HAUSDORFF DIMENSION AND STABILITY FOR THE p-OBSTACLE PROBLEM $(2 < p < \infty)$

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Abstract. In this paper we analyze the free boundary for the inhomogeneous obstacle problem with zero obstacle governed by the degenerate operator

$$\operatorname{div}(\mathcal{A}(\nabla u)) = f\chi_{\{u>0\}},$$

where f is a positive, Lipschitz function, and $\mathcal{A}(\nabla u)$ is of the p-Laplacian type, i.e.,

$$\mathcal{A}(\nabla u) \approx |\nabla u|^{p-2}\nabla u, \qquad (2 < p < \infty).$$

As a result of our analysis we obtain stability and finite $(N-1)$-Hausdorff dimension of the free boundary. Our technique is a modified version of that of L. Caffarelli, who obtained similar results for $p = 2$.

1. INTRODUCTION

The obstacle problem, describing the equilibrium position of a stretched membrane, consisting of homogeneous material, is a well-known and well-studied problem. A particular form of this problem can be formulated as to find a function u which solves the following complementary problem

$$\begin{cases} \Delta u - f \le 0, \\ u \ge \psi, \\ (\Delta u - f)(u - \psi) = 0, \end{cases}$$

a.e. in Ω, and with the Dirichlet condition $u - g \in H_0^1(\Omega)$. Here f, ψ, and g are given functions with certain properties, and Ω is a bounded domain in $\mathbb{R}^N$.

This problem has been the subject of investigation in several decades. As a result, today, there is a complete theory available for this problem. The regularity of the solution-function u and the free boundary $\partial\{u > \psi\}$ is completely described and known, due to works of many

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mathematicians; here we particularly mention the work of L. Caffarelli and D. Kinderlehrer [CK], for $C^{1,1}$ regularity of u, and also the work of L. Caffarelli [C1], for the regularity of the free boundary $\partial\{u > \psi\}$. However, a particular work that has been less in focus is a short note by L. Caffarelli [C2] which asserts that the free boundary in the above-mentioned obstacle problem has finite $(N-1)$-Hausdorff dimension, provided f is a positive Lipschitz function, and $\psi = 0$.

Although the case of the Laplacian operator, above, is a good model for describing variational problems with constraints, that appear in nature, there are many other nonlinear phenomena of the obstacle type that can't be linearized and/or approximated by linear models. One such problem arises when the material density (or any other physical quantity involved) is inhomogeneous. The problem becomes more intrigue when the inhomogeneity is such that the operator becomes degenerate; the singular case is not treated in this paper. The specific type of operator we have in mind, in this paper, is the following

$$\operatorname{div}(\mathcal{A}(\nabla u)) = f\chi_{\{u>0\}}, \tag{1.1}$$

where f is a positive Lipschitz function, and $\mathcal{A}(\nabla u)$ is of the p-Laplacian type, i.e.,

$$\mathcal{A}(\nabla u) \approx |\nabla u|^{p-2}\nabla u, \qquad (2 < p < \infty).$$

We refer to the book [HKM] for backgrounds and the type of operators that can be treated with our techniques. Basically, we need the operator to satisfy a strong maximum principle (this is for Lemma 2.1 below), and a uniform $C^{1,\alpha}$ estimate (this is for the compactness arguments used in the analysis). Two more properties that are needed are the homogeneity

$$\mathcal{A}(\lambda\nabla u) = \lambda^{p-1}\mathcal{A}(\nabla u), \qquad (\text{for } \lambda > 0),$$

and the differentiability of $\mathcal{A}$. However, for clarity and simplicity we'll only treat the case

$$\mathcal{A}(\nabla u) = |\nabla u|^{p-2}\nabla u, \qquad (2 < p < \infty),$$

and

$$f(x) \equiv 1.$$

The general case follows in a similar fashion and with obvious changes.

A fundamental problem that arises with the study of the obstacle problem for degenerate operators as in (1.1) is the optimal regularity of the solution u. The solutions are known to be $C^{1,\alpha}$, for some $\alpha > 0$, but the exact value for α is unknown. Of course the example

$$u(x) = c(\max(x_1, 0))^{\frac{p}{p-1}},$$

for appropriate c, shows that the best expected α can't exceed $1/(p-1)$. In [KKPS], the authors obtained the correct growth rate for u away from the free boundary, and as expected it is of order $\frac{p}{p-1}$.

