ON PHRAGMEN LINDELÖF PRINCIPLE

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Abstract. The well-known maximum principle asserts that if $u(z)$ is a subharmonic function in the bounded domain D and

$$\limsup_{D\ni z\to y} u(z) \leq 1$$

for every point $y \in \partial D$, then $u(z) \leq 1$ everywhere in D. In this paper we prove that the same conclusion holds under weaker asumptions than the above.

1. Introduction

The well-known maximum principle asserts that if $u(z)$ is a subharmonic function on the bounded domain D and

$$\limsup_{D\ni z\to y} u(z) \leq 1 \tag{1}$$

for every point $y \in \partial D$, then $u(z) \leq 1$ everywhere in D.

In this paper we consider the problem of weakening condition (1). Our first main result is the following.

Theorem 1. Let $G(r) > 1, \quad 0 \leq r < 1$, be continuous, increasing function such that for some $M < \infty$ we have

$$G\left(\frac{1+r}{2}\right) \leq MG(r), \quad 0 \leq r < 1$$

and

$$\int_0^1 G(r)dr < \infty.$$

Let $u(z)$ be a harmonic function defined on the unit disk D and

$$u(z) \leq \exp\left\{\exp\left\{G(|z|)\right\}\right\}.$$

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If there is a number $\varepsilon > 0$ such that for arbitrary $e^{ix} \in \partial D$ the inequality

$$\liminf_{r\to 1^-} \left(\max\{u(re^{iy}), |x-y| \leq \varepsilon\}\right) \leq 1 \tag{2}$$

holds, then everywhere on the unit disk D we have $u(z) \leq 1$.

2. The two dimensional case

In the proof of Theorem 1 we need the following classical result, see [3], p.77-80.

Lemma 1. Let $C(x), \quad x \geq 0$, be an increasing, continuously differentiable function, $C(0) = 0, \quad C'(x)x \leq MC(x), \quad 0 < x < \infty$, and

$$\int_1^\infty \frac{C(x)}{x^2} dx < \infty.$$

Then for an arbitrary positive number a there is a nonnegative function $f(x) \not\equiv 0$, which vanishes outside the interval $(-a, a)$ and

$$|\hat{f}(x)| \leq \exp\left(-C(|x|)\right), \quad x \in (-\infty, \infty),$$

where $\hat{f}(x)$ is the Fourier transform of the function $f(x)$.

Proof of Theorem 1. Without loss of generality we can assume that $G(r) \geq (1-r)^{-\varepsilon_0}$ $(0 < r < 1)$ for some $\varepsilon_0 > 0$.

Let us define the function $C(x)$ as follows:
$C(0) = 0$,
$C'(x) = 2G(1/2)$ for $0 \leq x \leq 1$,
$C'(x)x = 2G(1 - 1/(2x))$ for $x \geq 1$.
It is easy to check that $C'(x)x = C(x)$ for $0 \leq x \leq 1$ and

$$C'(x)x \leq 2MG(1-1/x) \leq \frac{2M}{\log 2}\int_{x/2}^{x} \frac{G(1-1/2t)}{t} dt \leq 2M\frac{C(x)}{\log 2}$$

for $x \geq 1$. Let us note that $C(x) \geq x^{\varepsilon_0}, \quad x > 1$. Now let $z = re^{ix}, \quad 1/2 < r < \rho < 1$. Then we have

$$u(z) + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\rho^2 - r^2}{|\rho e^{it} - z|^2} u^-(\rho e^{it}) dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\rho^2 - r^2}{|\rho e^{it} - z|^2} u^+(\rho e^{it}) dt,$$

where $u^+ = \max\{u, 0\}, \quad u^- = \max\{-u, 0\}$.

Consequently

$$0 \leq u(z) + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\rho^2 - r^2}{|\rho e^{it} - z|^2} u^-(\rho e^{it}) dt \leq u(z) + \frac{\rho + r}{2\pi(\rho - r)}\int_{-\pi}^{\pi} u^-(\rho e^{it}) dt =$$

$$= u(z) + \frac{\rho + r}{\rho - r}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} u^{+}(\rho e^{it})dt - u(0)\right) \leq$$

$$\leq u(z) + \frac{\rho + r}{\rho - r}\left(\exp\left\{\exp\left\{C\left(\frac{1}{(1-\rho)}\right)\right\}\right\} - u(0)\right)$$

since

$$G(1 - 1/x)\log 2 \leq \int_{x/2}^{x} \frac{G(1 - 1/2t)}{t} dt \leq C(x), \quad x > 1.$$

Choosing $\rho = \frac{1+r}{2}$ we obtain

$$|u(z)| \leq \frac{4}{1-r}\exp\left\{\exp\left\{C\left(\frac{2}{(1-|z|)}\right)\right\}\right\}, \tag{3}$$

where $z \in D$.

