows that

$$\int_{\mathbb{C}} \chi_{0,\tau}^2 |F_{n,m}^{(\kappa)}|^2 e^{-2mQ} dA = 1 + O(m^{-\kappa - \frac{1}{3}}),$$

as required by property (i) of Definition 3.1.2.

We turn to property (i) of Definition 3.1.2, the approximate orthogonality property. For a polynomial $p \in \text{Pol}_n$ of degree at most $n-1$, we put $g = \mathbf{\Lambda}_{n,m}^{-1}[p]$. The function $f_{n,m}^{(\kappa)}$ is zero-free in a neighbourhood of the exterior disk \mathbb{D}_e , which we may assume to be a fixed exterior disk $\mathbb{D}_e(0, \rho_0)$ for some fixed $\rho_0 < 1$ for all large enough n and m with $\tau = \frac{n}{m} \in I_{\epsilon_0}$, and by the isometric property of $\mathbf{\Lambda}_{n,m}$, we find that

$$\begin{aligned} (4.8.4) \quad \int_{\mathbb{C}} \chi_{0,\tau} p \overline{F_{n,m}^{(\kappa)}} e^{-2mQ} dA &= m^{\frac{1}{4}} \int_{\mathbb{C}} \chi_{1,\tau} g \overline{f_{n,m}^{(\kappa)}} e^{-2mR_\tau} dA(z) \\ &= m^{\frac{1}{4}} \int_{\mathcal{D}_{n,m}} \frac{g}{f_{n,m}^{(\kappa)}} |f_{n,m}^{(\kappa)}|^2 e^{-2mR_\tau} dA + O(m^{-\frac{\alpha_0}{2} \log m + \frac{3}{4}} \|p\|_{2mQ}), \end{aligned}$$

where we are required to justify the indicated error term estimate. To do this, we need Proposition 2.2.2, or more accurately, Lemma 3.5 in [1], which gives the estimate for $p \in \text{Pol}_n$

$$(4.8.5) \quad |p| \leq C_1 m^{\frac{1}{2}} \|p\|_{2mQ} e^{m\hat{Q}_\tau}$$

in the whole plane \mathbb{C} for some constant C_1 , independent of $\tau = \frac{n}{m} \in I_{\epsilon_0}$. The missing term on the right-hand side of (4.8.4) equals

$$m^{\frac{1}{4}} \int_{\mathbb{C} \setminus \mathcal{D}_{n,m}} \chi_{1,\tau} g \overline{f_{n,m}^{(\kappa)}} e^{-2mR_\tau} dA = \int_{\mathbb{C} \setminus \phi_\tau^{-1}(\mathcal{D}_{n,m})} \chi_{0,\tau} p \overline{F_{n,m}^{(\kappa)}} e^{-2mQ} dA,$$

and if we apply the pointwise estimate (4.8.5), we obtain

$$\begin{aligned} \int_{\mathbb{C} \setminus \phi_\tau^{-1}(\mathcal{D}_{n,m})} \chi_{0,\tau} |p F_{n,m}^{(\kappa)}| e^{-2mQ} dA &\leq C_1 m^{\frac{1}{2}} \int_{\mathbb{C} \setminus \phi_\tau^{-1}(\mathcal{D}_{n,m})} \chi_{0,\tau} |F_{n,m}^{(\kappa)}| e^{-2mQ + m\hat{Q}_\tau} dA \\ &= C_1 m^{\frac{3}{4}} \int_{\mathbb{C} \setminus \mathcal{D}_{n,m}} \chi_{1,\tau} |f_{n,m}^{(\kappa)}| e^{m(\hat{Q}_\tau - Q) \circ \phi_\tau^{-1} - mR_\tau} dA \leq C_0 C_1 m^{\frac{3}{4}} \int_{\mathbb{D}_e(0, \rho_0) \setminus \mathcal{D}_{n,m}} e^{-mR_\tau} dA, \end{aligned}$$

where in the last step, we applied the estimate (4.8.2) and the fact that $\hat{Q}_\tau \leq Q$. The rest of the argument that gives (4.8.4) involves splitting the domain of integration using the set \mathcal{D}_τ , and proceeds as in (4.8.3). This establishes (4.8.4), although we still need to control the main term on the right-hand side. To this end, we denote by h the ratio $h = g/f_{n,m}^{(\kappa)}$. Then h is holomorphic outside \mathcal{K}_τ , and vanishes at infinity, since $f_{n,m}$ is zero-free there. Using the foliation flow as coordinates on $\mathcal{D}_{n,m}$ in terms of $(t, \zeta) \in [-\delta_m, \delta_m] \times \mathbb{T}$, we find as in the above

proof of Lemma 4.1.2 that

$$\begin{aligned}
(4.8.6) \quad & m^{\frac{1}{4}} \int_{\mathcal{D}_{n,m}} h(z) |f_{n,m}^{(\kappa)}(z)|^2 e^{-2mR_\tau(z)} dA(z) \\
&= 2m^{\frac{1}{4}} \int_{-\delta_m}^{\delta_m} \int_{\mathbb{T}} h \circ \psi_{n,m,t}(\zeta) |f_{n,m}^{(\kappa)} \circ \psi_{n,m,t}(\zeta)|^2 e^{-2mR_\tau \circ \psi_{n,m,t}(\zeta)} \\
&\quad \times \operatorname{Re} \left\{ -\bar{\zeta} \partial_t \psi_{n,m,t}(\zeta) \overline{\psi'_{n,m,t}(\zeta)} \right\} ds(\zeta) dt \\
&= 2m^{\frac{1}{4}} \int_{-\delta_m}^{\delta_m} \int_{\mathbb{T}} h \circ \psi_{n,m,t}(\zeta) \left\{ (4\pi)^{-\frac{1}{2}} e^{-mt^2} + O(m^{-\kappa-\frac{1}{3}} e^{-mt^2}) \right\} ds(\zeta) dt \\
&= O \left(m^{-\kappa-\frac{1}{12}} \int_{-\delta_m}^{\delta_m} \int_{\mathbb{T}} |h \circ \psi_{n,m,t}(\zeta)| ds(\zeta) e^{-mt^2} dt \right).
\end{aligned}$$

