EXISTENCE OF A MAXIMAL GLOBALLY HYPERBOLIC DEVELOPMENT

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Abstract. Using local existence and uniqueness theorems for the Einstein-non-linear scalar field system, we prove the existence of a maximal globally hyperbolic development, given initial data.

1. Introduction

Given initial data to the Einstein-non-linear scalar field equations, there is a unique maximal globally hyperbolic development (MGHD). This result is due to the work of Yvonne Choquet-Bruhat and Robert Geroch; cf. [3] (strictly speaking, Choquet-Bruhat and Geroch discussed the vacuum case, but the modifications required to deal with the Einstein-non-linear scalar field case are minimal). However, since the treatment of some of the technical issues that arise in the course of the argument are somewhat brief in the presentation of [3], it is of interest to provide more details. This was the objective motivating the appearance of [8]. However, there is a mistake in the proof of the relevant theorem, [8, Theorem 16.6, p. 177]. Let us give an outline of the proof of the existence of an MGHD, in the course of which we shall describe where the error of [8] occurs.

1.1. **Existence of a globally hyperbolic development which is maximal in the set theoretic sense of the word.** Given initial data to the Einstein-non-linear scalar field system, the collection of isometry classes of globally hyperbolic developments of the data, say $\mathcal{M}$, can be viewed as a partially ordered set; cf. Section 4. The partial ordering, say $\leq$, arises from the concept of isometric embedding given in Definition 4.1. It can be argued that every totally ordered subset in $\mathcal{M}$ has an upper bound, cf. Section 5, so that $\mathcal{M}$ contains a maximal element (assuming we are prepared to accept the axiom of choice); cf. Theorem 3.4. In other words, given the initial data, there is a globally hyperbolic development, say (M, g, φ), such that any other globally hyperbolic development, say $(\hat{M}, \hat{g}, \hat{\varphi})$, with the property that (M, g, φ) can be isometrically embedded into $(\hat{M}, \hat{g}, \hat{\varphi})$, must be isometric to (M, g, φ). In this sense, (M, g, φ) is maximal. However, there could in principle be many non-isometric globally hyperbolic developments which are maximal in this sense.

1.2. **Distinction between the set theoretic notion of maximality and the concept of a maximal globally hyperbolic development.** The above concept of maximality is quite different from the ususal definition of a maximal globally hyperbolic development; cf. Definition 2.5: an MGHD, say (M, g, φ), is a globally hyperbolic development such that any other globally hyperbolic development can be isometrically embedded into it. Clearly, this is a much stronger concept of maximality, and it immediately leads to the conclusion that an MGHD is unique up to isometry; cf. Lemma 2.8.

1.3. **Bridging the gap; constructing a common extension of two developments.** Due to the above observations, it is not sufficient to observe that $\mathcal{M}$ contains a maximal element in order to be allowed to conclude that there is an MGHD. Let, nevertheless, (M, g, φ) be a globally hyperbolic development of a given set of initial data, say $(\Sigma, \bar{g}, \bar{k}, \bar{\varphi}_0, \bar{\varphi}_1)$, such that the corresponding isometry class is a maximal element of $\mathcal{M}$. Let, moreover, i denote the associated embedding from Σ to M. Finally, let $(\hat{M}, \hat{g}, \hat{\varphi})$ be a globally hyperbolic development of the same data (with embedding $\hat{i}$). We wish to prove that $(\hat{M}, \hat{g}, \hat{\varphi})$ can be isometrically embedded into (M, g, φ). Define, to this end, the set $C(\hat{M}, M)$, which consists of pairs $(\hat{U}, \psi)$, where

- $\hat{U}$ is an open subset containing $\hat{i}(\Sigma)$ such that $\hat{i}(\Sigma)$ is a Cauchy hypersurface in $(\hat{U}, g)$,
- ψ is an isometry from $\hat{U}$ to an open subset of M such that $\psi \circ \hat{i} = i$.

If $(\hat{U}_j, \psi_j)$, $j = 1, 2$, are two elements of $C(\hat{M}, M)$, it can be argued that $\psi_1 = \psi_2$ on $\hat{U}_1 \cap \hat{U}_2$. As a consequence, there is a partial ordering on $C(\hat{M}, M)$, given simply by set inclusion, and it is possible to argue that $C(\hat{M}, M)$ has a unique maximal element; cf. Lemma 6.4. Let $(\hat{U}, \psi)$ be the maximal element of $C(\hat{M}, M)$ and let $\tilde{M}$ be the topological space obtained by taking the disjoint union of M and $\hat{M}$ and identifying $\hat{p} \in \hat{U}$ with $\psi(\hat{p})$. Since ψ is an isometry, it is possible to define a Lorentz metric, say $\tilde{g}$, and a smooth function, say $\tilde{\varphi}$, on $\tilde{M}$. In fact, if $\tilde{M}$ is Hausdorff, it is possible to argue that $(\tilde{M}, \tilde{g}, \tilde{\varphi})$ is a globally hyperbolic development of the data; cf. Lemma 6.10. Moreover, this development is, by construction, an extension of both $(\hat{M}, \hat{g}, \hat{\varphi})$ and (M, g, φ). Due to the fact that the isometry class of (M, g, φ) is a maximal element of $\mathcal{M}$, it can then be argued that it has to be possible to embed $(\hat{M}, \hat{g}, \hat{\varphi})$ isometrically into (M, g, φ).

1.4. **Strategy for proving that the common extension is Hausdorff.** Due to the above argument, it is clear that the central problem is that of establishing that $\tilde{M}$ is Hausdorff. To begin with, let us try to develop some intuition for the possible obstructions by considering a simple example. Let us, furthermore, drop the assumption that $(\hat{U}, \psi)$ be maximal in the construction of $\tilde{M}$. Consider the initial data induced on the $t = 0$ hypersurface in $n + 1$-dimensional Minkowski space. If $(\hat{M}, \hat{g}, \hat{\varphi})$ and (M, g, φ) are both $n + 1$-dimensional Minkowski space, $\hat{U}$ is the subset defined by $t \in (-1, 1)$ and ψ is the identity, then $\tilde{M}$ is clearly not Hausdorff. On the other hand, $(\hat{U}, \psi)$ is in this case clearly not maximal. As a consequence, we see that we must make use of the maximiality of $(\hat{U}, \psi)$ in order to be able to prove that $\tilde{M}$ is Hausdorff. In fact, the strategy will be to assume that $\tilde{M}$ is not Hausdorff and to conclude that $(\hat{U}, \psi)$ is not maximal.

1.5. **Elementary consequences of non-Hausdorffness.** Assuming that $\tilde{M}$ is not Hausdorff, the problematic points must correspond to elements of $\partial\hat{U}$ and $\partial\psi(\hat{U})$. In fact, let $\tilde{p}, \tilde{q} \in \tilde{M}$, $\tilde{p} \neq \tilde{q}$, be a non-Hausdorff pair in the sense that if $\tilde{V}$ and $\tilde{W}$ are arbitrary open neighbourhoods of $\tilde{p}$ and $\tilde{q}$ respectively, then the intersection of $\tilde{V}$ and $\tilde{W}$ is non-empty. Then we may, without loss of generality, assume that $\tilde{p}$ corresponds to a point in $\partial\hat{U}$ and that $\tilde{q}$ corresponds to a point in $\partial\psi(\hat{U})$. Furthermore (since we assume all developments to be time oriented and ψ to be time orientation preserving), we may, without loss of generality, restrict our attention to the causal future of the initial hypersurface. Finally, given $\tilde{p}$, the point $\tilde{q}$ is uniquely determined and vice versa. These statements are justified in Section 8. In particular, it is thus sufficient to study points $\hat{p} \in \partial\hat{U} \cap J^+[\hat{\imath}(\Sigma)]$ (corresponding to a $\tilde{p} \in \tilde{M}$) such that there is a point, say $q \in \partial\psi(\hat{U}) \cap J^+[i(\Sigma)]$ (corresponding to a $\tilde{q} \in \tilde{M}$), with the property that $\tilde{p}$, $\tilde{q}$ is a non-Hausdorff pair. We shall denote this set of points by $\hat{H}^+$. Note that, in the above context, q is uniquely determined by $\hat{p}$, and we shall write $q = \hat{\mathcal{H}}(\hat{p})$. The set $\hat{H}^-$ can be defined similarly (simply replace $J^+[\hat{\imath}(\Sigma)]$ with $J^-[\hat{\imath}(\Sigma)]$), and we shall use the notation $\hat{H}$ for the union of $\hat{H}^+$ and $\hat{H}^-$. If $\hat{H}$ is empty, $\tilde{M}$ is Hausdorff.

1.6. **Constructing an extension of $(\hat{U}, \psi)$; perspective 1.** How are we to construct an extension of $(\hat{U}, \psi)$? Following [3], the idea is to find a point $\hat{p} \in \hat{H}^+$ and a spacelike hypersurface, say $\bar{S}$, in $\hat{M}$ such that $\bar{S} - \{\hat{p}\} \subset \hat{U}$. Given such a point and such a hypersurface, it can be argued that the union of $\psi(\bar{S} - \{\hat{p}\})$ and $q = \hat{\mathcal{H}}(\hat{p})$ is a spacelike hypersurface in M, say $\bar{S}_1$. Moreover, ψ can be extended to a smooth map from $\bar{S}$ to $\bar{S}_1$ which takes the initial data induced on $\bar{S}$ by $(\hat{g}, \hat{\varphi})$ to the initial data induced on $\bar{S}_1$ by (g, φ); cf. [8, p. 182]. Combining this observation with local uniqueness (in the form of, e.g., Theorem 2.4) leads to the conclusion that the isometry ψ can be extended. As a consequence, we obtain a contradition to the maximality of $(\hat{U}, \psi)$. What remains is to prove that if $\hat{H}^+$ is non-empty, then there is a point $\hat{p} \in \hat{H}^+$ and a spacelike hypersurface $\bar{S}$ in $\hat{M}$ with the above properties. Let us quote the passage in [3] ensuring the existence of these objects (cf. [3, p. 333]):

> Now H is certainly open in ∂U. Furthermore, given any null geodesic in H, its endpoint in ∂U must also be in H, for the corresponding null geodesic in $\partial(\psi(U))$ must have an endpoint in M. It follows from these two properties of H that we may find a point $p' \in H$ and a spacelike 3-surface T' through p' such that $T' - p' \subset U$.

Since the notation in [3] is somewhat different from the notation we use here, let us observe that H and U in [3] correspond to $\hat{H}$ and $\hat{U}$ in the present paper. We shall devote large parts of the present paper to justify the first two sentences; cf., in particular, Lemma 8.8 and Lemma 9.1. In the end, we shall be able to justify the last sentence as well. However, we shall only be able to do so after having developed enough knowledge concerning $\hat{H}^+$ that we are able to construct an extension of $(\hat{U}, \psi)$ without using this information; in practice, we shall use the third perspective described below.

1.7. **Constructing an extension of $(\hat{U}, \psi)$; (erroneous) perspective 2.** Let us turn to a description of the error occurring in [8]. The idea in [8] was to obtain the point $\hat{p}$ and surface $\bar{S}$ by changing the definition of $C(\hat{M}, M)$ to include an additional condition. In fact, the elements $(\hat{U}, \psi)$ were required to be such that for each point $\hat{p} \in \partial\hat{U}$ there is a spacelike hypersurface $\bar{S}$ such that $\bar{S} - \{\hat{p}\} \subset \hat{U}$. There are such elements $(\hat{U}, \psi)$. Moreover, the condition is preserved upon taking unions. However, it is easy to see that the condition is not preserved when taking arbitrary unions. As a consequence, it is not true that every totally ordered subset of the corresponding $C(\hat{M}, M)$ has an upper bound, contrary to what is claimed in [8, p. 178, l. -13 and -14]. In particular, it is obvious that the perspective taken in [8] is fundamentally flawed.

1.8. **Constructing an extension of** $(\hat{U}, \psi)$**; perspective 3.** Even though we shall in the end be able to verify that the first perspective described above can be used to construct an extension, it turns out that there is another way to approach the problem. It is of interest to note that the brief proofs of the existence of a maximal globally hyperbolic development presented in [4] and [5] do not contain claims concerning the existence of a point $\hat{p}$ and spatial hypersurface $\bar{S}$ with the properties described at the beginning of Subsection 1.6. Let us quote the relevant sentence from [5, p. 107]:

> Suppose that x' is a spatial point for $\subset U'$, then x is a spatial point for $\subset \psi(U')$: considerations of spatial surfaces lying in $\bar{U}'$ and $\overline{\psi(U')}$ enables [sic] one to enlarge the isometry from U' into M to a larger subset $U' \cup D$, contradicting the maximality of U'.

Again, there is the issue of notation; U' should be translated to $\hat{U}$. Furthermore, we need to clarify what is meant by the concept "spatial point". For our purposes, we can define $\hat{p} \in \partial\hat{U} \cap J^+[\hat{\imath}(\Sigma)]$ to be a *spatial point* of $\partial\hat{U} \cap J^+[\hat{\imath}(\Sigma)]$ if

$$J^-(\hat{p}) \cap \partial\hat{U} \cap J^+[\hat{\imath}(\Sigma)] = \{\hat{p}\}. \tag{1}$$

In order to verify that there is a spatial point in $\hat{H}^+$, it is sufficient to check that the first two sentences of the quote from [3] are correct. As already mentioned, we shall do so below. In order to be able to extend the isometry ψ beyond $\hat{U}$, we shall use the fact that (1) holds to construct a spacelike hypersurface, say $\bar{S}$, in the closure of $\hat{U}$, such that the local uniqueness theorem can be applied. However, the surface is allowed to intersect $\partial\hat{U}$ at more than one point. We provide the details at the end of the paper.

