Gamma limits in some Bernoulli free boundary problem

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26th October 2004

Abstract

We study the limit cases $p \to \infty$ and $p \to 1$ of the functionals

$$E_p(u) := \int_{\mathbb{R}^n} \left\{ \frac{1}{p} \left(\frac{|\nabla u|}{a} \right)^p + \frac{p-1}{p} \, \chi_{\{u>0\}} \right\} \, dx, \tag{1}$$

where $u \equiv 1$ on a given compact set $K \subset \mathbb{R}^n$, and $a > 0$ is also given. Minimizers u_p of these functionals have uniformly bounded support $\Omega_p := \{u_p > 0\}$ and satisfy

$$-\Delta_p u_p = 0 \quad \text{in } \Omega_p, \qquad u_p \equiv 1 \quad \text{on } K, \qquad |\nabla u_p| = a \quad \text{on } \partial\Omega_p. \tag{2}$$

Keywords: Bernoulli free boundary problem, Gamma convergence, p-harmonic, viscosity solution.

AMS classification: 35R35, 35J60, 49J45.

1 Introduction

For $p = 2$ this problem is known as Bernoulli's free boundary problem, and since the early treatments of Friedrichs [11] and Beurling [5] this problem and its generalizations have repeatedly attracted the attention of mathematicians, e.g. in [2], [13], [16] etc.

The problem has several applications in that it models non-Newtonian fluid flow problems, galvanization processes and so on. A list of applications and appropriate references can be found in [1].

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Of some interest was the question if the shape of K is somehow reflected in the shape of Ω_p. If K is a ball, then so is Ω_p, if K is starshaped, then so is Ω_p, see for instance [20], [1], and if K is convex, then so is Ω_p, see e.g. [1] or [14]. As soon as $\partial\Omega_p$ is smooth enough in the sense that it satisfies a certain flatness condition from [2], it has a uniquely defined normal and the boundary condition $|\nabla u| = a$ is satisfied in the classical sense

$$\lim_{\Omega_p \ni y \to x \in \partial\Omega_p} |\nabla u(y)| = a.$$

But non-smooth free boundaries can also occur for non-starlike K, and then this boundary condition can only be derived in its weak form, see [2], [16]

$$\lim_{\varepsilon \searrow 0} \int_{\partial\{u>\varepsilon\}} \{|\nabla u|^p - a^p\} \ \eta \cdot \nu \ d\mathcal{H}^{n-1} = 0 \tag{3}$$

for every vector field $\eta \in C_0^\infty(\Omega_p; \mathbb{R}^n)$. Here ν denotes the exterior normal to $\partial\{u > \varepsilon\}$.

From comparison results it follows that the sets Ω_p are all contained in a $\frac{1}{a}$–neigbourhood of the convex hull of K, so that all domains of integration can be limited to a sufficiently large ball $B \subset \mathbb{R}^n$. Throughout this paper B is fixed. In [22] the authors investigated the limits $p \to \infty$ and $p \to 1$ for problem (2) in the case of *convex* K. In this case the solutions of (2) are known to be unique, and thus they are also unique minimizers of E_p. Moreover, they have convex level sets, and this implies that the sequence u_p is pointwise monotone nondecreasing in p. So its pointwise limits $u_\infty = \lim_{p\to\infty} u_p$ and $u_1 = \lim_{p\to 1} u_p$ exist, and they were identified as

$$u_\infty(x) = (1 - a \operatorname{dist}(x, K))^+ \quad \text{and} \quad u_1(x) = \chi_K(x). \tag{4}$$

In the present paper we study the case of general, i.e. also *non-convex* K, in which problem (2) can have more than one solution, and we focus on the minimizers of E_p rather than on (2). We show that after extending their domain of definition, the functionals E_p Γ–converge, as $p \to \infty$ to

$$E_\infty(u) := \int_B \left\{ I_{[0,a]}(|\nabla u(x)|) + \chi_{\{u>0\}}(x) \right\} \ dx \tag{5}$$

for any $q > n$ on $W_0^{1,q}(B) \cap \{u \equiv 1 \text{ on } K\}$, and as $p \to 1$ the functionals E_p Γ–converge in $L^1(B) \cap \{u \equiv 1 \text{ on } K\}$ to

$$E_1(u) := \tfrac{1}{a} \int_B |Du(x)| \ dx. \tag{6}$$

To avoid misunderstandings, let us recall that the indicator function I_C of a set C vanishes on C and is $+\infty$ elsewhere, while the characteristic function χ_C is identically 1 on C and vanishes elsewhere. An inspection of these limiting functionals shows that minimizing (5) amounts to minimizing the support of u under the side constraint $|\nabla u| \leq a$ a.e., so that $u_\infty(x) = 1 - dist(x, K)$ is one (of possibly several) minimizers of E_∞. For details we refer to Section 2. So in this limit problem, a volume is minimized.

