CONVEXITY OF FREE BOUNDARIES WITH BERNOULLI TYPE BOUNDARY CONDITION

Antoine HENROT *
Dept of Mathematics
and URA 741 CNRS
Université de Franche-Comté
25030 Besancon Cedex France
e-mail henrot@grenet.fr

Henrik SHAHGHOLIAN
Dept of Mathematics
Royal Institute of technology
100 44 Stockholm Sweden
e-mail henriks@math.kth.se

1 Introduction

The aim of this paper is to prove convexity results for two (exterior and interior) classical free boundary problems which arise in various physical situations such as electrochemical machining and potential flow in fluid mechanics. For more details on the physical background and an exhaustive bibliography we refer to [F] and [Fas].

Let us describe the problems that we consider.

The exterior free boundary problem:

Let K be a bounded (regular) domain in R^N. The problem of finding a function u and a domain $\Omega \supset K$ such that the overdetermined problem

$$(1.1) \qquad \begin{cases} \Delta u = 0 & \text{in } \Omega \setminus \overline{K} \\ u = 1 & \text{on } K \\ u = 0, \quad |\nabla u| = 1 & \text{on } \partial\Omega \end{cases}$$

has a solution, has been studied for several decades.

Existence of solutions can be obtained by various ways, e.g. by variational methods (see [A-C]) or by the method of sub- and supersolutions (see [Beur] for the two-dimensional case and [H-S] for the general case); cf. also Acker [Ack89-2] for an approach based on iterative methods. Now assume that K is convex. Can we prove that Ω is also convex? This question has already been considered, and answered in the affirmative, in the two-dimensional case, first by Tepper [Tep] and then by Hamilton [Ham], both using conformal mappings. Kawohl [Kaw86] (see also [Kaw85]) proves the same result without using conformal mappings. Also this doesn't generalize to higher dimensions. In this paper, we give a positive answer to the above question in arbitrary dimension. In order to be complete, let us point out that several other geometric results for the free boundary have been obtained : see, for example Acker [Ack84] and [Ack86] or Lancaster [Lan].

The second problem is similar though it has been less studied.

*The first author wants to thank Swedish Natural Science Research Council (NFR) for a visiting appointment

The interior free boundary problem:

Let K be a bounded (regular) domain in R^N, we look for a function u and an open set $\omega \subset K$ such that the overdetermined problem

$$(1.2) \qquad \begin{cases} \Delta u = 0 & \text{in } K \setminus \overline{\omega} \\ u = 1 & \text{on } \partial K \\ u = 0 \quad |\nabla u| = 1 & \text{on } \partial\omega, \end{cases}$$

has a solution.

Neither existence nor uniqueness are always true in this case. We thus consider a domain K for which (1.2) admits a solution and want to show that convexity of K implies that of ω. It should be remarked that using conditions on the gradient of u on ∂K, A. Acker (see [Ack89]) states some interesting uniqueness results for the interior problem. Also two-dimensional existence results for this problem were obtained by Lavrentiev [Lav], Beurling [Beur] and Daniljuk [Dan]. For the N-dimensional case, we refer to the paper of Alt and Caffarelli [A-C].
At last, in the fourth section we will investigate the nonlinear case for both problems.

Remarks:
The results stated here seems likely to be proved using techniques of Caffarelli and Spruck (see [C-S]). However, the advantage of our method is its simplicity and straightforwardness.
When completing the writing of this paper, we learned from A. Acker and R. Meyer that they recently suceeded in proving the same convexity result than ours for the exterior problem using a method based on domain perturbation operators.

Acknowledgement: We would like to thank Prof. B. Gustafsson for reading the manuscript and giving helpful comments.

