THE SINGULAR SET FOR THE COMPOSITE MEMBRANE PROBLEM

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Abstract. In this paper we study the behavior of the singular set

$$\{u = |\nabla u| = 0\},$$

for solutions u to the free boundary problem

$$\Delta u = f\chi_{\{u\geq 0\}} - g\chi_{\{u<0\}},$$

with $f(x) + g(x) < 0$, $f, g \in W^{1,p} \cap C^0$. Such problems arises in an eigenvalue optimization for composite membranes. Here we show that if for a singular point $z \in \{u = \nabla u = 0\}$, the density is positive

$$|\{u \geq 0\} \cap B_r(z)| \geq c_0 r^n, \qquad \text{for some } c_0 > 0,$$

then z is isolated. The density assumption can be motivated by the following example

$$u = x_1^2, \quad f = 2, \quad g < -2, \qquad \text{and } \{u < 0\} = \emptyset.$$

1. Introduction and Background

In this paper we analyze properties of singular sets of solutions of a certain eigenvalue optimization problem. This problem has been very much in focus lately (see [CGK], [CGIKO]). The problem, stated in physical terms, amounts to building a body of a prescribed shape out of given materials of varying densities, in such a way that the body has a prescribed mass and with the property that the fundamental frequency of the resulting membrane (with fixed boundary) is lowest possible. The reformulated and slightly more general mathematical problem is as follows. A bounded Lipschitz domain $\Omega \subset \mathbf{R}^n$ is given. Also given are two numbers $\alpha > 0$, and $A \in [0, |\Omega|]$, ($|A|$ denotes the Lebesgue measure of Ω). Let λ be the lowest eigenvalue of the problem,

$$-\Delta v + \alpha\chi_D v = \lambda v \quad \text{in } \Omega \qquad v = 0 \quad \text{on } \partial\Omega. \tag{1.1}$$

and set

$$\Lambda_\Omega(\alpha, A) = \inf_{D\subset\Omega, |D|=A} \lambda_\Omega(\alpha, D)$$

Then one is interested in the optimal pair (v, D), solving the above problem. The existence of such an optimal pair is shown using minimization of the corresponding functional. Moreover it is known that any optimal pair has the property that $D = \{v \leq t\}$, for some t.

Now, after rewriting the equation $u := t - v$, and taking into consideration that $D = \{v \leq t\}$, one arrives at

$$\Delta u = (\alpha\chi_{\{u\geq 0\}} - \lambda)(u - t). \tag{1.2}$$

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One of the main questions that has puzzled several mathematicians is whether the singular set $S_u := \{u = |\nabla u| = 0\}$ is isolated, or small enough.

In this paper we consider a more general problem of the type

$$\Delta u = f\chi_{\{u\geq 0\}} - g\chi_{\{u<0\}}, \tag{1.3}$$

where f, g are $W^{1,p} \cap C^0$-functions.

Throughout this paper we assume

$$f + g < 0 \qquad \text{on the singular set } S_u \tag{1.4}$$

The case $f+g \geq 0$, can be handled easier, due to strong tools such as Alt-Caffarelli-Friedman monotonicity formula [ACF], see [SUW1]-[SUW2] for a general treatment of this case.

Our main result is the following.

Theorem 1.1. *Let u solve problem (1.3) in a bounded domain D (with given boundary condition) and suppose condition (1.4) is satisfied at $z \in S_u$*

$$f(z) + g(z) < 0. \tag{1.5}$$

Suppose further that for $z \in S_u := \{u = |\nabla u| = 0\}$, we have positive constants c_0, r_0 such that

$$|\{u \geq 0\} \cap B_r(z)| \geq c_0 r^n, \tag{1.6}$$

for all $0 < r < r_0$. Then z is an isolated point of S_u.

The proof uses a simple blow-up argument in combination with monotonicity of certain energy functionals, due to G.S. Weiss.

Define

$$u_r(x) := \frac{u(rx+z)}{r^2}, \qquad f_r(x) = f(rx+z).$$

Then the following function (due to Weiss [W])

$$W(r) = W(r,u,z) := \int_{B_1(0)} \left(|\nabla u_r|^2 + 2f_r u_r^+ + 2g_r u_r^-\right) - 2\int_{\partial B_1(0)} u_r^2,$$

has the property that

$$W(r) + C_0 r^{1/2} \qquad \textit{is monotone increasing in } r. \tag{1.7}$$

Here C_0 depends on the $W^{1,p}$ norms of f, g. If f, g are constants then we may take $C_0 = 0$ and also claim that the function

$$W(r) \quad \textit{is strictly monotone unless } u \textit{ is homogeneous of degree two,} \tag{1.8}$$

(see [W]).

First we state the following lemma.

Lemma 1.2. *If u is a degree two homogeneous solution to our problem (1.3), with $f = f_0, g = g_0$ constants, and $f_0 + g_0 < 0$, then either $S_u = \{0\}$ or (after rotation) $S_u = \{x_1 = 0\}$.*

The reader may found a proof of this in [B], where the author tacitly assumes $\{u < 0\} \neq \emptyset$. The case $\{u < 0\} = \emptyset$ was unfortunately forgotten to be considered in [B].

The proof of this, however, follows by straight-forward computations, or simply by the fact that due to homogeneity, if $z \in S_u$ and $z \neq 0$, then the ray $L_z := \{tz,\ t > 0\} \in S_u$. We may rotate and assume L_z is the positive x_1-axis. Now, if the set $\{u <$

$0\}$ is empty then $f_0x_1^2/2$ is a solution. Otherwise, to obtain a contradiction, apply the Hopf's boundary lemma to u in one of the half-balls $\{|x-(1,0)|<r_0,\ x_1>0\}$, or the lower one $\{|x-(1,0)|<r_0,\ x_1<0\}$, for some small r_0.

