

Planar orthogonal polynomials and related determinantal processes: random normal matrices and arithmetic jellium

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21st February 2019

Orthogonal polynomials in the plane

Let μ be a finite Borel measure on the plane \mathbb{C} , and consider the space $L^2(\mathbb{C}, \mu)$ of measurable functions $f : \mathbb{C} \rightarrow \mathbb{C}$ with

$$\|f\|_{L^2(\mu)}^2 := \int_{\mathbb{C}} |f|^2 d\mu < +\infty.$$

Associated with the Hilbert space norm we have also the inner product $\langle \cdot, \cdot \rangle_{L^2(\mu)}$. For the following definition to make sense, we require that all polynomials are in $L^2(\mathbb{C}, \mu)$, or at least all polynomials up to some given degree.

DEFINITION

The orthogonal polynomials in $L^2(\mathbb{C}, \mu)$ is a sequence of polynomials p_0, p_1, p_2, \dots such that p_j has degree j , belongs to $L^2(\mathbb{C}, \mu)$, and $p_j \perp p_k$ for $j \neq k$. They are unique up to a unimodular constant multiple. In particular, if we require that the leading coefficient is positive, the orthogonal polynomials are unique.

Szegő's theorem

In 1921, Szegő considered the orthogonal polynomials with respect to the measure ds_Γ , the arc length measure along a smooth closed loop Γ . We normalize arc length so that the unit circle gets length 1 (i.e. we divide by 2π).

SZEGŐ'S THEOREM

The orthogonal polynomials in $L^2(\mathbb{C}, ds_\Gamma)$ have the asymptotics

$$p_n(z) = \sqrt{\phi'(z)}[\phi(z)]^n(1 + o(1)) \quad \text{as } n \rightarrow +\infty.$$

This formula holds in the domain Ω_e , the unbounded component of $\mathbb{C} \setminus \Gamma$, and, moreover, ϕ is the conformal mapping $\Omega_e \rightarrow \mathbb{D}_e$ with $\phi(\infty) = \infty$ and $\phi'(\infty) > 0$.

Here, \mathbb{D}_e is the exterior disk of points with $|z| > 1$.

Carleman's theorem

In 1922, Carleman considered instead the orthogonal polynomials with respect to $1_{\Omega}dA$, the area measure restricted to a bounded domain Ω . We normalize area measure so that the unit disk \mathbb{D} gets area 1 (i.e. we divide by π).

CARLEMAN'S THEOREM

Suppose $\partial\Omega$ is a real-analytic closed loop. Then the orthogonal polynomials with respect to $1_{\Omega}dA$ have the asymptotics

$$p_n(z) = \sqrt{n+1} \phi'(z) [\phi(z)]^n (1 + O(e^{-\epsilon n})) \quad \text{as } n \rightarrow +\infty,$$

where ϕ is the conformal mapping $\Omega_e \rightarrow \mathbb{D}_e$, with $\phi(\infty) = \infty$ and $\phi'(\infty) > 0$. Here, Ω_e is the unbounded component of $\mathbb{C} \setminus \partial\Omega$, and ϵ is a positive constant. This asymptotics is valid on a fixed neighborhood of $\bar{\Omega}_e$.

The exponential decay of the error term is a miracle involving the Dirichlet integral.

Suetin's theorem

Later on, following Carleman, Suetin considered more general measures $1_{\Omega}\varrho dA$, where ϱ is a smooth positive weight function. In this setting the exponential decay of Carleman's theorem cannot be expected to hold.

SUETIN'S THEOREM

Let L denote the bounded holomorphic function in the exterior domain Ω_e with $\operatorname{Re} L = \log \varrho$ on $\partial\Omega$ and $L(\infty) \in \mathbb{R}$. Then the orthogonal polynomials in $L^2(\mathbb{C}, 1_{\Omega}\varrho dA)$ have the asymptotics

$$p_n(z) = \sqrt{n+1} e^{-L(z)} \phi'(z) [\phi(z)]^n (1 + o(n^{-\beta}))$$

in Ω_e for some constant $\beta = \beta(\alpha) > 0$, provided that $\partial\Omega$ is a Hölder- α smooth closed loop with $\alpha > 0$.

The probability density associated with an orthogonal polynomial

If p_0, p_1, p_2, \dots are the orthogonal polynomials of $L^2(\mathbb{C}, \mu)$, then we call

$$|p_n|^2 d\mu$$

the *probability density* associated with the orthogonal polynomial p_n . In Carleman's and Suetin's theorems, this density has a sharp cut-off at the edge $\partial\Omega$, and we might characterize both as extreme hard-edge cases.

OBSERVATION

In the setting of Suetin's theorem (and hence Carleman's as well), we have the convergence

$$|p_n|^2 \varrho \mathbf{1}_\Omega dA \rightarrow d\omega_\infty^{\partial\Omega},$$

where $\omega_\infty^{\partial\Omega}$ is harmonic measure for the point at infinity in the domain Ω_e complementary to Ω .