In contrast to this, minimizing E_1 amounts to finding sets $D \supset K$ of minimal perimeter, because according to the coarea formula the characteristic function of such sets minimize E_1. Clearly, if K is convex, ∂K is the only minimal surface that encloses K, and this recovers the result from [22], but for nonconvex simply connected K and $n = 2$ characteristic function of the convex hull of K constitutes the unique minimizer of E_1. There are also cases of nonuniqueness described in Section 3. So in this limit problem, a surface area is minimized.

It is interesting that studying the limit problems leads to such simple geometric questions. A similar effect occurred in the study of optimal Poincaré constants Λ_p in the estimate $||\nabla u||_p \geq \Lambda_p ||u||_p$ for functions in $W_0^{1,p}(\Omega)$. Clearly Λ_p depends on Ω, but $\Lambda_\infty(\Omega)$ is the inverse of the radius of the largest ball inside Ω, a simple geometric quantity (see [19] and [4]), and $\lambda_1(\Omega) = \inf_{D \subset\subset \Omega} \frac{|\partial D|}{|D|}$ is the so-called Cheeger constant of Ω which involves only perimeter and volume of subsets, see [10].

The usefulness of our Γ-convergence results are apparent when we recall the definition of- and a principal result on Γ–convergence, see [8] or [6].

Let X be a metric space and $F_\varepsilon : X \mapsto [0, \infty]$ a family of mappings. Then F **is the Γ-limit of** F_ε as $\varepsilon \to 0$, if and only if the following statements **a)** and **b)** hold.

a) For every $u \in X$ and every sequence $u_\varepsilon \to u$ in X

$$\liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) \geq F(u) \ . \tag{7}$$

b) For every $u \in X$ there exists a sequence u_ε such that $u_\varepsilon \to u$ in X and

$$\limsup_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) \leq F(u) \ . \tag{8}$$

Theorem 1 *[[6]-[8]] If F is Γ–limit of F_ε and if u_ε is a minimizer of F_ε, then every cluster point u of $\{u_\varepsilon\}_{\varepsilon>0}$ minimizes F.*

The proof of Γ–convergence or the existence of a cluster point can be difficult, as we shall see. In our situation, however, the following observation will be very helpful.

Proposition 1 *The family of functionals E_p is monotone nondecreasing in p, that is $E_p(v) \le E_q(v)$ for $q \ge p$.*

This follows from a simple application of Young's inequality $AB \le \frac{A^r}{r} + \frac{B^s}{s}$ with $s = r/(r-1)$ and the identification $A = (|\nabla v|/a)^p$, $B = 1$ and $r = q/p$.

2 The case $p \to \infty$

In this case we fix $q > n$, choose $X_q = \{v \in W_0^{1,q}(B);\ v \equiv 1 \text{ on } K\}$ and define E_p as follows

$$E_p(u) = \begin{cases} \int_B \left\{ \frac{1}{p} \left(\frac{|\nabla u|}{a} \right)^p + \frac{p-1}{p}\ \chi_{\{u>0\}} \right\}\ dx & \text{if } u \in W_0^{1,p}(B) \cap X_q, \\ +\infty & \text{if } u \in X_q \setminus W_0^{1,p}(B). \end{cases} \tag{9}$$

To prove Γ-convergence of E_p to E_∞ let us verify the inequalities (7) and (8) with $\varepsilon = 1/p$.

To verify (7) let $u \in W_0^{1,q}(B)$ and suppose that u_p is a sequence converging to u in $W_0^{1,q}(B)$. If $|\nabla u| \le a$ a.e. in B, then $E_\infty(u) = |\{u > 0\}|$ and $E_\infty(u) \le \liminf E_p(u_p)$ provided $\liminf |\{u_p > 0\}| \ge |\{u > 0\}|$. But this is clearly so, since if x is in the support of u, and u_p converges uniformly to u, then x is in the support of u_p for sufficiently large p. If, however, $|\nabla u| > a + \varepsilon$ on a set of positive measure, then since ∇u_p converges in L^q to ∇u, also $\nabla u_p > a + \varepsilon/2$ on a set of positive measure uniformly for large p. Therefore the left hand side in (7) becomes infinite.