Finally, with the above remark in mind, we describe the results obtained in this paper. To do so, we'll consider the following (local) formulation of the obstacle problem.

Definition 1.1. *We say that a function u in $W^{1,p}(B_1)$, where $B_1 = B_1(0)$ is the unit ball in $\mathbb{R}^N$, belongs to the class $\mathcal{G} = \mathcal{G}(p,N)$ ($2 < p < \infty$) if*

$$(\mathbf{FB}) \qquad \begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \chi_{\{u>0\}}, & \text{in } B_1 \\ 0 \le u \le 1 & \text{in } B_1, \\ 0 \in \partial\{u > 0\}\ . \end{cases}$$

Here the first equation is in the weak sense

$$\int |\nabla u|^{p-2}\nabla u \cdot \nabla\phi \, dx = -\int \phi\chi_{\{u>0\}}\, dx$$

for all $\phi \in W^{1,p}_o(B_1)$, were the latter is the completion of $C^\infty_o(B_1)$-functions in the $W^{1,p}$-norm. It will be apparent from the proofs, presented here, that we can replace B_1 in (FB) with any bounded domain Ω by using a finite covering of the set

$$\Omega_{-\epsilon} := \{x \in \Omega\ : \operatorname{dist}(x, \partial\Omega > \epsilon\}\,.$$

It is also noteworthy that our approach works perfectly for the case $p = 2$. But for clarity of the argument we exclude this case; which is already proved by L. Caffarelli [C2].

With (FB) as our departing point we'll show that the volume of the set $\{u < \epsilon^{\frac{p}{p-1}}\} \cap B_r(x)$, for any x on the free boundary, can be estimated in terms of $\epsilon\mathcal{H}^{N-1}(\partial B_r(x))$, where $\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure. From here a standard covering argument implies that the free boundary has finite $(N-1)$-Hausdorff measure. A second implication is the stability of the coincidence set $\{u = 0\}$. Namely, we show that the L^∞-norm of two solutions in $\mathcal{G}$ will control the measure theoretical difference of their coincidence sets.

2. Growth rates

In this section, we establish the correct growth rates for $u(x)$, and $\nabla u(x)$, away from the free boundary. We also show that

$$\frac{1}{|B_r(0)|}\int_{B_r(0)} [|\nabla u(x)|^{p-2}|D^2u(x)|]^2 dx \le M,$$

which together with a non-degeneracy argument (Lemma 2.5) will pave the way for the estimate of the volume of the set $\{u \le \epsilon^{\frac{p}{p-1}}\}$, later in Section 3.

We will employ standard homogeneous blow-ups of the solution that was introduced in [KS], and later developed in [KKPS]. Cf. also [CKS] and [LS]. Now, for a non-negative function v, we define

$$S(r, v, z) = \sup_{x \in B_r(z)} v(x), \qquad \text{and} \qquad S(r, v) = S(r, v, 0).$$

In [KKPS] it was shown that for functions u in the class $\mathcal{G}$ one has a growth rate

$$S(r, u) \le C_0 r^{\frac{p}{p-1}}.$$

Using a similar argument as in [KKPS], we will obtain a growth rate of order $1/(p-1)$ for the gradient of functions in $\mathcal{G}$. Indeed, we'll show that there exists a constant C_0 such that for all positive integers j one has

$$S(2^{-j-1}, |\nabla u|) \le \max\left(C_0 2^{-j(1/(p-1))}, 2^{-1/(p-1)} S(2^{-j}, |\nabla u|)\right).$$

By iteration then we'll have the actual growth rate.

Remark 2.1. We need to mention that capital letters such as M, C, C_0, C_1 are generic constants changing their value from one appearence to another. They all depend on the quantities p, and N.

Lemma 2.2. *[KKPS] There is a positive constant $M = M(p, N)$ such that for every $u \in \mathcal{G}$, there holds*

$$|u(x)| \le M|x|^{p/(p-1)} \qquad \forall x \in B_1. \tag{2.1}$$

Using this we can prove the growth rate for the gradient.

