UNIQUENESS OF A FREE BOUNDARY PROBLEM OF p-PARABOLIC TYPE

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ABSTRACT. Given a bounded domain $\Omega \subset \mathbb{R}^n \times (0,\infty)$ and an L^∞ -function $\mu \geq 0$ with compact support ($\subset \overline{\Omega}$). Assume that there exists a function u, satisfying the following overdetermined (free) boundary value problem

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) - D_t u = -\mu(x,t) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \cap \{t > 0\}, \\ -\partial u/\partial\nu = 1 & \text{on } \partial\Omega \cap \{t > 0\}, \\ u(x,0) = f(x), \end{cases}$$

where $1 < p < \infty$, $f \in C^0(\mathbb{R}^n)$ and ν is the spatial outward unit normal vector on $\partial\Omega$.

Under certain geometrical conditions we prove that the above problem has at most one solution (u,Ω), if any at all.

1. INTRODUCTION

1.1. **Problem setting.** In this paper we consider a free boundary problem of p-parabolic type related to heat combustion with power law nonlinearity. Our purpose is to obtain some uniqueness results inforced by geometric features of solutions of our problem. Let us denote $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0,\infty)$ and consider the following question: For a given bounded function $\mu \geq 0$, and $f \in C^0(\mathbb{R}^n)$ find and describe properties of a domain $\Omega \subset \mathbb{R}^{n+1}_+$ and a function

$$u \in C(0,T; L^2(\mathbb{R}^n)) \cap L^p(0,T; W^{1,p}(\mathbb{R}^n))$$

(see [D; page 2 and 7] for a definition of these spaces), satisfying the following overdetermined boundary value problem

$$\begin{cases} \Delta_p u - D_t u = -\mu(x,t) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \cap \{t > 0\}, \\ -\partial u/\partial\nu = 1 & \text{on } \partial\Omega \cap \{t > 0\}, \\ u(x,0) = f(x), \\ \operatorname{supp}\mu \subset \overline{\Omega}, \end{cases} \tag{1.1}$$

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where

$$\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u), \qquad (1 < p < \infty),$$

is the p-Laplace operator and ν is the spatial outward unit normal vector on $\partial\Omega$ (we assume $\partial\Omega$ to be C^1 in spatial direction). Here the differential equation is in the following weak sense: For all $T > 0$ and any function

$$v \in W^{1,2}(0,T; L^2(\mathbb{R}^n)) \cap L^p(0,T; W_o^{1,p}(\mathbb{R}^n))$$

(see [D; page 2 and 7] for a definition of these spaces)

$$\int\int_\Omega (|\nabla u|^{p-2}\nabla u \cdot \nabla v - uD_t v)dxdt - \int_{\Omega\cap\{t=0\}} fv\ dx$$
$$= \int_{\partial\Omega\cap\{\mathbb{R}^n\times(0,T)\}} |\nabla u|^{p-2} v d\sigma(cos\alpha) + \int\int_\Omega \mu v\ dx\ dt,$$

where $d\sigma$ is the area element on $\partial\Omega \cap \{\mathbb{R}^n \times (0,T)\}$ and α is the angle formed by the outward normal $\nu(x,t)$ at a point $(x,t) \in \partial\Omega \cap \{\mathbb{R}^n \times (0,T)\}$ and the hyperplane $\mathbb{R}^n \times \{t\}$.

The questions of existence, uniqueness and regularity of solutions to several related problems have been studied by many authors during few years. In particular, when $p = 2$, $\mu \equiv 0$, and $u \geq 0$, the existence of weak solutions has been proven in [C-V]. The equation $\Delta u + \sum a_i u_{x_i} - u_t = 0$ was studied in [L-V-W], and some uniqueness results were shown under certain conditions. Similar results are known in the elliptic case. When the operator is the ordinary Laplacian one can find in [GS] and [Sh] some results concerning the existence of solutions, depending on μ. In addition, some uniqueness results have been obtained in [H-S1] and [H-S2] for p-Laplacian $(1 < p < \infty)$.

