GLOBAL SOLUTIONS OF AN OBSTACLE-PROBLEM-LIKE EQUATION WITH TWO PHASES

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ABSTRACT. Concerning the obstacle-problem-like equation $\Delta u = \frac{\lambda_+}{2}\chi_{\{u>0\}} - \frac{\lambda_-}{2}\chi_{\{u<0\}}$, where $\lambda_+ > 0$ and $\lambda_- > 0$, we give a complete characterization of all global two-phase solutions with quadratic growth both at 0 and infinity.

1. INTRODUCTION

Whereas the regularity in one-phase free boundary problems has by now been extensively studied, one-phase methods prove in many cases to be unsuitable for the corresponding two-phase problems. Here we study the regularity of the obstacle-problem-like equation

$$\Delta u = \frac{\lambda_+}{2}\chi_{\{u>0\}} - \frac{\lambda_-}{2}\chi_{\{u<0\}} , \tag{1.1}$$

where $\lambda_+ > 0$ and $\lambda_- > 0$. The equation arises by minimizing the cost functional

$$\int_\Omega \left(|\nabla u|^2 + \lambda_+ \max(u,0) + \lambda_- \max(-u,0)\right) dx,$$

over an appropriate space. Possible applications of this functional may come in several problems when the external force is a function of u itself, in this case the external force is

$$\lambda_+ H(u) - \lambda_- H(-u).$$

As a specific example, imagine a membrane under the influence of an electric or a magnetic field of the form

$$F = \Lambda_+ \chi_{\{x_3>0\}} \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + \Lambda_- \chi_{\{x_3<0\}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} .$$

If we assume the membrane to be modeled by a graph in the x_3-direction and to be clamped in at the boundary, then the equilibrium state would correspond to the minimizer of our functional.

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One of the difficulties one confronts in this problem is that the interface $\{u = 0\}$ consists in general of two parts – one where the gradient of u is nonzero and one where the gradient of u vanishes. Close to points of the latter part we expect the gradient of u to have linear growth. However, because of the decomposition into two different types of growth, it is not possible to derive a growth estimate by for example a Bernstein technique.

Using a monotonicity formula and frequency estimates, G.S. Weiss derived (for more general coefficients λ_+ and λ_-) an estimate for the quadratic growth of u near the set $\Omega \cap \{u = 0\} \cap \{\nabla u = 0\}$ which leads to Hausdorff dimension estimates ([7]). Moreover N. Uraltseva succeeded in proving local $H^{2,\infty}$-regularity via an application of the monotonicity formula by H.W. Alt-L.A. Caffarelli-A. Friedman (see [4]).

In this paper we are interested in the true two-phase part of the free boundary with vanishing gradient, i.e. $\Omega \cap \partial\{u > 0\} \cap \partial\{u < 0\} \cap \{\nabla u = 0\}$. As a first step towards regularity, we give a complete characterization of two-phase solutions with quadratic growth at 0 and infinity: it turns out that each such solution coincides after rotation with the one-dimensional solution $u(x) = \frac{\lambda_+}{4} \max(x_n, 0)^2 - \frac{\lambda_-}{4} \min(x_n, 0)^2$. In particular this implies that each blow-up limit u_0 at $\Omega \cap \partial\{u > 0\} \cap \partial\{u < 0\} \cap \{\nabla u = 0\}$ is after rotation of the form $u_0(x) = \frac{\lambda_+}{4} \max(x_n, 0)^2 - \frac{\lambda_-}{4} \min(x_n, 0)^2$.

2. Notation

Throughout this article $\mathbf{R}^n$ will be equipped with the Euclidean inner product $x \cdot y$ and the induced norm $|x|$, and $B_r(x_0)$ will denote the open n-dimensional ball of center x_0 , radius r and volume $r^n \, \omega_n$.

We will use $\partial_e u = \nabla u \cdot e$ for the directional derivative.

When considering a set A , χ_A shall stand for the characteristic function of A , while ν shall typically denote the outward normal to a given boundary.