Any harmonic function $u(z)$ defined in the unit disc D can be represented in the form

$$u(z) = \sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{inx}, \quad z = re^{ix}.$$

For the coefficients a_n we have

$$|a_n| \leq r^{-|n|} \max_{|x| \leq \pi} |u(re^{ix})|, \quad 0 < r < 1.$$

Hence for r near 1

$$\log|a_n| \leq |n| \ln\frac{1}{r} + \log\frac{4}{1-r} + \exp\left\{C\left(\frac{2}{(1-r)}\right)\right\} \leq$$

$$\leq |n|\frac{(1-r)}{r} + \log\frac{4}{1-r} + \exp\left\{C\left(\frac{2}{(1-r)}\right)\right\}.$$

For large $|n|$ let us set

$$r_n = 1 - \frac{2}{C^{-1}(\log C(n))}.$$

Then we have

$$\log|a_n| \leq \frac{4|n|}{C^{-1}(\log C(n))} + C(n) + \log\left(2C^{-1}(\log C(n))\right), \tag{4}$$

where $|n| > n_0$ and $r_{n_0} > 0.5$. From here it follows that

$$\sum_{n=n_0}^{\infty} \frac{\log|a_n|}{n^2} \leq \int_1^{\infty} \frac{4dx}{xC^{-1}(\log C(x))} +$$

$$+ \int_1^{\infty} \frac{C(x)}{x^2}dx + \int_1^{\infty} \frac{\log 2C^{-1}(\log C(x))}{x^2}dx \leq$$

$$\leq \int_1^{x_0} \frac{2dx}{xC^{-1}(\log C(x))} + \int_{x_0}^{\infty} \frac{2dx}{xC^{-1}(\varepsilon_0 \log x)} +$$
$$+ \int_1^{\infty} \frac{C(x)}{x^2} dx + \int_1^{\infty} \frac{\log 2C^{-1}(\log C(x))}{x^2} dx,$$

where x_0 is a large positive number.

We observe that

$$\frac{C(x)}{x} \leq C(x) \int_x^{\infty} \frac{dt}{t^2} \leq \int_x^{\infty} \frac{C(t)}{t^2} dt.$$

Also $C(x)$ being an increasing function gives the inequality

$$\log \left(2C^{-1}(\log C(x)) \right) \leq \log 2x, \quad 1 \leq x.$$

Consequently,

$$\sum_{n=2}^{\infty} \frac{\log |a_n|}{n^2} \leq B + \int_{x_0}^{\infty} \frac{2dx}{xC^{-1}(\varepsilon_0 \log x)} +$$
$$+ \int_1^{\infty} \frac{C(x)}{x^2} dx + \int_1^{\infty} \frac{\log(2x)}{x^2} dx \leq$$
$$\leq B + \frac{2}{\varepsilon_0} \int_1^{\infty} \frac{C'(y)}{y} dy + \int_1^{\infty} \frac{C(x)}{x^2} dx + \int_1^{\infty} \frac{\log(2x)}{x^2} dx < \infty \ ,$$

where in the first integral we change the variable x to y by formula $C(y) = \varepsilon_0 \log x$ and x_0, y_0 are positive numbers for which $C(y_0) = \varepsilon_0 \log x_0$.

Let $0 < \delta < \varepsilon$ be a fixed number. By Lemma 1 there is a nonnegative function $\varphi_\delta(x)$ with support in the interval $(-\delta, \delta)$ and such that

$$\frac{1}{2\pi} \int_{-\delta}^{\delta} \varphi_\delta(x) dx = 1,$$

and

$$|c_n| \leq \exp \left\{ -2 \left(C(n) + \frac{|n|}{C^{-1}(\log C(n))} + 3 \log n \right) \right\},$$

where $\{c_n\}$ are Fourier coefficients of the function $\varphi_\delta(x)$.

In particular by (4) we have

$$\sum_{n=-\infty}^{\infty} |a_n||c_n| < \infty.$$

Let us consider the function

$$u_\delta(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(ze^{-ix}) \varphi_\delta(x) dx.$$

This function is harmonic and has continuous continuation up to the boundary of D. From conditions in Theorem 1 it follows that for arbitrary point $w \in \partial D$ there is a sequence $z_n, \quad n = 1, 2, \ldots$ from D tending to w and such that

$$\lim_{D \ni z_n \to w} u_\delta(z_n) \leq 1.$$

Since $u_\delta(z)$ is continuous up to the boundary we'll have,

$$\lim_{z \to \partial D} u(z) \leq 1.$$

By the maximum principle $u_\delta(z) \leq 1$ for $z \in D$. Hence

$$u(z) = \lim_{\delta \to 0} u_\delta(z) \leq 1.$$

Theorem 2. There is an analytic function defined on the unit disk for which

$$|f(z)| \leq \exp\left\{\exp\left\{\frac{C}{(1-|z|)}\right\}\right\}, \quad |z| < 1,$$

and for some $\delta > 0$ and for every $e^{iy} \in \partial D$ the inequality

$$\liminf_{r \to 1-0} \left(\max_{|t| \leq \delta} |f(re^{i(y+t)})| \right) \leq 1,$$

holds, however $f(z)$ is an unbounded function.