1.9. **Outline of the paper.** In Section 2, we begin by writing down the equations we shall be interested in and by giving the definitions of the basic concepts; initial data, globally hyperbolic development etc. Moreover, we demonstrate that if there is a maximal globally hyperbolic development, then it is unique up to isometry. In order to prove that there is a maximal element (in the set theoretic sense of the word), we need some results and terminology from set theory. The required background is provided in Section 3. In order to be allowed to apply the general theory, we need to prove that the collection of isometry classes of globally hyperbolic developments of a given set of initial data can be considered to be a partially ordered set, say $\mathcal{M}$; this is the subject of Section 4. In order to conclude that there is a maximal element (in the set theoretic sense of the word), it is sufficient to prove that every totally ordered subset of $\mathcal{M}$ contains an upper bound. The construction is based on carrying out identifications in the disjoint union of a collection of developments representing the isometry classes of the totally ordered subset. Since the number of developments involved need not be countable, it is of particular interest to ensure that the resulting topological space is second countable. We provide the necessary details in Section 5. The natural next step is to construct the common extension of a given pair of globally hyperbolic developments. This is the subject of Section 6, in which we also prove that the common extension is a globally hyperbolic development of the given initial data, assuming that it is Hausdorff. Before proceeding, we then, in Section 7, record some general facts concerning the type of boundaries we shall need to study in what follows. Having reduced the problem to analyzing whether the common extension is Hausdorff or not, we derive some basic consequences of the assumption that it is not Hausdorff in Section 8. As was indicated in connection with the quote from [3], null geodesics contained in $\hat{H}^+$ are of special importance, and we devote Section 9 to this topic. After these preparations, we are finally able to prove that there is a maximal globally hyperbolic development in Section 10. At the end of the paper, we also justify the last sentence of the above quote from [3].

2. Equations and basic definitions

We are interested in the Einstein-non-linear scalar field system, given by

$$\mathrm{Ric} - d\varphi \otimes d\varphi - \frac{2}{n-1} V(\varphi)\, g = 0, \tag{2}$$

$$\Box_g \varphi - V'(\varphi) = 0. \tag{3}$$

Here Ric is the Ricci tensor of an $n+1$-dimensional Lorentz manifold (M, g) and $V \in C^\infty(\mathbb{R})$.

Let us begin by recalling the definition of initial data and of developments of initial data.

Definition 2.1. *Initial data* for (2) and (3) consist of an n dimensional manifold Σ, a Riemannian metric $\bar{g}$, a covariant 2-tensor field $\bar{k}$ and two functions $\bar{\varphi}_0$ and $\bar{\varphi}_1$ on Σ, all assumed to be smooth and to satisfy

$$\bar{r} - \bar{k}_{ij}\bar{k}^{ij} + (\mathrm{tr}\bar{k})^2 = \bar{\varphi}_1^2 + \overline{\nabla}^i \bar{\varphi}_0 \overline{\nabla}_i \bar{\varphi}_0 + 2V(\bar{\varphi}_0), \tag{4}$$

$$\overline{\nabla}^j \bar{k}_{ji} - \overline{\nabla}_i(\mathrm{tr}\bar{k}) = \bar{\varphi}_1 \overline{\nabla}_i \bar{\varphi}_0, \tag{5}$$

where $\overline{\nabla}$ is the Levi-Civita connection of $\bar{g}$, $\bar{r}$ is the associated scalar curvature and indices are raised and lowered by $\bar{g}$. Given initial data, the *initial value problem* is that of finding an $n+1$ dimensional manifold M with a Lorentz metric g and a $\varphi \in C^\infty(M)$ such that (2) and (3) are satisfied, and an embedding $i : \Sigma \to M$ such that $i^* g = \bar{g}$, $\varphi \circ i = \bar{\varphi}_0$, and if N is the future directed unit normal and κ is the second fundamental form of $i(\Sigma)$, then $i^*\kappa = \bar{k}$ and $(N\varphi) \circ i = \bar{\varphi}_1$. Such a triple (M, g, φ) is referred to as a *development* of the initial data, the existence of an embedding i being tacit. If, in addition, $i(\Sigma)$ is a Cauchy hypersurface in (M, g), then (M, g, φ) is said to be a *globally hyperbolic development*.

Remark 2.2. Developments are always tacitly assumed to be time oriented.

Let us note the there is always a globally hyperbolic development of given initial data; cf. [8, Theorem 14.2, p. 156].

Theorem 2.3. *Let* $(\Sigma, \bar{g}, \bar{k}, \bar{\varphi}_0, \bar{\varphi}_1)$ *be initial data for (2) and (3). Then there is a globally hyperbolic development of the data.*

Furthermore, given two developments of the same initial data, there is a common development in the following sense; cf. [8, Theorem 14.3, p. 158].

Theorem 2.4. *Let* $(\Sigma, \bar{g}, \bar{k}, \bar{\varphi}_0, \bar{\varphi}_1)$ *be initial data for (2) and (3). Assume that there are two developments* (M_j, g_j, φ_j)*,* $j = 1, 2$*, with corresponding embeddings* $i_j : \Sigma \to M_j$*. Then there is a globally hyperbolic development* (M, g, φ) *with corresponding embedding* $i : \Sigma \to M$ *and smooth time orientation preserving maps* $\psi_j : M \to M_j$*,* $j = 1, 2$*, which are diffeomorphisms onto their images, such that* $\psi_j^* g_j = g$ *and* $\psi_j^* \varphi_j = \varphi$*,* $j = 1, 2$*. Finally,* $\psi_j \circ i = i_j$*,* $j = 1, 2$*.*

What we are interested in here is in proving the existence of a maximal globally hyperbolic development, defined as follows.

Definition 2.5. Let $(\Sigma, \bar{g}, \bar{k}, \bar{\varphi}_0, \bar{\varphi}_1)$ be initial data for (2) and (3). A globally hyperbolic development (M, g, φ) (with embedding $i : \Sigma \to M$) is called a *maximal globally hyperbolic development* of the initial data, if for every other globally hyperbolic development $(\hat{M}, \hat{g}, \hat{\varphi})$ (with embedding $\hat{i} : \Sigma \to \hat{M}$) there is a map $\psi : \hat{M} \to M$ which is a time orientation preserving diffeomorphism onto its image such that $\psi^* g = \hat{g}$, $\psi^* \varphi = \hat{\varphi}$ and $\psi \circ \hat{i} = i$.

Remark 2.6. As was noted in the introduction, this concept of maximality is quite different from that commonly used in set theory; cf. Definition 3.3 and Remark 4.5.

It will be convenient to introduce the following terminology.

Definition 2.7. Let (M, g, φ) and $(\hat{M}, \hat{g}, \hat{\varphi})$ be two developments of the same initial data (with embeddings i and $\hat{i}$ respectively). If there is a smooth map $\psi : \hat{M} \to M$ which is

- a diffeomorphism onto its image,
- time orientation preserving,
- such that $\psi^* g = \hat{g}$, $\psi^* \varphi = \hat{\varphi}$ and $\psi \circ \hat{i} = i$,

then ψ is said to be an *isometric embedding* of $(\hat{M}, \hat{g}, \hat{\varphi})$ into (M, g, φ), and $(\hat{M}, \hat{g}, \hat{\varphi})$ is said to be *isometrically embedded* into (M, g, φ). If ψ is surjective, the developments (M, g, φ) and $(\hat{M}, \hat{g}, \hat{\varphi})$ are said to be *isometric*.

It is of interest to note that the following uniqueness result concerning maximal globally hyperbolic developments is an immediate consequence of the definition.

Lemma 2.8. *Two maximal globally hyperbolic developments of the same initial data are isometric.*

Proof. Assume (M, g, φ) and $(\hat{M}, \hat{g}, \hat{\varphi})$ to be two maximal globally hyperbolic developments of the same data, say $(\Sigma, \bar{g}, \bar{k}, \bar{\varphi}_0, \bar{\varphi}_1)$, and denote the associated embeddings by i and $\hat{i}$ respectively. Then there are maps $\psi : \hat{M} \to M$ and $\hat{\psi} : M \to \hat{M}$ with properties as in the statement of Definition 2.5 (note, however, that we do not yet know ψ and $\hat{\psi}$ to be surjective). In particular $(\psi \circ \hat{\psi})^* g = g$ and $\psi \circ \hat{\psi}$ is the identity map on $i(\Sigma)$. Let $p \in J^+[i(\Sigma)]$ and let γ be a future directed inextendible timelike geodesic with $\gamma(0) = p$. Then there is an $s_0 < 0$ such that $\gamma(s_0) \in i(\Sigma)$. Thus $\psi \circ \hat{\psi} \circ \gamma(s_0) = \gamma(s_0)$. Furthermore, $\psi \circ \hat{\psi}$ sends a tangent vector to $i(\Sigma)$ to itself and since it is a time orientation preserving isometry, it sends the future directed unit normal to itself. In other words, $(\psi \circ \hat{\psi})_*$ is the identity on T_pM for $p \in i(\Sigma)$. We conclude that $(\psi \circ \hat{\psi} \circ \gamma)'(s_0) = \gamma'(s_0)$ so that $\psi \circ \hat{\psi} \circ \gamma = \gamma$. Thus $\psi \circ \hat{\psi}(p) = p$. The argument is the same for $p \in J^-[i(\Sigma)]$ and we conclude that $\psi \circ \hat{\psi} = \text{Id}$. Similarly $\hat{\psi} \circ \psi = \text{Id}$. Thus ψ and $\hat{\psi}$ are surjective, and the statement follows. □

3. Elements of Set Theory

The proof of the existence of a maximal globally hyperbolic development is partly based on some observations from set theory. Let us therefore recall the terminology and the results that we shall need.

Definition 3.1. A *partial ordering* on a set X is a relation $\leq$ on X such that:

- $a \leq a$ for all $a \in X$; i.e., the relation is *reflexive*,
- $a \leq b$ and $b \leq a$ implies $a = b$; i.e., the relation is *antisymmetric*,
- $a \leq b$ and $b \leq c$ implies $a \leq c$; i.e., the relation is *transitive.*

A set together with a partial ordering is called a *partially ordered set.*

Definition 3.2. A partially ordered set is said to be *totally ordered* if $a, b \in X$ implies that either $a \leq b$ or $b \leq a$.

Definition 3.3. If $(X, \leq)$ is a partially ordered set and $A \subseteq X$, then $x \in X$ is an *upper bound* for A if $a \in A$ implies $a \leq x$. A *maximal element* of X is an $x \in X$ such that $x' \geq x$ implies $x' = x$.

We shall here assume that the axiom of choice holds, and we shall make use of the following consequence; cf. [1, Theorem B.18, p. 526].

Theorem 3.4. *The following statements are equivalent:*

- *(Maximality principle). If X is a partially ordered set such that every totally ordered subset has an upper bound, then X has a maximal element.*
- *(Axiom of choice). If $\{S_\alpha | \alpha \in A\}$ is an indexed family of nonempty sets S_α then there exists a function $f : A \to \cup S_\alpha$ such that $f(\alpha) \in S_\alpha$ for all $\alpha \in A$.*

4. Partial ordering of isometry classes of developments

The concept of an isometric embedding leads to a relation on the collection of isometry classes of globally hyperbolic developments.

Definition 4.1. Given two globally hyperbolic developments (M, g, φ) and $(\hat{M}, \hat{g}, \hat{\varphi})$ of the same initial data, the relation $(\hat{M}, \hat{g}, \hat{\varphi}) \leq (M, g, \varphi)$ is said to hold if and only if $(\hat{M}, \hat{g}, \hat{\varphi})$ can be isometrically embedded into (M, g, φ). Two globally hyperbolic developments (of fixed initial data) will be said to be equivalent if they are isometric, and the notation $[M, g, \varphi]$ will be used to denote the equivalence class of globally hyperbolic developments isometric to (M, g, φ).

Remark 4.2. Clearly, the relation $\leq$ is well defined on the collection of equivalence classes of globally hyperbolic developments.

Remark 4.3. In practice, we shall often denote an equivalence class by $[M]$ instead of by $[M, g, \varphi]$, and we shall often write $\hat{M} \leq M$ and $[\hat{M}] \leq [M]$ instead of $(\hat{M}, \hat{g}, \hat{\varphi}) \leq (M, g, \varphi)$ and $[\hat{M}, \hat{g}, \hat{\varphi}] \leq [M, g, \varphi]$ respectively.

Lemma 4.4. *Let $(\Sigma, \bar{g}, \bar{k}, \bar{\varphi}_0, \bar{\varphi}_1)$ be initial data for (2) and (3). Then the collection of equivalence classes of globally hyperbolic developments thereof together with the relation $\leq$ is a partially ordered set, which will be denoted by $\mathcal{M}$.*

Remark 4.5. Combining the observation of the lemma with Definition 3.3, we obtain a natural concept of maximality. However, this concept does not automatically guarantee uniqueness of maximal elements, and is in this sense weaker than the one given in Definition 2.5.

Proof. It is perhaps not completely obvious that it is reasonable to consider the collection of equivalence classes of globally hyperbolic developments to be a set. Let us therefore justify this statement. Given initial data $(\Sigma, \bar{g}, \bar{k}, \bar{\varphi}_0, \bar{\varphi}_1)$, let (M, g, φ) be a globally hyperbolic development with embedding i. Due to [8, Proposition 11.3, p. 112], there is then a diffeomorphism

$$\chi_1 : \mathbb{R} \times i(\Sigma) \to M,$$

since $i(\Sigma)$ is a smooth spacelike Cauchy hypersurface. Considering the proof of [8, Proposition 11.3, p. 112], it is clear that we can assume χ_1 to be such that $\chi_1(0, \bar{x}) = \bar{x}$. Since i is a diffeomorphism from Σ to $i(\Sigma)$, we can furthermore define a diffeomorphism

$$\chi_2 : \mathbb{R} \times \Sigma \to \mathbb{R} \times i(\Sigma)$$

by $\chi_2(t, \bar{x}) = (t, i(\bar{x}))$. Let $\chi = \chi_1 \circ \chi_2$. Then χ is a diffeomorphism from $\mathbb{R} \times \Sigma$ to M and $\chi(0, \bar{x}) = i(\bar{x})$. Pulling back the development (M, g, φ) using χ, we obtain an isometric globally hyperbolic development $(\hat{M}, \hat{g}, \hat{\varphi})$ with $\hat{M} = \mathbb{R} \times \Sigma$, $\hat{g} = \chi^* g$, $\hat{\varphi} = \chi^* \varphi$ and associated embedding $\hat{\imath} : \Sigma \to \hat{M}$ given by $\hat{\imath}(\bar{x}) = (0, \bar{x})$. Since we are only interested in equivalence classes of developments, it is thus sufficient to consider globally hyperbolic developments (M, g, φ) such that $M = \mathbb{R} \times \Sigma$ with associated embedding i given by $i(\bar{x}) = (0, \bar{x})$.