Exponentially varying weights

We will study a family of exponentially varying weights e^{-2mQ} , where Q is a given potential and m is a real parameter that we will let tend to infinity. The corresponding planar measure is

$$d\mu_{mQ} := e^{-2mQ} dA,$$

and we will require that

$$Q(z) \gg \log |z|$$

near infinity. The orthogonal polynomials in $L^2(\mathbb{C}, \mu_{mQ})$ are denoted $p_{0,m}, p_{1,m}, p_{2,m}, \dots$

PROBLEM

Describe asymptotically $p_{n,m}$ when $m, n \rightarrow +\infty$ in a proportional fashion (so that the ratio $\tau = \frac{n}{m} > 0$ is kept fixed, essentially).

The probability density for exponentially varying weights

If we compare with Suetin's theorem, we might be tempted to believe that the probability measure $|p_{n,m}|^2 e^{-2mQ} dA$ must escape to infinity as $n \rightarrow +\infty$, since the domain \mathbb{C} has no boundary in the plane (except for the point at infinity on the extended complex plane). However, if we let $m \rightarrow +\infty$ with $\tau := \frac{n}{m}$ fixed, the potential Q acts as a countervailing force, and we instead get the following [AHM2, AHM3]. The set \mathcal{S}_τ is a by definition the contact set for an obstacle problem, and we refer to it as the *spectral droplet*. In the range of τ we consider, \mathcal{S}_τ is compact.

ONP WAVE THEOREM

Suppose $\Delta Q > 0$ in a neighborhood of \mathcal{S}_τ and that Q is real-analytically smooth there and that $\partial\mathcal{S}_\tau$ consists of a single real-analytically smooth Jordan curve. Then for $n = m\tau$,

$$|p_{n,m}|^2 e^{-2mQ} dA \rightarrow d\omega_\infty^{\partial\mathcal{S}_\tau},$$

as $m \rightarrow +\infty$, in the sense of weak-star convergence of measures.

Spectral droplets: underlying potential theory

For compactly supported Borel probability measures σ , we consider the associated energy

$$I_Q[\sigma] := \int_{\mathbb{C}} \int_{\mathbb{C}} \log \frac{1}{|\xi - \eta|} d\sigma(\xi) d\sigma(\eta) + 2 \int_{\mathbb{C}} Q d\sigma.$$

Next, we consider for $\tau > 0$ the problem of minimizing the energy

$$\min_{\sigma} I_{\tau^{-1}Q}[\sigma].$$

It turns out that the minimum is attained for a unique probability measure $\hat{\sigma}_{\tau}$. We call this measure the **equilibrium measure**.

Obstacle problem and the equilibrium measure

We consider the obstacle problem

$$\hat{Q}_\tau(z) := \sup\{q(z) : q \leq Q \text{ on } \mathbb{C}, q \in \text{Subh}_\tau(\mathbb{C})\},$$

where $\text{Subh}_\tau(\mathbb{C})$ denotes the convex set of subharmonic functions $u : \mathbb{C} \rightarrow [-\infty, +\infty[$ with

$$u(z) \leq \tau \log^+ |z| + O(1).$$

For a measure σ , its logarithmic potential U^σ is

$$U^\sigma(\xi) := \int_{\mathbb{C}} \log \frac{1}{|\xi - \eta|} d\sigma(\eta).$$

Frostman's Theorem

For some constant c ,

$$\hat{Q}_\tau = c - \tau U^{\hat{\sigma}_\tau}.$$

Spectral droplet and equilibrium measure

Let

$$\mathcal{S}_\tau := \{z \in \mathbb{C} : \hat{Q}_\tau(z) = Q(z)\}.$$

This coincidence set is the **spectral droplet**.

Kinderlehrer-Stampacchia theory

Under smoothness on Q , we have

$$\Delta \hat{Q}_\tau = 1_{\mathcal{S}_\tau} \Delta Q,$$

so that

$$d\hat{\sigma}_\tau = \frac{1_{\mathcal{S}_\tau} \Delta Q}{2\pi\tau} d\text{vol}_2 = \frac{1_{\mathcal{S}_\tau} \Delta Q}{2\tau} dA.$$

Remark

It follows that the study of the dynamics of the equilibrium measures $\hat{\sigma}_\tau$ reduces to the study of the supports \mathcal{S}_τ . This is in contrast with the 1D theory.

An illustration of an ONP wave

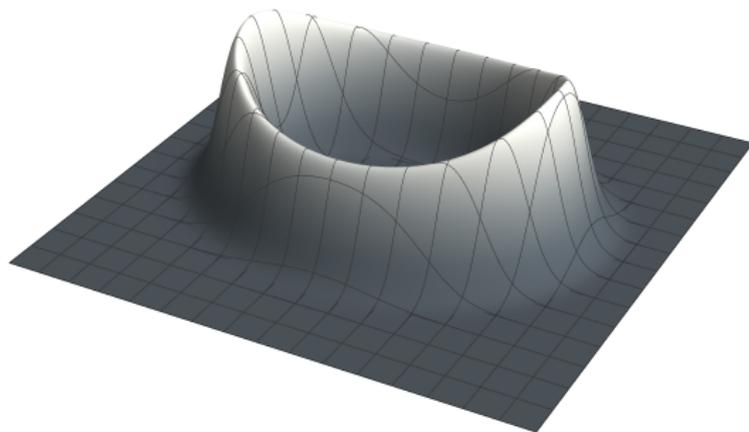


Figure: The orthogonal polynomial density $|p_{n,m}(z)|^2 e^{-2mQ(z)}$ for $n = 6$, $m = 20$ and $Q(z) = \frac{1}{2}|z|^2 - \operatorname{Re}(tz^2)$, where $t = 0.4$.