3. Existence, Regularity and Non-Degeneracy

Let $\lambda_+ > 0$ and $\lambda_- > 0$, $n \geq 2$, let Ω be a bounded open subset of $\mathbf{R}^n$ with Lipschitz boundary and assume that $u_D \in H^{1,2}(\Omega)$. From [7] we know then that there exists a "solution", i.e. a function $u \in H^{2,2}(\Omega)$ solving the strong equation $\Delta u = \frac{\lambda_+}{2} \chi_{\{u>0\}} - \frac{\lambda_-}{2} \chi_{\{u<0\}}$ a.e. in Ω, and attaining the boundary data u_D in L^2 . The boundary condition may be replaced by other, more general boundary conditions.

Our tools are two powerful monotonicity formulae. One is a monotonicity formula introduced in [6] by one of the authors for a class of semilinear free boundary problems (see also [5]). The second monotonicity formula has been introduced by

H.W. Alt-L.A. Caffarelli-A. Friedman in [1] and proved in [1]. What we are actually going to apply in section 4 is a stronger statement than that of the following monotonicity formula.

For the sake of completeness let us state both of them here. First is the two-phase obstacle problem.

Theorem 3.1 (Weiss's Monotonicity Formula). *Suppose that* $B_\delta(x_0) \subset \Omega$. *Then for all* $0 < \rho < \sigma < \delta$ *the function*

$$\Phi_{x_0}(r) := r^{-n-2} \int_{B_r(x_0)} \Big(|\nabla u|^2 \;+\; \lambda_+ \max(u,0) \;+\; \lambda_- \max(-u,0) \Big)$$

$$- \, 2\, r^{-n-3} \int_{\partial B_r(x_0)} u^2 \, d\mathcal{H}^{n-1} \;,$$

defined in $(0,\delta)$, *satisfies the monotonicity formula*

$$\Phi_{x_0}(\sigma) \;-\; \Phi_{x_0}(\rho) \;=\; \int_\rho^\sigma r^{-n-2} \int_{\partial B_r(x_0)} 2 \left(\nabla u \cdot \nu - 2\, \frac{u}{r} \right)^2 \, d\mathcal{H}^{n-1} \, dr \;\geq 0 \;\; .$$

For a proof see [6].

Next comes the Alt-Caffarelli-Friedman monotonicity formula.

Theorem 3.2 (The Alt-Caffarelli-Friedman Monotonicity Formula). *Let* h_1 *and* h_2 *be continuous non-negative subharmonic* $H^{1,2}$*-functions in* $B_R(z)$ *satisfying* $h_1 h_2 = 0$ *in* $B_R(z)$ *as well as* $h_1(z) = h_2(z) = 0$.
Then the function

$$\Psi_z(r) := r^{-4} \int_{B_r(z)} \frac{|\nabla h_1(x)|^2}{|x-z|^{n-2}}\, dx \int_{B_r(z)} \frac{|\nabla h_2(x)|^2}{|x-z|^{n-2}}\, dx$$

is a non-decreasing function of r *in* $(0,R)$.

For a proof see [1]. We also bring the readers attention to the, readily verified, fact that for degree-one-homogeneous functions h_1, h_2 the function Ψ is constant.

It is noteworthy that

$$\Psi_z(r,u) = \Psi_0(1,u_r), \qquad \Phi_z(r,u) = \Phi_0(1,u_r),$$

where

$$u_r(x) = \frac{u(rx+z)}{r^2}.$$

The way we apply the function $\Psi(r)$ is to the positive and negative part of directional derivatives of u . This is possible by the following lemma due to N. Uraltseva ([4]):

Lemma 3.3. *For each* $e \in \partial B_1(0)$ *the functions* $\max(\partial_e u, 0)$ *and* $-\min(\partial_e u, 0)$ *are subharmonic in* Ω .