Lemma 2.3. *There is a positive constant $M = M(p, N)$ such that for every $u \in \mathcal{G}$, there holds*

$$|\nabla u(x)| \le M|x|^{\frac{1}{p-1}} \qquad \forall x \in B_1. \tag{2.2}$$

Proof. As remarked earlier it suffices to show

$$S(2^{-j-1}, |\nabla u|) \le \max\left(C_0 2^{-j(1/(p-1))}, 2^{-1/(p-1)} S(2^{-j}, |\nabla u|)\right), \tag{2.3}$$

for all positive integers j and a large constant $C_0 = C_0(p, N)$. To prove this we argue by contradiction. So suppose (2.3) fails. Then there exists $u_j \in \mathcal{G}$ such that

$$S(2^{-j-1}, |\nabla u_j|) \ge \max\left(j 2^{-j(1/(p-1))}, 2^{-1/(p-1)} S(2^{-j}, |\nabla u_j|)\right), \tag{2.4}$$

Set

$$\tilde{u}_j = \frac{u_j(2^{-j}x)}{2^{-j} S(2^{-j-1}, |\nabla u_j|)},$$

where $x \in B_1(0)$. Then by (2.1), and (2.4)

$$|\tilde{u}_j| \le \frac{M(2^{-j})^{\frac{p}{p-1}}}{2^{-j}S(2^{-j-1}, |\nabla u_j|)} \le \frac{M}{j}$$

in $B_1(0)$. On the other hand

$$\sup_{B_{\frac{1}{2}}} |\nabla \tilde{u}_j| = \sup_{B_{\frac{1}{2}}} \frac{|\nabla u(2^{-j}x)|}{S(2^{-j-1}, |\nabla u_j|)} = 1,$$

$$\sup_{B_1} |\nabla \tilde{u}_j| = \sup_{B_1} \frac{|\nabla u(2^{-j}x)|}{S(2^{-j-1}, |\nabla u_j|)} = \frac{S(2^{-j}, |\nabla u_j|)}{S(2^{-j-1}, |\nabla u_j|)} \le 2^{\frac{1}{p-1}}.$$

One also has

$$\|\Delta_p \tilde{u}_j\|_\infty \le j^{1-p}.$$

Hence the uniform C^1-estimate for the p-Laplacian implies that

$$1 = \sup_{B_{\frac{1}{2}}} |\nabla \tilde{u}_j| \le C(\sup_{B_1} \tilde{u}_j + j^{1-p}) \le C(j^{-1} + j^{1-p}),$$

which makes a contradiction for a large $j > 0$. □

A similar technique works in estimating the L^2-norm of

$$|\nabla u(x)|^{p-2}|D^2u(x)|,$$

which doesn't necessarily exist pointwise.

Lemma 2.4. *There is a positive constant $M = M(p, N)$ such that for every $u \in \mathcal{G}$, and $x_0 \in \partial\{u > 0\} \cap B_{1/2}$ there holds*

$$\frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} [|\nabla u(x)|^{p-2}|D^2u(x)|]^2 dx \le M \qquad \forall\ r \le \frac{1}{2}. \tag{2.5}$$

Proof. By translation and scaling we may assume x_0 is the origin. Set $\tilde{S}(r, u) = \int_{B_1(0)} [|\nabla u(rx)|^{p-2}|D^2u(rx)|]^2 dx$. Then it suffices to show that there exists a constant C_0 such that, for all non-negative integers k, there holds

$$\tilde{S}(2^{-k-1}, u) \le \max\left(C_0, \tilde{S}(2^{-k}, u)\right). \tag{2.6}$$

As in Lemma 2.2 the proof is based on a contradictory argument. So once again let's assume that there are $u_k \in \mathcal{G}$, and positive integers j_k such that (2.6) fails, i.e.,

$$\tilde{S}(2^{-j_k-1}, u_k) \ge \max\left(k, \tilde{S}(2^{-j_k}, u_k)\right). \tag{2.7}$$

Set now

$$\tilde{u}_k(x) = \frac{u_k(2^{-j_k}x)}{\tilde{S}(2^{-j_k-1}, u_k)},$$

where $x \in B_1(0)$. Then one easily verifies

$$\|\tilde{u}_k\|_{B_1,\infty} \leq \frac{M}{k}, \quad \tilde{S}(\frac{1}{2}, \tilde{u}_k) = 1, \quad \tilde{S}(1, \tilde{u}_k) \leq 1, \quad \|\Delta_p \tilde{u}_k\| \leq k^{1-p}.$$