Here, the crucial reduction in the last step of (4.8.6) is based on the fact that the function $h \circ \psi_{n,m,t}$ is holomorphic in \mathbb{D}_e and vanishes at infinity, so that by the mean value property

$$\int_{\mathbb{T}} h \circ \psi_{n,m,t} ds = 0.$$

Now that (4.8.6) is established, we need to simplify the error term further. We will use the observation that all the steps before the last in (4.8.6) apply to a fairly general sufficiently integrable function in place of h , for instance $|h|$ will work. It then follows from (4.8.6) with $|h|$ instead that large enough m , we have

$$\begin{aligned}
& \int_{-\delta_m}^{\delta_m} \int_{\mathbb{T}} |h \circ \psi_{n,m,t}(\zeta)| e^{-mt^2} ds(\zeta) dt \leq 2 \int_{\mathcal{D}_{n,m}} |h(z)| |f_{n,m}^{(\kappa)}(z)|^2 e^{-2mR_\tau(z)} dA(z) \\
&= 2 \int_{\mathcal{D}_{n,m}} |g(z) f_{n,m}^{(\kappa)}(z)| e^{-2mR_\tau(z)} dA(z) \leq 2C_0 \int_{\mathcal{D}_{n,m}} |g(z)| e^{-2mR_\tau(z)} dA(z),
\end{aligned}$$

where in the last step we applied the bound (4.8.2). Finally, we apply the Cauchy-Schwarz inequality, and recall that recall that $g = \Lambda_{n,m}^{-1}[p]$ where $\Lambda_{n,m}$ has the isometry property of Proposition 3.1.5:

$$\begin{aligned}
(4.8.7) \quad & \int_{-\delta_m}^{\delta_m} \int_{\mathbb{T}} |h \circ \psi_{n,m,t}(\zeta)| e^{-mt^2} ds(\zeta) dt \leq 2C_0 \int_{\mathcal{D}_{n,m}} |g(z)| e^{-2mR_\tau(z)} dA(z) \\
&\leq 2C_0 \|g\|_{L^2(\mathcal{D}_{n,m}, e^{-2mR_\tau})} \left\{ \int_{\mathcal{D}_{n,m}} e^{-2mR_\tau} dA \right\}^{1/2} = O(m^{-\frac{1}{4}} \|p\|_{2mQ}).
\end{aligned}$$

Here, we used a simple decay estimate of the integral of the ‘‘Gaussian ridge’’ e^{-2mR_τ} . Next, we write $g/f_{n,m}^{(\kappa)}$ in place of h , and combine the estimates (4.8.6) and (4.8.7), and arrive at

$$\begin{aligned}
(4.8.8) \quad & m^{\frac{1}{4}} \int_{\mathcal{D}_{n,m}} g \overline{f_{n,m}^{(\kappa)}} e^{-2mR_\tau(z)} dA(z) = m^{\frac{1}{4}} \int_{\mathcal{D}_{n,m}} h(z) |f_{n,m}^{(\kappa)}(z)|^2 e^{-2mR_\tau(z)} dA(z) \\
&= O(m^{-\kappa-\frac{1}{3}} \|p\|_{2mQ}).
\end{aligned}$$

In view of (4.8.4) and (4.8.8), we find that for all polynomials $p \in \operatorname{Pol}_n$,

$$(4.8.9) \quad \int_{\mathbb{C}} \chi_{0,\tau} p \overline{F_{n,m}^{(\kappa)}} e^{-2mQ} dA = O(m^{-\kappa-\frac{1}{3}} \|p\|_{2mQ}),$$

as required. Since in addition, $f_{n,m}^{(\kappa)}(\infty) > 0$ by construction, and we have made sure that $Q_\tau(\infty) \in \mathbb{R}$ as well as $\phi'_\tau(\infty) > 0$, the leading coefficient of the quasipolynomial $F_{n,m}^{(\kappa)}$ is now positive, which settles property (iii) of Definition 3.1.2 as well. This completes the proof. \square

4.9. Polynomialization of the quasipolynomials. We have by now constructed our quasipolynomials $F_{n,m}^{(\kappa)}$, of degree n and accuracy κ , and shown that they are approximately orthogonal and normalized. It remains to show that they are indeed good approximations of the true normalized orthogonal polynomials $P_{n,m}$.

Proof of Theorem 1.3.3. We retain the above notation, and consider the $\bar{\partial}$ -problem

$$\bar{\partial}_z u(z) = F_{n,m}^{(\kappa)}(z) \bar{\partial}_z \chi_{0,\tau}(z).$$

In view of Corollary 2.4.2, the $L_{2mQ,n}^2$ -norm minimal solution u_0 , which then has the growth $u_0(z) = O(|z|^{n-1})$ near infinity, enjoys the norm bound

$$(4.9.1) \quad \int_{\mathbb{C}} |u_0|^2 e^{-2mQ} dA \leq \frac{1}{\alpha_1 m} \int_{\mathcal{S}_\tau} |F_{n,m}^{(\kappa)}|^2 |\bar{\partial}_z \chi_{0,\tau}|^2 e^{-2mQ} dA,$$

where $\alpha_1 > 0$ stands for the minimum of ΔQ on the biggest droplet \mathcal{S}_τ with $\tau \in I_{\epsilon_0}$ (which is attained for the rightmost endpoint $\tau = 1 + \epsilon_0$). Next, given that the quasipolynomials of degree n are of the form $F_{n,m}^{(\kappa)} = m^{\frac{1}{2}} \mathbf{\Lambda}_{n,m}[f_{n,m}^{(\kappa)}]$, where the functions $f_{n,m}^{(\kappa)}$ are uniformly bounded in $\mathbb{D}_e(0, \rho_0)$ for some radius $\rho_0 < 1$, we find that

$$(4.9.2) \quad \int_{\mathcal{S}_\tau} |F_{n,m}^{(\kappa)}|^2 |\bar{\partial}_z \chi_{0,\tau}|^2 e^{-2mQ} dA = m^{\frac{1}{2}} \int_{\mathbb{D}} |f_{n,m}^{(\kappa)}|^2 |\bar{\partial}_z \chi_{1,\tau}|^2 |\phi'_\tau \circ \phi_\tau^{-1}|^2 e^{-2mR_\tau} dA \\ = O(m^{\frac{1}{2}} e^{-\alpha_2 m})$$

for some $\alpha_2 > 0$ such that $2R_\tau \geq \alpha_2$ on the support of $\bar{\partial}_z \chi_{1,\tau}$. This exponential decay estimate is possible since the support of $\bar{\partial}_z \chi_{1,\tau}$ is located inside \mathbb{D} away from the boundary. Note that in the context of the estimate (4.9.2) it is important as well that the expression $|\phi'_\tau \circ \phi_\tau^{-1}|^2$ is uniformly bounded on the support of $\bar{\partial}_z \chi_{1,\tau}$ as well. If we combine the above estimates (4.9.1) and (4.9.2), we find that

$$(4.9.3) \quad \int_{\mathbb{C}} |u_0|^2 e^{-2mQ} dA = O(m^{-\frac{1}{2}} e^{-\alpha_2 m}),$$

as $m \rightarrow \infty$ while $\tau = \frac{n}{m} \in I_{\epsilon_0}$, with a uniform implicit constant. Next, we put

$$P_{n,m}^* := F_{n,m}^{(\kappa)} \chi_{0,\tau} - u_0(z)$$

which is then automatically a polynomial of degree n , since the function is entire and grows $\asymp |z|^n$ near infinity. Moreover, in view of (4.9.3), this polynomial is very close to the function $F_{n,m}^{(\kappa)} \chi_{0,\tau}$ in the norm of $L^2(\mathbb{C}, e^{-2mQ})$:

$$(4.9.4) \quad \int_{\mathbb{C}} |P_{n,m}^* - F_{n,m}^{(\kappa)} \chi_{0,\tau}|^2 e^{-2mQ} dA = \int_{\mathbb{C}} |u_0|^2 e^{-2mQ} dA = O(m^{-\frac{1}{2}} e^{-\alpha_2 m}).$$