For proof see Lemma 2 in [4]. An informal proof can be given as follows. First rewrite the equation as

$$\Delta u = (\lambda_+ H(u) - \lambda_- H(-u))/2,$$

where H denotes the Heavyside step function. Next differentiate in direction e to obtain

$$\Delta\partial_e u = \frac{\partial_e u}{2|\nabla u|}\left(\lambda_+\mathcal{H}^{n-1}\lfloor\partial\{u>0\} + \lambda_-\mathcal{H}^{n-1}\lfloor\partial\{u<0\}\right) .$$

In section 4 we are going to need the following stronger version of the Alt-Caffarelli-Friedman monotonicity formula:

Theorem 3.4. *Let h_1 and h_2 be continuous non-negative subharmonic $H^{1,2}$-functions in $B_R(z)$ satisfying $h_1 h_2 = 0$ in $B_R(z)$ as well as $h_1(z) = h_2(z) = 0$.*
Suppose for $0 < \rho < r < \sigma < R$, $\Psi_z(\rho) = \Psi_z(\sigma)$. Then either of the following holds:
(A) $h_1 = 0$ in $B_\sigma(z)$ or $h_2 = 0$ in $B_\sigma(z)$,
(B) for $i = 1, 2$, and $\rho < r < \sigma$, supp $(h_i) \cap \partial B_r$ is a half-spherical cap and $h_i \Delta h_i = 0$ in $B_\sigma(z) \setminus B_\rho(z)$ in the sense of measures.

Proof. The proof of this theorem follows the same lines as the original proof of [1]. The only differences are that we need to keep the terms that are thrown out during the estimates in [1]. We carry out some details. The following calculations can be justified regularizing as in [1]. Let us assume (A) does not hold. Set

$$I_i(r) = \int_{B_r(z)} \frac{|\nabla h_i(x)|^2}{|x-z|^{n-2}}\, dx.$$

Then

$$\Psi(r) = r^{-4} I_1(r) I_2(r).$$

Upon differentiation we obtain

$$\Psi' = \frac{2\Psi}{r}\left(\frac{rI_1'}{2I_1} + \frac{rI_2'}{2I_2} - 2\right).$$

Next, estimating $I_i(r)$ we have

$$2I_i(r) = \int_{B_r(z)} \frac{\Delta h_i^2 - 2h_i\Delta h_i}{|x-z|^{n-2}}\, dx = \int_{B_r(z)} \frac{\Delta h_i^2}{|x-z|^{n-2}}\, dx - g_i(r),$$

where

$$g_i(r) = \int_{B_r(z)} \frac{2h_i\Delta h_i}{|x-z|^{n-2}}\, dx.$$

Now, as in [1], we can estimate

$$\int_{B_r(z)} \frac{\Delta h_i^2}{|x-z|^{n-2}}\, dx$$

from above by I_i', and arrive at

$$\frac{I_i'(r)}{2I_i(r)} \geq \Lambda_i(r)\left(1 + \frac{g_i(r)}{2I_i(r)},\right)$$

where $-\Lambda_i(r)$ (< 0) is the corresponding first Dirichlet eigenvalue of the one-dimensional Ornstein-Uhlenbeck operator (see [2] for more details).

Using this and the expression for Ψ' we conclude

$$\Psi' \geq \frac{2\Psi}{r}\left(\Lambda_1(r) + \Lambda_2(r) - 2 + \Lambda_1(r)\frac{g_1(r)}{2I_1(r)} + \Lambda_2(r)\frac{g_2(r)}{2I_2(r)}\right).$$

Now according to results of Beckner-Pipher-Kenig (unpublished, see [CK] for another proof)

$$\Lambda_1(r) + \Lambda_2(r) - 2 \geq 0 ,$$

and the strict inequality holds if supp $h_i \cap \partial B_r$ digresses from a half-spherical cap by positive area. This shows the first part of (B). If the second statement in (B) fails, then one of g_i will be nonzero and we'll have $\Psi' > 0$, which contradicts the assumption in the theorem. □

A quadratic growth estimate near the set $\Omega \cap \{u = 0\} \cap \{\nabla u = 0\}$ had already been proved in [7] for more general coefficients λ_+ and λ_- , but local $H^{2,\infty}$- or $C^{1,1}$-regularity of the solution has been shown for the first time in [4]. Cf. also [3].

Theorem 3.5 (Regularity). $u \in H^{2,\infty}_{\text{loc}}(\Omega)$.

A consequence of the quadratic growth estimate near the set $\Omega \cap \{u = 0\} \cap \{\nabla u = 0\}$ and Weiss's monotonicity formula is now that each blow-up limit at a point of this set is a homogeneous function. In the following lemma we include also the case of a "blow-down".