Uniform C^1-estimates and L^2-bound of the second derivatives imply

$$\sup_{B_{\frac{1}{2}}} |\nabla \tilde{u}_k| \leq C(k^{-1} + k^{1-p}), \quad \text{and} \quad \int_{B_1(0)} |D^2 \tilde{u}_k(x)|^2 dx < C < \infty,$$

which, for large $k > 0$ and $p > 2$, results in the following contradiction

$$1 = \tilde{S}(\frac{1}{2}, \tilde{u}_k) < C(k^{-1} + k^{1-p})^{p-2} < 1. \qquad \square$$

We would like to remark that the final line of the proof of Lemma 2.3, and also later in Lemma 3.1, are the only places, in this paper, were the restriction $p > 2$ has been used. It seems that the generalization to the case below the value $p = 2$ needs a refinement the argument and most probably to use a different norm in Lemma 2.3. Note also that by known results $u \in W^{2,p}_{loc}$ when $p < 2$. This somehow suggests the change of the above-mentioned norm.

Our next lemma shows that the norm $\tilde{S}(r, u)$ is uniformly bounded away from zero.

Lemma 2.5. *(Non-degeneracy) For every $u \in \mathcal{G}$, there holds*

$$\frac{1}{(p-1)^2} \leq [|\nabla u(x)|^{p-2} |D^2 u(x)|]^2, \qquad a.e. \text{ in } \{u > 0\}. \tag{2.8}$$

Proof. We notice that the L^2-bound of $D^2 u$ implies the a.e. existence of $D^2 u$; see [T]. Hence

$$\begin{aligned}
1 &= \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} (\Delta_p u(x))^2 dx \\
&= \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} \left[\left(|\nabla u(x)|^{p-2} \delta_{ij} + (p-2) |\nabla u(x)|^{p-4} u_i(x) u_j(x) \right) u_{ij}(x) \right]^2 dx \\
&\leq \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} \left[|\nabla u(x)|^{p-2} |\triangle u| + (p-2) |\nabla u(x)|^{p-4} |u_i(x)| |u_j(x)| |u_{ij}(x)| \right]^2 dx \\
&\leq (p-1)^2 \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} [|\nabla u(x)|^{p-2} |D^2 u(x)|]^2 dx \ . \qquad \square
\end{aligned}$$

3. Hausdorff dimension and stability

In this section we will establish an estimate on $\{u(x) \leq \varepsilon^{\frac{p}{p-1}}\}$ based on the growth rates given in Section 2. For this purpose we let

$$O_\varepsilon = \{|\nabla u(x)| \leq \varepsilon^{\frac{1}{p-1}}\} \quad \text{and} \quad O^i_\varepsilon = \{|u_{x_i}(x)| \leq \varepsilon^{\frac{1}{p-1}}\}.$$

Denote also by $\mathcal{L}^N$ the N-dimensional Lebesgue measure. Then we have the following lemma.

To state our first lemma in this section we need to notify the reader about a general property shared by the solutions to the obstacle problem. Namely, if $x_0 \in \partial\{u > 0\} \cap B_{1-\epsilon}$ then there exists $y_0 \in \{u > 0\}$ and $c > 0$ ($c = c(N, p)$) such that

$$B_{c\epsilon}(y_0) \subset B_\epsilon(x_0) \cap O_\epsilon. \tag{3.1}$$

For $p = 2$ we refer to [C2], the general case follows in the same fasion.

Lemma 3.1. *For any ball $B_r(x_0) \subset B_1$ with $x_0 \in \partial\{u > 0\} \cap B_{1/2}$ and $r < 1/2$, there holds*

$$\int_0^1 \mathcal{L}^N(O_\varepsilon \cap B_{rs}(x_0)ds \le C\varepsilon r^N, \tag{3.2}$$

were $\varepsilon > 0$ is arbitrary.

