AN OVERDETERMINED PROBLEM IN NONLINEAR PARABOLIC POTENTIAL THEORY; UNIQUENESS

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Abstract. In this paper we study the question of uniqueness of an inverse problem, arising in the (thermal) linear and/or nonlinear potential theory. The overdetermined problem we'll study is represented by

$$\left(\operatorname{div}(|\nabla u|^{p-2}\nabla u) - D_t u - \chi_\Omega + \mu\ \right) u = 0,$$

where

$$\operatorname{supp}(\mu) \subset \Omega \subset \mathbb{R}^n \times (0,\infty), \quad 1 < p < \infty, \quad \mu \in L^\infty,$$

and $\Omega \cap \{t = \tau\}$ is bounded for $\tau > 0$.

The problem has applications in shape-recognition in underground water/oil recovery, subject to shape-change during time intervals. The particular case $u \geq 0$, $D_t u \geq 0$ is an example of the well-known Stefan problem with nonlinear governing equation.

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1. Introduction

1.1. **Problem setting.** Suppose we are given two (unknown) domains Ω_1, Ω_2 in $\mathbb{R}^n \times (0,\infty)$, with bounded t-sections, i.e., $\overline{\Omega}_i \cap \{t = \tau\}$ is bounded for all $\tau \geq 0$. Assume also both domains enjoy the property that their exterior thermal volume-potentials, with uniform distribution, coincide outside their union as well as at time $t = 0$. Can we conclude that $\Omega_1 = \Omega_2$? Obviously a question of such a general nature is very hard to answer and one expects to find examples violating the above conclusion. This would then suggest to restrict the problem to the class of domains having certain geometric configurations, such as convexity or starshapedness in the space variables, say. These restrictions have successfully been considered in the linear elliptic case by many mathematicians in the eastern part of the Europe; see [I1-2], [P-O-V], and the references therein. On the other side both the parabolic

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and nonlinear cases have gain little favor. This of course depends on the peculiarity that nonlinear/parabolic case introduces.

For the above domain-problem, in the elliptic case, there is a different approach, based on partial differential equations [G], [S], [B-S]. Interestingly, this approach leads to free boundary problems of obstacle type in the absence of obstacle. To explain this in more details let us set

$$U^{\Omega}(x,t) = \int\int K(x-y,t-s)\chi_{\Omega}dyds,$$

where

$$K(z,\tau) = (4\pi\tau)^{-n/2}\exp\left(-|z|^2/(4\tau)\right),$$

for $\tau > 0$, and $K(z,\tau) = 0$ for $\tau \leq 0$. Next suppose we are given the domains Ω_1, Ω_2 such that

$$U^{\Omega_1}(x,t) = U^{\Omega_2}(x,t), \qquad \text{in } \mathbb{R}^n \times [0,\infty) \setminus (\Omega_1 \cup \Omega_2). \tag{1.1}$$

Now let us set $U(x,t) = U^{\Omega_i}(x,t)$ in $\mathbb{R}^n \times [0,\infty) \setminus \Omega_i$ with $i = 1, 2$. Obviously this definition is consistent due to the assumption (1.1). Next, extend U to the $\mathbb{R}^n \times [0,\infty)$ as a continuous function, and set $-\mu = \Delta U - D_t U$. Then μ has support in $\overline{\Omega}_1 \cap \overline{\Omega}_2$. Since the potentials U^{Ω_j} are C^1 in the spatial variable we can assume that this continuation is C^1 in the spatial variables and in some neighborhood of $\partial\Omega_i \cap \Omega_j$ $(i \neq j)$. Finally, defining $u_j = U - U^{\Omega_j}$, for $j = 1, 2$, we see that u_j satisfies the following overdetermined problem

$$\begin{cases} \Delta u_j - D_t u_j = \chi_{\Omega_j} - \mu & \text{in } \mathbb{R}^{n+1}_+, \\ u_j = 0 & \text{in } \mathbb{R}^{n+1}_+ \setminus \Omega_j, \\ u_j(x,0) = 0, & \\ \text{supp}\mu \subset \overline{\Omega}_j, & \end{cases} \tag{1.2}$$

where

$$\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0,\infty),$$

and the differential equation is in the weak or distributional sense.