2 The exterior problem

Let us denote by K a bounded (regular) domain in R^N, $N \geq 2$. In this section, we denote by Ω a solution of the free boundary problem (1.1), i.e. Ω is an open set in R^N which contains K and such that the solution u of

$$(2.1) \qquad \begin{cases} \Delta u = 0 & \text{in } \Omega \setminus \overline{K} \\ u = 1 & \text{on } K \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

exists and satisfies

$$(2.2) \qquad \liminf_{\substack{y \to x \\ y \in \Omega}} |\nabla u(y)| = \limsup_{\substack{y \to x \\ y \in \Omega}} |\nabla u(y)| = 1 \quad \text{for every } x \in \partial\Omega.$$

We want to point out that, here, $\partial\Omega$ is not assumed to be regular. Nevertheless the information that the function $|\nabla u|$, which is C^∞ in $\Omega \setminus \overline{K}$, is continuous up to the boundary (by (2.2)) allows us to prove that the solution u is classical, i.e. u is continuous up to the boundary (see [H-S]).

Now we claim the main result of this section.

Theorem 2.1: In (2.1) let K be convex and assume (u, Ω) is a solution to (2.1), (2.2). Then Ω is also convex and there are no other solutions.

Proof: Let us denote by $\Omega^\star$ the convex hull of Ω and by $u^\star$ the (classical) solution of (2.1) with Ω replaced by $\Omega^\star$. We first assert:

Assertion:

$$|\nabla u^\star(x)| \geq 1 \qquad \forall x \in \Omega^\star \setminus \overline{K}.$$

Assume this for the moment. In order to prove the theorem we need to show that $\Omega^\star \subset \Omega$. For this we use the Lavrentiev principle.

We assume, for simplicity, that K contains the origin. In $\Omega_t^\star := \{x : x/t \in \Omega^\star\}$, we define

$$v_t(x) = u^\star(\frac{x}{t}),$$

where $0 < t < 1$.

Now suppose $\Omega^\star \setminus \Omega \neq \emptyset$. Then

$$t_0 = \sup\{t : \Omega_t^\star \subset \Omega\} < 1. \tag{2.3}$$

We also have, due to the convexity of K, that $K_{t_0} = \{x : x/t_0 \in K\} \subset K$. Since $v_{t_0} - u$ is harmonic in $\Omega_{t_0}^\star \setminus K$ and takes nonpositive values on $\partial(\Omega_{t_0}^\star \setminus K)$ we can apply the maximum principle to obtain $v_{t_0} \leq u$ in $\Omega_{t_0}^\star \setminus K$.

Now let $x_0 \in \partial\Omega_{t_0}^\star \cap \partial\Omega$ ($\neq \emptyset$ by (2.3)). Then by Assertion and (2.2) it follows that

$$\frac{1}{t_0} \leq \frac{1}{t_0} \limsup_{\substack{y/t_0 \to x_0 \\ y/t_0 \in \Omega^\star \setminus \overline{K}}} |\nabla u^\star(y/t_0)| = \limsup_{\substack{y \to x_0 \\ y \in \Omega_{t_0}^\star \setminus \overline{K}}} |\nabla v_{t_0}(y)| \leq \limsup_{\substack{y \to x_0 \\ y \in \Omega_{t_0}^\star \setminus \overline{K}}} |\nabla u(y)| = 1,$$

which contradicts (2.3). (The third step can be verified by working with the level sets of u (see [Shah, Lemma 2]).) This finishes the proof of the first part.

As to the second part, let Ω_1, Ω_2 be two different solutions. Then a similar argument as above (the Lavrentiev principle) applied to Ω_1 and Ω_2 shows that $\Omega_1 \subset \Omega_2 \subset \Omega_1$.

<u>Remark:</u>
In [H-S] (cf. also [Beur]) using the method of sub- and supersolutions the authors are able to obtain solutions to problem (2.1), (2.2). The method is simply to consider the class of all domains Ω ($\supset K$) for which the solution to (1.1) satisfies

$$\limsup_{\substack{y \to x \\ y \in \Omega}} |\nabla u(y)| \leq 1 \quad \text{for every } x \in \partial\Omega.$$

Since this class is closed under intersection we can obtain a solution by taking the intersection of all such domains. Of course lots of technical details are to be done.

Now if in the above argument we make the additional restriction that the domains should be convex then, since convexity is preserved under intersection, we can still (by some efforts) obtain a solution and then conclude, by uniqueness (in the convex case), the convexity of the original free boundary. This would give us a constructive proof.