To verify (8) we set $u_\varepsilon = u$. If $|\nabla u| > a$ on a set of positive measure, then $E_\infty(u) = +\infty$ and there is nothing to prove, and if $|\nabla u| \le a$ a.e. in B then $E_\infty(u) = |\{u > 0\}|$ is the volume of the support of u and

$$E_p(u) \le \tfrac{1}{p} |\{u > 0\} \setminus K| + \tfrac{p-1}{p} |\{u > 0\}| < |\{u > 0\}|.$$

This proves (8), and hence we have shown the following theorem.

Theorem 2 *As $p \to \infty$, for each $q > n$ the functionals E_p defined by* (9) *on X_q Γ-converge to the functional E_∞ given by* (5) *on X_q.*

In view of Theorem 1 it is instructive to study minimzers of E_∞. They satisfy $|\nabla u| \le a$ a.e. in B and they must minimize the volume of their support under this constraint in X_q. One minimizer is given by $u_\infty(x) = (1 - a\,\mathrm{dist}(x, K))^+$, but this is not necessarily the only minimizer. To see that E_∞ can in general have more than one minimizer, suppose that $n = 2$ and that K is the union of two disjoint balls B_1 and B_2 of distance $b < 1/a$.

If $u_1 = (1 - a\,\mathrm{dist}(x, B_1))^+$ and $u_2 = (1 - a\,\mathrm{dist}(x, B_2))^+$, then $u_\infty = (1 - a\,\mathrm{dist}(x, K))^+ = \max\{u_1(x), u_2(x)\}$.

Now consider the set $D := \{u_1 > 0\} \cap \{u_2 > 0\}$ where the supports of u_1 and u_2 overlap each other. In this set we can modify u_∞ to $v_\infty := u_\infty(x) + \varepsilon\eta(x)$ with $\eta \in C_0^\infty(D)$ nonnegative, and still get a minimizer, because the support of v_∞ and u_∞ coincide and v_∞ still satisfies the gradient constraint $|\nabla u| \leq a$ a.e. in B.

If, however, the union U of all fall lines of u_∞ emanating from ∂K and ending in a boundary point of its support equals $\{u_\infty > 0\} \setminus K$, as is the case for convex K, then E_∞ has only u_∞ as a minimizer.

To apply Theorem 1 we should check if the family of minimizers u_p of E_p has a cluster point in X_q.

First we observe that

$$E_p(u_p) \leq E_p(u_\infty) = \tfrac{1}{p}|\{u_\infty > 0\} \setminus K| + \tfrac{p-1}{p}|\{u_\infty > 0\}| < |\{u_\infty > 0\}|,$$

so that

$$||\nabla u_p||_p \leq a\ (p|\{u_\infty > 0\}|)^{1/p} \quad \to a \qquad \text{as } p \to \infty. \tag{10}$$

This proves that the sequence u_p is uniformly bounded in every $W^{1,r}(B)$ for sufficiently large $p > r$. In fact, using Cauchy Schwarz inequality and (10)

$$||\nabla u_p||_r \leq ||\nabla u_p||_p\ |B|^{\frac{1}{r}-\frac{1}{p}} \quad \to a\ |B|^{\frac{1}{r}} \text{ as } p \to \infty. \tag{11}$$

Therefore $\{u_p\}$ has a subsequence that converges weakly in $W^{1,q}(B)$ and strongly in $C(B)$ to a limit v.

Notice that v is NOT necessarily a cluster point of u_p in X_q, because the sequence does not converge in the strong topology of X_q. Therefore we cannot apply Theorem 1, and changing the definition of X_q to $C(B)$ might be helpful here, but creates problems when checking Γ-convergence.

What can be said about v, anyway? Since the bound (11) is uniform as $r \to \infty$, for any $\varepsilon > 0$ and any sufficiently large r we obtain $||\nabla v||_q \leq a + \varepsilon$, i.e. v satisfies the gradient constraint $|\nabla v| \leq a$ a.e. in B for minimizers of E_∞ so that $E_\infty(v)$ is finite.

Is E_∞ minimal at v? To see this we observe

$$\begin{aligned} E_p(u_p) &\leq E_p(v) \\ &\leq \tfrac{1}{p}|\{v > 0\} \setminus K| + \tfrac{p-1}{p}|\{v > 0\}| \\ &< |\{v > 0\}| \\ &\leq \textstyle\int_B \chi_{\{u_p > 0\}} dx + \varepsilon \\ &= E_p(u_p) - \tfrac{1}{p}\textstyle\int_B \left(\frac{|\nabla u_p|}{a}\right)^p\ dx + \varepsilon. \end{aligned} \tag{12}$$

This chain of inequalities holds for sufficiently large p and it shows that for any $u \in X_q$

$$E_\infty(v) \leq \liminf_{p\to\infty} E_p(u_p) \leq \liminf_{p\to\infty} E_p(u).$$

Together with Proposition 1 we may conclude that $E_\infty(v) \leq E_\infty(u)$ for any $u \in X_q$, that is v minimizes E_∞. This proves the first part of the following result.