In this paper we'll apply some of the techniques from those papers, adopted and modified for the current purposes. Let us recall the important details of the methods, used in this paper: 1) A Strong comparison principle, and 2) Hopf boundary point lemma. For 1) we note that a Strong comparison principle doesn't hold for singular/degenerate operators, such as the p-parabolic one (see for instance [B]). However, in our situation we can use boundary gradient condition in (1.1), which forces the equation to be nondegenerate in a local sense near the boundary. Therefore, a local Strong comparison principle holds.

In this paper we will not consider the question of existence or regularity of solutions to (1.1). We only analyze the question of uniqueness of the solutions, provided they exist. By imposing geometric conditions, such as convexity in space and monotonicity (in time direction) of the domains, we will be able to prove some uniqueness results for our problem.

2. Main Result

2.1. In this section we state and prove our main result.

First we need some notations: We set $\Omega_i(\tau) := \Omega_i \cap \{t = \tau\}$ and throughout this paper we assume $\Omega_1(\tau) \cap \Omega_2(\tau)$ (for $\tau > 0$) to be convex, $\partial\Omega$ to be C^1 in spatial direction. For a point

$$(x^1, t^1) \in \Omega_i(t^1) \cap \partial\Omega_j(t^1), \qquad (i \neq j; i, j = 1, 2)$$

we denote by $\Pi(x^1, t^1)$ the supporting plane to $\Omega_1(t^1) \cap \Omega_2(t^1)$ at (x^1, t^1). We also set $\Pi^+(x^1, t^1)$ to be the n-dimensional half-space in $\mathbb{R}^{n+1}$ that has $\Pi(x^1, t^1)$ as its boundary and such that it doesn't intersect the set $\Omega_1(t^1) \cap \Omega_2(t^1)$.

Let us notice that in this paper each Ω_j is assumed to be non-decreasing in t, i.e. $\Omega_i(t_1) \subseteq \Omega_i(t_2)$, when $t_1 \leq t_2$.

For a bounded domain $Q \subset \mathbb{R}^n$ and for $T > 0$ we will denote by Q_T the cylindrical domain $Q \times (0, T)$. We also define the parabolic boundary $\partial_p\Omega$ of a domain $\Omega \subset \mathbb{R}^{n+1}$ to be the set of all points $(x, t) \in \partial\Omega$ such that for any $\varepsilon > 0$, the cylinder $B(x, \varepsilon) \times (-\varepsilon + t, t)$ contains points not in Ω.

Theorem 2.1. *Let $\mu \geq 0$ be a bounded function with compact support and suppose that functions $u_j \in C(0, T; L^2(\mathbb{R}^n)) \cap L^p(0, T; W^{1,p}(\mathbb{R}^n))$ $(1 < p < \infty; j = 1, 2)$ are solutions to (1.1), and $\overline{\Omega}_j(\tau)$ bounded, for all $\tau \geq 0$. Assume moreover that each Ω_j is non-decreasing in t, then the following hold*

a) If $\Omega_1(\tau)$ is convex for all $\tau > 0$, then $\Omega_2 \subset \Omega_1$.

b) If $\Omega_1(\tau) \cap \Omega_2(\tau)$ is convex for all $\tau > 0$, then $\Omega_1 \equiv \Omega_2$.

c) If both $\Omega_1(\tau)$ and $\Omega_2(\tau)$ are convex for all $\tau > 0$, then $\Omega_1 \equiv \Omega_2$ and $u_1 \equiv u_2$.

It is known, that condition (1.1) implies that $u_j(x, t)$ are $C_x^{1,\alpha} \cap C_t^{0,\alpha}(\Omega_j)$ (for some $0 < \alpha < 1$, $j = 1, 2$); see [D].

Some Lemmas. In this paper we will repeatedly use the following lemma.