Given initial data $(\Sigma, \bar{g}, \bar{k}, \bar{\varphi}_0, \bar{\varphi}_1)$, let $\mathcal{D}$ denote the set whose elements are given by a time oriented Lorentz metric g and a smooth function φ on $M := \mathbb{R} \times \Sigma$ such that

- (M, g, φ) is a solution to (2) and (3),
- $\{0\} \times \Sigma$ is smooth spacelike Cauchy hypersurface in (M, g),
- if $i : \Sigma \to M$ is given by $i(\bar{x}) = (0, \bar{x})$ and N is the future directed unit normal and κ the second fundamental form of $\{0\} \times \Sigma$ in (M, g), then $i^* g = \bar{g}$, $i^* \kappa = \bar{k}$, $i^* \varphi = \bar{\varphi}_0$ and $i^*(N\varphi) = \bar{\varphi}_1$.

Then the quotient of $\mathcal{D}$ by the set of diffeomorphisms of M which equal the identity on $\{0\} \times \Sigma$ (the diffeomorphisms acting by pullback) is a set which can be identified with the collection of isometry classes of globally hyperbolic developments of the given initial data.

The verification of the fact that $\leq$ defines a partial ordering is left to the reader; the proof of the antisymmetry of $\leq$ is quite similar to the proof of Lemma 2.8. $\square$

5. Existence of a maximal element

The proof of the existence of a maximal globally hyperbolic development will proceed in two steps. It is natural to begin by demonstrating that the partially ordered set $\mathcal{M}$ (cf. the terminology introduced in the statement of Lemma 4.4) has a maximal element (in the sense of Definition 3.3). The second step then consists in proving that the maximal element satisfies the requirements of Definition 2.5. The first step, which we intend to take in the present section, is based on the maximality principle. However, this principle does not provide any help in taking the second step. We thus need to develop different tools, and this is the subject of the remaining sections.

In order to be allowed to apply the maximality principle, we need to show that every totally ordered subset in $\mathcal{M}$ has an upper bound. Given a totally ordered subset, let us write its elements $[N_\alpha]$ for α in some index set $\mathcal{A}$. Choosing a specific representative for each isometry class, say N_α for $\alpha \in \mathcal{A}$, we can think of the totally ordered set as consisting of developments. Moreover, we can think of the developments as having the same topology; cf. the proof of Lemma 4.4. However, even so, there are potentially uncountably many embeddings involved which relate the different developments. To construct the upper bound, we take the disjoint union of N_α for $\alpha \in \mathcal{A}$ (note that this topological space is typically not second countable), and identify points that are related via isometric embeddings. It is quite straightforward to verify that the resulting object has most of the desired properties. However, it is somewhat less clear that it is second countable. Naively, this may seem strange, since the constructed object is globally hyperbolic, which would lead us to expect it to be diffeomorphic to the real numbers in Cartesian product with the initial hypersurface. However, this expectation is based on the presupposition that there is a timelike vectorfield. Even though the object we construct is time oriented, the step from a time orientation to a timelike vector field is usually based on the existence of a partition of unity; cf., e.g., [7, Lemma 32, p. 145]. The existence of a partition of unity is, in its turn, usually based on the assumption of paracompactness. Moreover, in the topological characterization of globally hyperbolic manifolds given in [8, Proposition 11.3, p. 112], use is made of the fact that there is a complete timelike vectorfield, and in the arguments of [8], the existence of such a vector field uses the second countability of the underlying manifold. The above observations do not imply that it is impossible to address the issue of the second countability of the constructed object using a topological characterization. Nevertheless, they indicate that it is not necessarily trivial to do so. In the proof below, we shall demonstrate second countability using the geodesic flow. However, it is clearly necessary to be careful when appealing to standard results on semi-Riemannian geometry, since they are often tacitly based on the assumption of second countability. As a consequence, we shall be very careful when referring to standard results, and we shall develop the necessary tools from scratch whenever possible.

Proposition 5.1. *Let $(\Sigma, \bar{g}, \bar{k}, \bar{\varphi}_0, \bar{\varphi}_1)$ be initial data for (2) and (3). Then the corresponding partially ordered set $\mathcal{M}$ contains a maximal element.*

Remark 5.2. The notation $\mathcal{M}$ was introduced in the statement of Lemma 4.4

Remark 5.3. The argument presented below is to a large extent identical to the one given in [8, pp. 178–181].

Proof. In view of Theorem 3.4, all we need to prove is that every totally ordered subset of $\mathcal{M}$ has an upper bound.

Let $\{[N_\alpha]\}$, $\alpha \in \mathcal{A}$, be a totally ordered subset of $\mathcal{M}$ (for the sake of brevity, we here write $[N_\alpha]$ instead of $[N_\alpha, g_\alpha, \varphi_\alpha]$). Choose developments $(N_\alpha, g_\alpha, \varphi_\alpha)$ (N_α for short) representing the isometry classes; in practice, we shall work with the developments rather than the isometry classes. If $N_\alpha \leq N_\beta$, there is an isometric embedding $\psi_{\beta\alpha} : N_\alpha \to N_\beta$. By arguments similar to the ones given in the proof of Lemma 2.8, $\psi_{\gamma\beta} \circ \psi_{\beta\alpha} = \psi_{\gamma\alpha}$, assuming $N_\alpha \leq N_\beta \leq N_\gamma$. Define K to be the disjoint union of the N_α:

$$K = \coprod_{\alpha \in \mathcal{A}} N_\alpha.$$

We define an equivalence relation on K by saying that $p_\alpha \in N_\alpha$ is equivalent to $p_\beta \in N_\beta$, written $p_\alpha \sim p_\beta$, if one of the following two statements are true:

$$N_\alpha \leq N_\beta, \ \psi_{\beta\alpha}(p_\alpha) = p_\beta, \quad \text{or} \quad N_\beta \leq N_\alpha, \ \psi_{\alpha\beta}(p_\beta) = p_\alpha$$

(note that $\psi_{\alpha\alpha} = \mathrm{Id}$, so that $p, q \in N_\alpha$ are equivalent if and only if they are equal). Let Q be the quotient of K under this equivalence relation.

Q is a Hausdorff topological space. We shall write $[p_\alpha]$ for the equivalence class containing $p_\alpha \in N_\alpha$, and we define $\pi : K \to Q$ by $\pi(p_\alpha) = [p_\alpha]$. Clearly, K is a topological space, and we endow Q with the quotient topology. In other words, a subset U of Q is open if and only if $\pi^{-1}(U)$ is open. Note that if $U_\alpha \subseteq N_\alpha$, then $\pi(U_\alpha)$ is open. The reason for this is that

$$N_\beta \cap \pi^{-1}\left(\pi(U_\alpha)\right) = \psi_{\alpha\beta}^{-1}(U_\alpha)$$

if $N_\beta \leq N_\alpha$ and

$$N_\beta \cap \pi^{-1}\left(\pi(U_\alpha)\right) = \psi_{\beta\alpha}(U_\alpha)$$

if $N_\alpha \leq N_\beta$. To prove that Q is Hausdorff, let $q_1, q_2 \in Q$ be such that $q_1 \neq q_2$. Let $p_\alpha \in q_1$, $r_\beta \in q_2$ and assume, without loss of generality, that $N_\alpha \leq N_\beta$. Then $[\psi_{\beta\alpha}(p_\alpha)] = q_1$ and $\psi_{\beta\alpha}(p_\alpha) \neq r_\beta$. Thus there are open neighbourhoods U_β and V_β of $\psi_{\beta\alpha}(p_\alpha)$ and r_β respectively such that U_β and V_β are disjoint. Then $\pi(U_\beta)$ and $\pi(V_\beta)$ are disjoint open neighbourhoods containing q_1 and q_2 respectively. Thus Q is Hausdorff.

Q has a differentiable structure. If $q \in Q$, then there is a $p_\alpha \in q$. Let $(\mathsf{x}_\alpha, U_\alpha)$ be a coordinate chart on N_α such that $p_\alpha \in U_\alpha$. Since, given any $q \in \pi(U_\alpha)$, there is a unique $q_\alpha \in U_\alpha$ such that $\pi(q_\alpha) = q$, we can define an injective map y_α on $\pi(U_\alpha)$ by $\mathsf{y}_\alpha([q_\alpha]) = \mathsf{x}_\alpha(q_\alpha)$. We would like the y_α to constitute coordinate charts. First we need to prove that y_α is a homeomorphism. Since $\mathsf{y}_\alpha^{-1}(W) = \pi[\mathsf{x}_\alpha^{-1}(W)]$, we see that y_α is continuous. Let $W \subset Q$ be an open set. Then $\pi^{-1}(W) \cap U_\alpha$ is open and

$$W \cap \pi(U_\alpha) = \pi(\pi^{-1}(W) \cap U_\alpha).$$

Thus any open subset of $\pi(U_\alpha)$ can be written $\pi(V_\alpha)$ for some open subset $V_\alpha \subseteq U_\alpha$. Consequently, y_α takes open sets to open sets, so that it is a homeomorphism. Let y_β be defined similarly on $\pi(U_\beta)$ and assume that $\pi(U_\alpha) \cap \pi(U_\beta)$ is non-empty. Assuming, without loss of generality, that $N_\alpha \leq N_\beta$, we get $\mathsf{y}_\beta \circ \mathsf{y}_\alpha^{-1} = \mathsf{x}_\beta \circ \psi_{\beta\alpha} \circ \mathsf{x}_\alpha^{-1}$

so that $\mathsf{y}_\beta \circ \mathsf{y}_\alpha^{-1}$ and $\mathsf{y}_\alpha \circ \mathsf{y}_\beta^{-1}$ are both smooth. Thus $(\mathsf{y}_\alpha, \pi(U_\alpha))$ constitutes an atlas which can be extended to become maximal; cf., e.g., [6, Lemma 1.10, p. 14].

Existence of a Lorentz metric on Q. In order to define a Lorentz metric on Q, let $\pi_\alpha : N_\alpha \to Q$ be defined by $\pi_\alpha(p_\alpha) = [p_\alpha]$. Note that π_α is injective, and since $\mathsf{y}_\alpha \circ \pi_\alpha \circ \mathsf{x}_\alpha^{-1}$ equals the identity when it is defined, π_α is a local diffeomorphism. We conclude that π_α is a diffeomorphism onto its image. Given $p \in Q$ and $X, Y \in T_pQ$, let p_α be such that $p = [p_\alpha]$ and $X_\alpha, Y_\alpha \in T_{p_\alpha}N_\alpha$ be such that $\pi_{\alpha*}X_\alpha = X$ and similarly for Y. Note that X_α and Y_α are unique since π_α is a diffeomorphism. Define

$$g(X, Y) = g_\alpha(X_\alpha, Y_\alpha).$$

We need to prove that this definition makes sense. Assume $[p_\beta] = p$ and define X_β, Y_β analogously to the definition of X_α, Y_α. We can, without loss of generality, assume that $N_\alpha \le N_\beta$. By uniqueness, $\psi_{\beta\alpha*}X_\alpha = X_\beta$ and similarly for Y_β. Thus

$$g_\beta(X_\beta, Y_\beta) = g_\beta(\psi_{\beta\alpha*}X_\alpha, \psi_{\beta\alpha*}Y_\alpha) = \psi_{\beta\alpha}^* g_\beta(X_\alpha, Y_\alpha) = g_\alpha(X_\alpha, Y_\alpha).$$

The smoothness of g is immediate since

$$g(\partial_{\mathsf{y}_\alpha^\lambda}\big|_p, \partial_{\mathsf{y}_\alpha^\nu}\big|_p) = g_\alpha(\partial_{\mathsf{x}_\alpha^\lambda}\big|_{\pi_\alpha^{-1}(p)}, \partial_{\mathsf{x}_\alpha^\nu}\big|_{\pi_\alpha^{-1}(p)}).$$

Note that, as a consequence, $\pi_\alpha^* g = g_\alpha$. Since $\varphi_\alpha \circ \psi_{\beta\alpha} = \varphi_\beta$ if $N_\beta \le N_\alpha$, it is clear that we can define a smooth function φ on Q. We can define an embedding $i : \Sigma \to Q$ by $i = \pi_\alpha \circ i_\alpha$ (where i_α is the embedding of the initial hypersurface Σ into N_α). Then $i^*g = \bar{g}$, $\varphi \circ i = \bar{\varphi}_0$, and if N is the future directed unit normal and κ is the second fundamental form of $i(\Sigma)$, then $i^*\kappa = \bar{k}$ and $(N\varphi) \circ i = \bar{\varphi}_1$ (since we assume all the maps $\psi_{\alpha\beta}$ to be time orientation preserving and orientation preserving, we can assume Q to be time oriented and oriented). Furthermore, $i(\Sigma)$ is spacelike. In order to prove that it is a Cauchy hypersurface, let γ be an inextendible timelike curve in Q. Then γ has to intersect, say, $\pi(N_\alpha)$. Let I be a connected component of $\gamma^{-1}(\pi(N_\alpha))$. Then I is an open interval and there is a timelike curve $\gamma_\alpha : I \to N_\alpha$ such that $\pi_\alpha \circ \gamma_\alpha = \gamma$. Moreover, γ_α is inextendible. As a consequence, γ_α must intersect $i_\alpha(\Sigma)$, so that γ intersects $i(\Sigma)$. Assume that γ intersects $i(\Sigma)$ twice; say that $\gamma(t_j) \in i(\Sigma)$ for $t_j \in I$, $j = 1, 2$, where $t_1 < t_2$. Then $\gamma([t_1, t_2])$ is a compact set and $\pi_\alpha(N_\alpha)$, $\alpha \in \mathcal{A}$, is an open covering. There is thus a finite subcovering, which can be reduced to one element due to the total ordering. We may thus assume that $\gamma([t_1, t_2]) \subset \pi_\alpha(N_\alpha)$. This leads to a contradiction with the fact that $i_\alpha(\Sigma)$ is a Cauchy hypersurface in N_α. We conclude that $i(\Sigma)$ is a Cauchy hypersurface in Q. Assuming Σ to be connected, we conclude that Q is connected.