Proof. Let us consider the function

$$w(z) = z^i \exp\left\{-\frac{1}{z}\right\},$$

defined on the strip

$$S = \{z = x + iy; \quad a < x, \quad |y| < \delta\}$$

where $a > 1$ is a fixed number and $0 < \delta < 1$. We have

$$1 - |w(z)| = 1 - \exp\left(-\arctan\frac{y}{x} - \frac{x}{x^2 + y^2}\right) \leq \frac{C}{|z|}.$$

For large x we have

$$w(x + iy) = \exp\left\{-\frac{1+y}{x} + i\log x + O\left(\frac{1}{x^2}\right)\right\}.$$

If $x_1 < x_2$ and

$$w(x_1 + iy_1) = w(x_2 + iy_2)$$

then

$$-\frac{1+y_1}{x_1} + i\log x_1 = -\frac{1+y_2}{x_2} + i(\log x_2 - 2\pi n) + O\left(\frac{1}{x_1^2}\right),$$

for some natural number n. Moreover, we have

$$\frac{1+y_1}{x_1}+O\left(\frac{1}{x_1^2}\right)=\frac{1+y_2}{x_2}<\frac{1+\delta}{x_2}<\frac{1+y_1}{x_1}\left(\frac{1+\delta}{1-\delta}\right)\frac{x_1}{x_2}.$$

Hence

$$\frac{1+y_1}{x_1}\left(1-\left(\frac{1+\delta}{1-\delta}\right)\frac{x_1}{x_2}\right)<O\left(\frac{1}{x_1^2}\right).$$

Consequently

$$1-\delta-(1+\delta)e^{-2\pi}<O\left(\frac{1}{x_1}\right).$$

If the inequality $1+\delta<(1-\delta)e^{2\pi}$ holds we get the contradiction for large a.

This contradiction shows that $w(z)$, for $z\in S$ is a one-to-one map from S into the unit disk, provided a is large enough.

Let us denote by $\Omega=\{w;\quad w=w(z),\quad z\in S\}$. For $w\in\Omega$ let $z=z(w)$ denote the inverse to $w=w(z)$. Now set $\varphi(w)=\frac{\pi}{2\delta}$ and

$$f(z)=\frac{1}{2\pi i}\int_E\frac{\exp\{\exp\{\varphi(w)\}\}}{(w-z)\varphi^2(w)}dw \tag{5}$$

where $E=\partial\Omega\cap\{w;|w|<1\}$. This function is defined for arbitrary $w\in D\setminus\bar{\Omega}$. For each x_0 with $0<x_0<1$, let us set

$$\Omega_{x_0}=\{w\in\Omega,\quad x_0<|w|\}.$$

Using Lindelöf method, (see [5], p. 144), we prove that $f(z)$ permits an analytic continuation onto the unit disk D.

First note that the following representation

$$f(z)=\frac{1}{2\pi i}\int_{E_{x_0}}\frac{\exp\{\exp\{\varphi(w)\}\}}{(w-z)\varphi^2(w)}dw$$

holds, where $E_{x_0}=\partial\Omega_{x_0}\cap\{w;|w|<1\}$. Consequently, the function $f(z)$ permits an analytic continuation on the unit disk. Moreover for $\frac{1}{2}<|z|<1$ there holds

$$|f(z)|\le\frac{1}{2\pi}\int_{E_{x_0}}\frac{\exp\{Re(\exp\{\varphi(w)\})\}}{|w-z||\varphi(w)|^2}|dw|+$$

$$\le\exp\left\{\exp\left\{C\left(\frac{\pi}{2\delta(1-|z|)}\right)\right\}\right\}.$$

Let us also note that the function $f(z)$ is bounded on the domain

$$D\setminus\bar{S}.$$

By the formula (5) we define two functions: one defined in S and the second defined on $D \setminus \bar{S}$. Let us denote

$$f_-(w) = f(w), \quad w \in S, \qquad f_+(w) = f(w), \quad w \in D \setminus \bar{S}.$$

For arbitrary $z \in E$ we have

$$\lim_{S \ni w \to z} f_-(w) - \lim_{D \setminus S \ni w \to z} f_+(w) = \frac{\exp\{\exp\{\varphi(z)\}\}}{\varphi^2(z)}.$$

Since $f_-(w)$ is bounded in S, the analytic continuation of the function $f_+(w)$ cannot be bounded.