It now follows from (4.8.9) and (4.9.4) that of the function for all polynomials $p \in \text{Pol}_n$ of degree $\leq n-1$,

$$(4.9.5) \quad \int_{\mathbb{C}} P_{n,m}^* \bar{p} e^{-2mQ} dA = O(m^{-\kappa - \frac{1}{3}} \|p\|_{2mQ}),$$

while

$$(4.9.6) \quad \int_{\mathbb{C}} |P_{n,m}^*|^2 e^{-2mQ} dA = 1 + O(m^{-\kappa - \frac{1}{3}}).$$

We observe also that by duality, (4.9.5) asserts that

$$(4.9.7) \quad \|\mathbf{P}_{n,m} P_{n,m}^*\|_{2mQ} = O(m^{-\kappa - \frac{1}{3}}),$$

where $\mathbf{P}_{n,m}$ denotes the orthogonal projection in $L^2(\mathbb{C}, e^{-2mQ})$ onto the subspace Pol_n of polynomials of degree $\leq n-1$. If we use this to correct the polynomial $P_{n,m}^*$, and put $\tilde{P}_{n,m} := \mathbf{P}_{n,m}^\perp P_{n,m}^* = P_{n,m}^* - \mathbf{P}_{n,m} P_{n,m}^*$, then automatically $\tilde{P}_{n,m}$ has degree n and it is also orthogonal to all the lower degree polynomials. As a consequence, $\tilde{P}_{n,m}$ must be a scalar multiple of $P_{n,m}$,

the orthogonal polynomial we are looking for, which we write as $\tilde{P}_{n,m} = cP_{n,m}$ for a constant c . Putting things together so far, we have obtained that

$$(4.9.8) \quad \|\tilde{P}_{n,m} - F_{n,m}^{(\kappa)}\chi_{0,\tau}\|_{2mQ} = O(m^{-\kappa-\frac{1}{3}})$$

with a uniform implied constant. Moreover, by (4.9.6) and (4.9.7), the norm of $\tilde{P}_{n,m}$ equals

$$(4.9.9) \quad |c| = \|cP_{n,m}\|_{2mQ} = \|\tilde{P}_{n,m}\|_{2mQ} = 1 + O(m^{-\kappa-\frac{1}{3}}),$$

Next, by Proposition 2.2.2, it follows from (4.9.8) that

$$(4.9.10) \quad |cP_{n,m} - F_{n,m}^{(\kappa)}| = |\tilde{P}_{n,m} - F_{n,m}^{(\kappa)}| = O(m^{-\kappa+\frac{1}{6}}e^{m\hat{Q}_\tau})$$

holds in $\mathbb{C} \setminus \mathcal{S}_\tau$. Since by construction

$$\lim_{|z| \rightarrow \infty} z^{-n} F_{n,m}^{(\kappa)}(z) > 0,$$

and the function $e^{m\hat{Q}_\tau}$ grows like $|z|^{-n}$ as $|z| \rightarrow \infty$, it follows from (4.9.10) that the above constant $c = c_{n,m}$ must have

$$\frac{\operatorname{Im} c}{\operatorname{Re} c} = O(m^{-\kappa-\frac{1}{12}}),$$

as $n, m \rightarrow \infty$ while $\tau = \frac{n}{m} \in I_{\epsilon_0}$. Hence we may write $c = \gamma c'$, where c' is real and positive and $\gamma \in \mathbb{C}$ with $\gamma = 1 + O(m^{-\kappa-\frac{1}{12}})$. Taking into account the relation (4.9.9) as well, we find that $c' = 1 + O(m^{-\kappa-\frac{1}{12}})$. It now follows from this observation combined with (4.9.8) that

$$\|P_{n,m} - \chi_{0,\tau} F_{n,m}^{(\kappa)}\|_{2mQ} = O(m^{-\kappa-\frac{1}{12}}).$$

This falls slightly short of allowing us to obtain Theorem 1.3.3 right away. The problem is that our error term is larger than what is claimed. However, since the precision κ is arbitrary, we might as well replace κ by $\kappa + 1$ and see what we get. This would give that

$$(4.9.11) \quad \|P_{n,m} - \chi_{0,\tau} F_{n,m}^{(\kappa+1)}\|_{2mQ} = O(m^{-\kappa-1-\frac{1}{12}}).$$

By analyzing a single term in the asymptotic expansion, it is easy to verify that

$$\|\chi_{0,\tau} F_{n,m}^{(\kappa+1)} - \chi_{0,\tau} F_{n,m}^{(\kappa)}\|_{2mQ} = O(m^{-\kappa-1}),$$

and hence the assertion of the theorem immediate from this estimate and (4.9.11). \square

Proof of Theorem 1.3.5. The quasipolynomials $F_{n,m}^{(\kappa)}$ that we obtained above can be written in the form

$$F_{n,m}^{(\kappa)} = \left(\frac{m}{2\pi}\right)^{\frac{1}{4}} \sqrt{\phi'_\tau} [\phi_\tau]^n e^{mQ_\tau} \sum_{j=0}^{\kappa} m^{-j} \mathcal{B}_{j,\tau},$$

where $\mathcal{B}_{j,\tau}$ are uniformly bounded, and holomorphic in a fixed neighbourhood of the exterior disk $\bar{\mathbb{D}}_e$. To obtain the theorem, we need to show that $F_{n,m}^{(\kappa)}$ is close to $P_{n,m}$ pointwise in some domains $\mathcal{K}_{m,\tau}^c$, where \mathcal{K}_τ are compact subsets of \mathcal{S}_τ that remain far enough away from the boundary $\partial\mathcal{S}_\tau$. To this end, we consider the level sets $\gamma_{t,\tau}$

$$\gamma_{t,\tau} := \{z \in \mathcal{S}_\tau : \operatorname{dist}_{\mathbb{C}}(z, \partial\mathcal{S}_\tau) = t\},$$

and observe that by the implicit function theorem, these sets are closed simple curves, provided that t is small enough. Next, we define the compact sets $\mathcal{K}_{m,\tau}$ as the closure of the bounded component of $\mathbb{C} \setminus \gamma_{t_m,\tau}$, where $t_m = (m^{-1} \log \log m)^{\frac{1}{2}}$. On the set $\mathbb{C} \setminus \mathcal{K}_{m,\tau}$, we have the estimate

$$0 \leq m(\hat{Q}_\tau - \check{Q}_\tau)(z) \leq D \log \log m,$$

where D is some positive constant, which is uniformly bounded while $\tau \in I_{\epsilon_0}$. Thus