Lemma 1.3. *Let u be a solutions to our problem (1.3), with $f,g\in W^{1,p}$ $(p>1)$, and suppose that for some sequence $r_j\searrow 0$,*

$$\frac{u(r_jx)}{r_j^2}$$

converges to a function u_0. Then u_0 is a degree two homogeneous function, solving our problem.

The proof of this follows from Weiss monotonicity function. Indeed,

$$W(sr_j,u)=W(s,u_j)$$

converges to $W(0^+,u)=C(u)$ as r_j tend to zero, and we obtain $W(s,u_0)=C(u)$. In particular u_0 solves our problem with f,g constants, and the Weiss function is constant. Hence the monotonicity theorem tells us that u_0 is homogeneous of degree two.

2. Proof Theorem 1.1.

Let us assume z is the origin, and that is is not an isolated point of S_u, i.e. there exists $x^j\in S_u$, with $r_j:=|x^j|\to 0$. We have two possibilities.
(A) There exists a constant M such that

$$M_j\le Mr_j^2,\qquad \text{for } j=1,2,\cdots$$

(B) There exists a sequence α_j, tending to infinity, such that

$$M_j\ge \alpha_j r_j^2,\qquad \text{for } j=1,2,\cdots \tag{2.1}$$

Now if (A) above is true then

$$u_j(x):=\frac{u(r_jx)}{r_j^2}$$

is bounded. In particular for a subsequence u_j converges to a limit function u_0, solving our problem with constant f,g

$$\Delta u_0=f(0)\chi_{\{u_0\ge 0\}}-g(0)\chi_{\{u_0<0\}}.$$

Using the monotonicity function of Weiss

$$W(0^+,u)=\lim_j W(sr_j,u)=\lim_j W(s,u_j)=W(s,u_0)$$

for any constant $s<1$. By Weiss monotonicity argument (1.7)-(1.8), u_0 is degree two homogeneous global solution to our problem.

Now at the same time $\tilde{x}^j=x^j/r_j\in S_{u_j}$ converges (for yet another subsequence) to a point $x^0\in S_{u_0}$ and with $|x^0|=1$. Since condition (1.6) is stable under scaling

$$|\{u_j\ge 0\}\cap B_r(0)|\ge c_0r^n,$$

we conclude by Lemma 1.2 that $S_{u_0}=\{0\}$. A contradiction.

In Case (B) we use a homogeneous scaling at $r_j = |x^j|$, $u_j(x) = u(r_j x)/M_j$. Te idea is to use the argument in [W] (Proof of Proposition 4.1). Obviously u_j are bounded in $L^2_{\partial B_1}$. Let now

$$f_j := \frac{r_j^2}{M_j} f(r_j x), \qquad g_j := \frac{r_j^2}{M_j} g(r_j x).$$

Then by the monotonicity function

$$W(1, u_j, f_j, g_j) = \left(\frac{r_j^2}{M_j}\right)^2 W(r_j, u, f, g) \leq \left(\frac{r_j^2}{M_j}\right)^2 (W(1, u, f, g) + C) \quad \to 0$$

as j tends to infinity. In particular

(2.2)

$$\int_{B_1} |\nabla u_j|^2 \leq 2 \int_{\partial B_1} u_j^2 + \left(\frac{r_j^2}{M_j}\right)^2 (W(1, u, f, g) + C) + \left(\frac{r_j^2}{M_j}\right) (\|f\| + \|g\|) \left(\int_{B_1} |u_j|\right).$$

Since $\Delta |u_j| \geq -C$ (for some constant C), it is not hard to see (using monotonicity of the integral $\int_{\partial B_t} h$ for subharmonic functions h) that

$$\int_{B_1} |u_j| \leq \int_{\partial B_1} |u_j| + C.$$

Putting this into estimate (2.2), and using Hölder's inequality, we conclude $u_j \in W^{1,2}(B_1)$. Hence there is a subsequence of u_j converging weakly in $W^{1,2}$ to a limit function u_0. Now the compact embedding on the boundary (i.e. the trace theorem) implies that $\|u_0\|_{L^2(\partial B_1)} = 1$. Moreover

$$|u_0(0)| = |\nabla u_0(0)| = 0, \tag{2.3}$$

and due to the assumption in (B)

$$M_{k_j} \geq \alpha_k 4^{-k_j},$$

which leads to

$$|\Delta u_j| \leq \frac{4^{-k_j}}{M_{k_j}} \leq \frac{1}{\alpha_j}, \quad \to \quad 0, \tag{2.4}$$

i.e., u_0 is harmonic. It also follows from inequality (2.2) that

$$\int_{B_1} |\nabla u_0|^2 \leq 2 \int_{\partial B_1} u_0^2. \tag{2.5}$$

Using (2.3)-(2.4)-(2.5) in conjunction with Lemma 4.1 in [W] we conclude that u_0 is a degree two homogeneous harmonic function.

Now on the other hand we have that the sequence $\tilde{x}^j = x^j/r_j \in S_{u_j}$ (where $|\tilde{x}^0| = 1$) and hence there is limit point $x^0 \in S_{u_0}$, with $|x^0| = 1$. This of course is a contradiction, as for any degree two homogeneous harmonic function h we must have $S_h = \{0\}$.

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