This new formulation gives us a better starting point, since the entire machinery of partial differential equations are now available. Another, and a more important, advantage of this formulation is that we can now consider general operators, linear as well as nonlinear. This formulation, subject to the restriction $u \geq 0$, is also known as a variational inequality or complementary problem, expressed in Euler-Lagrange equations.

It is also noteworthy that (1.2), with or without the non-negativity assumption u, appears in several applications such as mathematical physics, fluid dynamics, mathematical finance (American put option), filtration problem in reservoirs, and many other problems.

1.2. **Some basic questions.** With this formulation at hand one can ask several questions concerning existence, regularity and stability as well as geometric properties inherited by the solutions.

Q1: Given a bounded function $\mu(x,t) \geq 0$ with compact support, can we find a domain Ω admitting a solution u to (1.2)?

Readers, familiar to the theory of variational inequality, will see immediately that we can, indeed, solve the problem provided the function μ is concentrated enough. In other words if the source function μ is large enough at its support. The only difference between our problem (1.2) and the standard parabolic variational inequality (see [F]) is the extra requirement on the support of μ. It is also noteworthy that variational techniques produce nonnegative unique solution, which is not required by our problem.

Q2: A more general problem is the consideration of the initial data $u(x,0) = f(x)$ with f not necessarily zero, and a weight function $g(x,t) > 0$

$$\begin{cases} \Delta u - D_t u = g(x,t)\chi_\Omega - \mu & \text{in } \mathbb{R}^{n+1}_+, \\ u = 0 & \text{in } \mathbb{R}^{n+1}_+ \setminus \Omega, \\ u(x,0) = f(x), & \\ \text{supp}\mu \subset \overline{\Omega}. & \end{cases} \tag{1.3}$$

The reader should notice that when the function μ can be split into two parts, one supported in $\overline{\Omega}$ and the other one supported on $\{t = 0\}$ then it automatically creates the initial data mentioned above. In the above, one may also allow the source functions μ to be identically zero, provided $f \not\equiv 0$.

An interesting case is when $\mu = 0$, $g = 1$, and $f(x)$ is a multiple of the Dirac mass concentrated at the origin, $C\delta_0(x)$ $(C > 0)$. Obviously, if we could find such a domain Ω and a solution u to (1.2), then using integration by parts (Green's identity) we can have the mean value property

$$\int\int K(x-y, t-s)\chi_\Omega \, dyds = CK(x,t) \qquad (x,t) \notin \Omega,$$

where K is the fundamental solution introduced earlier. Observe that for $t < 0$ both sides of the above equality are zero.

Q3: Other questions such as the regularity of the solution u near the boundary, away from the support of μ, is also of fundamental importance. Indeed, by classical results, one knows that $u \in C^{1,\alpha}_x \cap C^{0,\alpha}_t$ $(0 < \alpha < 1)$, locally and away from the support of μ, (actually also on the support of μ provided μ is a bounded function). A question that arises directly is whether $u \in C^{1,1}_x \cap C^{0,1}_t$. When, in addition to (1.2),

$u \geq 0$ then one can apply Harnack's inequality to obtain this optimal regularity. A recent result of the second author, L. Caffarelli, and A. Petrosyan [C-P-S] shows that even without the non-negativity assumption for u one can deduce the optimal regularity mentioned above.
Q4: A final question we'd like to address is the question of the regularity of the so-called free boundary $\partial\Omega$ away from the support of μ. The particular case $u \geq 0$ and $D_t u \geq 0$, which amounts to the melting of ice (the Stefan problem), has been considered earlier by several authors. It is known [C] that for $(x,t) \in \partial\Omega$ there is a universal neighborhood $Q_r(x,t)$ such that $\partial\Omega$ is regular, provided the complement of Ω is thick enough near the point (x,t).