$$e^{m(\hat{Q}_\tau - \check{Q}_\tau)} \leq e^{D \log \log m} = (\log m)^D,$$

which grows slower than m^ε for arbitrarily small ε , and in particular slower than $m^{\frac{1}{6}}$. In view of Theorem 1.3.3, and the pointwise estimate of Proposition 2.2.2 applied to the intermediate set \mathcal{X}_τ between \mathcal{K}_τ and \mathcal{S}_τ^c where the cut-off function $\chi_{0,\tau}$ assumes the value 1, we find that

$$|P_{n,m}(z) - F_{n,m}^{(\kappa)}(z)| = O(m^{-\kappa-\frac{1}{2}}e^{m\hat{Q}_\tau(z)}) = O(m^{-\kappa-\frac{1}{3}}e^{m\check{Q}_\tau(z)}), \quad z \in \mathcal{K}_{m,\tau}^c,$$

where the implicit constant again is uniform in the relevant parameter range. We may rephrase this as saying that

$$P_{n,m}(z) = F_{n,m}^{(\kappa)}(z) + O(m^{-\kappa-\frac{1}{3}}e^{m\check{Q}_\tau(z)}) = \left(\frac{m}{2\pi}\right)^{\frac{1}{4}} \sqrt{\phi'_\tau}[\phi_\tau]^n e^{m\mathcal{Q}_\tau} \left(\sum_{j=0}^{\kappa} \mathcal{B}_{j,\tau} + O(m^{-\kappa-\frac{7}{12}}) \right),$$

for $z \in \mathcal{K}_{m,\tau}^c$. This essentially proves the theorem, except that the error term is now slightly worse than claimed. However, we may fix this by replacing κ by $\kappa + 1$ in the above argument, to obtain

$$\begin{aligned} P_{n,m}(z) &= \left(\frac{m}{2\pi}\right)^{\frac{1}{4}} \sqrt{\phi'_\tau}[\phi_\tau]^n e^{m\mathcal{Q}_\tau} \left(\sum_{j=0}^{\kappa+1} m^{-j} \mathcal{B}_{j,\tau} + O(m^{-\kappa-\frac{19}{12}}) \right) \\ &= \left(\frac{m}{2\pi}\right)^{\frac{1}{4}} \sqrt{\phi'_\tau}[\phi_\tau]^n e^{m\mathcal{Q}_\tau} \left(\sum_{j=0}^{\kappa} m^{-j} \mathcal{B}_{j,\tau} + O(m^{-\kappa-1}) \right), \end{aligned}$$

where the last step follows since the function $\mathcal{B}_{\kappa+1,\tau}$ is bounded in the relevant region. The proof is complete. \square

5. BOUNDARY UNIVERSALITY IN THE RANDOM NORMAL MATRIX MODEL

5.1. Uniform asymptotics near $\tau = 1$. We take as our starting point the first term of the asymptotic expansion of Theorem 1.3.5. Recall from Theorem 1.3.3 that $\mathcal{K}_{m,\tau}$ is a compact subset of \mathcal{S}_τ with $\text{dist}_{\mathbb{C}}(\mathcal{K}_{m,\tau}, \mathcal{S}_\tau^c) \geq (m^{-1} \log \log m)^{1/2}$.

Corollary 5.1.1. *Let $\mathcal{H}_{Q,\tau}$ be the bounded holomorphic function in the set \mathcal{K}_τ^c with real part $\text{Re } \mathcal{H}_{Q,\tau} = \frac{1}{4} \log(2\Delta Q)$ on the boundary $\partial\mathcal{S}_\tau$, which is real-valued at infinity. Then, in the limit as $m, n \rightarrow \infty$ while $\tau = \frac{n}{m} \in I_{\epsilon_0}$, we have the asymptotics*

$$|P_{n,m}(z)|^2 e^{-2mQ(z)} = \sqrt{\frac{m}{\pi}} |\phi'_\tau(z)| e^{-2m(Q-\check{Q}_\tau)(z)} \left(e^{2\text{Re } \mathcal{H}_{Q,\tau}(z)} + O(m^{-1}) \right),$$

where the implied constant is uniform for $z \in \mathcal{K}_{m,\tau}^c$.

Proof. We recall that

$$\check{Q}_\tau = \text{Re } \mathcal{Q}_\tau + \tau \log |\phi_\tau| = \text{Re } \mathcal{Q}_\tau + \frac{n}{m} \log |\phi_\tau|,$$

and in view of Theorems 1.3.5 and 1.3.7, we may write

$$\begin{aligned} |P_{n,m}|^2 &= \sqrt{\frac{m}{\pi}} |\phi'_\tau(z)| |\phi_\tau|^{2n} e^{2m \text{Re } \mathcal{Q}_\tau} |\mathcal{B}_{0,\tau} + O(m^{-1})|^2 \\ &= \sqrt{\frac{m}{\pi}} |\phi'_\tau(z)| e^{2m\check{Q}_\tau} \left(e^{2\text{Re } \mathcal{H}_{Q,\tau}(z)} + O(m^{-1}) \right), \end{aligned}$$

and the assertion follows. \square

We now obtain Theorem 1.4.1.

5.2. Error function asymptotics. In view of Corollary 5.1.1, we observe that the probability density $|P_{n,m}|^2 e^{-2mQ}$ resembles a Gaussian wave which crests around the boundary $\partial\mathcal{S}_\tau$ of the droplet, where $\tau = \frac{n}{m}$. As a consequence, we expect the density to be obtained as the sum of such Gaussians. Near the droplet boundary, this effect is the strongest, and adding a large but finite number of such Gaussian waves crested along boundary curves $\partial\mathcal{S}_\tau$ which move with the degree parameter n results in error function asymptotics.

Proof of Theorem 1.4.1. We recall the rescaled variable from the introduction

$$z_m(\xi) = z_0 + n \frac{\xi}{\sqrt{2m\Delta Q(z_0)}},$$

where $z_0 \in \partial\mathcal{S}_\tau$ and n is the outward unit normal to \mathcal{S}_τ at z_0 , and the rescaled density $\rho_m(\xi)$ given by (1.2.2). In terms of orthogonal polynomials, the object of study is the function

$$\rho_m(\xi) = \frac{1}{2m\Delta Q(z_0)} \sum_{n=0}^{m-1} |P_{n,m}(z_m(\xi))|^2 e^{-2mQ(z_m(\xi))}.$$

We begin by noting that $z_m(\xi)$ is in the set $\mathcal{K}_{m,1}^c$ (see Theorem 1.3.5), provided that ξ is confined to the disk $\mathbb{D}(0, r_m)$, where $r_m = \sqrt{2\Delta Q(z_0)} \log \log m$. We shall assume throughout that $\xi \in \mathbb{D}(0, r_m)$.

Next, we write

$$\rho_{m_1,m}(\xi) = \frac{1}{2m\Delta Q(z_0)} \sum_{n=0}^{m_1-1} |P_{n,m}(z_m(\xi))|^2 e^{-2mQ(z_m(\xi))}$$

and split accordingly for $m_1 < m$

$$(5.2.1) \quad \rho_m(\xi) = \frac{1}{2m\Delta Q(z_0)} \sum_{n=m_1}^{m-1} |P_{n,m}(z_m(\xi))|^2 e^{-2mQ(z_m(\xi))} + \rho_{m_1,m}(\xi).$$

We choose m_1 to be the integer part of $m - m^{\frac{1}{2}} \log m$.