3. The higher dimensional case and comparison with the classical results

Let us note that the Theorem 1 for higher dimensions is true. In this section we let $x \in \mathbf{R}^n$ with $x = (x_1, \cdots, x_n)$. Now we formulate our first theorem in this section.

Theorem 3. Let $G(r) > 1$, with $0 \le r < 1$, be increasing function and

$$G\left(\frac{1+r}{2}\right) \le MG(r), \quad 0 \le r < 1,$$

and

$$\int_0^1 G(r)dr < \infty.$$

Let $u(z)$ be a harmonic function defined on the unit ball $D \in \mathbf{R}^n$ $(n \ge 2)$ with the following property

$$u(z) \le \exp\left\{\exp\left\{G(|z|)\right\}\right\}.$$

If there is a number $\varepsilon > 0$ such that for arbitrary $y \in \partial D$ the inequality

$$\liminf_{r \to 1^-} \left(\max\{u(rz) : \ |z-y| \le \varepsilon, \quad z \in \partial D\}\right) \le 1$$

holds, then everywhere on the unit ball D we have $u(z) \le 1$.

Proof. In the general case the estimate (3) for the modulus of $u(z)$ similar to the 2-dimensional case can be proved, using integral representation.

Next we use the representation (see [4], corollary 5.23),

$$u(x) = \sum_{m=0}^{\infty} p_m(x)$$

for all $|x| < 1$, where $p_m(x)$ is a homogeneous harmonic polynomial of degree m and the series converges absolutely on compact subsets of the unit ball. Moreover, we have

$$|p_m(x)| \leq Cm^{n-2}|x|^m \int_{|y|=1} |u(y)|d\sigma(y)$$

where $d\sigma(y)$ is the surface area element.

At last, instead of the function $\varphi_\delta(t)$, with $t \in \mathbf{R}$, we take the function $c_n\varphi_\delta(|x|), \quad x \in \mathbf{R}^n$, where the number c_n is chosen such that

$$c_n \int_{\mathbf{R}^n} \varphi_\delta(|x|)dx = 1. \qquad \square$$

In Theorem 1 we put very specific condition on the set in D, where we put the condition (2). Now we present some other conditions.

Definition. Let D be unit disk, E a closed subset in D and $\sigma > 0$ a constant. We say that E is σ-connected if for arbitrary continuous curve l in $D \setminus E$ the inequality

$$|l| \geq C\frac{\max_{z\in l}|z| - \min_{z\in l}|z|}{(1-\max_{z\in l}|z|)^\sigma}$$

is valid, where $|l|$ is the length of the curve l and $C > 0$ is a fixed constant.

For the next theorem we need a result due to A. Beurling's (see [2], p.377).

Theorem 4. Let D be a simply connected domain having finite total area and let z_0 be a fixed point in D. For $r > 0$ denote by D_r the subdomain of D consisting of points that can be connected to z_0 by a curve in D of length $\leq r$. Let $f(z)$ be an analytic function on D with

$$\limsup_{z\to w} |f(w)| \leq 1, \quad w \in \partial D.$$

Then either

$$|f(z)| \leq 1, \quad z \in D,$$

or

$$\lim_{r\to\infty} \frac{1}{r^2} \sup_{D_r}\{\log^+ \log |f(z)|\} = \infty.$$

Theorem 5. Let $u(z)$ be harmonic function defined on the unit disk D and σ be positive number. Let E be closed subset in D and $D \setminus E$

be $\frac{\sigma}{2}$ - connected. If

$$u(z) \leq \exp\left\{\frac{A}{(1-|z|)^{\sigma}}\right\}, \quad z \in D$$

and

$$\limsup_{z\in E, |z|\to 1} u(z) \leq 1,$$

then $u(z) \leq 1$ for all $z \in D$.

Proof. Let us assume that the theorem is not true. Then the function $u(z)$ must be unbounded. So, in one of the connected components of the open set $D \setminus E$, (let us denote it by Ω), the function $u(z)$ must be unbounded. Using Theorem 4 for the function $f(z) = \exp(u(z)+iv(z))$, where $v(z)$ is complex conjugate to $u(z)$, we get

$$\limsup_{r\to\infty} \frac{1}{r^2} \max_{z\in\Omega_r} \log^+ u(z) = \infty.$$

This inequality contradicts the upper bound estimate for $u(z)$.

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