Lemma 3.6. *Let u be a global solution such that $D^2u \in L^\infty(\mathbf{R}^n)$ and let $r_0 \in \{0, +\infty\}$. Then each limit u_0 of $u_r(\cdot) = \frac{u(x_0+r\cdot)}{r^2}$ as a sequence $r_m \to r_0$ $(\in \{0, \infty\})$ is a homogeneous function of degree 2 .*

Proof. For $0 < R < S < +\infty$, $r_m \searrow 0$, let $m > k$ so that $Sr_m \leq Rr_k$ and for $r_m \nearrow 0$, let $m < k$ so that $Sr_m \leq Rr_k$. Then we have $\Phi_{x_0}(Rr_m) \leq \Phi_{x_0}(Sr_m) \leq \Phi_{x_0}(Rr_k)$. Thus

$$S^{-n-2}\int_{B_S(0)}\left(|\nabla u_{r_m}|^2 \;+\; \lambda_+ \max(u_{r_m}, 0) \;+\; \lambda_- \max(-u_{r_m}, 0)\right)$$

$$-\,2\,S^{-n-3}\int_{\partial B_S(0)} {u_{r_m}}^2 \, d\mathcal{H}^{n-1}$$

converges to a constant function of S as $m, k \to \infty$, and Theorem 3.1 implies that u_{r_0} is a homogeneous function of degree 2 . □

A non-degeneracy lemma has already been proven in [7], however we need the following stronger statement on the true two-phase free boundary.

Lemma 3.7 (Non-Degeneracy). *For every $x_0 \in \Omega \cap (\partial\{u > 0\} \cap \partial\{u < 0\})$ and every $B_{2r}(x_0) \subset \Omega$ the following estimates hold:*

$$\sup_{\partial B_r(x_0)} u \;\geq\; \frac{1}{4n}\lambda_+ \, r^2 \, ,$$

$$-\inf_{\partial B_r(x_0)} u \geq \frac{1}{4n}\lambda_- r^2 .$$

Proof. First note that for any ball B and strong sub/supersolutions v, w of (1.1) in B satisfying $v \leq w$ on ∂B we may multiply the differential inequalities by $\max(v-w, 0)$ and integrate to obtain the comparison principle

$$\int_B |\nabla(v-w)|^2 \leq -\frac{\lambda_+}{2}(\chi_{\{v>0\}} - \chi_{\{w>0\}}) \max(v-w,0)$$

$$+\frac{\lambda_-}{2}(\chi_{\{v<0\}} - \chi_{\{w<0\}}) \max(v-w,0) \leq 0$$

Concerning the proof of the lemma, we choose a sequence $\{u>0\} \ni x_m \to x_0$ as $m \to \infty$. Supposing that $\sup_{\partial B_r(x_m)} u \leq \frac{1}{4n}\lambda_+ r^2$, the comparison principle yields that $u(x) \leq v(x) := \frac{1}{4n}\lambda_+ |x-x_m|^2$ in $B_r(x_m)$, a contradiction to the fact that $u(x_m) > 0$.
The estimate for $-\inf_{\partial B_r(x_0)} u$ is obtained the same way, replacing u by $-u$ and λ_+ by λ_-. □

4. Global Solutions

Lemma 4.1. *Let u be a global solution such that $u(x_0) = 0$ for some $x_0 \in \mathbf{R}^n$, $\nabla u = 0$ on $\{u = 0\}$ and $|D^2 u| \leq C$ in $\mathbf{R}^n$. Then $\max(u, 0)$ and $-\min(u, 0)$ are convex functions.*