By Proposition 2.2.2 that for $n \leq m_1$,

$$(5.2.2) \quad |P_{n,m}(z)|^2 e^{-2mQ(z)} \leq C m e^{-2m(Q - \hat{Q}_{\tau_1})(z)}, \quad z \in \mathbb{D}(z_0, \alpha_0 \delta_m),$$

where $\tau_1 = m_1/m$. By Taylor's formula applied to $Q - \hat{Q}_{\tau_1} = R_{\tau_1} \circ \phi_{\tau_1}$ in $\mathcal{S}_{\tau_1}^c$ (Proposition 3.1.3), it follows that

$$(5.2.3) \quad (Q - \hat{Q}_{\tau_1})(z) \geq \beta_0 \text{dist}_{\mathbb{C}}(z, \partial\mathcal{S}_{\tau_1})^2$$

for some constant $\beta_0 > 0$, provided that $z \in \mathcal{S}_{\tau_1}^c$ is close enough to $\partial\mathcal{S}_{\tau_1}$. For instance, this estimate holds for $z \in \mathcal{S}_1 \setminus \mathcal{S}_{\tau_1}$. Moreover, as $\tau_1 = \frac{m_1}{m}$ eventually is in I_{ϵ_0} , the function $Q - \hat{Q}_{\tau_1}$ does not vanish on $\mathcal{S}_{\tau_1}^c$, and tends to infinity at infinity. The latter observation shows that further away from the boundary $\partial\mathcal{S}_{\tau_1}$, the right-hand side of (5.2.2) decays exponentially.

If $n \leq m_1$ and $\tau = \frac{n}{m}$, then $1 - \tau \geq m^{-\frac{1}{2}} \log m = \delta_m$. As a consequence of Lemma 2.3.1 we obtain that the boundary $\partial\mathcal{S}_\tau$ moves at a positive speed in τ . In particular, for $\tau = \frac{n}{m}$ where $n \leq m_1$ we have that the distance $\text{dist}_{\mathbb{C}}(\partial\mathcal{S}_\tau, \partial\mathcal{S}_1)$ is at least $2\alpha_0\delta_m$, for some fixed positive α_0 . Since $\text{dist}_{\mathbb{C}}(\partial\mathcal{S}_{\tau_1}, \partial\mathcal{S}_1)$ is at least $2\alpha_0\delta_m$, we have that

$$(5.2.4) \quad \text{dist}_{\mathbb{C}}(z, \partial\mathcal{S}_{\tau_1}) \geq \alpha_0\delta_m, \quad z \in \mathbb{D}(z_0, \alpha_0\delta_m).$$

Next, we note that if $\zeta \in \mathbb{D}(0, r_m)$, then for large enough m we have $z_m(\zeta) \in \mathbb{D}(z_0, \alpha_0\delta_m)$. This follows from the obvious fact that $\log \log m = o(\log m)$. By a combination of (5.2.3) and (5.2.4) it follows that

$$(Q - \hat{Q}_{\tau_1})(z_m(\zeta)) \geq \beta_0 \alpha_0^2 \delta_m^2.$$

Now, it follows from the above estimates (5.2.2) and (5.2.3) that for $n \leq m_1$

$$|P_{n,m}(z_m(\xi))|^2 e^{-2mQ(z_m(\xi))} = O(m e^{-2\beta_0 \alpha_0^2 (\log m)^2}),$$

where the constant $c_0 > 0$ can be taken to be independent of $\xi \in \mathbb{D}(0, r_m)$. It follows that

$$\rho_{m_1, m}(\xi) = O(m^2 e^{-\beta_0 \alpha_0^2 (\log m)^2}), \quad \xi \in \mathbb{D}(0, r_m)$$

which shows in particular $\rho_{m_1, m}(\xi) = O(m^{-M})$ for arbitrarily large M .

As a result of the above considerations, it follows that we may focus on the remaining sum in (5.2.1) over the degrees n with $m_1 \leq n \leq m-1$, that is, $\tau = \frac{n}{m}$ with $\frac{m_1}{m} \leq \tau \leq 1$. In particular, the asymptotics of Corollary 5.1.1 applies for the whole range. Set $\tau(j) = \tau_m(j) = 1 - \frac{j}{m}$, where j ranges from 1 to $m - m_1$, which is approximately $m^{\frac{1}{2}} \log m$. We obtain

$$(5.2.5) \quad \rho_m(\xi) = \frac{(\pi m)^{-\frac{1}{2}}}{2\Delta Q(z_0)} \sum_{j=1}^{m-m_1} |\phi'_{\tau(j)}(z_m(\xi))| e^{-2m(Q - \check{Q}_{\tau(j)})(z_m(\xi)) + 2\operatorname{Re} \mathcal{H}_{Q, \tau(j)}(z_m(\xi))} + O(m^{-M}).$$

By Taylor expansion, it follows that

$$|\phi'_{\tau(j)}(z_m(\xi))| = |\phi'_1(z_0)| + O(m^{-1/2} \log \log m),$$

and by the same token that

$$2\operatorname{Re} \mathcal{H}_{Q, \tau(j)}(z_m(\xi)) = \frac{1}{2} \log \Delta Q(z_0) + O(m^{-1/2} \log \log m)$$

as $m \rightarrow \infty$ for all $j \leq m - m_1$. The next thing to consider is the movement of $\partial \mathcal{S}_\tau$, for $\tau = \tau(j)$ as j increases. Recalling that \mathbf{n} denotes the outward pointing unit normal to $\partial \mathcal{S}_1$ at the point z_0 , Lemma 2.3.1 tells us that the line $z_0 + \mathbf{n}\mathbb{R}$ intersects $\partial \mathcal{S}_{\tau(j)}$ at the nearest point

$$z_j = z_0 - \mathbf{n} \frac{j}{m} \frac{|\phi'_1(z_0)|}{4\Delta Q(z_0)} + O\left(\left(\frac{j}{m}\right)^2\right),$$

and the outer unit normal \mathbf{n}_j to $\partial \mathcal{S}_{\tau(j)}$ at the point z_j will satisfy

$$\mathbf{n}_j = \mathbf{n} + O\left(\frac{j}{m}\right) = \mathbf{n} + O(m^{-\frac{1}{2}} \log m).$$

We may hence write

$$(Q - \check{Q}_{\tau(j)})(z_m(\xi)) = (Q - \check{Q}_{\tau(j)})\left(z_j + \mathbf{n}_j \frac{\xi + \frac{j}{2} \frac{|\phi'_1(z_0)|}{\sqrt{2m\Delta Q(z_0)}} + O(m^{-\frac{1}{2}} (\log m)^2)}{\sqrt{2m\Delta Q(z_0)}}\right).$$