Q is second countable. We need to prove that Q is second countable. We shall do so using the geodesic flow. Considering the section entitled "Geodesics" in [7, pp. 67–70], it can be verified that the results do not depend on the assumption of second countability (though the Hausdorff property is required). As a consequence, we are allowed to use [7, Proposition 28, p. 70]. There is thus a vectorfield, say G, on TQ such that the projection $\pi_{TQ} : TQ \to Q$ yields a one to one correspondence between (maximal) integral curves of G and (maximal) geodesics in Q. At $v \in TQ$, G_v is given by the initial velocity of the curve $\gamma_v'(s)$, where γ_v is the geodesic satisfying $\gamma_v(0) = \pi_{TQ}(v)$ and $\gamma_v'(0) = v$. Considering the section entitled "The global flow of a vector field" in [2, p. 53–55], it can, again, be verified that the results do not depend on the assumption of second countability. In particular, we may thus

use the conclusions of [2, Theorem 4.26, p. 54]. Given $v \in TQ$, let I_v denote the maximal interval of existence of the integral curve, say ξ, of G with $\xi(0) = v$. Define

$$\mathcal{D} = \{(t, v) \in \mathbb{R} \times TQ : t \in I_v\}.$$

Then, according to [2, Theorem 4.26, p. 54], $\mathcal{D}$ is open and the flow of G, say $\Phi : \mathcal{D} \to TQ$, is smooth. In particular, $\Phi'(t, v) = G_{\Phi(t,v)}$, $\pi_{TQ} \circ \Phi(t, v) = \gamma_v(t)$ and $\Phi(t, v) = \gamma_v'(t)$ with the above notation. It is also of interest to note that, for a fixed $v \in TQ$, $\pi_{TQ} \circ \Phi(\cdot, v)$ is an inextendible geodesic. Let

$$E = \bigcup_{p \in i(\Sigma)} (\mathbb{R} \times C_p Q) \cap \mathcal{D},$$

where $C_p Q$ is the set of timelike vectors in $T_p Q$. Then E is second countable since Σ is second countable. If Σ is n dimensional, E is a $2n + 2$ dimensional manifold. Let W_j be a countable basis for the topology of E and let $O_j = \pi_{TQ} \circ \Phi(W_j)$. We claim that the O_j form a basis for the topology of Q. Let $q \in Q$ be contained in an open neighbourhood U. In order to prove that there is a $(t, v) \in E$ such that $\pi_{TQ} \circ \Phi(t, v) = q$, let γ be a maximal timelike geodesic passing through q. That γ is inextendible, considered as a timelike curve, is a consequence of [7, Lemma 8, p. 130] (note that the proof does not make any use of second countability). Thus γ must intersect the Cauchy hypersurface $i(\Sigma)$. Since γ' is an integral curve of G, the desired statement follows. By the continuity of $\pi_{TQ} \circ \Phi$, we conclude that there is a W_j such that $(t, v) \in W_j$ and $\pi_{TQ} \circ \Phi(W_j) \subseteq U$. All that remains to be proved is that the O_j are open. Before we do so, let us note the following fact. If $v \in TQ$ is a timelike vector, let γ_v be the maximal geodesic with initial velocity v. Since $i(\Sigma)$ is a Cauchy hypersurface, there is a unique $s_v \in \mathbb{R}$ such that $\gamma_v(s_v) \in i(\Sigma)$. In this way we obtain a map $\sigma : \mathcal{T} \to \mathbb{R}$ given by $\sigma(v) = s_v$, where $\mathcal{T} \subset TQ$ is the set of timelike vectors. We wish to prove that σ is continuous. Let $v_j \to v$ and assume that v is future directed. Assume $\sigma(v_j) \leq \sigma(v) - \epsilon$ for some $\epsilon > 0$ and all j that are large enough. Then

$$\gamma_{v_j}[\sigma(v_j)] \leq \gamma_{v_j}[\sigma(v) - \epsilon] \ll \gamma_v[\sigma(v)]$$

for j large enough, and we can choose ϵ to be small enough that $\gamma_{v_j}[\sigma(v) - \epsilon]$ is well defined. The reason for the last inequality is the fact that $\gamma_{v_j}[\sigma(v) - \epsilon]$ converges to $\gamma_v[\sigma(v) - \epsilon]$ (note that this is a consequence of the continuity of Φ). Since $\gamma_{v_j}[\sigma(v_j)]$ and $\gamma_v[\sigma(v)]$ both belong to $i(\Sigma)$ we obtain a timelike curve with endpoints in $i(\Sigma)$, contradicting the fact that $i(\Sigma)$ is a Cauchy hypersurface. Note that this conclusion uses results from causal theory which might, potentially, depend on the assumption of second countability. However, an argument similar to the one carried out in the proof of the fact that $i(\Sigma)$ is a Cauchy hypersurface can be used to obtain the conclusion that the curves in question can be considered to take values in some N_α. Similarly, we cannot have $\sigma(v_j) \geq \sigma(v) + \epsilon$. We conclude that σ is continuous.

Let $q \in O_j$. Then there is a $(t, v) \in E$ such that $q = \pi_{TQ} \circ \Phi(t, v)$. Let $w = \Phi(t, v)$, let O be an open neighbourhood of w in TQ and assume O to be small enough that it only consists of timelike vectors. In particular, $\pi_{TQ}(O)$ is an open neighbourhood of q. Note that as O shrinks, $\Phi(\sigma(u), u)$ converges to $\Phi(\sigma(w), w) = v$ and $\sigma(u)$ converges to $-t$ (for $u \in O$). Thus

$$[-\sigma(u), \Phi(\sigma(u), u)] \tag{6}$$

converges to (t, v). For O small enough (6) is thus contained in W_j, so that $\pi_{TQ}(O)$ is contained in O_j and O_j is open. We conclude that Q is second countable.

Existence of a maximal element. Since Q is a development and $N_\alpha \leq Q$ for all $\alpha \in A$, we conclude that $[Q]$ is an upper bound for $\{[N_\alpha]\}$, $\alpha \in A$. In other words, each totally ordered subset of $\mathcal{M}$ has an upper bound. We conclude that $\mathcal{M}$ has a maximal element $[M]$. □

6. Constructing an extension of two developments

Given two globally hyperbolic developments, say (M, g, φ) and $(\hat{M}, \hat{g}, \hat{\varphi})$, of a given set of initial data, we wish to construct a common extension. If $(\hat{M}, \hat{g}, \hat{\varphi})$ can be isometrically embedded into (M, g, φ), nothing needs to be done. However, in general such a relation will not hold. In fact, take two arbitrary open subsets, say U_1 and U_2, of $n+1$-dimensional Minkowski space, both containing the $t = 0$ hypersurface as a Cauchy hypersurface. Then the union is also a globally hyperbolic development (of the $t = 0$ initial data) extending both U_1 and U_2. On the other hand, we need not have $U_1 \subseteq U_2$ or vice versa. In general, there is of course also the issue that the relevant isometry need not be the identity. The idea is then to try to find the largest piece of $\hat{M}$ which can be isometrically embedded into M (i.e., the largest common development; cf. Theorem 2.4) and to construct a common extension by identifying the points in the disjoint union of $\hat{M}$ and M related by the isometry.

6.1. Finding the largest common development.

In order to implement the above idea, we first need to find "the largest common development". Let us begin with a definition.

Definition 6.1. Let $(\Sigma, \bar{g}, \bar{k}, \bar{\varphi}_0, \bar{\varphi}_1)$ be initial data for (2) and (3), and let (M, g, φ) and $(\hat{M}, \hat{g}, \hat{\varphi})$ be two globally hyperbolic developments with embeddings i and $\hat{i}$ respectively. Then $C(\hat{M}, M)$ denotes the collection of $(\hat{U}, \psi)$ such that

- $\hat{U}$ is an open subset of $\hat{M}$ containing $\hat{i}(\Sigma)$,
- $\hat{i}(\Sigma)$ is a Cauchy hypersurface in $(\hat{U}, \hat{g})$,
- ψ is a time orientation preserving diffeomorphism from $\hat{U}$ to an open subset of M,
- $\psi^* g = \hat{g}$, $\psi^* \varphi = \hat{\varphi}$ and $\psi \circ \hat{i} = i$.

If $(\hat{U}_j, \psi_j) \in C(\hat{M}, M)$, $j = 1, 2$, then the relation $(\hat{U}_1, \psi_1) \leq (\hat{U}_2, \psi_2)$ is said to hold if and only if $\hat{U}_1 \subseteq \hat{U}_2$ and $\psi_2 = \psi_1$ on $\hat{U}_1$.

Remark 6.2. Due to Theorem 2.3, there is a globally hyperbolic development, given initial data. Furthermore, given two developments (M, g, φ) and $(\hat{M}, \hat{g}, \hat{\varphi})$, the set $C(\hat{M}, M)$ is non-empty; cf. Theorem 2.4.

Remark 6.3. If $(\hat{U}_j, \psi_j) \in C(\hat{M}, M)$, $j = 1, 2$, and $\hat{U}_1 \subseteq \hat{U}_2$, then $\psi_2 = \psi_1$ on $\hat{U}_1$ by an argument similar to the one given in the proof of Lemma 2.8; cf. also [7, Proposition 62, p. 91]. Thus the relation $\leq$ simply corresponds to inclusion of the corresponding sets.

Let us make the following observation.

Lemma 6.4. *Let (M, g, φ) and $(\hat{M}, \hat{g}, \hat{\varphi})$ be two globally hyperbolic developments of the same initial data. Then the relation $\leq$ on $C(\hat{M}, M)$ is a partial ordering and every totally ordered subset in $C(\hat{M}, M)$ has an upper bound. In fact, $C(\hat{M}, M)$ has a unique maximal element.*

Remark 6.5. This is the stage at which the error in the proof of [8, Theorem 16.6, p. 177] occurs (to be more precise, it occurs on lines -14 and -13 of [8, p. 178]). The set $C(\hat{M}, M)$ is defined in a different way in [8]. Most of the arguments of the proof below still go through given the different definition. However, the crucial property that every totally ordered subset of $C(\hat{M}, M)$ have an upper bound fails; cf. the introduction for more details.

Remark 6.6. If $(\hat{U}, \psi)$ is the maximal element of $C(\hat{M}, M)$, then $(\psi(\hat{U}), \psi^{-1})$ is the maximal element of $C(M, \hat{M})$.

Proof. That the relation $\leq$ is a partial ordering is a consequence of the fact that it simply corresponds to set inclusion; cf. Remark 6.3 and Definition 3.1. If $(\hat{U}_j, \psi_j) \in C(\hat{M}, M)$, $j = 1, 2$, then $(\hat{U}_1 \cap \hat{U}_2, \hat{g})$ and $(\hat{U}_1 \cup \hat{U}_2, \hat{g})$ are globally hyperbolic with $\hat{i}(\Sigma)$ as a Cauchy hypersurface; we leave the verification of this statement to the reader. By an argument similar to the one given in the proof of Lemma 2.8, we then conclude that $\psi_1 = \psi_2$ on $\hat{U}_1 \cap \hat{U}_2$; cf. also [7, Proposition 62, p. 91]. As a consequence, we can define ψ on $\hat{U} = \hat{U}_1 \cup \hat{U}_2$ by $\psi(p) = \psi_j(p)$ for $p \in \hat{U}_j$, $j = 1, 2$.

Let us now assume that we have a totally ordered subset, say $\mathcal{A}$, of $C(\hat{M}, M)$. Let $\hat{U}$ be the union of the sets corresponding to the elements in $\mathcal{A}$ and let $\psi : \hat{U} \to M$ be defined as follows: if $p \in \hat{U}$, there is a $(\hat{V}, \zeta) \in C(\hat{M}, M)$ such that $p \in \hat{V}$, and we let $\psi(p) = \zeta(p)$; the choice of $(\hat{V}, \zeta) \in C(\hat{M}, M)$ such that $p \in \hat{V}$ is irrelevant due to the above argument. We leave it to the reader to verify that $(\hat{U}, \psi) \in C(\hat{M}, M)$. Clearly, $(\hat{U}, \psi)$ is an upper bound of $\mathcal{A}$. In other words, every totally ordered subset of $C(\hat{M}, M)$ has an upper bound.

Combining the above information with Theorem 3.4, we conclude that $C(\hat{M}, M)$ contains a maximal element. If $(\hat{U}_j, \psi_j) \in C(\hat{M}, M)$, $j = 1, 2$, are two maximal elements, then the above argument yields a ψ such that $(\hat{U}, \psi) \in C(\hat{M}, M)$, where $\hat{U} = \hat{U}_1 \cup \hat{U}_2$. Thus $(\hat{U}, \psi)$ is an extension of $(\hat{U}_j, \psi_j)$, $j = 1, 2$. By the definition of a maximal element, we conclude that

$$(\hat{U}_1, \psi_1) = (\hat{U}, \psi) = (\hat{U}_2, \psi_2).$$

The lemma follows. □

6.2. **Definition and basic properties of the common extension.** Recall that, given initial data, $\mathcal{M}$ contains a maximal element. In order to prove that there is a maximal globally hyperbolic development (given initial data), it is thus sufficient to demonstrate that, given two developments, there is a globally hyperbolic development which is an extension of both of them. In other words, that there is a common extension. The following definition introduces notation for an object we shall later demonstrate to be a common extension of two developments.