Proof. Let

$$G(\eta) = \begin{cases} \varepsilon & \eta > \varepsilon^{\frac{1}{p-1}}, \\ |\eta|^{p-1}\mathrm{sign}(\eta) & -\varepsilon^{\frac{1}{p-1}} \le \eta \le \varepsilon^{\frac{1}{p-1}}, \\ -\varepsilon & \eta < -\varepsilon^{\frac{1}{p-1}} \ . \end{cases} \tag{3.3}$$

Then $G'(\eta) = (p-1)|\eta|^{p-2}\chi_{\{|\eta|<\varepsilon^{\frac{1}{p-1}}\}}$. Since, in $\{u > 0\}$, $\Delta_p u = 1$ in the weak sense, we'll have

$$D_e \Delta_p u = 0 \qquad \text{in } \{u > 0\};$$

again in the weak sens. Here D_e is a directional derivative. Expanding this, in the weak sense, we'll have

$$0 = \nabla \cdot (|\nabla u|^{p-2}\nabla u_e + (p-2)|\nabla u|^{p-4}\nabla u \nabla u \cdot \nabla u_e),$$

which, after multiplication by $G(u_e)$ and integration (by parts) over $B_{rs}(x_0)$, results in

$$\begin{aligned} &\int_{B_{rs}(x_0)} \left[|\nabla u|^{p-2}\nabla u_e + (p-2)|\nabla u|^{p-4}\nabla u \nabla u \cdot \nabla u_e\right] \cdot \nabla G(u_e) \\ &= \int_{\partial B_{rs}(x_0)} \left[|\nabla u|^{p-2} D_\nu u_e + (p-2)|\nabla u|^{p-4}\nabla_\nu u \nabla u \cdot \nabla u_e\right] G(u_e) dS \ . \end{aligned} \tag{3.4}$$

Here we have used the notations $u_e = D_e u$, and D_ν as the outward normal derivative. Next integrating the right hand side over the variable

$s \in (0,1)$ and using the Schwarz inequality, and Lemma 2.4, we arrive at

(3.5)

$$\int_0^1 \int_{\partial B_{rs}(x_0)} \left[|\nabla u|^{p-2} D_\nu u_e + (p-2)|\nabla u|^{p-4} \nabla_\nu u \nabla u \cdot \nabla u_e\right] G(u_e) dS ds$$
$$\leq (p-1) \int_{B_r(x_0)} |\nabla u|^{p-2} |D^2 u| |G(u_e)| dx$$
$$\leq (p-1)\varepsilon r^{N/2} \left(\int_{B_r(x_0)} \left[|\nabla u|^{p-2}|D^2 u|\right]^2 dx \right)^{\frac{1}{2}} \leq \varepsilon C r^N,$$

were $C = C(N,p)$. Here we have also used the fact that $D^2 u \in L^2_{loc}$, so that the integral

$$\int_{\partial B_{rs}(x_0)} \left[|\nabla u|^{p-2} D_\nu u_e + (p-2)|\nabla u|^{p-4} \nabla_\nu u \nabla u \cdot \nabla u_e\right] G(u_e) dS,$$

exists for a.e. $s \in (0,1)$.

Now the left hand side in (3.4) can be estimated from below. For this purpose we take $e = e_i$ for $i = 1, \cdots, N$, and with e_i directed in the standard coordinate axis. Then we have

(3.6)

$$\sum_{i=1}^N \left[\int_{B_{rs}(x_0)} |\nabla u|^{p-2} \nabla u_{x_i} + (p-2)|\nabla u|^{p-4} \nabla u \nabla u \cdot \nabla u_{x_i} \right] \cdot \nabla G(u_{x_i})$$
$$= (p-1) \int_{B_{rs}(x_0) \cap O_{\varepsilon_i}} |\nabla u|^{2(p-2)} \left(|D^2 u|^2 + (p-2) \sum_{i=1}^N \left(\frac{\nabla u}{|\nabla u|} \cdot \nabla u_{x_i} \right)^2 \right)$$
$$= (p-1)^2 \int_{B_{rs}(x_0) \cap O_{\varepsilon_i}} \left[|\nabla u|^{p-2}|D^2 u|\right]^2 dx \geq$$
$$\geq \mathcal{L}^N(B_{rs}(x_0) \cap O_{\varepsilon_i}) \geq \mathcal{L}^N(B_{rs}(x_0) \cap O_\varepsilon),$$

were in the final steps we have used Lemma 2.5, and the fact that $O_\epsilon \subset O_{\epsilon_i}$. Now putting (3.4)-(3.6) together we arrive at (3.2). □