A simple Taylor series expansion in normal and tangential coordinates at the point z_j gives that

$$(Q - \check{Q}_{\tau(j)})(z_j + \mathbf{n}_j \eta) = 2\Delta Q(z_j) (\operatorname{Re} \eta)^2 + O(|\eta|^3) = 2\Delta Q(z_0) \operatorname{Re}(\eta)^2 + O\left(|\eta|^2 \frac{j}{m} + |\eta|^3\right),$$

for η close to 0. From this we deduce that for η with $|\eta| = O(\log m)$ we have

$$2m(Q - \check{Q}_{\tau(j)})\left(z_{\tau_j} + \mathbf{n}_j \frac{\eta}{\sqrt{2m\Delta Q(z_0)}}\right) = \frac{1}{2} (2\operatorname{Re} \eta)^2 + O(m^{-1/2} (\log m)^3), \quad m \rightarrow \infty.$$

We apply this with η given by

$$\eta = \xi + \frac{j}{2} \frac{|\phi'_1(z_0)|}{\sqrt{2m\Delta Q(z_0)}} + O(m^{-\frac{1}{2}} (\log m)^2),$$

which then gives that

$$(2\operatorname{Re} \eta)^2 = \left(2\operatorname{Re} \xi + j \frac{|\phi'_1(z_0)|}{\sqrt{2m\Delta Q(z_0)}}\right)^2 + O(m^{-\frac{1}{2}} (\log m)^3).$$

Putting these asymptotic relations together, we find that

$$(5.2.6) \quad \rho_m(\xi) = \frac{1}{\sqrt{2\pi}} \left(1 + O(m^{-\frac{1}{2}} (\log m)^3)\right) \\ \times \sum_{j=1}^{m-m_1} \frac{|\phi'_1(z_0)|}{\sqrt{2m\Delta Q(z_0)}} \exp\left\{-\frac{1}{2} \left(2\operatorname{Re} \xi + j \frac{|\phi'_1(z_0)|}{\sqrt{2m\Delta Q(z_0)}}\right)^2\right\} + O(m^{-M}).$$

We recognize immediately (5.2.6) as an approximate Riemann sum for

$$\operatorname{erf}(2 \operatorname{Re} \xi) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{1}{2}(2 \operatorname{Re} \xi + t)^2} dt$$

with respect to a partition of the interval $[0, \gamma_0 \log m]$, with step length $m^{-\frac{1}{2}} \gamma_0$, where

$$\gamma_0 = \frac{|\phi'(z_0)|}{\sqrt{2\Delta Q(z_0)}}.$$

Since such Riemann sums converge to the corresponding integral with small error, this implies that

$$\lim_{m \rightarrow \infty} \rho_m(\xi) = \operatorname{erf}(2 \operatorname{Re} \xi),$$

which completes the proof. \square

5.3. Convergence of correlation kernels. Finally, we turn to the convergence of the rescaled kernels $k_m(z_m(\xi), z_m(\eta))$ as $m \rightarrow \infty$. In principle, this should follow from our expansion of the orthogonal polynomials, but to do this directly seems a bit tricky. However, given the work of Ameur, Kang, and Makarov [4], it turns out to be enough to obtain the more straightforward diagonal convergence of the correlation kernel.

Proof of Corollary 1.4.2. As in the introduction, we denote by $G(z, w)$ the Ginibre- ∞ kernel

$$G(\xi, \eta) = e^{\xi \bar{\eta} - \frac{1}{2}(|\xi|^2 + |\eta|^2)},$$

which is the correlation kernel of a translation invariant planar point process. We now present some material from [4]. An important concept is that of cocycles. We recall Theorem 2.6.1, which tells us that there exists a sequence of continuous functions $c_m : \mathbb{C} \rightarrow \mathbb{T}$ such that, for any subsequence \mathcal{N} of the natural numbers \mathbb{N} , there exists a Hermitian entire function $F(\xi, \eta)$ and a further subsequence $\mathcal{N}^* \subset \mathcal{N}$ such that

$$(5.3.1) \quad c_m(\xi) \bar{c}_m(\eta) k_m(z_m(\xi), z_m(\eta)) \rightarrow G(\xi, \eta) F(\xi, \eta), \quad m \in \mathcal{N}^*, m \rightarrow \infty,$$

where the convergence is uniform on compact subsets of \mathbb{C}^2 . Here, we recall the familiar notion that a function $F(\xi, \eta)$ is Hermitian entire if it is an entire function of the two variables $(\xi, \bar{\eta})$ with the symmetry property $F(z, w) = \bar{F}(w, z)$. For such functions, the diagonal restriction $F(\xi, \xi)$ determines the function uniquely. Indeed, the polarization of the diagonal restriction gives back our function $F(\xi, \eta)$. We denote by $\rho(\xi)$ the limiting density

$$\rho(\xi) = \lim_{m \rightarrow \infty, m \in \mathcal{N}^*} k_m(z_m(\xi), z_m(\xi)) = G(\xi, \xi) F(\xi, \xi),$$

and since $G(\xi, \xi) \equiv 1$, it follows that $F(\xi, \xi) = \rho(\xi)$. By Theorem 1.4.1, it follows that

$$\rho(\xi) = \operatorname{erf}(2 \operatorname{Re} \xi).$$

By the uniqueness property of diagonal restriction, the only possibility for the entire Hermitian kernel is

$$F(\xi, \eta) = \operatorname{erf}(\xi + \bar{\eta}).$$

This shows that the limit along some subsequence of any given sequence of positive integers is always the same. We claim that this means that the whole sequence converges. Before we turn to this, we need to observe that the method of [4] gives a locally uniform bound on the blow-up correlation kernels, which in our setting means that the entire Hermitian kernels that converge to $F(\xi, \xi)$ on the diagonal form a normal family on \mathbb{C}^2 . In case the convergence (5.3.1) were to fail along the positive integers, by normal families, we could distill a sequence \mathcal{N}_0 such that the left-hand side of (5.3.1) would converge to something else along the subsequence \mathcal{N}_0 . This would contradict what we have already established, which is that we have convergence along a subsequence \mathcal{N}_0^* of \mathcal{N}_0 . The assertion of the corollary follows. \square

6. CONNECTION WITH THE MATRIX $\bar{\partial}$ -PROBLEM OF ITS AND TAKHTAJAN

6.1. Matrix $\bar{\partial}$ -problems and orthogonal polynomials. Given the successful application of Riemann-Hilbert problem methods to the study of orthogonal polynomials in the context of the real line and the unit circle, it has been proposed that the planar orthogonal polynomials should be approached in a similar fashion. Following Its and Takhtajan [30], we consider a matrix $\bar{\partial}$ -problem (or a *thick Riemann-Hilbert problem*) and see how it fits in with our orthogonal foliation flow.