Definition 6.7. Let $(\Sigma, \bar{g}, \bar{k}, \bar{\varphi}_1, \bar{\varphi}_0)$ be initial data for (2) and (3), let (M, g, φ) and $(\hat{M}, \hat{g}, \hat{\varphi})$ be two globally hyperbolic developments thereof, and let $(\hat{U}, \psi)$ be the maximal element of $C(\hat{M}, M)$. Define an equivalence relation on the disjoint union $\hat{M} \amalg M$ by requiring $q \sim q$ for all q, $q \sim \psi(q)$ for $q \in \hat{U}$ and $\psi(q) \sim q$ for $q \in \hat{U}$. Taking the quotient of $\hat{M} \amalg M$ under this equivalence relation leads to a topological space, say $\tilde{M}$, with a projection from $\hat{M} \amalg M$ to $\tilde{M}$, say $\tilde{\pi}$. Then $\tilde{M}$ has a differentiable structure, and since $\psi^* g = \hat{g}$ and $\psi^* \varphi = \hat{\varphi}$, there is a Lorentz metric and a smooth function on $\tilde{M}$, say $\tilde{g}$ and $\tilde{\varphi}$ respectively; cf. Remarks 6.8. The triple $(\tilde{M}, \tilde{g}, \tilde{\varphi})$ will be referred to as *the common extension of* (M, g, φ) and $(\hat{M}, \hat{g}, \hat{\varphi})$.

Remarks 6.8. The topology of $\tilde{M}$ is defined by declaring a subset $\tilde{U} \subseteq \tilde{M}$ to be open if and only if $\tilde{\pi}^{-1}(\tilde{U})$ is open. If $\hat{V} \subseteq \hat{M}$ is open, then $\tilde{\pi}(\hat{V})$ is open, since

$$\tilde{\pi}^{-1}(\tilde{\pi}(\hat{V})) = \hat{V} \amalg \psi(\hat{V} \cap \hat{U}).$$

Similarly, if $V \subseteq M$ is open, then $\tilde{\pi}(V)$ is open. If (V, x) is a coordinate chart on M, we define a coordinate chart on $\tilde{\pi}(V)$ by mapping $\tilde{\pi}(p)$ (where $p \in V$) to $\mathsf{x}(p)$; cf. the proof of Proposition 5.1. Similarly, we define coordinate charts on $\tilde{M}$ using coordinate charts on $\hat{M}$. Since ψ is smooth, we obtain a differentiable structure on $\tilde{M}$. Due to the fact that ψ is an isometry onto its image, it is also clear that we obtain a Lorentz metric $\tilde{g}$ and a smooth function $\tilde{\varphi}$.

Remark 6.9. If the developments (M, g, φ) and $(\hat{M}, \hat{g}, \hat{\varphi})$ are clear from the context, we shall simply speak of the common extension.

Lemma 6.10. *Let $(\Sigma, \bar{g}, \bar{k}, \bar{\varphi}_1, \bar{\varphi}_0)$ be initial data for (2) and (3), let (M, g, φ) and $(\hat{M}, \hat{g}, \hat{\varphi})$ be two globally hyperbolic developments thereof, and let $(\tilde{M}, \tilde{g}, \tilde{\varphi})$ be the common extension. If $\tilde{M}$ is Hausdorff, then $(\tilde{M}, \tilde{g}, \tilde{\varphi})$ is a globally hyperbolic development of the initial data such that (M, g, φ) and $(\hat{M}, \hat{g}, \hat{\varphi})$ can be isometrically embedded into it.*

Proof. We already know that $\tilde{M}$ is a Hausdorff topological space with a differentiable structure. Since $\hat{M}$ and M have countable bases for their topology, we obtain a countable basis for the topology of $\tilde{M}$ by using the projection $\tilde{\pi}$; cf. Remarks 6.8. Thus, $\tilde{M}$ is a smooth manifold. It is clear that $\tilde{g}$ is a smooth Lorentz metric (and that there is a time orientation), that $\tilde{\varphi}$ is a smooth function and that $(\tilde{g}, \tilde{\varphi})$ satisfy (2) and (3). Furthermore, they induce the correct initial data (the relevant embedding is given by $\tilde{\pi} \circ i$). We need to prove that

$$\tilde{\pi} \circ i(\Sigma) = \tilde{\pi} \circ \hat{i}(\Sigma)$$

is a Cauchy hypersurface (here we, by abuse of notation, write i instead of the composition of i with the inclusion from M to $\hat{M} \amalg M$ and similarly for $\hat{i}$). Let γ be an inextendible timelike curve in $\tilde{M}$. Restricting γ to a connected subinterval of $\gamma^{-1}[\tilde{\pi}(\hat{U})]$, we obtain a curve which can be considered to be an inextendible timelike curve in $\hat{U}$ (if γ never intersects $\tilde{\pi}(\hat{U})$, then the image of γ is contained in $\tilde{\pi}(M)$ (or in $\tilde{\pi}(\hat{M})$) and can be considered to be an inextendible curve in M (or $\hat{M}$) so that it has to intersect $i(\Sigma)$ (or $\hat{i}(\Sigma)$), in contradiction to the supposition that γ never intersects $\tilde{\pi}(\hat{U})$). It must thus intersect $\hat{i}(\Sigma)$. As a consequence, γ must intersect $\tilde{\pi} \circ \hat{i}(\Sigma)$. In order to prove that γ cannot intersect $\tilde{\pi} \circ \hat{i}(\Sigma)$ twice, say that

$$\gamma(t_0) \in \tilde{\pi} \circ \hat{i}(\Sigma),$$

and say that γ is future oriented. If γ enters $\tilde{\pi}(\hat{M}-\hat{U})$ to the future, it cannot leave this set again to the future, since we would otherwise obtain an inextendible timelike curve in $\hat{U}$ which would not intersect $\hat{i}(\Sigma)$. A similar statement holds concerning $\tilde{\pi}(M-\psi(\hat{U}))$. As a consequence, $\gamma(t)$ belongs either to $\tilde{\pi}(\hat{M})$ or $\tilde{\pi}(M)$ for $t \geq t_0$, so that γ cannot intersect $\tilde{\pi}\circ\hat{i}(\Sigma)$ for $t > t_0$. The argument concerning the past is similar. Thus $\tilde{\pi}\circ i(\Sigma)$ is a Cauchy hypersurface. That (M,g,φ) and $(\hat{M},\hat{g},\hat{\varphi})$ can be isometrically embedded into the common extension is an immediate consequence of the construction. □

7. Properties of boundaries of globally hyperbolic regions

Let $(\Sigma,\bar{g},\bar{k},\bar{\varphi}_0,\bar{\varphi}_1)$ be initial data for (2) and (3) and let $[M,g,\varphi]$ be a maximal element of $\mathcal{M}$; due to Proposition 5.1, such an element exists. What remains is to prove that if $(\hat{M},\hat{g},\hat{\varphi})$ is a globally hyperbolic development of the initial data, then $(\hat{M},\hat{g},\hat{\varphi})$ can be isometrically embedded into (M,g,φ). To this end, we construct the common extension $(\tilde{M},\tilde{g},\tilde{\varphi})$; cf. Definition 6.7. If we can prove that $\tilde{M}$ is Hausdorff, then $(\tilde{M},\tilde{g},\tilde{\varphi})$ is a globally hyperbolic development extending (M,g,φ); cf. Lemma 6.10. Due to the maximality of $[M,g,\varphi]$, $(\tilde{M},\tilde{g},\tilde{\varphi})$ and (M,g,φ) must then be isometric. As a consequence, it can be argued that it must be possible to embed $(\hat{M},\hat{g},\hat{\varphi})$ isometrically into (M,g,φ). In other words, the main task is to prove that $\tilde{M}$ is Hausdorff, and the remaining sections are devoted to this goal. Recall that a basic ingredient of the construction of $\tilde{M}$ is the maximal element, say $(\hat{U},\psi)$, of $C(\hat{M},M)$. In order to establish that $\tilde{M}$ is Hausdorff, a careful study of $\partial\hat{U}$ is required. This is what we wish to initiate in the present section.

Lemma 7.1. *Let (M,g) be a time oriented Lorentz manifold and assume that it admits a smooth spacelike Cauchy hypersurface Σ. Assume that U is an open subset of M which contains Σ and is such that Σ is a Cauchy hypersurface in (U,g). Then*

- *if $p \in \partial U \cap J^+(\Sigma)$, then $I^-(p)\cap J^+(\Sigma) \subset U$,*
- *$\partial U \cap J^+(\Sigma)$ is achronal,*
- *if $p_i \in \partial U \cap J^+(\Sigma)$, $i = 1,2$, are such that $p_1 < p_2$, then there is a null geodesic $\alpha : [0,1] \to M$ such that $\alpha(0) = p_1$, $\alpha(1) = p_2$ and $\alpha(t) \in \partial U \cap J^+(\Sigma)$ for $t \in [0,1]$,*
- *if p_i, $0 \leq i \in \mathbb{Z}$, is a collection of points, not all equal, such that $p_i \in \partial U \cap J^+(\Sigma)$ and $p_{i+1} \leq p_i$ for all $i \geq 0$, then there is a null geodesic, say α, defined on an interval I such that $\alpha(I) \subset \partial U \cap J^+(\Sigma)$ and $p_i \in \alpha(I)$ for all $i \geq 0$,*
- *if $p_i \in \partial U \cap J^+(\Sigma)$, $i = 1,\dots,k$, $3 \leq k \in \mathbb{Z}$, are such that $p_i < p_{i+1}$, $i = 1,\dots,k-1$, and α is an inextendible null geodesic containing two distinct points in the set $\{p_2,\dots,p_k\}$, then the range of α contains $\{p_1,\dots,p_k\}$.*

Remark 7.2. Similar statements hold concerning $\partial U \cap J^-(\Sigma)$.

Remark 7.3. Strictly speaking, we shall not need the fourth statement. However, it is useful for the sake of developing an intuition for the propertis of boundaries.

Proof. Let $p \in \partial U \cap J^+(\Sigma)$. Assume that there is a $q \in I^-(p)\cap J^+(\Sigma)$ which does not belong to U. Then there is no $r \in I^+(q)\cap U$, since there is otherwise an inextendible timelike curve in U which does not intersect Σ, in contradiction to the fact that Σ is a Cauchy hypersurface in (U, g). Thus $I^+(q)$ is an open neighbourhood of p which does not intersect U, in contradiction with the fact that $p \in \partial U$. This proves the first statement.

The second statement follows from the first; if there is a future directed timelike curve from p_1 to p_2, with $p_i \in \partial U \cap J^+(\Sigma)$, $i = 1, 2$, then, by the above, $p_1 \in U$, so that $p_1 \notin \partial U$, a contradiction.

In order to prove the third statement, assume that $p_i \in \partial U \cap J^+(\Sigma)$, $i = 1, 2$, are such that $p_1 < p_2$. Since (M, g) is globally hyperbolic, there is a length maximizing causal geodesic connecting p_1 and p_2; cf. [7, Proposition 19, p. 411] and the definition of global hyperbolicity; cf. [7, Definition 20, p. 412] or [8, Subsection 10.2.6, pp. 108–109]. We already know that this geodesic cannot be timelike, so it must be null. Let us call it α and assume that it is parametrized in such a way that $\alpha(0) = p_1$ and $\alpha(1) = p_2$. Assume that there is a $t \in (0, 1)$ such that $\alpha(t) \in U$. Since U is open, there is thus a point $q \in U$ such that $\alpha(t) \ll q$. However, $p_1 \leq \alpha(t) \ll q$ implies that $p_1 \ll q$; cf. [7, Corollary 1, p. 402]. Since $p_1 \in \partial U \cap J^+(\Sigma)$, we thus obtain an inextendible timelike curve in U which does not intersect Σ: a contradiction. Thus $\alpha(t) \notin U$ for all $t \in [0, 1]$. Assume now that there is a $t \in (0, 1)$ such that $\alpha(t) \in M - \bar{U}$, where $\bar{U}$ denotes the closure of U. Since $M - \bar{U}$ is an open set, we then obtain a $q \in (M - \bar{U}) \cap J^+(\Sigma)$ such that $q \ll \alpha(t)$. Thus $q \in I^-(p_2) \cap J^+(\Sigma)$, but $q \notin U$, in contradiction to the first conclusion. To conclude, $\alpha(t) \in \bar{U} - U$ for all $t \in [0, 1]$, and the third statement follows.

Consider the fourth statement. If there are only two distinct points, the third statement yields the desired conclusion. We may thus, without loss of generality, assume that $p_3 < p_2 < p_1$. Due to the third statement, we know that p_2 and p_1 are connected by a null geodesic contained in $\partial U \cap J^+(\Sigma)$, say α. Let $i \geq 3$. Again, we know that there is a null geodesic connecting p_i and p_2 contained in $\partial U \cap J^+(\Sigma)$, say β. If the tangent vectors of α and β do not match in p_2, there is a timelike curve from p_i to p_1; cf. [7, Proposition 46, p. 294]. Thus the null curve α continues on to p_i (and is contained in $\partial U \cap J^+(\Sigma)$). The curve α, restricted to $\alpha^{-1}(\partial U \cap J^+(\Sigma))$ (note that this set is by necessity an interval due to the proof of the third statement), is the desired null geodesic. The fourth statement follows.