1.3. **The p-parabolic case.** To come back to the main topic of this paper "nonlinear potential theory", we depart from the formulation given in (1.3), and consider a more general type of operators with power-law nonlinearity. Namely, the p-parabolic operator. In this case we'll have the following problem: For given bounded functions μ, $g > 0$ and f find

$$u \in C(0,T; L^2(\mathbb{R}^n)) \cap L^p(0,T; W^{1,p}(\mathbb{R}^n)); \qquad \text{for all } T > 0$$

(see [D; page 2 and 7] for a definition of these spaces) such that

$$\begin{cases} \Delta_p u - D_t u = g(x,t)\chi_\Omega - \mu(x,t) & \text{in } \mathbb{R}^{n+1}_+, \\ u = 0 & \text{in } \mathbb{R}^{n+1}_+ \setminus \Omega, \\ u(x,0) = f(x), \\ \operatorname{supp}\mu \subset \overline{\Omega}, \end{cases} \tag{1.4}$$

where

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u), \qquad (1 < p < \infty).$$

Here the differential equation is in the following weak sense

$$\begin{aligned} &\left[-\int_{\mathbb{R}^n} uv\right]_{t=0}^{t=T} - \int_0^T \int_{\mathbb{R}^n} \left(|\nabla u|^{p-2}\nabla u \cdot \nabla v - uD_t v\right)\, dx\, dt \\ &= \int_0^T \int_{\mathbb{R}^n} (g\chi_\Omega - \mu)\, v\, dx\, dt\ , \end{aligned} \tag{1.5}$$

for all $T > 0$ and any function

$$v \in W^{1,2}(0,T; L^2(\mathbb{R}^n)) \cap L^p(0,T; W^{1,p}(\mathbb{R}^n))$$

(see [D; page 2 and 7] for a definition of these spaces).

We refer the reader to the book of E. DiBenedetto [D] for backgrounds about this operator. The basic properties that we will require for any operator, in order for our technique to go through, is the weak maximum principle and the rotation invariance in x. We also need to find barriers such as that in Lemma 2.2, below. The strong maximum principle (Lemma 2.3 below) is then obtained as a consequence of a barrier argument. Observe that even though there is no strong maximum principle for the p-parabolic operator we obtain this using the weak maximum principle and a barrier function. The existence of the barrier function, in turn, is due to the positivity of g.

2. Uniqueness

The question of uniqueness to problem (1.4), provided a solution exists, is the main topic of our study. In general, this is a very hard question and by no means elementary; at least our experiences say so. In the elliptic case such questions have been studied very thoroughly during several decades. There have also been several successful approaches to the study of the uniqueness property. It is our intention in this section to apply some of the techniques, adopted and modified from the elliptic case; see [B-S].

By imposing geometric conditions, such as convexity, we hope to be able to prove some uniqueness results for our problem.

An undesirable situation that may happen with solutions to (1.4) is the case when locally the boundary is flat in t-direction. By this we mean that the solution behaves like $-(t-t^0)^+$ near a point $(x^0,t^0) \in \partial\Omega$. This behavior is usually avoided by a simple assumption such as: For each $r>0$, and small enough, we assume

$$\sup_{Q_r^-(x^0,t^0)} u > 0 \qquad \forall\ (x^0,t^0) \in \partial\Omega, \tag{2.1}$$

where $Q_r^-(x^0,t^0) = B(x^0,r) \times (-r+t^0,t^0)$. The reason for this statement is the non-degeneracy lemma below (Lemma 2.4). Also, in the presence of strong maximum principle for subsolutions (e.g., when $p=2$) this assumption is superfluous.

Notation: We set $\Omega_i(\tau) = \Omega_i \cap \{t=\tau\}$ and recall that $\Omega_1(\tau)\cap\Omega_2(\tau)$ (for $\tau>0$) is assumed to be convex in this paper. For a point

$$(x^1,t^1) \in \Omega_i(t^1) \cap \partial\Omega_j(t^1), \qquad (i \neq j; i,j=1,2)$$