Proof. As $\max(u,0)$ and $-\min(u,0)$ are in this case solutions of the one-phase obstacle problem, we can apply the well-known blow-up arguments: it is sufficient to show that $\partial_{ee} u \geq 0$ in $\{u>0\}$ for each $e \in \partial B_1(0)$. Suppose towards a contradiction that $L := \inf_{\{u>0\}} \partial_{ee} u(x) < 0$, let $(x_m)_{m\in\mathbf{N}} \subset \{u>0\}$ be a sequence such that $\lim_{m\to\infty} \partial_{ee} u(x_m) = L$ and let $r_m = \operatorname{dist}(x_m, \{u \leq 0\}) \to R \in \{0, +\infty\}$ (the case $R \in (0,+\infty)$ is much easier). Defining $u_m(x) := \frac{u(x_m + r_m x)}{{r_m}^2}$ and passing if necessary to a subsequence, we obtain from the assumptions that $u_m \to u_0$ weakly-* in $H^{2,\infty}_{\text{loc}}(\mathbf{R}^n)$ and strongly in $C^{1,\alpha}_{\text{loc}}(\mathbf{R}^n) \cap C^{2,\alpha}_{\text{loc}}(B_1(0))$ as $m \to \infty$. It follows that $u_0 \geq 0$ in $B_1(0)$ and $\nabla u_0 = 0$ on $\{u_0 = 0\}$, that $\Delta u_0 = \frac{\lambda_+}{2}$ in $B_1(0)$ and that $L = \partial_{ee} u_0(0) = \inf_{B_1(0)} \partial_{ee} u_0$. Hence, by the strong maximum principle, $\partial_{ee} u_0 = L < 0$ in $B_1(0)$ and in a neighborhood of $A = \{te : t \in \mathbf{R} \text{ and } u_0(se) > 0 \text{ for } s \in [0,t]\}$. We observe that the non-negativity of u_0 in $B_1(0)$ and $\partial_{ee} u_0(0) < 0$ imply that A is non-empty and bounded. Thus $\phi(t) = u_0(te)$ is in $I = \{t \in \mathbf{R} : u_0(se) > 0 \text{ for } s \in [0,t]\}$ a parabola satisfying $\phi'' = L < 0$. We obtain that $\phi' \neq 0$ at the boundary points ℓ_- and ℓ_+ of the interval I. This implies $\nabla u_0(\ell_- e) \neq 0$, contradicting the fact that $\nabla u_0 = 0$ on $\{u_0 = 0\}$. □

Corollary 4.2. *Let u be a global solution such that $\{u>0\}$ and $\{u<0\}$ are both non-empty, that $\nabla u = 0$ on $\{u = 0\}$ and $|D^2 u| \leq C$ in $\mathbf{R}^n$. Then u is after a translation and rotation for some $\ell \in (-\infty, 0]$ of the form $u(x) = -\frac{\lambda_-}{4}\min(x_n - \ell, 0)^2 + \frac{\lambda_+}{4}\max(x_n, 0)^2$.*

Proof. It follows from Lemma 4 that the sets $E_+ := \{u \geq 0\}$ and $E_- := \{u \leq 0\}$ are convex. At any point $x^0 \in \partial E_-$ there exists a supporting plane $T \subset E_+$. Thus E_+ is a half-space and the same is true for E_-. As $E_+ \cup E_- = R^n$ it follows that ∂E_- is parallel to ∂E_+. Rotating and translating we obtain $\partial E_- = \{x_n = 0\}$, $\partial E_+ = \{x_n = l\}$ with $l \leq 0$. □

Theorem 4.3. *Let u be a global solution such that $x_0 \in \partial\{u > 0\} \cap \partial\{u < 0\}$ and $\nabla u(x_0) = 0$ for some $x_0 \in \mathbf{R}^n$ and that $|D^2 u| \leq C$ in $\mathbf{R}^n$. Then u is after a translation and rotation of the form $u(x) = -\frac{\lambda_-}{4}\min(x_n, 0)^2 + \frac{\lambda_+}{4}\max(x_n, 0)^2$.*

Proof. By a translation we may assume that $x_0 = 0$. We will use the notation $\Gamma^* := \{u = 0\} \cap \{\nabla u \neq 0\}$.
Step 1: We show that the theorem holds in the case that u is homogeneous of degree two.
To this end, we prove that Γ^* is in this case empty whereupon the statement in the Theorem follows from Corollary 4.2.
First, we apply the Alt-Caffarelli-Friedman monotonicity formula Theorem 3.2 for fixed $e \in \partial B_1(0)$ to the directional derivative $\partial_e u$. Since $\partial_e u$ is a homogeneous function of degree 1, the function of the monotonicity formula $\Psi_0(r)$ is constant in r. From Theorem 3.4 we obtain therefore that either
(A) $\partial_e u \geq 0$ in $\mathbf{R}^n$ or $\partial_e u \leq 0$ in $\mathbf{R}^n$
or
(B) $\max(\partial_e u, 0)\Delta\max(\partial_e u, 0) = 0$ in $\mathbf{R}^n$ and $\min(\partial_e u, 0)\Delta\min(\partial_e u, 0) = 0$ in $\mathbf{R}^n$ in the sense of measures.