Proof of the Assertion: For $0 < s < 1$ denote by $\mathcal{L}_s$ the level sets of $u^\star$, i.e.

$$\mathcal{L}_s = \{x \in \Omega^\star : u^\star(x) > s\}.$$

By results of Lewis [Le] (cf. also [C-S]) these level sets are convex. Let y be any point on $\partial\mathcal{L}_s$. Then, by rotation and translation, we may assume that y is the origin and that $e_1 = (1, 0, ..., 0)$ is the exterior normal vector to $\partial\mathcal{L}_s$ at y so that $\{x_1 = 0\}$ is a supporting hyperplane to $\mathcal{L}_s$ at y.

Let us take $z \in \partial\Omega^\star \cap \{x_1 > 0\}$ with largest distance to $\{x_1 = 0\}$. By geometric properties of the convex hull, z can be chosen to belong to $\partial\Omega \cap \partial\Omega^\star$. On the open set $\Omega^\star \cap \{x_1 > 0\}$ consider the harmonic function $v(x) = u^\star(x) + ax_1$ where $0 < a < 1$. Since v is harmonic in $\Omega^\star \cap \{x_1 > 0\}$ its maximum is achieved on the boundary of $\Omega^\star \cap \{x_1 > 0\}$. Hence either at z or at the origin.

Now Let B be the ball with center $z + e_1$ and radius 1. Then by [A-C, 4.10] (see the remark below) there is a sequence $z^k \in \Omega$ converging to z such that

$$\lim \frac{u(z^k)}{d(z^k, B)} \geq 1. \tag{2.4}$$

Suppose v takes maximum at z. Then, since $\Omega \subset \Omega^\star$, by the maximum principle we have $u^\star \geq u$ on $\Omega \setminus K$. Hence

$$(2.5) \qquad 0 \geq \frac{v(z^k) - v(z)}{d(z^k, B)} = \frac{u^\star(z^k) - u^\star(z)}{d(z^k, B)} + a\frac{(z_1^k - z_1)}{d(z^k, B)} \geq \frac{u(z^k)}{d(z^k, B)} + a\frac{(z_1^k - z_1)}{d(z^k, B)}.$$

Since $z_1 - z_1^k = d(z^k, \{x_1 = z_1\}) \leq d(z^k, B)$, (2.5) implies that $\frac{u(z^k)}{d(z^k,B)} \leq a$. Letting k go to infinity we obtain (by (2.4)) $a \geq 1$, which is a contradiction. Therefore v achieves its maximum at the origin. Hence

$$-\frac{\partial v}{\partial x_1}(0) \geq 0,$$

i.e.

$$|\nabla u^\star(0)| \geq a.$$

As $a \in (0, 1)$ was arbitrary we obtain the desired result.

Remark:
In this remark we use notations adopted by [A-C]. As one may check the requirements in Lemma 4.10 in [A-C] are that [A-C, 4.3] is satisfied and there is an exterior ball touching the boundary at x_0 (z in our case). The latter is obvious because of the situation of z. [A-C, 4.3] is needed, so that u is Lipschitz, nondegenerate and satisfies the representation $\Delta u = H^{N-1}\lfloor\partial\Omega$ i.e. the restriction of the $(N-1)$-dimensional Hausdorff measure to $\partial\Omega$. It is obvious that u is Lipschitz. For the representation we should take a different 'blow-up' sequence

$$u_k(x) = \frac{1}{d_k}u_k'(x_k + d_k x),$$

where $u_k' = (u - u(y_k))_+$. Then $q_{u_k} = |\nabla u_k|\lfloor\partial\{u_k > 0\}$ (in [A-C, 4.10) will converge to one continuously, according to (2.2). The nondegeneracy can be proven by taking the function u^2 and observing that $\Delta u^2 = 2|\nabla u|^2 \geq 1/2$ in $B = B(x_0, r)$, provided r is small enough. By a theoerm of Caffarelli (see [C, Lemma 1]) $\sup_B u^2 \geq Cr^2$, where $C = C(N)$. This gives the nondegeneracy needed to "blow-up", otherwise the blow-up may become more and more flat, and vanish in the limit.