Lemma 2.2. *Let $Q_T \subset \mathbb{R}^{n+1}_+$ be a cylindrical domain, and $v_1, v_2 \in C(0, T; L^2(Q'_T)) \cap L^p(0, T; W^{1,p}(Q'_T))$; $\overline{Q}_T \subset Q'_T$, with*

$$\Delta_p v_1 - D_t v_1 \leq \Delta_p v_2 - D_t v_2 \text{ in } Q_T. \tag{2.1}$$

Then the following hold:

a) *(**The Weak Comparison Principle**) If $v_1 \geq v_2$ on $\partial_p Q_T$ then $v_1 \geq v_2$ in Q_T.*

b) *(***Hopf's Comparison Principle***) Suppose $v_1 > v_2$ in Q_T, $v_1(x_0,t_0) = v_2(x_0,t_0)$ for some $(x_0,t_0) \in \partial_p Q_T$. If in addition $|\nabla v_2| > 0$ in Q_T, then $\frac{\partial v_1}{\partial \nu}(x_0,t_0) < \frac{\partial v_2}{\partial \nu}(x_0,t_0)$, where ν is the unit outward normal vector on ∂Q_T, at (x_0,t_0).*
c) *(***Strong Comparison Principle***) If $v_1 \geq v_2$, $v_1 \not\equiv v_2$ on $\partial_p Q_T$ and $|\nabla v_2| > 0$ in Q_T, then $v_1 > v_2$ in Q_T.*

The proofs of propositions a)-c) of Lemma 2.2 are similar to the elliptic case (see [T], Lemma 3.2, Propositions 3.3.1, 3.3.2) and one can obtain those by appropriate changes. However, for the reader's convenience we give exact references for the proofs: Weak comparison principle is proven in [D], p. 160, Lemma 3.1. Hopf's and Strong comparison principles can be found in [A-G], Lemma 2.1 and Lemma 1.1.

Remark 2.3. It is crucial that the relation $|\nabla v_2| > 0$ holds for the function v_2. In our applications of this lemma we will always consider a small subdomain with (x_0,t_0) on its boundary. Since the magnitude of the gradient of any solution to (1.1) approaches one, continuously, the required condition in the lemma is fulfilled, near the boundary for solutions to (1.1). In the sequel we will omit mentioning this argument.

Next, we need to prove some lemmas.

Lemma 2.4. *Let (u,Ω) be a solution to (1.1). Consider a hyperplane H, which is ortogonal to $\mathbb{R}^n$ and cuts off Ω a cap Ω', such that $\overline{\Omega'} \cap supp(\mu) = \emptyset$, $\overline{\Omega'} \cap \{t = 0\} = \emptyset$. Then*

$$d(\tau) := \sup_{x \in \partial\Omega'(\tau)} dist(x,H) < \sup_{H \cap \{t \leq \tau\}} u. \tag{2.2}$$

Moreover, if $(x^0,t^0) \in H$ is such that $u(x^0,t^0) = \sup_{H\cap\{t\leq\tau\}} u$, then

$$\frac{\partial u}{\partial l}(x^0,t^0) < -1, \tag{2.3}$$

where l is the unit normal vector to H pointing inwards Ω'.

Proof. Since the problem is invariant under rotation around the t-axis and translation, we may assume that $H = \{x_1 = 0\}$, and $\Omega' = \{x_1 > 0\} \cap \Omega(\tau)$. Now let $(z,\tau) \in \partial\Omega'(\tau)$, be such that $d(\tau) = dist(z,H)$ and observe that by boundary condition of (1.1) $\frac{\partial u}{\partial x_1}(z,\tau) = -1$. Then define

$$h(x,t) = s(d(\tau) - x_1),$$

where

$$s := \frac{\sup_{H\cap\{t\le\tau\}} u}{d(\tau)}.$$

It is obvious that

$$\Delta_p h - D_t h = \Delta_p u - D_t u = 0 \text{ in } \Omega', \qquad u(z,\tau) = h(z,\tau) = 0. \tag{2.4}$$