we denote by $\Pi(x^1,t^1)$ a supporting plane (which might not be unique) to $\Omega_1(t^1)\cap\Omega_2(t^1)$ at (x^1,t^1). We also denote $\Pi^+(x^1,t^1)$ to be the n-dimensional half-space in $\mathbb{R}^n \times \{t=t^1\}$ that has $\Pi(x^1,t^1)$ as its boundary and such that it doesn't intersect the set $\Omega_1(t^1)\cap\Omega_2(t^1)$. Now for $(x^1,t^1) \in \Omega_i(t^1)\cap\partial\Omega_j(t^1), (i\neq j)$ we set

$$d_i(x^1,t^1) = \sup \operatorname{dist}\left((x,t^1), \Pi(x^1,t^1)\right),$$

where supremum has been taken over all points

$$(x,t^1) \in \Pi^+(x^1,t^1) \cap \Omega_i(t^1)$$

and all possible $\Pi(x^1, t^1)$ (in case $\Pi(x^1, t^1)$ is non-unique). Here, the distance is the usual Euclidean one in $\mathbb{R}^n$. We also set

$$d_i(t^1) = \sup_{(x,t)} d_i(x,t),$$

where supremum has been taken over all possible

$$(x,t) \in \Omega_i(t) \cap \partial\Omega_j(t) : \ 0 < t \leq t^1, \qquad (i \neq j).$$

Theorem 2.1. *Let μ be a bounded function with compact support and suppose that functions $u_j \in C(0,T; L^2(\mathbb{R}^{n+1})) \cap L^p(0,T; W^{1,p}(\mathbb{R}^{n+1}))$ $(1 < p < \infty; j = 1,2)$ are solutions to (1.4), with $g \equiv 1$, and $\overline{\Omega}_j(\tau)$ bounded, for all $\tau \geq 0$. Assume moreover that each Ω_j is non-decreasing in t, $\Omega_1(\tau) \cap \Omega_2(\tau)$ is convex for each $\tau > 0$, and (2.1) is verified. Then $\Omega_1 \equiv \Omega_2$ and $u_1 = u_2$.*

It is known, that solutions to (1.4) are $C_x^{1,\alpha} \cap C_t^{0,\alpha}$ in $\mathbb{R}^{n+1}_+$, for some $0 < \alpha < 1$, $j = 1,2$; see [D].

First we need to prove some lemmas. For simplicity we set $q = p/(p-1)$ (the conjugate of p).

It is well known that the strong maximum (and comparison) principle fails for the p-parabolic equation. The standard Hopf's boundary point lemma also fails for the p-parabolic case. However, in presence of a positive right hand side one may introduce barriers that can help us to prove such results. Here is how we do it.

Lemma 2.2. *(Hopf's boundary point lemma) Let $Q_{r,\tau} = B_r \times (0,\tau)$ be a cylinder in $\mathbb{R}^{n+1}$, where B_r is the ball of radius r and center at the origin. Let $u(x,t) \in C_x^{1,\alpha} \cap C_t^{0,\alpha}(\overline{Q}_{r,\tau})$ $(0 < \alpha < 1)$ satisfy $u \leq 0$ and $\Delta_p u - D_t u \geq \gamma$, for some $\gamma > 0$. Suppose also there is a point $z \in \partial B_r$ such that $u(z,\tau) = 0$. Let also ν be any unit (spatial) vector on ∂B_r at (z,τ), directed inward B_r. Then $\frac{\partial u}{\partial \nu}(z,\tau) < 0$, provided $r << \tau$. Indeed*

$$\tau > \frac{1}{q}\left(\frac{\gamma}{2n}\right)^{1/(p-1)} \frac{2r^q}{3\gamma}, \tag{2.2}$$

will suffice.

Proof. Define the function $h(x,t) = A(|x|^q - r^q) - B(t-\tau)$, where $A = \frac{1}{q}\left(\frac{\gamma}{2n}\right)^{\frac{1}{p-1}}$ and $B = \frac{3}{2}\gamma)$. One readily verifies that $\Delta_p h - D_t h = \gamma$. Thus in $Q_{r,\tau}$, $\Delta_p h - D_t h \leq \Delta_p u - D_t u$.