3 The interior problem

Let us denote by K a bounded domain in R^N, $N \geq 2$. In this section, we denote by ω a solution of the free boundary problem (1.2), that is to say :
ω is an open set in R^N contained in K such that the solution u to

$$(3.1) \qquad \begin{cases} \Delta u = 0 & \text{in } K \setminus \overline{\omega} \\ u = 1 & \text{on } \partial K \\ u = 0 & \text{on } \partial\omega \end{cases}$$

satisfies

$$(3.2) \qquad \liminf_{\substack{y \to x \\ y \in K \setminus \overline{\omega}}} |\nabla u(y)| = \limsup_{\substack{y \to x \\ y \in K \setminus \overline{\omega}}} |\nabla u(y)| = 1 \quad \text{for every } x \in \partial\omega.$$

We also assume from now on that the given domain K is convex and we want to study to what extent this implies that the free boundary ω is also convex. We give a positive answer in the two-dimensional case when ω is connected. In the N-dimensional case, with $N \geq 3$, we only succeed (under reasonable regularity condition) in proving that the mean curvature of $\partial\omega$ is positive for

any connected component.
Let us denote by $\omega^\star$ the convex hull of ω and by $u^\star$ the associated potential, solution of the system (3.1) posed on $K \setminus \overline{\omega^\star}$.

Our results will rely on the following lemma analogous to Assertion in Section 2.

Lemma 3.1:

$$|\nabla u^\star(x)| \leq 1 \qquad \forall x \in K \setminus \overline{\omega^\star}.$$

Proof: The proof is similar to that of the exterior case (see the proof of Assertion above). The only differences lie in the fact that we work with the function $v(x) = u^\star(x) + ax_1$ with $a > 1$ and we are interested in the minimum of v. We should also reverse inequalities (2.4), (2.5), and choose z^k in $K \setminus \overline{\omega^\star}$ on the x_1-axis. In this case (2.4) follows automatically with $d(z^k, B)$ replaced by $|z^k - z|$.

Remark:
A by-product of the proof (also in the exterior case) is an estimation of the distance between K and the free boundary $\partial\omega$. Since $v(y) < v(z)$, we easily obtain that this distance must be larger than one, whence a necessary condition for the existence of a solution to the free boundary problem (3.1), (3.2) is that the width of K in every direction is larger than two.

Now to prove convexity of the free boundary we cannot use the Lavrentiev principle as we did in the exterior case. Nevertheless, Lemma 3.1 allows us to conclude in the two-dimensional case using the perimeter :

Theorem 3.2: Let ω be a connected subset in R^2 with finite perimeter which admits a solution to (3.1), (3.2) with K convex. Then ω is also convex.

Proof: By Lemma 3.1 and the fact that $\partial\omega^\star$ is Lipschitz (due to the convexity of $\omega^\star$)

$$P(\omega^\star) = \int_{\partial\omega^\star} dH^1 \geq \int_{\partial\omega^\star} |\nabla u^\star| dH^1 = \int_{\partial K} |\nabla u^\star| dH^1, \tag{3.3}$$

where P means the perimeter and H^1 denotes the one dimensional Hausdorff measure. Since, by the maximum principle, $u^\star \leq u$ in $K \setminus \overline{\omega^\star}$ and $u^\star = u = 1$ on ∂K, we have $|\nabla u^\star| \geq |\nabla u|$ on ∂K. By (3.2) for any $\epsilon > 0$ there is $\delta_\epsilon \searrow 0$ such that $|\nabla u| > 1 - \delta_\epsilon$ in $\{u < \epsilon\}$. Hence (3.3) results in

$$P(\omega^\star) \geq \int_{\partial K} |\nabla u| dH^1 = \int_{\partial\{u<\epsilon\}} |\nabla u| dH^1 \geq (1 - \delta_\epsilon) P(\partial\{u < \epsilon\}).$$

Letting $\epsilon \to 0$ and using the lower-semicontinuity of the variation measure (perimeter) along with the fact that $\chi_{\{u<\epsilon\}} \to \chi_\omega$ in $L^1(K)$ (see the remark below) we obtain

$$P(\omega^\star) \geq P(\omega).$$

This is a contradiction in 2-dimensions for connected domains.