In order to prove the last statement, let us assume that α contains p_j and p_m, where $2 \leq j < m \leq k$. Let $1 \leq l < j$ and β be a null geodesic connecting p_l and p_j. If α and β cannot be combined to form a null geodesic containing $\{p_l, p_j, p_m\}$, then $p_l \ll p_m$, a contradiction. Thus the range of α contains $\{p_l, p_j, p_m\}$ for every $1 \leq l < j$. For similar reasons, the range of α contains $\{p_j, p_m, p_l\}$ for every $m < l \leq k$. The case that remains to be considered is $j < l < m$. However, we already know that $\{p_1, p_j\}$ is in the range of α. Thus p_l has to be in the range of α by an argument similar to the one given above. The desired conclusion follows. □

8. Properties of common extensions that are not Hausdorff

Due to Lemma 6.10, it is clear that it would be desirable to prove that $\tilde{M}$ Hausdorff. As we have mentioned above, the strategy is to prove that the assumption that $\tilde{M}$ is not Hausdorff leads to the conclusion that $(\hat{U}, \psi)$ is not maximal. In order to be able to construct an extension of $(\hat{U}, \psi)$, we need to obtain information concerning the non-Hausdorff points in $\tilde{M}$. This is the subject of the remaining sections. Let us begin by introducing some terminology.

Definition 8.1. Let $(\Sigma, \bar{g}, \bar{k}, \bar{\varphi}_1, \bar{\varphi}_0)$ be initial data for (2) and (3), let (M, g, φ) and $(\hat{M}, \hat{g}, \hat{\varphi})$ be two globally hyperbolic developments thereof, and let $(\tilde{M}, \tilde{g}, \tilde{\varphi})$ be the common extension. A pair of points $\tilde{p}, \tilde{q} \in \tilde{M}$ such that $\tilde{p} \neq \tilde{q}$ and such that for every pair of open neighbourhoods, say $\tilde{V}$ and $\tilde{W}$, of $\tilde{p}$ and $\tilde{q}$ respectively, the intersection $\tilde{V}$ and $\tilde{W}$ is non-empty will be called a *non-Hausdorff pair.* A point $\tilde{q} \in \tilde{M}$ will be called a *non-Hausdorff point* if there is a point $\tilde{p} \in \tilde{M}$ such that $\tilde{p}, \tilde{q}$ is a non-Hausdorff pair. The set of $\hat{p} \in \partial \hat{U} \cap J^{\pm}[\hat{i}(\Sigma)]$ such that $\tilde{\pi}(\hat{p})$ is a non-Hausdorff point will be denoted by $\hat{H}^{\pm}$. Similarly, the set of $p \in \partial \psi(\hat{U}) \cap J^{\pm}[i(\Sigma)]$ such that $\tilde{\pi}(p)$ is a non-Hausdorff point will be denoted by $H^{\pm}$.

Next, we wish to prove that

- $\tilde{M}$ is Hausdorff if and only if $\hat{H}^{+}$ or $\hat{H}^{-}$ is non-empty,
- to every point in $\hat{H}^{+}$ there is a uniquely associated point in H^{+} (and similarly for $\hat{H}^{-}$).

The reason this information is of interest is that it can be used as a starting point for constructing an extension of $(\hat{U}, \psi)$.

Lemma 8.2. *Let* $(\Sigma, \bar{g}, \bar{k}, \bar{\varphi}_1, \bar{\varphi}_0)$ *be initial data for (2) and (3), let* (M, g, φ) *and* $(\hat{M}, \hat{g}, \hat{\varphi})$ *be two globally hyperbolic developments thereof, and let* $(\tilde{M}, \tilde{g}, \tilde{\varphi})$ *be the common extension. Then the set of non-Hausdorff points equals the union of* $\tilde{\pi}(\hat{H}^{\pm})$ *and* $\tilde{\pi}(H^{\pm})$*. Furthermore, if* $\hat{p} \in \hat{H}^{+}$*, then there is a unique* $\tilde{p} \in \tilde{M}$ *such that* $\tilde{\pi}(\hat{p})$*,* $\tilde{p}$ *is a non-Hausdorff pair. In fact,* $\tilde{p} = \tilde{\pi}(p)$ *for a (unique)* $p \in H^{+}$*, and if* γ *is a future directed timelike curve in* $\hat{M}$ *such that* $\gamma(0) = \hat{p}$*, then*

$$\lim_{t \to 0-} \psi \circ \gamma(t) = p. \tag{7}$$

Remark 8.3. There are similar statements concerning the causal past; i.e., if $\hat{p} \in \hat{H}^{-}$, then there is a unique $\tilde{p} \in \tilde{M}$ such that $\tilde{\pi}(\hat{p})$, $\tilde{p}$ is a non-Hausdorff pair, and it is given by $\tilde{\pi}(p)$ for a (unique) $p \in H^{-}$. Furthermore, the relation can be turned around in the sense that if $p \in H^{+}$, then there is a unique $\hat{p} \in \hat{H}^{+}$ such that $\tilde{\pi}(p)$, $\tilde{\pi}(\hat{p})$ is a non-Hausdorff pair etc.

Remark 8.4. In the situation considered in the lemma, $\gamma(t) \in \hat{U}$ for small enough negative t due to Lemma 7.1. Thus the limit (7) is well defined.

Proof. Say that $\tilde{p} \in \tilde{M}$ is a non-Hausdorff point. Choose a $\tilde{q} \in \tilde{M}$ such that $\tilde{p}$, $\tilde{q}$ is a non-Hausdorff pair. If both points are in $\tilde{\pi}(\hat{M})$ or in $\tilde{\pi}(M)$, they cannot be non-Hausdorff points. We must thus have $\tilde{p} \in \tilde{\pi}(\hat{M}) - \tilde{\pi}(M)$ and $\tilde{q} \in \tilde{\pi}(M) - \tilde{\pi}(\hat{M})$ or vice versa. Let us assume that $\tilde{p} \in \tilde{\pi}(\hat{M}) - \tilde{\pi}(M)$ and $\tilde{q} \in \tilde{\pi}(M) - \tilde{\pi}(\hat{M})$; the argument in the other case is similar. Then $\tilde{p} = \tilde{\pi}(\hat{p})$ for some $\hat{p} \in \hat{M} - \hat{U}$ and $\tilde{q} = \tilde{\pi}(q)$ for some $q \in M - \psi(\hat{U})$. If $\hat{p} \notin \partial \hat{U}$, then there is an open neighbourhood, say $\hat{V}$, of $\hat{p}$ such that $\hat{V}$ does not intersect $\hat{U}$. Let V be any open neighbourhood of q. Then $\tilde{\pi}(\hat{V})$ and $\tilde{\pi}(V)$ are open neighbourhoods of $\tilde{p}$ and $\tilde{q}$ respectively which do not intersect. Thus $\hat{p} \in \partial \hat{U}$. Similarly, we must have $q \in \partial \psi(\hat{U})$. In particular, the set of non-Hausdorff points equals the union of $\tilde{\pi}(\hat{H}^{\pm})$ and $\tilde{\pi}(H^{\pm})$.

Let $\hat{p} \in \hat{H}^+$. Since ψ preserves the time orientation, the above argument leads to the conclusion that there is a $p \in H^+$ such that $\tilde{\pi}(\hat{p})$, $\tilde{\pi}(p)$ is a non-Hausdorff pair. Let $\hat{q} \in \hat{U}$ and $q \in M$ be such that $\hat{q} \ll \hat{p}$ and $p \ll q$. Then $I^+(\hat{q})$ and $I^-(q)$ are open neighbourhoods of $\hat{p}$ and p respectively, and the projections thereof are open neigbourhoods of $\tilde{\pi}(\hat{p})$ and $\tilde{\pi}(p)$ respectively. As a consequence, there is a $\hat{q}_1 \in I^+(\hat{q})$ such that $\psi(\hat{q}_1) \ll q$. Thus $\psi(\hat{q}) \ll q$. Letting q_l be a sequence such that $p \ll q_l$ and $q_l \to p$, we conclude that $\psi(\hat{q}) \leq p$, since the relation $\leq$ is closed on a globally hyperbolic Lorentz manifold. On the other hand, we can carry out the same argument for a $\hat{q}_1 \in \hat{U}$ such that $\hat{q} \ll \hat{q}_1 \ll \hat{p}$ in order to obtain $\psi(\hat{q}) \ll \psi(\hat{q}_1) \leq p$. To conclude, if $\hat{q} \ll \hat{p}$, $\hat{q} \in \hat{U}$, then $\psi(\hat{q}) \ll p$. Conversely, if $q \ll p$, $q \in \psi(\hat{U})$, then $\psi^{-1}(q) \ll \hat{p}$.

Let γ be a future directed timelike curve such that $\gamma(0) = \hat{p}$. Then $\gamma(t) \in \hat{U}$ for small enough negative t due to Lemma 7.1. Let, furthermore, $q \in \psi(\hat{U})$ be such that $q \ll p$. Then $\psi^{-1}(q) \ll \hat{p}$ so that $\psi^{-1}(q) \ll \gamma(t) \ll \hat{p}$ for $t < 0$ close enough to zero. There is thus an $\epsilon > 0$ such that $q \ll \psi \circ \gamma(t) \ll p$ for all $t \in (-\epsilon, 0)$. Since, for every neighbourhood V of p, there is a $q \ll p$ such that $J^+(q) \cap J^-(p)$ is contained in V (this statement is based on the fact that (M, g) is globally hyperbolic, in particular on the fact that the strong causality condition holds), we conclude that

$$\lim_{t \to 0-} \psi \circ \gamma(t) = p.$$

The converse statement also holds.

Given $\hat{p} \in \hat{H}^+$, there is, due to the above argument, a unique $p \in H^+$ such that $\tilde{\pi}(\hat{p})$ and $\tilde{\pi}(p)$ is a non-Hausdorff pair. The statements of the lemma follow. □

Due to the above result, we are allowed to make the following definition.

Definition 8.5. Let $(\Sigma, \bar{g}, \bar{k}, \bar{\varphi}_1, \bar{\varphi}_0)$ be initial data for (2) and (3), let (M, g, φ) and $(\hat{M}, \hat{g}, \hat{\varphi})$ be two globally hyperbolic developments thereof, and let $(\tilde{M}, \tilde{g}, \tilde{\varphi})$ be the common extension. Assume that $\hat{H}^{\pm}$ is non-empty and that $\hat{p} \in \hat{H}^{\pm}$ and $p \in H^{\pm}$ are such that $\tilde{\pi}(\hat{p})$, $\tilde{\pi}(p)$ is a non-Hausdorff pair. Then p will be denoted by $\hat{\mathcal{H}}(\hat{p})$ and $\hat{p}$ will be denoted by $\mathcal{H}(p)$.

It is of interest to make the following observation.

Lemma 8.6. *Let $(\Sigma, \bar{g}, \bar{k}, \bar{\varphi}_1, \bar{\varphi}_0)$ be initial data for (2) and (3), let (M, g, φ) and $(\hat{M}, \hat{g}, \hat{\varphi})$ be two globally hyperbolic developments thereof, and let $(\tilde{M}, \tilde{g}, \tilde{\varphi})$ be the common extension. Assume that $\hat{H}^+$ is non-empty and that $\hat{p}, \hat{q} \in \hat{H}^+$ are such that $\hat{p} < \hat{q}$. Then $\hat{\mathcal{H}}(\hat{p}) < \hat{\mathcal{H}}(\hat{q})$.*

Remark 8.7. Similar statements hold concerning $\hat{H}^-$ and $H^{\pm}$.

Proof. Let us use the notation $p = \hat{\mathcal{H}}(\hat{p})$ and $q = \hat{\mathcal{H}}(\hat{q})$. Let γ be a future directed timelike geodesic with $\gamma(0) = p$. Then $\psi^{-1} \circ \gamma(t) \to \hat{p}$ as $t \to 0-$ due to Lemma 8.2. In particular, $\psi^{-1} \circ \gamma(t) \ll \hat{p} < \hat{q}$ for $t < 0$. Thus $\psi^{-1} \circ \gamma(t) \ll \hat{q}$ for $t < 0$. Due to the arguments given in the proof of Lemma 8.2, we conclude that $\gamma(t) \ll q$ for $t < 0$. Since the relation $\leq$ is closed on a globally hyperbolic manifold, we obtain $p \leq q$. However, we know that $p \neq q$. Thus $p < q$. □

In the end, we wish to show that the assumption that one of the sets $\hat{H}^{\pm}$ is non-empty leads to the possibility of extending $(\hat{U}, \psi)$; i.e., a contradiction. The extension will be based on the existence of an element in $\hat{H}^{\pm}$ with special properties. In order to prove that such an element exists, assuming, say, $\hat{H}^{+}$ to be non-empty, we need to derive some additional properties of $\hat{H}^{+}$ such as, e.g., the fact that it is an open subset of $\partial\hat{U}$.

Lemma 8.8. *Let $(\Sigma, \bar{g}, \bar{k}, \bar{\varphi}_1, \bar{\varphi}_0)$ be initial data for (2) and (3), let (M, g, φ) and $(\hat{M}, \hat{g}, \hat{\varphi})$ be two globally hyperbolic developments thereof, and let $(\tilde{M}, \tilde{g}, \tilde{\varphi})$ be the common extension. Assume $\hat{H}^{+}$ to be non-empty, let $\hat{p} \in \hat{H}^{+}$ and $p = \hat{\mathcal{H}}(\hat{p})$. Then there is an open neighbourhood $\hat{V}$ of $\hat{p}$ such that*

$$\hat{V} \cap \partial\hat{U} \subset \hat{H}^{+}.$$

Furthermore, there is an open neighbourhood V of p and a diffeomorphism $\zeta : \hat{V} \to V$ such that $\zeta = \psi$ on $\hat{V} \cap \hat{U}$. Finally, if $\hat{q} \in \hat{H}^{+} \cap \hat{V}$, then $\zeta(\hat{q}) \in H^{+} \cap V$ and $\zeta(\hat{q}) = \hat{\mathcal{H}}(\hat{q})$.

Remark 8.9. Similar statements hold concerning $\hat{H}^{-}$ and $H^{\pm}$.