Gaussian ONP wave conjecture

Gaussian ONP wave conjecture

The ONP waves $|p_n|^2 e^{-2mQ}$ converge to harmonic measure as *Gaussian waves*, as $\tau = \frac{n}{m}$ is fixed and $m \rightarrow +\infty$.

Remark

A more precise version of the conjecture would of course ask for more details on the convergence.

The asymptotic expansion of ONP

We consider as always $\tau = \frac{n}{m}$ fixed and let $m \rightarrow +\infty$. Let Q_τ denote the bounded holomorphic function in S_τ^c whose real part equals Q along the loop ∂S_τ , which is real-valued at infinity. We extend it analytically across ∂S_τ . Let $\phi_\tau : S_\tau^c \rightarrow \mathbb{D}_e$ be the conformal mapping which sends infinity to infinity with positive derivative $\phi'_\tau(\infty) > 0$.

THEOREM

We have an asymptotic expansion

$$p_n(z) \sim m^{\frac{1}{4}} [\phi'_\tau(z)]^{\frac{1}{2}} [\phi_\tau(z)]^n e^{mQ_\tau(z)} (\mathcal{B}_{0,\tau}(z) + m^{-1}\mathcal{B}_{1,\tau}(z) + \cdots).$$

Here, the functions $\mathcal{B}_{j,\tau}$ are bounded holomorphic functions in S_τ^c that extend across the boundary. For instance, $\mathcal{B}_{0,\tau} = \pi^{-\frac{1}{4}} e^{H_\tau}$, where H_τ is bounded and holomorphic with real part $\operatorname{Re} H_\tau = \frac{1}{4} \log \Delta Q$ on ∂S_τ and $H_\tau(\infty) \in \mathbb{R}$.

Comments on the asymptotic expansion of ONP

We need to get more specific:

- How small is the error term in the asymptotic expansion? POINTWISE AND IN THE WEIGHTED L^2 -SENSE $O(m^{-\kappa-1})$, FOR GIVEN PRECISION.
- Where is the asymptotic expansion valid? POINTWISE: WITHIN $O(m^{-1/2}\sqrt{\log m})$ DISTANCE OF $\partial\mathcal{S}_\tau$ AND IN THE WHOLE COMPLEMENT \mathcal{S}_τ^c . IN THE WEIGHTED L^2 -SENSE: NEED TO INTRODUCE A SMOOTH CUT-OFF FUNCTION.
- How do we compute the coefficient functions $\mathcal{B}_{j,\tau}$? ALGORITHMICALLY.
- Does it resolve the Gaussian wave conjecture? YES.

Underlying ideas

- Hörmander-type estimates of solutions of the $\bar{\partial}$ -equation to localize ($\bar{\partial}$ -surgery).
- The *canonical positioning operator* to turn $\partial\mathcal{S}_\tau$ into the unit circle \mathbb{T} .
- The *weighted Laplacian growth flow* which gives the evolution of the shape of the droplet \mathcal{S}_τ as τ varies.
- The *orthogonal foliation flow* around $\partial\mathcal{S}_\tau$ or alternatively around \mathbb{T} .

Canonical positioning operator

Canonical positioning operator

$$\Lambda_{n,m}[f](z) := \phi'_\tau(z) [\phi_\tau(z)]^n e^{mQ_\tau(z)} (f \circ \phi_\tau)(z), \quad \tau = \frac{n}{m}.$$

It maps *isometrically* from $L^2(e^{-2mR_\tau})$ to $L^2(e^{-2mQ})$, where

$$R_\tau := (Q - \check{Q}_\tau) \circ \phi_\tau^{-1}.$$

Here, \check{Q}_τ denotes the harmonic extension across $\partial\mathcal{S}_\tau$ of the solution to the obstacle problem. The weight e^{-2mR_τ} is a Gaussian wave with ridge along the unit circle. It allows us to localize the problem around the standard setting of the circle \mathbb{T} .

The algorithmic aspect

The coefficients $\mathcal{B}_{j,\tau}$

After canonical positioning, we apply the steepest descent method (Laplace's method) in the radial direction to figure out the coefficients. Need to transfer some radial derivatives to tangential derivatives on the circle, which is done with methods that standard for pseudodifferential operators. Finally obtain the Toeplitz kernel equations:

$$B_{j,\tau}|_{\mathbb{T}} \in H_-^2 \cap e^{2\operatorname{Re}H_{R\tau}}(-F_{j,\tau} + H^2)$$

where $F_{0,\tau} = 0$ and generally $F_{j,\tau}$ is obtained algorithmically from previous $B_{k,\tau}$, $k = 0, \dots, j-1$. These equations are then solved. Finally,