Corollary 3.2. Retain the hypothesis of Lemma 3.1. Then

$$\mathcal{L}^N(O_\varepsilon \cap B_r(x_0)) \leq C\varepsilon r^{N-1} \qquad \text{for all } r < \frac{1}{4}.$$

Proof. If the conclusion of the corollary fails, then there exists $B_r(x_0)$ with center on the free boundary and such that

$$\mathcal{L}^N(O_\varepsilon \cap B_r(x_0)) \geq C_0 \varepsilon r^{N-1},$$

with C_0 arbitrarily large. Now by Lemma 3.1 we have

$$C\varepsilon r^N \geq \int_0^1 \mathcal{L}^N(O_\varepsilon \cap B_{2rs}(x_0))ds \geq \frac{1}{2}\mathcal{L}^N(O_\varepsilon \cap B_r(x_0)) \geq C_0\varepsilon r^{N-1}$$

which is a contradiction for large C_0. □

Theorem 3.3. *For $u \in \mathcal{G}$, $x_0 \in \partial\{u > 0\} \cap B_{1/2}$, and $0 < r < 1/4$, there holds*

$$\mathcal{H}^{N-1}(\partial\{u > 0\} \cap B_r(x_0)) \leq C_1 r^N,$$

for a generic constant $C_1 = C_1(p, N)$.

Proof. Let $\{B_\epsilon(x^i)\}_{i\in I}$ be a finite covering of $\partial\{u > 0\} \cap B_r(x_0)$ with $x^i \in \partial\{u > 0\}$, with at most n overlappings at each point. Then, by (3.1) and Corollar 3.2

$$\sum_{i\in I} \epsilon^N \leq C \sum_{i\in I} \mathcal{L}^N(O_\epsilon \cap B_\epsilon(x^i)) \leq nC\mathcal{L}^N(O_\epsilon \cap B_r(x_0)) \leq C_1\epsilon r^{N-1}.$$

This proves the theorem. □

Next we deduce a stability result. For this purpose we introduce the notation

$$\Lambda(u) = B_{1/2} \cap \{u = 0\}.$$

We also recall a general fact for solutions in the class $\mathcal{G}$, which states that for appropriate C

$$\{0 < u < C\epsilon^{\frac{p}{p-1}}\} \cap B_r(x_0) \subset O_\epsilon \cap B_r(x_0). \tag{3.7}$$

In fact this follows from Lemmas 2.2–2.3

Theorem 3.4. *Let $u_1, u_2 \in \mathcal{G}$ be two disjoint solutions satisfying*

$$\|u_1 - u_2\|_\infty \leq \epsilon^{\frac{p}{p-1}}. \tag{3.8}$$

Then

$$\mathcal{L}^N(\Lambda(u_1)\Delta\Lambda(u_2)) \leq C\epsilon,$$

and

$$(\Lambda(u_2))_{(-C\epsilon)} \subset \Lambda(u_1) \subset \{u_2 < \epsilon^{\frac{p}{p-1}}\}.$$

Here $C = C(N, p)$ is large enough, and

$$(\Lambda(u_2))_{(-C\epsilon)} = \{x \in \Lambda(u_2) : dist(x, \{u_2 > 0\}) > \epsilon\}.$$

Proof. The first statement follows from the facts

$$\mathcal{L}^N(\Lambda(u_1) \setminus \Lambda(u_2)) < C\epsilon, \qquad \Lambda(u_1) \subset \{u_2 < \epsilon^{\frac{p}{p-1}}\},$$

which in turn are consequences of (3.7) and (3.8) respectively.

For the second statement, we notice that if $x \in \overline{\{u_1 > 0\}}$ then by nondegeneracy lemma (Lemma 3.1 in [KKPS])

$$\sup_{B_{C\epsilon}(x)} u_1 \geq C_1(C\epsilon)^{\frac{p}{p-1}} > \epsilon^{\frac{p}{p-1}},$$

for large C. Hence

$$\sup_{B_{C\epsilon}(x)} u_1 > 0,$$

which implies $x \notin (\Lambda(u_2))_{(-C\epsilon)}$. $\square$

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