Proof. Before beginning the proof, let us make the following general comment: we shall consistently be working to the future of the Cauchy hypersurfaces $i(\Sigma)$ and $\hat{i}(\Sigma)$ in the present proof. In particular, neighbourhoods of points in $\hat{H}^{+}$ and H^{+} will always be assumed to be subsets of $I^{+}[\hat{i}(\Sigma)]$ and $I^{+}[i(\Sigma)]$ respectively.

Let $\hat{\gamma}$ be a timelike geodesic in $\hat{M}$ such that $\hat{\gamma}(0) = \hat{p}$. Let I be an open interval containing 0 such that $\hat{\gamma}$ is defined on I. Due to Lemma 8.2, $\psi \circ \hat{\gamma}$ is a timelike geodesic on $\hat{\gamma}^{-1}(\hat{U})$ such that

$$\lim_{t \to 0-} \psi \circ \hat{\gamma}(t) = p.$$

Thus there is a timelike geodesic, say γ, in M such that $\gamma(t) = \psi \circ \hat{\gamma}(t)$ for $t < 0$ and $\gamma(0) = p$; cf. [7, Lemma 8, p. 130]. Let us, for the sake of simplicity, restrict the domains of definition of γ and $\hat{\gamma}$ so that they are defined on a common interval, say $I_\epsilon = (-2\epsilon, 2\epsilon)$ for some $\epsilon > 0$. Let us also assume that ϵ is small enough that $\gamma(I_\epsilon)$ is contained in a convex neighbourhood, say W, of p and such that $\hat{\gamma}(I_\epsilon)$ is contained in a convex neighbourhood, say $\hat{W}$, of $\hat{p}$. Let $\hat{V} \subset \hat{W}$ be an open neighbourhood of $\hat{p}$ such that $\hat{V} \subset I^{+}[\hat{\gamma}(-\epsilon)]$ and such that

$$\psi_* \exp^{-1}_{\hat{\gamma}(-\epsilon)}(\hat{V}) \subset \exp^{-1}_{\gamma(-\epsilon)}(W).$$

For $\hat{q} \in \hat{V}$, we can thus define

$$\zeta(\hat{q}) = \exp_{\gamma(-\epsilon)} \psi_* \exp^{-1}_{\hat{\gamma}(-\epsilon)}(\hat{q}).$$

Clearly, ζ is a diffeomorphism from $\hat{V}$ onto its image. Moreover, the image contains p in its interior.

Assume now that $\hat{q} \in \hat{V} \cap \hat{U}$. Then there is a unique geodesic $\hat{\alpha}$ such that $\hat{\alpha}(0) = \hat{\gamma}(-\epsilon)$ and $\hat{\alpha}(1) = \hat{q}$. Furthermore,

$$\hat{\alpha}'(0) = \exp^{-1}_{\hat{\gamma}(-\epsilon)}(\hat{q}),$$

and since $\hat{q} \in I^+[\hat{\gamma}(-\epsilon)]$, $\hat{\alpha}$ is a timelike geodesic. We have to have $\hat{\alpha}([0,1]) \subset \hat{U}$, since there is otherwise an inextendible timelike geodesic in $\hat{U}$ which does not intersect $\hat{i}(\Sigma)$. In particular $\psi \circ \hat{\alpha}$ is a timelike geodesic in $\psi(\hat{U})$. Moreover,

$$\psi_* \hat{\alpha}'(0) = \psi_* \exp^{-1}_{\hat{\gamma}(-\epsilon)}(\hat{q}),$$

so that

$$\zeta(\hat{q}) = \exp_{\gamma(-\epsilon)} \psi_* \exp^{-1}_{\hat{\gamma}(-\epsilon)}(\hat{q}) = \exp_{\gamma(-\epsilon)}(\psi \circ \hat{\alpha})'(0) = \psi \circ \hat{\alpha}(1) = \psi(\hat{q}).$$

To conclude, ζ is a diffeomorphism from $\hat{V}$ to $\zeta(\hat{V})$ such that $\zeta = \psi$ on $\hat{U} \cap \hat{V}$, and $V = \zeta(\hat{V})$ is an open neighbourhood of p.

We now wish to demonstrate that $\hat{V} \cap \partial\hat{U} \subset \hat{H}^+$. Let $\hat{q} \in \hat{V} \cap \partial\hat{U}$. We know that there is a timelike geodesic $\hat{\alpha}$ with $\hat{\alpha}(0) = \hat{\gamma}(-\epsilon)$ and $\hat{\alpha}(1) = \hat{q}$. Furthermore, we know that $\hat{\alpha}([0,1)) \subset \hat{U}$. Clearly, $\psi \circ \hat{\alpha}(t)$ converges to $\zeta(\hat{q})$ as $t \to 1-$. Thus $\zeta(\hat{q})$ is such that $\tilde{\pi}(\hat{q})$, $\tilde{\pi} \circ \zeta(\hat{q})$ is a non-Hausdorff pair; for any pair of neighbourhoods, say $\tilde{X}$ and $\tilde{Y}$, of $\tilde{\pi}(\hat{q})$ and $\tilde{\pi} \circ \zeta(\hat{q})$ respectively, we clearly have $\tilde{\pi} \circ \hat{\alpha}(t) \in \tilde{X} \cap \tilde{Y}$ for $t < 1$ close enough to 1. □

9. Null geodesics in $\partial\hat{U}$

As we have already mentioned, the idea is to use a point in $\hat{H}^+$ (or $\hat{H}^-$) to construct an extension. However, if there is a null geodesic in $\partial\hat{U}$ through a point, say $\hat{p}$, in $\hat{H}^+$, then $\hat{p}$ is an unsuitable starting point for carrying out the extension. The purpose of the following lemma is to prove that not all elements of $\hat{H}^+$ have this property.

Lemma 9.1. *Let $(\Sigma, \bar{g}, \bar{k}, \bar{\varphi}_1, \bar{\varphi}_0)$ be initial data for (2) and (3), let (M, g, φ) and $(\hat{M}, \hat{g}, \hat{\varphi})$ be two globally hyperbolic developments thereof, and let $(\tilde{M}, \tilde{g}, \tilde{\varphi})$ be the common extension. Let $\hat{p}_j \in \partial\hat{U} \cap J^+[\hat{i}(\Sigma)]$, $j = 1, 2$, be such that $\hat{p}_2 < \hat{p}_1$ and such that $\hat{p}_1 \in \hat{H}^+$. Then there is an inextendible null geodesic $\hat{\alpha}$ containing $\hat{p}_2$ and $\hat{p}_1$ such that $\hat{\alpha}(0) = \hat{p}_1$. Let $\hat{K} = \hat{\alpha}^{-1}(\partial\hat{U} \cap J^+[\hat{i}(\Sigma)])$. Then $\hat{K}$ is a closed interval, and if t_0 is the left end point of $\hat{K}$, then $\hat{\alpha}([t_0, 0])$ is contained in H^+. Finally,*

$$J^-[\hat{\alpha}(t_0)] \cap \partial\hat{U} \cap J^+[\hat{i}(\Sigma)] = \{\hat{\alpha}(t_0)\}. \tag{8}$$

Remark 9.2. A similar statement holds concerning H^-.

Proof. Due to Lemma 7.1, we know that there is a null geodesic $\hat{\alpha}$ such that $\hat{\alpha}(-1) = \hat{p}_2$, such that $\hat{\alpha}(0) = \hat{p}_1$ and such that $\hat{\alpha}(t) \in \partial\hat{U} \cap J^+[\hat{i}(\Sigma)]$ for $t \in [-1, 0]$. We can assume it to be inextendible. Due to Lemma 8.8, we know that there is an $\epsilon > 0$ such that $\hat{\alpha}(t) \in \hat{H}^+$ for $t \in (-\epsilon, 0]$. Let $\hat{q} \in \hat{\alpha}[(-\epsilon, 0)]$, $q = \hat{\mathcal{H}}(\hat{q})$ and $p_1 = \hat{\mathcal{H}}(\hat{p}_1)$. Due to Lemma 8.6, $q < p_1$. Again, we know that there is a null geodesic, say α (which we shall assume to be inextendible), such that $\alpha(0) = p_1$ and q belongs to the image of α. Let $K = \alpha^{-1}(\partial\psi(\hat{U}) \cap J^+[i(\Sigma)])$. Note that, due to arguments of the type given in the proof of Lemma 7.1, K and $\hat{K}$ are connected. Thus they are closed intervals. Note, moreover, that since α and $\hat{\alpha}$ have to intersect $i(\Sigma)$ and $\hat{i}(\Sigma)$ to the past, respectively, K and $\hat{K}$ have to be bounded to the past. Let us denote the left endpoint of $\hat{K}$ by t_0.

It will be convenient to note that

$$\hat{\mathcal{H}}[\hat{\alpha}(s)] \in \alpha(K) \tag{9}$$

for $s \in (-\epsilon, 0]$. We know that $\hat{\mathcal{H}}(\hat{q}), \hat{\mathcal{H}}(\hat{p}_1) \in \alpha(K)$. In order to obtain the desired conclusion, it is thus sufficient to note that $\hat{\mathcal{H}}[\hat{\alpha}(s_1)] < \hat{\mathcal{H}}[\hat{\alpha}(s_2)]$ for $s_1 < s_2$, $s_j \in (-\epsilon, 0]$, $j = 1, 2$, and to refer to the last statement of Lemma 7.1.

Let

$$\mathcal{A} = \{t \in [t_0, 0] | \hat{\alpha}(s) \in \hat{H}^+, \ \hat{\mathcal{H}}[\hat{\alpha}(s)] \in \alpha(K) \ \forall s \in [t, 0]\}.$$

Note that $\mathcal{A}$ is connected by definition. Furthermore, $0 \in \mathcal{A}$. In fact, since $\hat{H}^+$ is open (considered as a subset of $\partial\hat{U}$) and (9) holds for $s \in (-\epsilon, 0]$, we know that there is a $t < 0$ such that $t \in \mathcal{A}$. Our goal is to prove that $\mathcal{A} = [t_0, 0]$. We shall do so by proving that $\mathcal{A}$ is open, closed, connected and non-empty (considered as a subset of $[t_0, 0]$). What we need to prove is that

- if $t \in (t_0, 0] \cap \mathcal{A}$, then there is a $t_1 \in [t_0, t) \cap \mathcal{A}$,
- if $t_j \in \mathcal{A}$, $t \in [t_0, 0]$, $t_j \to t$ and $t_j \geq t$, then $t \in \mathcal{A}$.

Let us assume that $t \in (t_0, 0] \cap \mathcal{A}$. Since we already know that $\mathcal{A}$ contains a negative number, we may assume that $t < 0$. Due to the fact that $\hat{H}^+$ is open in $\partial\hat{U}$, we know that there is a $t_1 \in [t_0, t)$ such that $\hat{\alpha}(s) \in \hat{H}^+$ for $s \in [t_1, 0]$. Combining the fact that (9) holds for $s \in (-\epsilon, 0]$ with Lemma 8.6 and the last statement of Lemma 7.1, we conclude that $\hat{\mathcal{H}}[\hat{\alpha}(s)] \in \alpha(K)$ for all $s \in [t_1, 0]$. Thus $t_1 \in \mathcal{A}$.

Let $t_j \in \mathcal{A}$, $t \in [t_0, 0]$, $t_j \to t$ and $t_j \geq t$. Since

$$\hat{\mathcal{H}}[\hat{\alpha}(t_j)] \in \alpha(K \cap (-\infty, 0])$$

for all j and since $\alpha(K \cap (-\infty, 0])$ is a compact set, we can (by choosing a suitable subsequence) assume that the sequence $\hat{\mathcal{H}}[\hat{\alpha}(t_j)]$ converges to an element, say r, of $\alpha(K)$. We wish to prove that $\hat{\alpha}(t) \in \hat{H}^+$ and that $\hat{\mathcal{H}}[\hat{\alpha}(t)] = r$. Let $\tilde{V}$ be an open neighbourhood of $\tilde{\pi}(\hat{\alpha}(t))$ and $\tilde{W}$ be an open neighbourhood of $\tilde{\pi}(r)$. Let

$$\hat{V} = \tilde{\pi}^{-1}(\tilde{V}) \cap \hat{M}, \quad V = \tilde{\pi}^{-1}(\tilde{W}) \cap M.$$

Then $\hat{V}$ is an open neighbourhood of $\hat{\alpha}(t)$ and V is an open neighbourhood of r. Choose a j large enough that $\hat{\alpha}(t_j)$ belongs to $\hat{V}$ and $\hat{\mathcal{H}}[\hat{\alpha}(t_j)]$ belongs to V. Let γ be a future directed timelike curve with $\gamma(0) = \hat{\alpha}(t_j)$. For $\tau < 0$ small enough, we then have $\gamma(\tau) \in \hat{V}$ and $\psi \circ \gamma(\tau) \in V$ due to Lemma 8.2. Thus

$$\tilde{\pi}(\gamma(\tau)) \in \tilde{V} \cap \tilde{W},$$

so that $\hat{\alpha}(t) \in \hat{H}^+$ and $\hat{\mathcal{H}}[\hat{\alpha}(t)] = r$. As a consequence of this fact and the existence of the sequence t_j, we conclude that $t \in \mathcal{A}$.