$$\mathcal{B}_{j,\tau} = \sqrt{\phi'_\tau} B_{j,\tau} \circ \phi_\tau.$$

$\bar{\partial}$ -surgery and Laplace's method

Let us say some words about Laplace's method. Using $\bar{\partial}$ -surgery inside the spectral droplet, we may forget about any fixed compact subset of the interior of \mathcal{S}_τ . For $l = 0, 1, 2, \dots$, the functions $\Lambda_{n,m}[z^{-l}]$ grow like $O(|z|^{n-l})$ near infinity, that is, like polynomials of degree $n - l$, and with $\bar{\partial}$ -surgery we may say they are very close to being polynomials of the same degree. If we let $q_{n,m} \in L^2(e^{-2mR_\tau})$ be such that $p_{n,m} = \Lambda_{n,m}[q_{n,m}]$, then the isometric property of $\Lambda_{n,m}$ and the fact that $p_{n,m}$ is orthogonal to lower degree polynomials tells us that we should have

$$\int_{|z|>1-\epsilon} \overline{z^{-l}} q_{n,m}(z) e^{-2mR_\tau(z)} dA(z) \sim 0, \quad l = 1, 2, 3, \dots$$

Using polar coordinates this amounts to, for $l = 1, 2, 3, \dots$,

$$\int_{-\pi}^{\pi} e^{ilt} \left(\int_{1-\epsilon}^{+\infty} r^{1-l} q_{n,m}(re^{it}) e^{-2mR_\tau(re^{it})} dr \right) dt \sim 0. \quad (1)$$

$\bar{\partial}$ -surgery and Laplace's method, cont

The function $q_{n,m}(z)$ is holomorphic and bounded and bounded away from 0 in $|z| > 1 - \epsilon$. If $q_{n,m}$ is assumed to have an asymptotic expansion

$$q_{n,m} \sim m^{\frac{1}{4}}(B_{0,\tau} + m^{-1}B_{1,\tau} + m^{-2}B_{2,\tau} + \dots),$$

where the $b_{j,\tau}$ are bounded and holomorphic in $|z| > 1 - \epsilon$, then we should have for $l = 1, 2, 3, \dots$,

$$\int_{-\pi}^{\pi} e^{ilt} \left(\int_{1-\epsilon}^{+\infty} r^{1-l} (B_{0,\tau}(re^{it}) + m^{-1}B_{1,\tau}(re^{it}) + \dots) e^{-2mR_{\tau}(re^{it})} dr \right) dt \sim 0.$$

Each term within the parenthesis may be evaluated with Laplace's method. The only problem is that it will depend perhaps polynomially on l . To get rid of this we transform such polynomial expressions into tangential angular differential operators with standard Fourier methods. Finally, we note that $\int_{-\pi}^{\pi} e^{ilt} f(e^{it}) dt = 0$ for $l = 1, 2, 3, \dots$ if and only if $f \in H^2(\mathbb{D})$. Modulo technical details this supplies the algorithm.

Why is the asymptotic expansion true?

Where do we stand after Laplace's method?

If we believe that the given asymptotic expansion of the ONP is correct, Laplace's method gives us the coefficients (after some work). But why should we believe this is so? Can we prove it?

The orthogonal foliation flow

In principle, we just need to prove that the approximate polynomial given by the formula is (almost) orthogonal to all the lower degree polynomials. But why would this be true? Here, the *orthogonal foliation flow* comes in. Along a flow loop, we have the needed orthogonality and by integrating in the flow variable we obtain orthogonality in the domain covered by the flow loops, as long as these behave nicely. The key difficulty is to prove that this flow is well-behaved.

The master equation for the orthogonal foliation flow

A simple underlying idea is to represent the orthogonal foliation flow by a family of conformal mappings $\psi_{s,t}(z)$ close to $z \mapsto z$ parametrized by s and t . Here, we use $s := m^{-1}$, and think of it as a *quantization parameter* (like Planck's constant), whereas t is the *flow parameter*. With

$$\psi_{s,t} \sim \psi_{0,t} + \sum_{j \geq 1, l \geq 0} s^j t^l \hat{\psi}_{j,l},$$

the **Master equation** is, with $f_s \sim \sum_{j=0}^{+\infty} s^j B_{j,\tau}$ (we suppress τ , and think of it as fixed), on the unit circle \mathbb{T} :

$$|f_s \circ \psi_{s,t}(\zeta)|^2 e^{-2s^{-1} R \circ \psi_{s,t}(\zeta)} \operatorname{Re} \left(-\bar{\zeta} \partial_t \psi_{s,t}(\zeta) \overline{\psi'_{s,t}(\zeta)} \right) \sim (4\pi)^{-\frac{1}{2}} e^{-t^2/s}.$$

The deeper inside the spectral droplet we may flow the better is our control. Letting $s \rightarrow 0^+$ in this equation we find that the “semi-classical limit” function $\psi_{0,t}$ is associated with the level curves of $R = R_\tau$.