Denote by Ω a bounded domain in \mathbb{C} (chosen to be sufficiently large), and let $\pi_{n,m}$ denote the monic orthogonal polynomial of degree n with respect to the measure $1_{\Omega} e^{-2mQ} dA$. That is, $\pi_{n,m}$ is given by

$$\pi_{n,m}(z) = \kappa_{n,m}^{-1} P_{n,m,\Omega}(z),$$

where $\kappa_{n,m}$ is the leading coefficient of the normalized orthogonal polynomial $P_{n,m,\Omega}$, taken with respect to $1_{\Omega} e^{-2mQ} dA$. The requirement on the bounded domain Ω is that it should contain the spectral droplet S_1 . Under this condition, the polynomials $P_{n,m,\Omega}$ can be shown to be close to the orthogonal polynomials $P_{n,m}$ considered previously in this work, with an exponentially decaying error term.

For a compactly supported function f , let $\mathbf{C}[f]$ be its Cauchy transform, given by

$$\mathbf{C}[f](z) = \int_{\mathbb{C}} \frac{f(w)}{z-w} dA(w).$$

The Cauchy transform may immediately be extended past compactly supported functions, e.g. to $L^p(\mathbb{C})$ for $1 \leq p < 2$. This is not optimal, but for our purposes there will be no need to push the matter further. The importance of the Cauchy transform comes from the fact that in the sense of distribution theory, $\bar{\partial}\mathbf{C}[f] = f$.

In [30], Its and Takhtajan propose to study the asymptotics of $\pi_{n,m}$ starting from the observation that the matrix-valued function

$$(6.1.1) \quad Y_{n,m}(z) = \begin{pmatrix} \pi_{n,m}(z) & -\mathbf{C}[1_{\Omega} \bar{\pi}_{n,m} e^{-2mQ}](z) \\ -\kappa_{n-1,m}^2 \pi_{n-1,m}(z) & \kappa_{n-1,m}^2 \mathbf{C}[1_{\Omega} \bar{\pi}_{n-1,m} e^{-2mQ}](z) \end{pmatrix}$$

solves the $\bar{\partial}$ -problem

$$(6.1.2) \quad \begin{cases} \bar{\partial}Y(z) = -\bar{Y}(z)W(z), & \text{for } z \in \mathbb{C}, \\ Y(z) = (I + O(z^{-1})) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, & \text{as } |z| \rightarrow +\infty, \end{cases}$$

where $W(z) = W_m(z)$ is the matrix-valued function

$$W(z) = \begin{pmatrix} 0 & 1_{\Omega}(z) e^{-2mQ(z)} \\ 0 & 0 \end{pmatrix}.$$

Moreover, the solution is unique, as shown in [30]. We remark that classical Riemann-Hilbert problems where a jump occurs on a curve may be phrased as $\bar{\partial}$ -problems where $\bar{\partial}Y(z)$ is understood as a measure supported on a curve Γ , and the above problem is a natural generalization to a more genuinely two-dimensional situation.

The idea that underlies the Its-Takhtajan approach, as well as the classical Riemann-Hilbert approach to orthogonal polynomials, is the expectation that one may constructively obtain an approximate solution $\tilde{Y} = \tilde{Y}_{n,m}(z)$ to the problem (6.1.1) (or the corresponding RHP), which should then produce a component $(\tilde{Y}_{n,m})_{1,1}$ approximately equal to $\pi_{n,m}(z)$.

6.2. Integration of Riemann-Hilbert problems along curve families. Unfortunately, it has proven difficult to solve the problem (6.1.1) constructively. The following simple observation shows how our orthogonal foliation flow reduces the $\bar{\partial}$ -problem to a family of classical Riemann-Hilbert problems along closed curves.

In order to describe this problem, we denote by $J(z)$ a 2×2 jump matrix and let Γ be an oriented smooth simple closed curve in \mathbb{C} . We denote by Ω^+ and Ω^- the interior and exterior

components of the complement $\mathbb{C} \setminus \Gamma$, respectively. If f is a function defined on $\mathbb{C} \setminus \Gamma$, which is continuous up to the boundary Γ as seen from each component we define the two boundary value functions f^+ and f^- on Γ by

$$f^\pm(\zeta) = \lim_{\substack{z \rightarrow \zeta \\ z \in \Omega^\pm}} f(z), \quad \zeta \in \Gamma.$$

We consider the Riemann-Hilbert problem of finding a 2×2 matrix-valued function $Y(z)$ which meets

$$(6.2.1) \quad \begin{cases} Y \text{ is holomorphic on } \mathbb{C} \setminus \Gamma, \\ Y^+(z) = Y^-(z) + \bar{Y}^-(z)J(z), & \text{for } z \in \Gamma, \\ Y(z) = (I + O(z^{-1})) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, & \text{as } |z| \rightarrow +\infty. \end{cases}$$

In order to analyze this problem, we need a variant of the Cauchy transform, which applies to functions defined on Γ . For smooth Γ , we write

$$\mathbf{C}_\Gamma[f](z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w - z} dw, \quad z \in \mathbb{C} \setminus \Gamma.$$

As is well-known, the classical Plemelj formula is a useful tool in the study of Riemann-Hilbert problems:

$$(6.2.2) \quad (\mathbf{C}_\Gamma[f])^+(z) = (\mathbf{C}_\Gamma[f])^-(z) + f(z).$$

We now connect the classical Riemann-Hilbert problem (6.2.1) with the matrix $\bar{\partial}$ -problem (6.1.2).

Proposition 6.2.1. *Let $\{\Gamma_t\}_{t \in I}$ be a smooth strictly expanding flow of positively oriented simple closed curves, and denote by \mathcal{D} the union $\mathcal{D} = \bigcup_{t \in I} \Gamma_t$. Let $\omega(z)$ denote a smooth positive function on \mathcal{D} , and denote by $\xi : \mathcal{D} \rightarrow \mathbb{C}$ the vector field $\nu\bar{\eta}$, where $\eta(z)$ denotes the outward unit normal field to the curve family and ν denotes the scalar normal velocity of the flow. Then, for each $t \in I$, there is a unique solution $Y_t(z)$ to the Riemann-Hilbert problem (6.2.1) with jump matrix*

$$J = \begin{pmatrix} 0 & 2\omega\xi \\ 0 & 0 \end{pmatrix}.$$

Moreover, if there exists a continuous positive function $\lambda(t)$ such that $(Y_t)_{1,1}$ and $\lambda(t)(Y_t)_{2,1}$ are independent of t , then the matrix-valued function

$$Y(z) = \Lambda_1^{-1} \int_I \Lambda(t) Y_t(z) dt \Lambda_2^{-1}$$

is the unique solution to (6.1.2), with $W = \begin{pmatrix} 0 & 1_{\mathcal{D}}\omega \\ 0 & 0 \end{pmatrix}$, provided that

$$\Lambda(t) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda(t) \end{pmatrix}, \quad \Lambda_1 = \begin{pmatrix} 1 & 0 \\ 0 & \int_I \lambda(t) dt \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} |I| & 0 \\ 0 & 1 \end{pmatrix}.$$