Remark:
According to (3.2) for ϵ small, $|\nabla u|^2 \geq 1/2$ on $\{0 < u < \epsilon\}$. Hence

$$\int_{\{0<u<\epsilon\}} dx \leq 2 \int_{\{0<u<\epsilon\}} |\nabla u|^2 = 2\epsilon \int_{\{u=\epsilon\}} |\nabla u| = 2\epsilon \int_{\{u=1/2\}} |\nabla u| \leq C\epsilon,$$

where C is independent of ϵ. This shows that $\chi_{\{u<\epsilon\}} \to \chi_\omega$ in $L^1(K)$.

Let us, now, consider the N-dimensional case ($N \geq 2$).

Theorem 3.3: Assume that ω is an open subset in R^N of class C^2 which is a solution to (3.1), (3.2) with K convex. Then the mean curvature of $\partial\omega$ is positive.

Proof: Since $u^\star \leq u$ on $K \setminus \overline{\omega^\star}$ (with $u^\star = u = 1$ on ∂K) we'll have $|\nabla u^\star| \geq |\nabla u|$ on ∂K. Hence, by Lemma 3.1

$$|\nabla u| \leq 1 \qquad \text{on } \partial K. \tag{3.4}$$

Set $v = |\nabla u|^2$. Then v is subharmonic in $K \setminus \overline{\omega}$ (where u is harmonic), $v = 1$ on $\partial\omega$ (by the assumption) and $v \leq 1$ on ∂K (by (3.4)). Now by the maximum principle v takes its maximum on the boundary and hence on $\partial\omega$. Then, by the Hopf's maximum principle, the inward normal derivative ($\frac{\partial}{\partial n}$) at each boundary point of ω must be negative, i.e.

$$\frac{\partial v}{\partial n} = 2\frac{\partial u}{\partial n}\frac{\partial^2 u}{\partial n^2} = 2\frac{\partial^2 u}{\partial n^2} < 0 \qquad \text{on } \partial\omega. \tag{3.5}$$

Using (3.2), (3.5) and the following well-known formula (see [Sp, page 62])

$$\Delta u = \Delta_s u + (N-1)H\frac{\partial u}{\partial n} + \frac{\partial^2 u}{\partial n^2},$$

where Δ_s is the induced Laplacian on $\partial\omega$ and H the mean curvature, we obtain

$$0 = (N-1)H\frac{\partial u}{\partial n} + \frac{\partial^2 u}{\partial n^2} < (N-1)H.$$

This is the desired result.

Remark:
In the same way as in the exterior case (see the remark following the proof of Theorem 2.1), we could prove here the existence of a convex solution to (3.1), (3.2) when K is convex, provided there already exists a solution ω or if the condition $u = 1$ is replaced by $u = \lambda$ for λ small enough. This can be done by taking the convex hull of ω (to begin with) and using Lemma 3.1 to show that the set (supersolutions)

$$\mathcal{M} := \{\text{subdomains of } K \text{ solving (3.1) with } |\nabla u| \leq 1 \text{ on } \partial\omega\ \},$$

is nonempty. Now the convex hull of the union of any two members of $\mathcal{M}$ is again in $\mathcal{M}$ (by Lemma 3.1). Hence we can construct a maximal convex domain ω in K with the property that the solution to (3.1) satisfies $|\nabla u| \leq 1$ in the usual limit sense. Now if at any point $z \in \partial\omega$ there were a strict inequality we could take a small ball B centered at z and such that $\omega \cup B$ is in $\mathcal{M}$ (a consequence of Schauder type estimates). Hence by Lemma 3.1 the convex hull of $\omega \cup B$ is in $\mathcal{M}$, contradicting the maximality of ω. Therefore $|\nabla u| = 1$ on $\partial\omega$ and u solves our problem.

4 Nonlinear equations

In this section we extend the above results and proofs to some nonlinear elliptic operators. We have in mind a general nonlinear operator of the form

$$Lu = F(x, u, \nabla u, \nabla^2 u)$$

The exterior case: Here we make the following assumptions on L:

(H1) L is weakly elliptic and then satisfies the comparison principle: if u_1 and u_2 are such that $Lu_1 \geq Lu_2$ in ω and $u_1 \leq u_2$ on $\partial\omega$ then $u_1 \leq u_2$ in ω.