As a consequence of the above observations, we know that $\mathcal{A} = [t_0, 0]$. What remains is to prove that (8) holds. Assume that there is an

$$x \in J^-[\hat{\alpha}(t_0)] \cap \partial\hat{U} \cap J^+[\hat{i}(\Sigma)]$$

such that $x \neq \hat{\alpha}(t_0)$. Then there is a null geodesic from x to $\hat{\alpha}(t_0)$, say β. If β and α could be combined to form a null geodesic, we would obtain a contradiction to the definition of t_0. If they cannot, we obtain the conclusion that $x \ll \hat{\alpha}(t_0)$, a contradiction. Thus, the assumption of the existence of an x as above leads to a contradiction, and the desired conclusion follows. □

Corollary 9.3. *Let $(\Sigma, \bar{g}, \bar{k}, \bar{\varphi}_1, \bar{\varphi}_0)$ be initial data for (2) and (3), let (M, g, φ) and $(\hat{M}, \hat{g}, \hat{\varphi})$ be two globally hyperbolic developments thereof, and let $(\tilde{M}, \tilde{g}, \tilde{\varphi})$ be the common extension. Assume $\hat{H}^+$ to be non-empty. Then there is a $\hat{p} \in \hat{H}^+$ such that*

$$J^-(\hat{p}) \cap J^+[\hat{i}(\Sigma)] \cap \partial\hat{U} = \{\hat{p}\}. \tag{10}$$

Proof. Let $\hat{p} \in \hat{H}^+$. If (10) holds, we are done. If it does not hold, Lemma 9.1 is applicable, and we obtain the desired point by appealing to Lemma 9.1. □

Lemma 9.4. *Let $(\Sigma, \bar{g}, \bar{k}, \bar{\varphi}_1, \bar{\varphi}_0)$ be initial data for (2) and (3), let (M, g, φ) and $(\hat{M}, \hat{g}, \hat{\varphi})$ be two globally hyperbolic developments thereof, and let $(\tilde{M}, \tilde{g}, \tilde{\varphi})$ be the common extension. Assume $\hat{H}^+$ to be non-empty, let $\hat{p} \in \hat{H}^+$ be such that*

$$J^-(\hat{p}) \cap J^+[\hat{i}(\Sigma)] \cap \partial\hat{U} = \{\hat{p}\},$$

let γ be a future directed timelike geodesic with $\gamma(0) = \hat{p}$ and let $\hat{W}$ be an open neighbourhood of $\hat{p}$. Then there is an $\epsilon > 0$ such that $I_\epsilon = (-\epsilon, \epsilon)$ is subset of the interval of existence of γ and such that for $t \in I_\epsilon$,

$$J^-[\gamma(t)] \cap J^+[\hat{i}(\Sigma)] \cap \hat{U}^c \subset \hat{W},$$

where $\hat{U}^c = \hat{M} - \hat{U}$ denotes the complement of $\hat{U}$.

Remark 9.5. A similar statement holds concerning $\hat{H}^-$.

Proof. Assume that the statement is not true. Then there is a sequence $t_j \to 0$ and points $\hat{p}_j$ such that

$$\hat{p}_j \in J^-[\gamma(t_j)] \cap \hat{U}^c \cap J^+[\hat{i}(\Sigma)] \cap \hat{W}^c.$$

Fix a $t > 0$ in the existence interval of γ. Then

$$\hat{p}_j \in J^-[\gamma(t)] \cap \hat{U}^c \cap J^+[\hat{i}(\Sigma)] \cap \hat{W}^c$$

for j large enough. Note that $\hat{U}^c$ and $\hat{W}^c$ are closed, so that

$$J^-[\gamma(t)] \cap \hat{U}^c \cap J^+[\hat{i}(\Sigma)] \cap \hat{W}^c$$

is compact due to [7, Lemma 40, p. 423]. We may thus assume the sequence $\hat{p}_j$ to converge to some point, say $\hat{q}$. Then $\hat{q} \leq \hat{p}$ since the relation $\leq$ is closed in the current setting, so that

$$\hat{q} \in J^-(\hat{p}) \cap \hat{U}^c \cap J^+[\hat{i}(\Sigma)] \cap \hat{W}^c.$$

Since $\hat{p} \in \hat{W}$, we thus have $\hat{q} < \hat{p}$. If $\hat{q} \in \partial\hat{U}$, we obtain a contradiction to the assumptions, so let us assume that this is not the case. Then $\hat{q}$ is in the interior of $\hat{U}^c$. Let α be a future directed inextendible timelike geodesic with $\alpha(0) = \hat{q}$. Since α has to intersect $\hat{i}(\Sigma)$ to the past, we have to have $\alpha(t) \in \partial\hat{U}$ for some $t < 0$. But then $\alpha(t) \ll \hat{q} \leq \hat{p}$, which implies that $\partial\hat{U} \cap J^+[\hat{i}(\Sigma)]$ is chronal, a contradiction. □

10. Existence of a maximal globally hyperbolic development

We are finally in a position to prove that there is a maximal globally hyperbolic development, given initial data.

Before stating the result which will yield us the desired conclusion, let us recall the definition of $\tau(p,q)$; cf. [7, Definition 15, p. 409]. Let (M,g) be a Lorentz manifold. For $p,q\in M$, we then let $\tau(p,q)$ denote the supremum of the lengths of future pointing causal curve segments from p to q. If the set of lengths is unbounded, $\tau(p,q)=\infty$ and if $q\notin J^+(p)$, then $\tau(p,q)=0$.

Lemma 10.1. *Let $(\Sigma,\bar{g},\bar{k},\bar{\varphi}_1,\bar{\varphi}_0)$ be initial data for (2) and (3), let (M,g,φ) and $(\hat{M},\hat{g},\hat{\varphi})$ be two globally hyperbolic developments thereof, and let $(\tilde{M},\tilde{g},\tilde{\varphi})$ be the common extension. Then $\tilde{M}$ is Hausdorff.*

Proof. Let us assume that $\hat{H}^+$ is non-empty in order to arrive at a contradiction (the argument in the case of a non-empty $\hat{H}^-$ being similar). Due to Corollary 9.3, there is then a point $\hat{p}\in\hat{H}^+$ such that

$$J^-(\hat{p})\cap J^+[\hat{i}(\Sigma)]\cap\partial\hat{U}=\{\hat{p}\}.$$

Due to Lemma 8.8, there is a convex open neighbourhood $\hat{V}$ of $\hat{p}$ (which we shall assume to be a subset of $I^+[\hat{i}(\Sigma)]$) such that

$$\hat{V}\cap\partial\hat{U}\subset\hat{H}^+,$$

an open neighbourhood V of $p:=\hat{\mathcal{H}}(\hat{p})$ and a diffeomorphism $\zeta:\hat{V}\to V$ such that $\zeta=\psi$ on $\hat{V}\cap\hat{U}$. Let $\hat{W}$ be an open neighbourhood of $\hat{p}$ with compact closure contained in $\hat{V}$ and let $\hat{q}$ be such that $\hat{p}\ll\hat{q}$ and such that

$$J^-(\hat{q})\cap\hat{U}^c\cap J^+[\hat{i}(\Sigma)]\subset\hat{W};\tag{11}$$

the existence of a $\hat{q}$ with these properties is guaranteed by Lemma 9.4. Let

$$\chi(\hat{r})=\tau(\hat{r},\hat{q}),$$

where τ was defined prior to the statement of the lemma. Due to [7, Lemma 21, p. 412], τ is continuous on $\hat{M}\times\hat{M}$, so that χ is a continuous function on $\hat{M}$. As a consequence, χ attains its maximum in the compact set

$$\hat{K}:=\hat{U}^c\cap J^+[\hat{i}(\Sigma)]\cap\mathrm{clos}(\hat{W}),$$

where $\mathrm{clos}(\hat{W})$ denotes the closure of $\hat{W}$. Let us call the maximum d_0. Since $\hat{p}\in\hat{K}$ and $\hat{p}\ll\hat{q}$, we conclude that $\tau(\hat{p},\hat{q})>0$, so that $d_0>0$. Let

$$\hat{r}\in\hat{U}^c\cap J^+[\hat{i}(\Sigma)].\tag{12}$$

If $\hat{r}\notin\hat{W}$, then $\chi(\hat{r})=0$ due to (11). As a consequence, $\chi(\hat{r})\le d_0$ for all $\hat{r}$ satisfying (12). Let $\hat{X}$ denote the set of $\hat{r}\in\hat{V}$ such that $\chi(\hat{r})>d_0$. Clearly, $\hat{X}$ is an open set. Furthermore, since we know that $\hat{V}\subset I^+[\hat{i}(\Sigma)]$, and since $\chi(\hat{r})\le d_0$ for all $\hat{r}$ satisfying (12), we know that $\hat{X}\subset\hat{U}$. Thus $\hat{X}\subset\hat{U}\cap\hat{V}$. We also wish to show that $\hat{X}$ is non-empty. Let $\hat{r}\in\hat{K}$ be such that $\chi(\hat{r})=d_0$ and let $\hat{\alpha}$ be the timelike geodesic with $\hat{\alpha}(-1)=\hat{r}$ and $\hat{\alpha}(0)=\hat{q}$. For $t<-1$ close enough to -1, we have $\chi[\hat{\alpha}(t)]>d_0$ and $\hat{\alpha}(t)\in\hat{V}$. Thus $\hat{X}$ is non-empty.

Pulling back the boundary of $\hat{X}$ in $\hat{V}$ to $T_{\hat{q}}\hat{M}$ using $\exp_{\hat{q}}$, we obtain part of a hyperboloid. As a consequence, the boundary is a smooth spacelike hypersurface.

Let us call it $\bar{S}$. Pulling back the solution on V to $\hat{V}$ using ζ, we obtain two solutions on $\hat{V}$. However, they coincide in $\hat{X}$. Since $\exp_{\hat{q}}^{-1}(\hat{X})$ is foliated by a family of partial hyperboloids which, in the appropriate limit, converges smoothly to the partial hyperbolid corresponding to the boundary of $\hat{X}$ in $\hat{V}$, the two solutions induce the same initial data on $\bar{S}$. Using Theorem 2.4, we conclude that the two solutions have to coincide in a neighbourhood of $\bar{S}$. Let $\hat{D}$ be an open neighbourhood of $\bar{S}$ such that the solutions coincide in $\hat{D}$ and such that $\bar{S}$ is a Cauchy hypersurface in $\hat{D}$. Clearly, we can extend ψ to be defined on $\hat{U}_{\rm ext} = \hat{U} \cup \hat{D}$, but we need to prove that $(\hat{U}_{\rm ext}, \hat{g})$ is globally hyperbolic with $\hat{\imath}(\Sigma)$ as a Cauchy hypersurface. Let γ be an inextendible timelike curve in $\hat{U}_{\rm ext}$. Since $\hat{U}_{\rm ext} \subset \hat{M}$, it is clear that γ cannot intersect $\hat{\imath}(\Sigma)$ twice. If γ intersects $\hat{U}$, then γ restricted to any connected component of $\gamma^{-1}(\hat{U})$ is an inextendible curve in $\hat{U}$ and must thus intersect $\hat{\imath}(\Sigma)$. Let us now assume that γ intersects $\hat{D}$. Then γ, restricted to any connected component of $\gamma^{-1}(\hat{D})$ is an inextendible curve which must therefore intersect $\bar{S}$. Thus γ intersects $\bar{S}$. If the point of intersection is in $\hat{U}$, we are done, so that we may assume that $\gamma(t_0) \in \partial\hat{U}$ for some t_0 in the domain of definition of γ. But then $\gamma(t) \in \hat{U}$ for $t < t_0$ close enough to t_0, due to Lemma 7.1. To conclude $\hat{\imath}(\Sigma)$ is a Cauchy hypersurface in $(\hat{U}_{\rm ext}, g)$. As a consequence, the isometry ψ can be extended to a larger domain, so that $(\hat{U}, \psi)$ was not the maximal element of $C(\hat{M}, M)$, a contradition. Thus $\hat{H}^+ = \emptyset$ and, by a similar argument, $\hat{H}^- = \emptyset$. Consequently, $\tilde{M}$ is Hausdorff. □

As a curiosity, let us mention that the above proof can be used to prove that if $\hat{H}^+$ is non-empty, then there is a point $\hat{p} \in \hat{H}^+$ and a spacelike hypersurface, say $\bar{T}$, in $\hat{M}$ such that $\bar{T} - \{\hat{p}\} \subset \hat{U}$ (since $\hat{H}^+$ is always empty, this statement might seem uninteresting, but we can construct $\tilde{M}$ using a non-maximal element $(\hat{U}, \psi)$ of $C(\hat{M}, M)$; this would lead to a different situation). The reason is that the surface $\bar{S}$ constructed at the end of the proof of Lemma 10.1 has to contain a point $\hat{p} \in \hat{H}^+$. Considering geodesic normal coordinates around $\bar{S}$ and pushing the surface to the past in a punctured neigbourhood of $\hat{p}$ (but not at $\hat{p}$), we obtain the desired surface.

Finally, we are in a position to prove the desired result.

Corollary 10.2. *Given initial data* $(\Sigma, \bar{g}, \bar{k}, \bar{\varphi}_1, \bar{\varphi}_0)$ *for (2) and (3), there is a maximal globally hyperbolic development.*

Proof. Let $[M, g, \varphi]$ be a maximal element of $\mathcal{M}$ and let $(\hat{M}, \hat{g}, \hat{\varphi})$ be a globally hyperbolic development of the given initial data. Let $(\tilde{M}, \tilde{g}, \tilde{\varphi})$ be the common extension of (M, g, φ) and $(\hat{M}, \hat{g}, \hat{\varphi})$. Due to Lemma 10.1, $\tilde{M}$ is then Hausdorff, so that $(\tilde{M}, \tilde{g}, \tilde{\varphi})$ is a globally hyperbolic development of the initial data due to Lemma 6.10. Moreover, (M, g, φ) can be isometrically embedded into $(\tilde{M}, \tilde{g}, \tilde{\varphi})$. Due to the maximality of $[M, g, \varphi]$, there is thus an isometry ξ from $(\tilde{M}, \tilde{g}, \tilde{\varphi})$ to (M, g, φ). Composing the isometric embedding of $(\hat{M}, \hat{g}, \hat{\varphi})$ into $(\tilde{M}, \tilde{g}, \tilde{\varphi})$ with this isometric embedding, we obtain an isometric embedding of $(\hat{M}, \hat{g}, \hat{\varphi})$ into (M, g, φ). In other words, the maximal element of $C(\hat{M}, M)$ is of the form $(\hat{U}, \psi)$ with $\hat{U} = \hat{M}$. The desired conclusion follows. □

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