It is obvious, that $h = -B(t-\tau) \geq 0 \geq u$ on $\partial B_r \times (0,\tau]$ and $h = A(|x|^q - r^q) + B\tau \geq -Ar^q + B\tau \geq 0 \geq u$ on $\overline{B}_r \times \{0\}$, provided (2.2) is satisfied. In particular $h \geq u$ on $\partial_p Q_{r,\tau}$ (parabolic boundary). Hence by comparison principle[1] $h \geq u$ in $Q_{r,\tau}$. Since $h(z,\tau) = u(z,\tau) = 0$, we obtain

$$\frac{\partial u}{\partial \nu}(z,\tau) \leq \frac{\partial h}{\partial \nu}(z,\tau) = Aqr^{q-2} z \cdot \nu = -Aqr^{q-1}|\cos(z,\nu)| < 0. \quad \square$$

As a consequence of the above lemma we obtain the strong maximum principle whose obvious proof is omitted.

Lemma 2.3. *(Strong Maximum Principle) Let α, γ be given constants with $0 < \alpha < 1$, and $\gamma > 0$, and $D \subset \mathbb{R}^{n+1}$ be an open set. Suppose $u(x,t) \in \left(C_x^{1,\alpha} \cap C_t^{0,\alpha}\right)(D)$ satisfy $\Delta_p u - D_t u \geq \gamma$ in D. Then u doesn't attain a local maximum value in D.*

Lemma 2.4. *Recall the hypotheses in Theorem 2.1, and assume, for some $t^1 > 0$, $\Omega_1(t^1) \setminus \Omega_2(t^1) \not\equiv \emptyset$. Fix $(x^1,t^1) \in \Omega_1(t^1) \cap \partial\Omega_2$, and define*

$$T = T(x^1,t^1) := \left\{(x,t) :\ x \in \Pi(x^1,t^1),\ 0 < t \leq t^1 \ \right\},$$

where $\Pi(x^1,t^1)$ denotes a supporting plane to the convex set $\Omega_1(t^1) \cap \Omega_2(t^1)$ at (x^1,t^1). Then

$$\sup_T u_1 \geq \frac{1}{q} d_1^q(x^1,t^1).$$

The same conclusion holds for u_2 analogously.

Proof. Let $(z^1,t^1) \in \partial\Omega_1 \setminus \overline{\Omega}_2$ be a point that realizes the distance $d_1(x^1,t^1)$. Let also $(z,\tau) \in \Omega_1 \setminus \Omega_2$ be a point close to (z^1,t^1) and such that $u_1(z,\tau) > 0$ (the existence of such a point is guaranteed by assumption (2.1)). Using translation and rotation around the t-axis, along with the convexity of $\Omega_1(t^1) \cap \Omega_2(t^1)$ and the non-decreasing character of the domains Ω_i $(i = 1,2)$ in the t-variable, we may, as we do, assume that

$$\Omega_1 \cap \Omega_2 \cap \{t < t^1\} \subset \{x_1 < 0\}.$$

Set $v(x) = (1/q)\,|x_1 - z_1|^q$, where x_1, z_1 are the first coordinate of x and z, respectively. Next, one verifies that $\Delta_p v - D_t v = 1$ and

$$v(x) \geq 0 > w(x,t) := u_1(x,t) - \frac{u_1(z,\tau)}{2}$$

on $\partial D \setminus \{x_1 = 0\}$, where D is the connected component of $\Omega_1 \cap \{x_1 > 0\}$ with (z^1,t^1) on its boundary. Now, if also

$$v(x) \geq w(x,t), \qquad \text{on } \{x_1 = 0\},$$

[1] A proof of this for cylindrical domains can be found in [KL]. The same proof works for domains that are increasing in time. Cf. also [D; Lemma 3.1].

then by the comparison principle $v \geq w$ in D. This contradicts the simple fact that $v(z) = 0 < w(z,\tau) = u_1(z,\tau)/2$. Therefore we conclude

$$v \leq \sup_D u_1(x,t) - \frac{u_1(z,\tau)}{2} \leq u_1(x^1,t^1) - \frac{u_1(z,\tau)}{2}.$$

