

1. PROBLEM SET 3

Problem 1. Let M be a smooth, second countable Hausdorff manifold. Then there is a countable collection of smooth functions ϕ_j , $j = 1, 2, \dots$, on M with the following properties:

- $\text{supp}\phi_i$ is compact for every i ,
- for every $p \in M$, there is an open neighbourhood U of p such that U only intersects finitely many of the sets $\text{supp}\phi_i$,
- $\sum_{j=1}^{\infty} \phi_j = 1$.

a. Show that the (smooth) function

$$\varrho = \sum_{j=1}^{\infty} j\phi_j$$

is a *proper* function, i.e. $\varrho^{-1}(C)$ is compact for every compact subset C of \mathbb{R} .

b. Let h be a Riemannian metric on M (such a metric exists due to Lemma 25, p. 140 of O'Neill's book) and define

$$g = h + d\varrho \otimes d\varrho.$$

Show that g is a complete Riemannian metric on M .

Problem 2. Let (\bar{M}, \bar{g}) be a Lorentz manifold and assume that there is a compact spacelike hypersurface M in \bar{M} . Show that there is an $\epsilon > 0$ and a smooth map $\phi : \bar{M}_\epsilon \rightarrow \bar{M}$, where $\bar{M}_\epsilon = (-\epsilon, \epsilon) \times M$, which is a diffeomorphism onto its image, such that $\phi(0, p) = p$ and such that

$$\phi^*\bar{g} = -dt \otimes dt + g,$$

where g is a family of Riemannian metrics on M parametrized by $t \in (-\epsilon, \epsilon)$. What geometric significance should be attached to $\dot{g}(0)$?

Problem 3. Solve problems 6 and 7 on p. 213-214 of O'Neill's book.