Suppose now that there exists a point $y_0 \in \Gamma^*$ and denote by ν the direction of gradient of u at y_0. There is a neighborhood $B_\rho(y_0)$ where $\partial_\nu u > 0$ and $\{u = 0\} \cap B_\rho(y_0)$ is a $C^{1+\alpha}$-surface. If $e \cdot \nu \neq 0$ then $\partial_e u(y_0) \neq 0$, and for sufficiently small δ we obtain

$$|\Delta\partial_e u|(B_\delta(y_0)) = \frac{|\lambda_+ + \lambda_-|}{2}\int_{\partial\{u=0\}\cap B_\delta(y_0)} |e\cdot\nu|\, d\mathcal{H}^{n-1}$$

$$= \frac{|\lambda_+ + \lambda_-|}{2}\int_{\partial\{u=0\}\cap B_\delta(y_0)} \frac{|e\cdot\nabla u|}{|\nabla u|}\, d\mathcal{H}^{n-1} \neq 0\,.$$

Thus (A) holds. More precisely, $\partial_e u \geq 0$ in $\mathbf{R}^n$ if $e\cdot\nu 0$ and $\partial_e u < 0$ in $\mathbf{R}^n$ if $e\cdot\nu < 0$. Hence $\partial_e u = 0$ in $\mathbf{R}^n$ for each $e \perp \nu$. By the assumption $\nabla u(x_0) = 0$, this implies that u must after rotation be of the form $u(x) = -\frac{\lambda_-}{4}\min(x_n, 0)^2 + \frac{\lambda_+}{4}\max(x_n, 0)^2$, which contradicts the assumption that Γ^* is non-empty.

Step 2: We are now ready to prove the theorem in the general case.
To this end, we consider blow-up limits and "blow-down" limits. Let $u_r(x) := \frac{u(rx)}{r^2}$. By the assumptions $(u_r)_{r\in(0,+\infty)}$ is bounded in $H^{2,\infty}_{\text{loc}}(\mathbf{R}^n)$. By the non-degeneracy

property Lemma 3.7,

$$\min(\sup_{\partial B_r(0)} u, -\inf_{\partial B_r(0)} u) \geq \frac{1}{4n}\min(\lambda_+, \lambda_-) r^2 \text{ for all } r \in (0, +\infty)$$

We find therefore two sequences $(u_{r_m})_{m\in\mathbf{N}}$ and $(u_{R_m})_{m\in\mathbf{N}}$ such that $r_m \to 0, R_m \to +\infty$ and $u_{r_m} \to u_0, u_{R_m} \to u_\infty$ weakly-* in $H^{2,\infty}_{\text{loc}}(\mathbf{R}^n)$ and strongly in $C^{1,\alpha}_{\text{loc}}(\mathbf{R}^n)$ as $m \to \infty$, $\nabla u_0(0) = \nabla u_\infty(0) = 0$ and $0 \in \partial\{u_0 > 0\} \cap \partial\{u_0 < 0\} \cap \partial\{u_\infty > 0\} \cap \partial\{u_\infty < 0\}$. Furthermore u_0 and u_∞ are by Lemma 3.6 homogeneous functions of degree 2 . From the result of Step 1 we infer therefore that there are rotations U_1 and U_2 such that $u_0(U_1 x) = u_\infty(U_2 x) = -\frac{\lambda_-}{4}\min(x_n, 0)^2 + \frac{\lambda_+}{4}\max(x_n, 0)^2$. But then the function of the monotonicity formula $\Phi_0(r)$, (applied to u), satisfies $\Phi_0(0+) = \Phi(+\infty)$ and Theorem 3.1 implies that u must have been a homogeneous function of degree 2 all along. The statement follows then from the result proved in Step 1. □

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