Proof. We first establish the existence of solutions to the problem (6.2.1) of Γ_t , which may be expressed in terms of a family of t -dependent orthogonal polynomials. We recall that ξ factors as $\nu\bar{\eta}$, where ν denotes the speed of the boundary in the normal direction while η denotes the outward pointing unit normal field. Since arc-length measure $|dz|$ on Γ_t relates to the complex line element dz by $dz = \tau|dz|$ where τ denotes the unit tangent vector field along Γ_t , it follows that

$$(6.2.3) \quad \frac{1}{2\pi i} dz = \frac{1}{2\pi} (-i\tau)|dz| = \eta ds$$

where we recall the convention $ds = \frac{|dz|}{2\pi}$. From this it follows that $(2\pi i)^{-1} \xi dz = \nu ds$, and we may consequently define an inner product by

$$\langle f, g \rangle_t := \int_{\Gamma_t} f(z) \bar{g}(z) \nu(z) ds(z) = \frac{1}{2\pi i} \int_{\Gamma_t} f(z) \bar{g}(z) \xi(z) dz.$$

Let $\{\pi_{n,t}^*\}_n$ denote the sequence of monic orthogonal polynomials with respect to this inner product, such that $\pi_{n,t}^*$ has degree n , and denote by $\kappa_{n,t}^*$ the sequence of leading coefficient of the corresponding normalized orthogonal polynomials $P_{n,t}^* = \kappa_{n,t}^* \pi_{n,t}^*$. It is straightforward to check that the function

$$\begin{pmatrix} \pi_{n,t}^* & 2\mathbf{C}_{\Gamma_t}[\bar{\pi}_{n,t}^* \omega \xi] \\ -\frac{1}{2}(\kappa_{n-1,t}^*)^2 \pi_{n-1,t}^* & -(\kappa_{n-1,t}^*)^2 \mathbf{C}_{\Gamma_t}[\bar{\pi}_{n-1,t}^* \omega \xi](z) \end{pmatrix}$$

supplies a solution to the Riemann-Hilbert problem (6.2.1).

Turning to unicity, it is clear from Plemelj's formula (6.2.2) and the jump condition that any solution $Y_t(z)$ must take the form

$$Y_t(z) = \begin{pmatrix} a_t(z) & u_t(z) + 2\mathbf{C}_{\Gamma_t}[\bar{a}_t \omega \xi](z) \\ b_t(z) & v_t(z) + 2\mathbf{C}_{\Gamma_t}[\bar{b}_t \omega \xi](z) \end{pmatrix},$$

where a_t, b_t, u_t, v_t are entire functions. From the growth constraint at infinity, we see that these four functions are all polynomials. Moreover, $u_t = v_t = 0$ for the same reason. A standard expansion of the Cauchy kernel at infinity shows that a_t is a monic polynomial of degree n which is orthogonal to the lower degree polynomials Pol_n with respect to $\omega \xi dz$ on Γ_t . It follows that $a_t = \pi_{n,t}^*$. Analogously, b_t is given by $b_t = -\frac{1}{2}(\kappa_{n-1,t}^*)^2 \pi_{n-1,t}^*$. We have established the unique solvability of the Riemann-Hilbert problem (6.2.1) with the given jump matrix.

Next, we turn to the connection with the $\bar{\partial}$ -problem (6.1.2). Under the assumption that $(Y_t)_{1,1} = a_t = A$ and $\lambda(t)(Y_t)_{2,1} = \lambda(t)b_t = B$ are independent of t , we may consequently write

$$\Lambda(t) Y_t(z) = \begin{pmatrix} A(z) & 2\mathbf{C}_{\Gamma_t}[\bar{A} \omega \xi](z) \\ B(z) & 2\mathbf{C}_{\Gamma_t}[\bar{B} \omega \xi](z) \end{pmatrix}.$$

Recall that we may integrate over the flow $\{\Gamma_t\}_t$ using the disintegration

$$\int_{t \in I} \left\{ 2 \int_{\Gamma_t} u(z) \nu(z) ds \right\} dt = \int_{\mathcal{D}} u(z) dA(z),$$

for functions u such that the indicated integrals have a well-defined meaning. It now follows that if $\langle \lambda \rangle_I = \int_I \lambda(t) dt$, the matrix-valued function

$$\begin{aligned} \hat{Y}(z) &:= \Lambda_1^{-1} \int_I \Lambda(t) Y_t(z) dt \Lambda_2^{-1} = \Lambda_1^{-1} \begin{pmatrix} |I| A(z) & -\mathbf{C}[\bar{A} \omega 1_{\mathcal{D}}](z) \\ |I| B(z) & -\mathbf{C}[\bar{B} \omega 1_{\mathcal{D}}](z) \end{pmatrix} \Lambda_2^{-1} \\ &= \begin{pmatrix} A(z) & -\mathbf{C}[\bar{A} \omega 1_{\mathcal{D}}](z) \\ (\langle \lambda \rangle_I)^{-1} B(z) & -(\langle \lambda \rangle_I)^{-1} \mathbf{C}[\bar{B} \omega 1_{\mathcal{D}}](z) \end{pmatrix} \end{aligned}$$

solves

$$\bar{\partial} \hat{Y}(z) = \begin{pmatrix} 0 & -\bar{A} \omega 1_{\mathcal{D}} \\ 0 & -(\langle \lambda \rangle_I)^{-1} \bar{B} \omega 1_{\mathcal{D}} \end{pmatrix} = -\overline{\hat{Y}(z)} \begin{pmatrix} 0 & \omega 1_{\mathcal{D}} \\ 0 & 0 \end{pmatrix}$$

with asymptotics

$$\hat{Y}(z) = (I + \mathcal{O}(z^{-1})) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, \text{ as } |z| \rightarrow +\infty,$$

as a consequence of the corresponding asymptotics of Y_t for each $t \in I$. \square

Remark 6.2.2. For the orthogonal foliation flow, in the context of a neighborhood of the boundary curve of the droplet \mathcal{S}_τ with $\tau = n/m$, the (approximate) orthogonal polynomial of degree n is also approximately orthogonal to the lower degree polynomials along the individual flow loops corresponding to $\omega = e^{-2mQ}$. So, in view of Proposition 6.2.1, the conditions

$$(6.2.4) \quad \partial_t (Y_t)_{1,1} = 0, \quad \partial_t (\lambda(t)(Y_t)_{2,1}) = 0$$

should be met at least approximately for some appropriate scalar-valued function $\lambda(t)$. Alternatively, we could use the (6.2.4) as a criterion to define a flow of curves. In the given setting, this should give us back our orthogonal foliation flow. In other words, (6.2.4) should be analogous to the condition (4.4.2), once the Riemann-Hilbert problems of Proposition 6.2.1 are approximately solved in a constructive fashion, and we would expect that in an approximate sense,

$$\Gamma_t \sim \phi_\tau^{-1}(\psi_{n,m,-t}(\mathbb{T})).$$

It is entirely possible that the conditions (6.2.4) would be more stable close to the zeros of the given orthogonal polynomial of degree n . For instance, this might be the case with a highly eccentric ellipse.

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