Theorem 3 *After passing to a subsequence, if needed, u_p converges weakly in $W^{1,q}(B)$ and strongly in $C(B)$ to a minimizer v of E_∞ as $p \to \infty$. Moreover, this minimizer is ∞–harmonic so that it satisfies the differential equation*

$$\Delta_\infty v := \sum_{i,j} v_{x_i} v_{x_j} v_{x_i x_j} = 0 \qquad \text{on } \{0 < v < 1\} \tag{13}$$

in the sense of viscosity solutions.

Proof. To prove that v is ∞–harmonic one can appeal to a stability result for viscosity solutions, which says that if u_p is a viscosity solution of $F_p(Du, D^2u :) = 0$ and both F_p and u_p converge to F_∞ and v, then v is a viscosity solution of F_∞ (see Exercise 8.2 in [7]). Another way of proving this, is to use direct computations as done in [18] (proof of Theorem 1.22), [22] (Theorem 9.1) or [17] (proof of Proposition 5.4).

□

3 The case $p \to 1$

In this case we set $Y := \{v \in L^1(B);\ v \equiv 1 \text{ on } K\}$ and extend the domain of definition of E_p to Y, so that under slight abuse of notation

$$E_p(u) = \begin{cases} \int_B \left\{ \frac{1}{p} \left(\frac{|\nabla u|}{a} \right)^p + \frac{p-1}{p}\ \chi_{\{u>0\}} \right\} dx & \text{if } u \in W_0^{1,p}(B) \cap Y \\ +\infty & \text{if } u \in Y \setminus W_0^{1,p}(B) \end{cases} \tag{14}$$

The limit functional

$$E_1(u) = \begin{cases} \frac{1}{a} \int_B |Du(x)|\ dx & \text{if } u \in BV(B) \cap Y \\ +\infty & \text{if } u \in Y \setminus BV(B) \end{cases} \tag{15}$$

can be rewritten, using the coarea formula (see [9]), as

$$E_1(u) = \frac{1}{a} \int_0^1 |\partial\{u > t\}|\ dt$$

and has minimizers, almost all of whose level sets have minimal perimeter among all subsets of B that contain K. To show that E_p is Γ–convergent to E_1, rather than checking (7) and (8) again, we can refer to [8], Proposition 5.7, which says in our case that a sequence E_p which decreases pointwise in Y to E_1, also Γ– converges to a limit functional, and that its Γ–limit can be identified as the lower semicontinuous envelope $\mathrm{sc}^- E_1$ of E_1.

So to prove Γ–convergence of E_p to E_1 as $p \to 1$ it suffices to show that E_1 is lower semicontinuous on $L^1(B)$. To this end suppose that $u \in L^1(B)$ and that there is a sequence $u_k \to u$ in $L^1(B)$. We have to show

$$E_1(u) \leq \liminf_{k \to \infty} E_1(u_k), \tag{16}$$

and without loss of generality we may assume that every element u_k is in $BV(B)$, because otherwise $E_1(u_k) = \infty$ and there is nothing to show. But then (16) is the well-known semicontinuity property of the BV-seminorm, see [12], p.7 or [9], p.172. This proves the following theorem.

Theorem 4 *As $p \to 1$ the functionals E_p defined by (14) on Y Γ-converge to the functional E_1 given by (15) on Y.*

Combining Theorem 4 with Theorem 1 we can now show

Theorem 5 *After passing to a subsequence, if needed, u_p converges strongly in $L^1(B)$ to a minimizer w of E_1 as $p \to 1$. Moreover, the boundary of almost each level set of w minimizes perimeter among sets containing K.*

Proof. To get uniform bounds on the minimizers u_p of E_p let $\hat{K}$ be a perimeter minimizing set containing K (there may be several) and set $u_1 = \chi_{\hat{K}}(x)$. We would like to estimate $E_p(u_p)$ by $E_p(u_1^\varepsilon)$, where u_1^ε is close to u_1 but in $W^{1,p}(B)$. Therefore we set $u_1^\varepsilon = (1 - \frac{1}{\varepsilon}\operatorname{dist}(x, \hat{K}))^+$ and find out that

$$E_p(u_p) \leq E_p(u_1^\varepsilon) = \int_{0<\operatorname{dist}(x,\hat{K})<\varepsilon} \tfrac{1}{p}\left(\tfrac{1}{a\varepsilon}\right)^p \, dx + \tfrac{p-1}{p}(|\hat{K}| + O(\varepsilon)).$$