Illustration of the orthogonal foliation flow

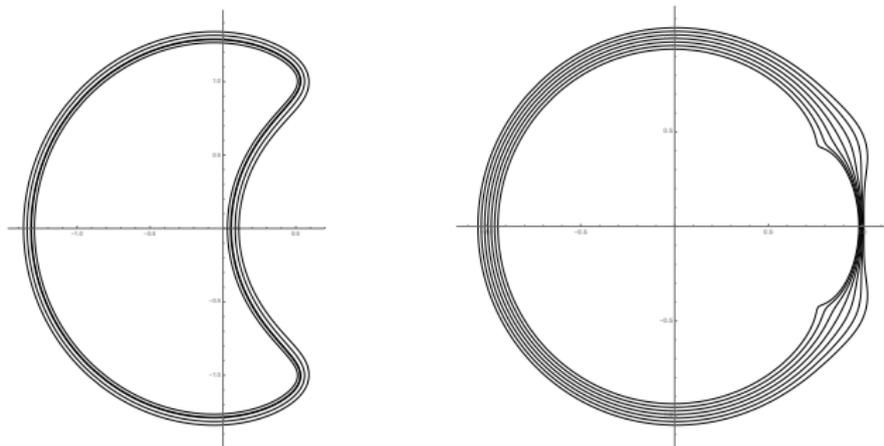


Figure: Approximate orthogonal foliation flow, near ∂S_1 (left) and near \mathbb{T} (right), associated with the potential $Q(z) = \frac{1}{2}|z|^2 - 2^{-\frac{1}{2}} \log |z - 1|$.

Coulomb gas. Gibbs model and inverse temperature

The asymptotics of the orthogonal polynomials is useful in the context of Coulomb gas with inverse temperature $\beta = 2$. We turn to the general Coulomb gas model in 2D, and consider n repelling particles in \mathbb{C} confined by a potential $V = 2mQ$. The interaction energy between the repelling particles is modelled by

$$\mathcal{E}_V^{\text{int}} := \sum_{j,k:j \neq k} \log \frac{1}{|z_j - z_k|},$$

where z_j denotes the position of the j -th particle, and the potential energy is given by

$$\mathcal{E}_V^{\text{pot}} := \sum_{j=1}^n V(z_j).$$

The total energy of a configuration $(z_1, \dots, z_n) \in \mathbb{C}^n$ is then given by

$$\mathcal{E}_V := \mathcal{E}_V^{\text{int}} + \mathcal{E}_V^{\text{pot}}.$$

Coulomb gas. Gibbs model and inverse temperature

In any reasonable gas dynamics model, the low energy states should be more likely than the high energy states. Fix a positive constant β , and let Z_n be the constant (“partition function”)

$$Z_n := \int_{\mathbb{C}^n} e^{-\frac{\beta}{2}\mathcal{E}_V} d\text{vol}_{2n},$$

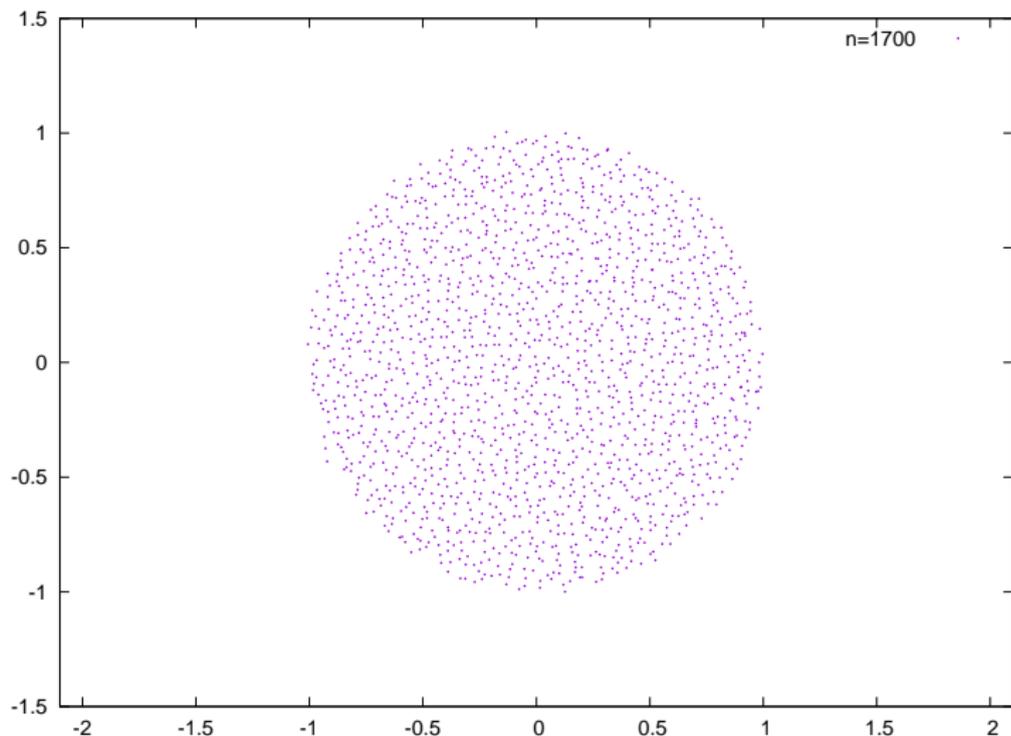
where vol_{2n} denotes standard volume measure in $\mathbb{C}^n \cong \mathbb{R}^{2n}$. Here, we need to assume that V grows at sufficiently at infinity to make the integral converge. The Gibbs model gives the joint density of states

$$\frac{1}{Z_n} e^{-\frac{\beta}{2}\mathcal{E}_V},$$

which we use to define a probability point process $\Pi_n \in \text{prob}(\mathbb{C}^n)$ by setting

$$d\Pi_n := \frac{1}{Z_n} e^{-\frac{\beta}{2}\mathcal{E}_V} d\text{vol}_{2n}.$$

Simulation of the Ginibre ensemble $V(z) = m|z|^2$
($n = m = 1700$)