Since also $h = \sup_{H\cap\{t\le\tau\}} u$ on H, $h \ge 0$ and $u = 0$ on $\partial\Omega$, we must have

$$h \ge u \qquad \text{on } \partial\Omega' \cap \{t \le \tau\}. \tag{2.5}$$

Now using Lemma 2.2 a)-c) we arrive at

$$-s = \frac{\partial h}{\partial x_1}(z,\tau) < \frac{\partial u}{\partial x_1}(z,\tau) = -1,$$

i.e.

$$\frac{\sup_{H\cap\{t\le\tau\}} u}{d(\tau)} = s > 1, \tag{2.6}$$

which proves (2.2). Next using (2.4)-(2.6) and the fact that $u(x^0,t^0) = h(x^0,t^0)$ (here $x^0 = 0, t^0 \le \tau$) we obtain

$$\frac{\partial u}{\partial l}(x^0,t^0) \le \frac{\partial h}{\partial l}(x^0,t^0) = \frac{\partial h}{\partial x_1}(x^0,t^0) = -s < -1,$$

i.e. (2.3) holds.

Lemma 2.5. *Let u be any solution to (1.1) and extend it to the entire $\mathbb{R}^{n+1}_+$ by defining it to be zero in $\mathbb{R}^{n+1}_+ \setminus \Omega$. Then*

$$\Delta_p(u-c) - D_t(u-c) \ge -\mu,$$

for any constant c.

Proof. Take a small neighborhood N of $\partial\Omega$ such that $\operatorname{supp}(\mu)\cap N = \emptyset$, and define

$$v = \begin{cases} \max(u,0) & \text{in } N\cap\Omega, \\ 0 & \mathbb{R}^{n+1}\setminus\Omega. \end{cases}$$

Then, v is p-subcaloric in N (see [D], p. 18). Hence $v-c$, is p -subcaloric in N, which is the desired result.

2.1. **Proof of Theorem 2.1.** Since the proofs of (a) and (b) are similar, we only prove (a); (c) follows from (a) or (b). Suppose $\Omega_2 \setminus \Omega_1 \neq \emptyset$. Extend u_j by zero to $\mathbb{R}^{n+1} \setminus \Omega_j$, for $j = 1, 2$, and let $(x^0, t^0) \in \partial\Omega_1$ be such that $u_2(x^0, t^0) = \sup_{\partial\Omega_1} u_2$. We see at once that $u_2(x^0, t^0) > 0$ by weak maximum principle applied to u_2 in $\Omega_2 \setminus \Omega_1$. Define now $w(x, t) = u_2(x, t) - u_2(x^0, t^0)$ in $\overline{\Omega}_1$. Then by Lemma 2.5 $\Delta_p w - D_t w \geq -\mu$. Hence

$$\Delta_p w - D_t w \geq \Delta_p u_1 - D_t u_1 \text{ in } \Omega_1, \quad w \leq u_1 \text{ on } \partial\Omega_1 \cap \{t > 0\}$$

and

$$w(x^0, t^0) = u_1(x^0, t^0) = 0.$$

We may thus apply Lemma 2.2 to deduce that

$$\frac{\partial u_2}{\partial \nu}(x^0, t^0) = \frac{\partial w}{\partial \nu}(x^0, t^0) > \frac{\partial u_1}{\partial \nu}(x^0, t^0) = -1$$

i.e.

$$\frac{\partial u_2}{\partial \nu}(x^0, t^0) > -1. \tag{2.7}$$

Now, using the convexity in space, non-decreasing property of $\Omega_1(t)$ and that $\mathrm{supp}(\mu) \subset \Omega_1 \cap \Omega_2$, we can take a supporting plane $\Pi(x^0, t^0)$ such that $\Omega_1(t_0) \cap \Omega_2(t_0) \subset \Pi^+(x^0, t^0)$, i.e. the assumptions of Lemma 2.4 are fulfilled (when $t \leq t^0$). But then, (2.7) contradicts (2.3). This completes the proof of the theorem.

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