Notice that the last term becomes smaller than any given δ as $p \to 1$, while the integral term can be estimated from above by

$$\tfrac{1}{p}(|\partial\hat{K}| + \delta)a^{-p}\varepsilon^{1-p}.$$

If we choose $\varepsilon = p - 1$, we see that

$$||\nabla u||_p \leq \tfrac{1}{a}(|\partial\hat{K}| + \delta) \tag{17}$$

provides a uniform bound for u_p as $p \to 1$. This bound implies in particular that u_p is bounded in $BV(B)$, because

$$\int_B |Du_p|\ dx \leq ||\nabla u_p||_p\ |B|^{\frac{p-1}{p}}$$

so that it has a weakly convergent subsequence and a limit w as $p \to 1$. Using the compact embedding of BV into L^1, for this subsequence $u_p \to w$ in $L^1(B)$ and thus the assumptions of Theorem 1 are verified and w must be a minimizer of E_1.

Incidentally, without having to appeal to Theorem 1 this estimate and and (17) show that

$$\limsup_{p\to 1^+}\ E_p(u_p) = \tfrac{1}{a}|\partial \hat{K}| = \inf_{v\in X} E_1(v), \tag{18}$$

so that w is indeed a minimizer of E_1. Consequently, the boundary of almost each level set of w minimizes perimeter among sets containing K. In particular, if there is only one set $\hat{K}$ that minimizes perimeter and contains K, as in the case where K is convex and $\hat{K} = K$, then $w(x) = \chi_{\hat{K}}(x)$. ▭

Remark. It should be noted that there are situations in which more than one set can minimize perimeter and contain K. Suppose that $n = 2$ and that K is the union of two disjoint unit balls of distance d from each other. For small d the convex hull $conv(K)$ of K will minimize perimeter, while for large d the set K will minimize perimeter. For continuity reasons there is a particular d at which both sets minimize perimeter. In that case it is conceivable (although unlikely) that the function w from above, which was the L^1–limit of a subsequence of u_p as $p \to 1$, could be a step function, e.g. $w = t\chi_{conv(K)} + (1-t)\chi_K$ with $t \in (0,1)$. The fact that this K has two components is not relevant here. Another example of nonuniqueness can be constructed in $\mathbb{R}^3$ by taking a torus and varying its radii.

It is natural to ask if the limit w of u_p satisfies the limit differential equation $\mathrm{div}(\nabla u/|\nabla u|) = 0$ by applying general stability results for viscosity solutions as in the proof of Theorem 3. Notice that p-harmonic functions u_p satisfy $F_p(Du_p, D^2u_p) = 0$ with

$$F_p(q,X) = \ -|q|^{p-4}\left\{(p-2)\left\langle Xq,q\right\rangle + |q|^2\ \mathrm{trace}X\right\} \tag{19}$$

and that F_p is not well-defined (and discontinuous) at $q = 0$. If we define $H_p(q,X) := |q|^{2+\varepsilon}F_p(q,X)$, then u_p solves also the equation

$$H_p(Du_p, D^2u_p) = 0$$

in its support and H_p is continuous at $q = 0$. Now we can apply a stability result from [7] (Proposition 8. 2) or [3] (Exercise on p. 74) to conclude that the upper semicontinuous function $\overline{w}$ is a viscosity subsolution of $H_1 = 0$, i.e. a solution of $H_1 \leq 0$, while the lower semicontinuous function $\underline{w}$ is a supersolution of $H_1 = 0$. Here the upper weak limit $\overline{w}(x)$ is defined as

$$\begin{aligned} \overline{w}(x) &= \limsup{}^*_{p\to 1} u_p(x) \\ &= \limsup{}_{r\to 1}\{u_p(y) : r \geq p,\ |y-x| \leq p-1,\ w(y) > 0,\ y \notin K\} \end{aligned}$$

and the lower weak limit $\underline{w}$ is given by $-(\overline{-w})$. It is in this sense that our sequence u_p converges to a particular minimizer w of E_1.

Acknowledgement: This research was partially supported by the ESF program "Global and geometric aspects of nonlinear partial differential equations". H. Shahgholian was supported in part by Swedish Research Council. We thank G.Buttazzo for bringing the manuscript [17] to our attention.

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