Electron cloud interpretation. Marginal measures

The process Π_n models a cloud of electrons in a confining potential. Clearly, Π_n is random probability measure on \mathbb{C}^n . In order to study this process as $n \rightarrow +\infty$, it is advantageous to introduce the marginal probability measures $\Pi_n^{(k)}$ (for $0 \leq k \leq n$) given by

$$\Pi_n^{(k)}(e) := \Pi_n(e \times \mathbb{C}^{n-k}),$$

for Borel measurable subsets $e \subset \mathbb{C}^k$. In particular, $\Pi_n^{(n)} = \Pi_n$. The associated measures

$$\Gamma_n^{(k)} := \frac{n!}{(n-k)!} \Pi_n^{(k)}$$

are called **intensity (or correlation) measures**. To simplify the notation, we write $\Gamma_n := \Gamma_n^{(n)}$.

Aggregation of quantum droplets. Monotonicity

It is of interest to analyze what the addition of one more particle means for the process.

Aggregation theorem

If $\beta = 2$, then

$$\forall k : \Gamma_n^{(k)} \leq \Gamma_{n+1}^{(k)}.$$

This means that for the special inverse temperature $\beta = 2$, the addition of a new particle monotonically increases all the intensities.

Remark

The assertion of the Aggregation theorem fails for $\beta > 2$. For $\beta < 2$, however, we conjecture that the assertion holds.

$\beta = 2$: random normal matrices

The proof of Theorem 1 (monotonicity) is based on the fact that the point process Π_n is **determinantal** for $\beta = 2$. To explain what this means, we need the space Pol_n of all polynomials in z of degree $\leq n - 1$. We equip Pol_n with the inner product structure of $L^2(\mathbb{C}, e^{-V})$. Then under standard assumptions on V , point evaluations are bounded, and we obtain an element $K_w \in \text{Pol}_n$ such that

$$p(w) = \langle p, K_w \rangle_{L^2(\mathbb{C}, e^{-V})} = \int_{\mathbb{C}} p \bar{K}_w e^{-V} dA,$$

where $dA = \pi^{-1} d\text{vol}_2$. The function $K(z, w) := K_w(z)$ may be written in the form

$$K(z, w) = \sum_{j=0}^{n-1} p_j(z) \bar{p}_j(w), \quad (2)$$

where the p_j form an ONB. It is called **the reproducing kernel**.

$\beta = 2$: random normal matrices (2)

The determinantal structure of the process for $\beta = 2$ has to do with the fact that the product of all the differences $z_j - z_k$ with $k < j$ can be expressed as the determinant of the matrix with entries z_j^l , where $j = 1, \dots, n$ and $l = 0, \dots, n - 1$, the famous **Vandermondian**. Now, the taking the product of the Vandermondian with its complex conjugate, while performing suitable row and column operations, we obtain the instance $k = n$ of (3) below. In terms of intensities, we have

$$d\Gamma_n^{(k)}(z) = e^{-\sum_j V(z_j)} \det[K(z_i, z_j)]_{i,j=1}^k dA^{\otimes k}(z), \quad (3)$$

where $dA^{\otimes k} = \pi^{-k} d\text{vol}_{2k}$. For $k = 1$, this reduces to

$$d\Gamma_n^{(1)}(z) = K(z, z) e^{-V(z)} dA(z).$$

$\beta = 2$: random normal matrices (3)

Density of states and the 1-point function

The function $K(z, z) e^{-V(z)}$ is the *density of states*. The corresponding probability density

$$u_n(z) := \frac{1}{n} K(z, z) e^{-V(z)}$$

is called the *1-point function*.

NOTE

The determinantal case $\beta = 2$ models the eigenvalues of **Random Normal Matrices**.

Scaling up the potential

To obtain a reasonable limit as $n \rightarrow +\infty$, we need to renormalize the potential. So we put $V := 2mQ$, where the parameter m is essentially proportional to n as n tends to infinity. Here, Q is a *fixed confining potential*.

NOTE

Note that in the determinantal case, we just need to analyze the reproducing kernels $K = K_{n,m}$ for the space of polynomials of degree $\leq n - 1$ with respect to the weight e^{-2mQ} in the plane \mathbb{C} . If we write $\tau := n/m$, we would need a condition like

$$\liminf_{|z| \rightarrow +\infty} \frac{Q(z)}{\log |z|} > \tau \quad (4)$$

to ensure that the polynomials of degree up to $n - 1$ are in $L^2(e^{-2mQ})$.

Johansson's marginal measure theorem

Marginal measure theorem

("bosonization") Under minimal growth and smoothness assumptions on Q , we have for fixed k that

$$\Pi_n^{(k)} \rightarrow \hat{\sigma}_\tau^{\otimes k} \quad \text{as } n \rightarrow +\infty, \quad \text{while } \frac{n}{m} \rightarrow \tau,$$

in the weak-star sense of measures.