(H2) The action of the group of direct similitudes on L is as followed: let u be a solution of $Lu = 0$ in ω, then

- for any homothety H_λ of (positive) ratio λ, we have

$$L(u \circ H_\lambda) \geq 0 \ \text{ if } \lambda \geq 1$$

$$L(u \circ H_\lambda) \leq 0 \ \text{ if } \lambda \leq 1$$

- for any displacement R (rotation and translation of R^N)

$$L(u \circ R) \leq 0$$

(H3) Let φ_1 denotes the first coordinate linear form ($\varphi_1(x) = x_1$), then $L(\varphi_1) \leq 0$.

(H4) Let $K \subset \Omega$ be two convex subsets in R^N and u the (nonlinear) capacitary potential associated to K and Ω, that is to say the solution of the problem

$$\begin{cases} Lu = 0 & \text{in } \Omega \setminus \overline{K} \\ u = 1 & \text{on } K \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

then we assume that the level sets of u are convex.

Remark: Obviously (H2) and (H3) imply that $L(h) \leq 0$ for every affine non expansive form (i.e. a function $h(x) = c + a.x$ where $c \in R$ and $a \in R^N, |a| \leq 1$)

Examples:
The assumptions (H1),(H2),(H3) and (H4) are, in particular satisfied by the p-Laplace operator $Lu = \Delta_p u = \nabla \cdot (|\nabla u|^{p-2} \nabla u)$ (the property (H4) is proved in [Lew]), or more generally for the operator

$$Lu = \Delta_p u - f(u)$$

where f is a continuous, non-decreasing function on R such that $f(x) \geq 0$ for all x (for this case the property (H4) is proved in [D-K] and partly in [C-S]).

Let us state now the convexity result in this case

Theorem 4.1: Let K be a convex domain in R^N such that there exists Ω a lipschitz domain containing K and a function u solution of

$$\begin{cases} Lu = 0 & \text{in } \Omega \setminus \overline{K} \\ u = 1 & \text{on } K \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

and

$$\liminf_{\substack{y \to x \\ y \in \Omega}} |\nabla u(y)| = \limsup_{\substack{y \to x \\ y \in \Omega}} |\nabla u(y)| = 1 \ \text{ for every } x \in \partial\Omega.$$

Then Ω is convex (and unique).

Remark: The assumption of regularity on Ω (Ω is lipschitz) can be removed if we assume, as in the proof of the theorem 2.1 that there exists a sequence $z_k \subset \Omega$ such that $z_k \to z$ and (2.4) is satisfied (what is true if Ω is lipschitz).

The proof of theorem 4.1 is almost the same than the one of theorem 2.1, let us just give now the changes:
according to (H1) and (H2) (homothety with ratio greater than 1), the Lavrentev principle applies and lead to the same conclusion. To prove the assertion, we need to choose another auxiliary function. Let us set (with the previous notations)

$$h(x) = \max(s, az_1) - ax_1$$

according to (H2) and (H3), we have

$$0 = Lu^\star \geq Lh \ \text{ in } \Omega^\star \cap \{x_1 > 0\}$$

and since $u^\star \leq h$ on $\partial(\Omega^\star \cap \{x_1 > 0\})$ by comparison principle, we have $u^\star \leq h$ on $\Omega^\star \cap \{x_1 > 0\}$ (we have also $u \leq u^\star$ in the same domain). Now if $az_1 \geq s$ then $h(z) = u^\star(z) = 0$. Hence as in (2.5) we obtain $a \geq 1$ which is a contradiction. Therefore $az_1 < s$. Since $h(0) = u^\star(0) = s$, we have

$$|\nabla u^\star(0)| \geq |\nabla h(0)| = a,$$

for all $a \in (0,1)$. This proves the assertion. Here we have used a rotation and translation as in the proof of the assertion. This is justified by assumption (H2).