Now, using (2.1), we let $(z,\tau) \to (z^1,t^1)$, to arrive at the conclusion of the lemma. $\square$

By varying the boundary point (x^1,t^1) in the set $D' := \partial\Omega_2 \cap \Omega_1 \cap \{t \leq t^1\}$, we can conclude

$$\sup_{D'} \geq \frac{1}{q} d_1^q(t^1). \tag{2.3}$$

Lemma 2.5. *Under the hypotheses of Theorem 2.1 and Lemma 2.4, suppose $\Omega_1(t^1) \neq \Omega_2(t^1)$ and $d_1(t^1) \leq d_2(t^1)$, for some $t^1 > 0$. Then there exists $(z^0,t^0) \in \Omega_1 \setminus \Omega_2$ ($t^0 \leq t^1$) such that*

$$-u_1(z^0,t^0) > \frac{1}{q} d_1^q(t^1) \quad \textit{and} \quad |\nabla u_1(z^0,t^0)| = 0.$$

Similar conclusions hold if $d_2(t^1) \leq d_1(t^1)$, analogously.

Proof. Let $D := (\Omega_1 \setminus \Omega_2) \cap \{t \leq t^1\}$, $a = -\inf_D u_1 \geq 0$ and $(z^0,t^0) \in \overline{\Omega}_1 \setminus \Omega_2$ be such that $u_1(z^0,t^0) = -a$. Then in the weak sense

$$\begin{aligned} \Delta_p(u_1+a) - D_t(u_1+a) \leq \Delta_p u_2 - D_t u_2 \qquad &\text{in } \Omega_2 \cap \{t \leq t^1\}, \\ \text{and } u_1 + a \geq u_2 \quad &\text{on } \partial_p\left(\Omega_2 \cap \{t < t^1\}\right). \end{aligned}$$

Therefore, by the comparison principle $u_1 + a \geq u_2$ in $\Omega_2 \cap \{t < t^1\}$. Also in $(\Omega_1 \setminus \Omega_2) \cap \{t < t^1\}$, $u_2 = 0$ and $u_1 + a \geq 0$. We thus observe that

$$u_1 + a - u_2 \geq 0 \qquad \text{in } (\Omega_1 \cup \Omega_2) \cap \{t \leq t^1\}. \tag{2.4}$$

In particular $u_2 \leq a$ in $(\Omega_2 \setminus \Omega_1) \cap \{t \leq t^1\}$ and consequently by the assumption (2.1) we must have $a > 0$. Hence $\sup_{\Omega_2 \setminus \Omega_1 \cap \{t<t^1\}} u_2 \leq a$. We now prove, that

$$\sup_{\Omega_2 \setminus \Omega_1 \cap \{t<t^1\}} u_2 < a. \tag{2.5}$$

Suppose on the contrary that there is a point $(z,\tau) \in \Omega_2 \setminus \Omega_1$ such that $u_2(z,\tau) = a$. Then by (2.4), $u_1 + a - u_2 \geq 0$ in Ω_2, and therefore $u_1 + a - u_2$ has a minimum value at (z,τ), and consequently a vanishing gradient at this point. Since $(z,\tau) \in \Omega_2 \setminus \Omega_1$, also u_1 must have a vanishing gradient at (z,τ). Therefore we conclude that $|\nabla u_2|(z,\tau) =$

0. By the non-decreasing property of Ω_1 and Ω_2 we may take a cylinder Q in $\Omega_2\setminus\Omega_1$ with (z,τ) on its boundary and apply Lemma 2.2 (to u_2-a) to obtain a contradiction. This proves (2.5).