Remark

In particular, the 1-point function converges to the equilibrium density. Theorem 3 was obtained by K. Johansson in the case of Coulomb gas on the real line [J1]. His techniques work also in the planar case, with some modifications [HM1].

Orthogonal polynomials and reproducing kernels

We recall that the reproducing kernel $K = K_n$ for the *polynomial Bergman space* Pol_n (consisting of polynomials of degree $\leq n - 1$) with respect to the inner product of $L^2(e^{-2mQ})$ has the form

$$K_n(z, w) = \sum_{j=0}^{n-1} p_j(z) \bar{p}_j(w),$$

where the polynomials p_0, \dots, p_{n-1} are normalized orthogonal polynomials.

Classical methods: bulk expansion of the polynomial kernel

Background

There is a long history of expansion of Bergman kernels, going back to work of Hörmander, Fefferman, Boutet de Monvel, Sjöstrand, Tian-Zelditch-Catlin, Berman-Berndtsson, etc. The local approach applies to the polynomial Bergman kernels as well, in the *bulk* of the droplet \mathcal{S}_τ , as was shown rigorously by Ameur-Hedenmalm-Makarov [AHM1], based on the approach of Berman-Berndtsson-Sjöstrand [BBS1].

Specific adaptations

What is needed to fit the specific requirements here is an adaptation using Hörmander's $\bar{\partial}$ -theorem involving two potentials and a function used for “peaking”.

Bulk expansion theorem

Bulk expansion theorem

We have, for fixed $\frac{n}{m} = \tau > 0$, the local expansion

$$\begin{aligned} K(z, w) e^{-mQ(z)-mQ(w)} \\ = mA_0(z, w) + A_1(z, w) + \dots + m^{-k+1}A_k(z, w) + O(m^{-k}), \end{aligned}$$

where it is assumed the points z and w are both close to a point z_0 in the interior of S_τ with $\Delta Q(z_0) > 0$. The leading terms A_0, A_1 have diagonal restriction

$$A_0(z, z) = 2\Delta Q(z), \quad A_1(z, z) = \frac{1}{2}\Delta \log \Delta Q(z),$$

if we use as Δ the quarter-Laplacian.

Boundary expansion of the polynomial Bergman kernel?

What about boundary points?

This is all very well for bulk points. But what about boundary points? Clearly, the local point process there is different! Just look at the simulation of the Ginibre ensemble. In terms of correlation kernels, this means that the above bulk expansion theorem has no direct analogue. But something must be going on. But what? We can use our orthogonal polynomial expansion theorem to obtain the boundary behavior for regular boundaries.

Interpretation of the orthogonal polynomials

We note that the orthogonal polynomial p_n arises from consideration of

$$K_{n+1}(z, z) - K_n(z, z) = |p_n(z)|^2,$$

and that

$$|p_n|^2 e^{-2mQ} \tag{5}$$

is a probability density. *What does it represent?* Since $K_n(z, z) e^{-2mQ(z)}$ is the (expected) density of particles when we have n particles, (5) is the *net effect of adding an additional particle to the given n -particle system.*

Boundary universality

Take a boundary point $z_0 \in \partial\mathcal{S}_\tau$ and blow up according to

$$z_j = z_0 + n \frac{\xi_j}{\sqrt{2m\Delta Q(z_0)}}.$$

In other words, we let the points z_j be the n random points given by the Gibbs model with $\beta = 2$ and $V = 2mQ$, with $\tau = \frac{n}{m}$ as before. The question appears as to what is the asymptotic probability law of the blow-up process with the points ξ_j .

Boundary Universality Theorem

The limit law for the points ξ_j is determinantal with correlation kernel

$$k(\xi, \eta) = e^{\xi\bar{\eta} - \frac{1}{2}(|\xi|^2 + |\eta|^2)} \operatorname{erf}(\xi + \bar{\eta}),$$

where

$$\operatorname{erf}(z) = (2\pi)^{-1/2} \int_z^{+\infty} e^{-t^2/2} dt$$

is the *error function*.

Illustration of the Boundary Berezin density (repulsive effect)

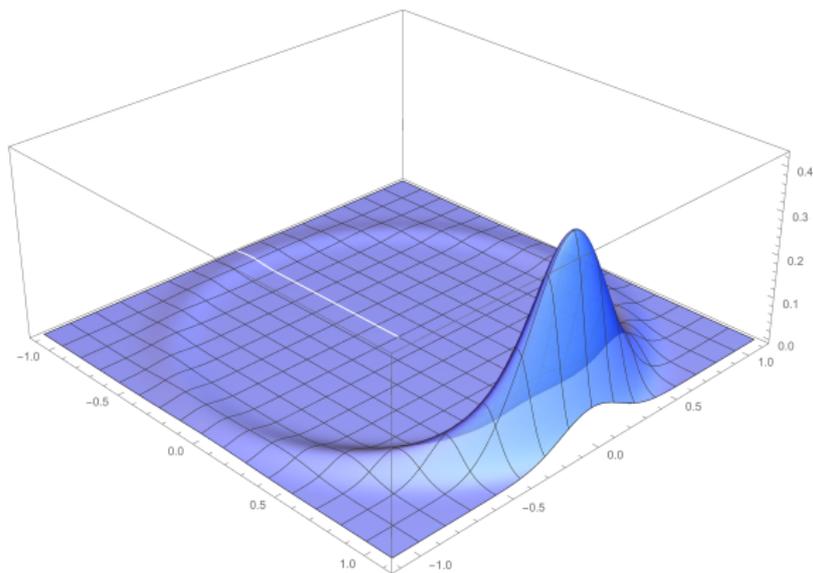


Figure: The boundary Berezin density, showing non-local higher order asymptotics (Ginibre case).