The Interior case: In this case, if we assume the operator $Lu = F(x, u, \nabla u, \nabla^2 u)$ to satisfy (H1),(H2) (but the case of homothety with ratio greater than 1 since we would not to apply Lavrentev principle),(H3) and (H4), we are still able to prove Lemma 3.1. Moreover, we do not need any regularity assumption on the domain ω since, in the interior case the above remark is automatically satisfied. Now, we can prove an analogous of Theorem 3.2, for example in the case of the p-Laplace operator $L = \Delta_p$ because the Green's formula

$$\int_{\partial K} |\nabla u|^{p-1} dH^1 = \int_{\partial \omega^\star} |\nabla u|^{p-1} dH^1$$

remains valid.

References

[Ack84] A. ACKER, On the geometric form of free boundaries satisfying a Bernoulli condition I, *Math. Methods Appl. Sci.*, **6** (1984), 449-456

[Ack86] A. ACKER, On the geometric form of free boundaries satisfying a Bernoulli condition II, *Math. Methods Appl. Sci.*, **8** (1986), 387-404.

[Ack89-1] A. ACKER, Uniqueness and monotonicity of solutions for the interior Bernoulli free boundary problem in the convex, n-dimensional case, *Nonlinear Analysis, T.M.A.*, vol.13 (1989), 1409-1425.

[Ack89-2] A. ACKER, On the convexity and on the successive approximation of solutions in a free boundary problem with two fluid phases, *Comm. Part. Diff. Eq.*, **14** (1989), 1635-1652.

[A-C] H. W. ALT, L. A. CAFFARELLI, Existence and regularity for a minimum problem with free boundary, *J. Reine angew. Math.*, **325** (1981), 105-144.

[Beur] A. BEURLING, On free boundary problems for the Laplace equation, Seminars on analytic functions I, Institute Advance Studies Seminars (1957) Princeton 248-263.

[C] L.A. CAFFARELLI, Compactness methods in free boundary problems, *Comm. Part. Diff. Eq.* **5** (1980), 427–448.

[C-S] L.A. CAFFARELLI, J. SPRUCK, Convexity properties of solutions of some classical variational problems, *Comm. Part. Diff. Eq.*, **7** (1982), 1337-1379.

[Dan] I.I. DANILJUK, On integral functionals with a variable domain of integration, *Proc. Steklov Inst. Math.* , **118** (1972) (English translation, AMS 1976)

[D-K] J.I. DIAZ, B. KAWOHL, On convexity and starshapedness of level sets for some nonlinear elliptic and parabolic problems on convex rings, *J.Math. Anal.Appl.*, **177** (1993), 263-286.

[Fas] A. FASANO, Some free boundary problems with industrial applications, in Shape Optimization and Free Boundaries, M. Delfour ed., NATO ASI Series, vol. 380, Kluwer 1992.

[F] A. FRIEDMAN, Variational principles and free boundary problems, Wiley, New York, 1982.

[Ham] R. S. HAMILTON, The inverse function theorems of Nash and Moser, *Bull. Amer. Math. Soc.* 7 (1982), 65-222.

[H-S] A. HENROT, D. SECK, Back to an old approach for a classical free boundary problem, to appear.

[Kaw85] B. KAWOHL, Rearrangements and convexity of level sets in PDE, Springer Lecture Notes in Math., **1150** (1985).

[Kaw86] B. KAWOHL, Some qualitative properties of nonlinear partial differential equations, in Proceedings of the MSRI microprogram on nonlinear diffusion equations and their equilibrium states, W.M. Ni, J. Serrin and L. Peletier eds, Berkeley (1986).

[Lan] K.E. LANCASTER, Qualitative behavior of solutions of elliptic free boundary problems, *Pacific J. of Math.*, **154** (1992), 297-316.

[Lav] M.A. LAVRENT'EV, Variational Methods, Noordhof, Groningen, 1963.

[Lew] J.L. LEWIS, Capacitary functions in convex rings, *Arch. Rat. Mech. Anal.*, **66** (1977), 201-224.

[Shah] H. SHAHGHOLIAN, Existence of quadrature surfaces for positive measures with finite support, Potential Anal. 3 (1994), 245–255.

[Sp] R. SPERB, Maximum Principles and Their Applications, Academic Press, New York, 1981.

[Tep] D. E. TEPPER, Free boundary problem, *SIAM J. Math. Anal.,*, **5** (1974), 841-846.