Now (2.5) in conjunction with (2.3) implies

$$a = -u_1(z^0,t^0) > \sup_{\Omega_2\setminus\Omega_1\cap\{t<t^1\}} u_2 \geq \frac{1}{q}d_2^q(t^1) \geq \frac{1}{q}d_1^q(t^1),$$

which proves the first part of the lemma. To prove the second statement, observe that by (2.4) $u_1+a-u_2 \geq 0$ in Ω_1. Also by the definition of a and since $(z^0,t^0) \notin \Omega_2$ we may conclude that u_1+a-u_2 has a minimum value at (z^0,t^0) and thus $\nabla u_1(z^0,t^0) = \nabla u_2(z^0,t^0) = 0$. $\square$

Remark 2.6. We remark that if $\Omega_1(T) \neq \Omega_2(T)$, for some $T>0$, then $\min(d_1(T),d_2(T)) > 0$. Indeed, if $\min(d_1(T),d_2(T)) = 0$ then $\Omega_1(T) \subset \Omega_2(T)$ (or the reverse). To see this observe that by (1.5)

$$\int_{\mathbb{R}^n\times T} u_1 + \int_0^T\int_{\mathbb{R}^n} g\chi_{\Omega_1} = \int_{\mathbb{R}^n\times T} u_2 + \int_0^T\int_{\mathbb{R}^n} g\chi_{\Omega_2}. \tag{2.6}$$

Next $\Omega_1(T) \subset \Omega_2(T)$ can be used in combination with the comparison principle to conclude $u_1 \leq u_2$ in $\Omega_2 \cap \{t<T\}$. Hence (2.6) can be reduced to

$$\int_0^T\int_{\mathbb{R}^n} g\chi_{\Omega_1} \geq \int_0^T\int_{\mathbb{R}^n} g\chi_{\Omega_2}.$$

Since $g>0$, we have a contradiction.

Proof of Theorem 2.1. Suppose the statement in the theorem fails. Then there exists $t^1>0$ such that the symmetric difference $\Omega_1(t^1)\triangle\Omega_2(t^1)$ is nonempty.

By Remark 2.6 we may, as we do, assume $0 < d_1(t^1) \leq d_2(t^1)$. Now by Lemma 2.5, there is a point $(z^0,t^0) \in \Omega_1\setminus\Omega_2$ ($t^0 \leq t^1$) such that

$$a := -\inf_{(\Omega_1\setminus\Omega_2)\cap\{t<t^1\}} u_1 = -u_1(z^0,t^0) > \frac{1}{q}d_1^q(t^1),$$

$$|\nabla u_1(z^0,t^0)| = 0. \tag{2.7}$$

By translation we may assume that (z^0,t^0) is the origin. By the convexity of $\Omega_1(t^0)\cap\Omega_2(t^0)$ and by the non-decreasing property of Ω_1 and Ω_2 we also assume, using rotation around the t-axis and translation in t-direction, that $\Omega_1\cap\Omega_2\cap\{t<0\} \subset \{x_1<0\}$. Now, choose $(z,\tau) \in \partial\Omega_1(0)\setminus\Omega_2(0)$ with largest distance to $\{x_1=0\}$. Then $z_1 \leq d_1(t^1)$ and by (2.7), $z_1^q/q < a$. Let $\varepsilon>0$ be such that $(z_1+\varepsilon)^q/q \leq a$, and define $w(x,t) = u_1(x,t)+a$, $v(x,t) = \frac{(x_1+\varepsilon)^q-\varepsilon^q}{q}$.

Then $\Delta_p w - D_t w = \Delta_p v - D_t v = 1$ in $D := \Omega_1 \cap \{x_1 > 0\} \cap \{t < 0\}$. It is obvious that $w \geq v$ on $\partial_p D$, because $w \geq 0 = v$ in $\{x_1 = 0\}$ and $w = a \geq v$ on $\partial\Omega_1 \cap \partial D$. Then we'll have (by comparison principle) $w \geq v$ in D. Since also $w(0,0) = 0 = v(0,0)$, it follows that

$$\frac{\partial w}{\partial x_1}(0,0) \geq \frac{\partial v}{\partial x_1}(0,0);$$

i.e.

$$\frac{\partial u_1}{\partial x_1}(0,0) \geq \frac{\partial v}{\partial x_1}(0,0) = \varepsilon^{q-1},$$

which contradicts (2.7). This proves the theorem in the case $d_1(t^1) \leq d_2(t^1)$. If $d_2(t^1) \leq d_1(t^1)$ we interchange Ω_1, Ω_2 and repeat the same argument. The proof is now completed. $\square$

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