The dependence on τ : Laplacian growth

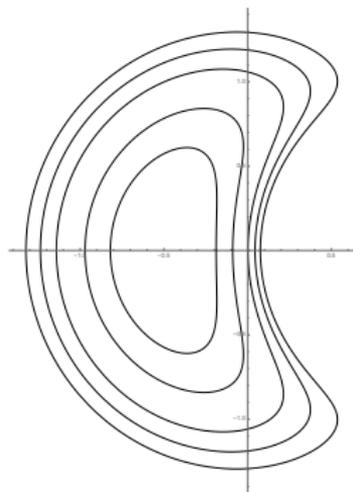


Figure: Laplacian growth of the compacts \mathcal{S}_τ for the potential $Q(z) = \frac{1}{2}|z|^2 - 2^{-\frac{1}{2}} \log |z - 1|$.

Porous and arithmetic jellium

JELLIUM

Jellium is the determinantal process with reproducing kernel

$$K(z, w) = \sum_{j \leq n} p_j(z) \overline{p_j(w)}.$$

POROUS JELLIUM

Porous jellium is the determinantal process with reproducing kernel

$$K(z, w) = \sum_{j \in I_n} p_j(z) \overline{p_j(w)},$$

where I_n is a subset of $\{0, \dots, n\}$.

Arithmetic jellium

There are many ways to obtain porous jellium. One particularly natural way is to use arithmetics.

ARITHMETIC JELLIUM

Aritmetic jellium is the determinantal process with reproducing kernel

$$K(z, w) = \sum_{j \leq n, j+q\mathbb{Z} \in A_q} p_j(z) \overline{p_j(w)},$$

where $A_q \subset \mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$.

Arithmetic jellium I (repulsion from a fixed particle)

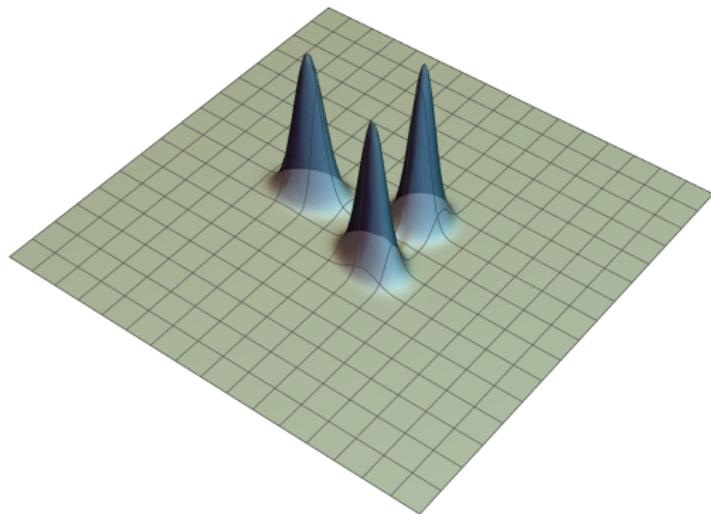


Figure: Berezin density of arithmetic jellium (elliptic), $q = 3$.

Arithmetic jellium II (repulsion from a fixed particle)



Figure: Berezin density of arithmetic jellium (elliptic), $q = 3$.

Arithmetic jellium III (repulsion from a fixed particle)

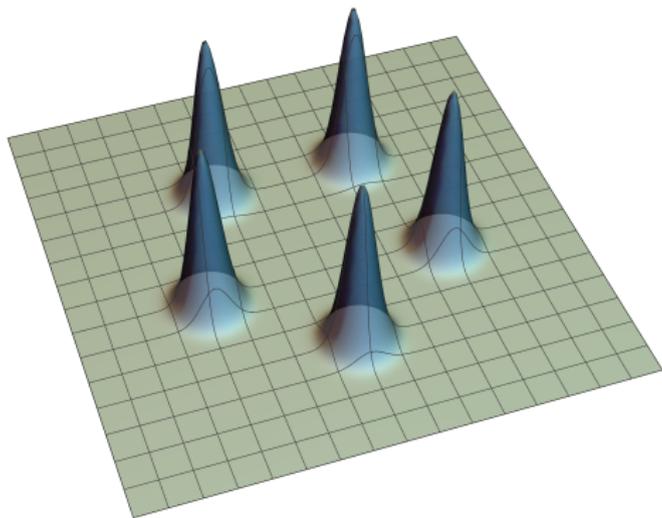


Figure: Berezin density of arithmetic jellium (Ginibre), $q = 5$.

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