

1. PROBLEM SET 4

Reading instructions: Read Chapter 8, Chapter 9, pp. 250-256, and Chapter 10.

Problem 1. Solve problems 3 and 7 on p. 232 of O’Neill.

Problem 2. Solve problems 9 and 10 on p. 259 of O’Neill.

Problem 3. Determine the Killing fields on the n -sphere and on n -dimensional hyperbolic space.

Problem 4. Solve problem 1 on p. 298.

Problem 5. Prove that a compact, oriented even-dimensional manifold with positive sectional curvature must be simply connected.

(Hint: If not there is a closed geodesic of minimal length and energy in a homotopy class. Study parallel transport around this geodesic to find a energy-decreasing variation, and a contradiction.)

Also, prove that a compact, odd-dimensional manifold with positive sectional curvature must be orientable.

Problem 6. Generalize the results in the section entitled “The endmanifold case”, pp. 280-281, to the situation where there are two endmanifolds (and give detailed proofs).

Problem 7. Let (M, g) be a time oriented Lorentz manifold, let $\Sigma_i, i = 1, 2$, be spacelike hypersurfaces and assume there is a future directed timelike curve $\sigma \in \Omega(\Sigma_1, \Sigma_2)$ (cf. the notation on p. 280 of O’Neill’s book) with constant speed $c > 0$ which has maximal length among the future oriented timelike curves in $\Omega(\Sigma_1, \Sigma_2)$. Then (due to problem 6) σ is a timelike geodesic which is normal to both Σ_1 and Σ_2 . Prove that

$$\frac{1}{c} \int_0^b \text{Ric}[\sigma'(s), \sigma'(s)] ds + \kappa_2[\sigma(b)] - \kappa_1[\sigma(0)] \leq 0,$$

where κ_i is the mean curvature of Σ_i .

Problem 8. Let (M, g) be a time oriented Lorentz manifold such that $\text{Ric}(v, v) \geq 0$ for all timelike vectors $v \in TM$. Assume (M, g) has two spacelike hypersurfaces $\Sigma_i, i = 1, 2$, and let κ_i be the associated mean curvatures. Prove that if

$$\sup_{p \in \Sigma_1} \kappa_1(p) < \inf_{p \in \Sigma_2} \kappa_2(p),$$

then there is no future directed timelike curve from Σ_1 to Σ_2 which has maximal length among the future oriented timelike curves in $\Omega(\Sigma_1, \Sigma_2)$.

Let us remark that if Σ_i are compact and the Lorentz manifold is globally hyperbolic, cf. Definition 20, p. 412 of O'Neill, then the assumption that there is a future oriented timelike curve in $\Omega(\Sigma_1, \Sigma_2)$ implies that there is a future directed timelike curve from Σ_1 to Σ_2 which has maximal length among the future oriented timelike curves in $\Omega(\Sigma_1, \Sigma_2)$.