CONVEXITY OF THE FREE BOUNDARY FOR AN EXTERIOR FREE BOUNDARY PROBLEM INVOLVING THE PERIMETER

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Abstract. We prove that if the given compact set K is convex then a minimizer of the functional

$$I(v) = \int_{B_R} |\nabla v|^p dx + \text{Per}(\{v > 0\}),\ 1 < p < \infty,$$

over the set $\{v \in H_0^1(B_R) |\ v \equiv 1 \text{ on } K \subset B_R\}$ has a convex support, and as a result all its level sets are convex as well. We derive the free boundary condition for the minimizers and prove that the free boundary is analytic and the minimizer is unique.

1. Introduction

1.1. **The Problem.** The following problem has been considered in [Maz]: given a bounded domain $E \subset B_R \subset \mathbb{R}^n$ (R large), satisfying the interior ball condition, find a (local) minimizer of the functional

$$I(v) = \int_{B_R} F(|\nabla v|)dx + \text{Per}(\{v > 0\}) \tag{1}$$

over the set of functions $\{v \in H_0^1(B_R) | v \equiv 1 \text{ on } E\}$, where $F \in C^1([0, +\infty))$ is a positive convex function, with $F(0) = 0$ and for some $1 < p < +\infty$ and $0 < \lambda < \Lambda < +\infty$

$$\lambda t^{p-1} \leq F'(t) \leq \Lambda t^{p-1}.$$

Here we set $\text{Per}(\{v > 0\}) = +\infty$ if $\chi_{\{u>0\}} \notin BV(\mathbb{R}^n)$. This problem is the one-phase exterior analogue of the problem introduced in [ACKS] for a functional with general convex function $F(t)$ in the first term (in [ACKS] they treat the case $F(t) = t^2$).

It is easy to show that such a minimizer u is H-harmonic in $\{u > 0\}$, i.e.

$$\Delta_H u := \text{div}(H(|\nabla u|)\nabla u) = 0,$$

where $H(t) := t^{-1}F'(t)$ if $t > 0$ and $H(0) := 0$.

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It is proved in [Maz] that the minimizers are Lipschitz continuous. As in [ACKS] this yields that the free boundary is an almost minimal surface (the blow-up is a minimal cone) and that the reduced boundary is $C^{1,\frac{1}{2}}$.

In this paper, except of Section 2 , we restrict ourselves mainly on the case $F(t) = t^p$, $p > 1$, i.e., the functional

$$I(v) = \int_{B_R} |\nabla v|^p dx + \mathrm{Per}(\{v > 0\}), \tag{2}$$

though we want to mention that the same ideas and methods will work in the general case (1) if we put some additional (rather weak) conditions on the function F.

The main result of this paper is the following theorem.

Theorem. *If K is a convex set with $C^{1,Dini}$ boundary and u is a minimizer of (2) then the set $\{v > 0\}$ is also convex.*

As we will see, this will prove that the free boundary is an analytic surface in case of convex K. We also prove the uniqueness of the minimizer.

1.2. **Notations.** In the sequel we use following notations:

$\mathbf{R}^n_+$	$\{x \in \mathbf{R}^n : x_1 > 0\}$
$B(z,r)$	$\{x \in \mathbf{R}^n : \lvert x - z\rvert < r\}$,
B_r	$B(0,r)$,
χ_D	characteristic function of the set D,
∂D	boundary of the set D,
Ω_u	$\{x \in \mathbf{R}^n : u\,(x) > 0\}$,
Γ_u	$\partial\Omega_u$ the free boundary,
Γ^*_u	$\partial^*\Omega_u$ the reduced boundary of Ω_u (see [EG]),
$\mathrm{cov}(U)$	the convex hull of the set U,
F^*	Legendre transform of the function F, see Section 2.

2. An energy estimate for H-harmonic extensions

Assume $E \Subset \Omega_1 \subset \Omega_2$, where E, Ω_1, Ω_2 are open and bounded subsets of $\mathbb{R}^n$, and that u_j minimizes the functional

$$J(v) = \int F(|\nabla v|)dx \tag{3}$$

in the class of functions $\{v \in H^1_0(\Omega_j)|\, v \equiv 1 \text{ on } E\}$ $(j = 1, 2)$. Then we say that u_2 is the H-harmonic extension of u_1 from Ω_1 to Ω_2.

We write

$$v\Delta_H u = -H(|\nabla u|)\nabla u \nabla v + \mathrm{div}(vH(|\nabla u|)\nabla u),$$

and using Gauss' theorem we obtain

$$\begin{aligned}(4)\quad &\int_{\Omega_2} H(|\nabla u_2|)\nabla u_2 \nabla(u_1-u_2)dx = \\ &-\int_{\Omega_2}(u_1-u_2)\Delta_H u_2 dx + \int_{\partial\Omega_2} H(|\nabla u_2|)(u_1-u_2)\partial_\nu u_2 dH^{n-1} = 0\end{aligned}$$

From here and that $H(t) = t^{-1}F'(t)$ we have

$$\begin{aligned}&\int_{\Omega_2} F(|\nabla u_1|) - F(|\nabla u_2|) = \\ &\int_{\Omega_2} [F'(|\nabla u_2|)|\nabla u_2| - F(|\nabla u_2|)] + [F(|\nabla u_1|) - \frac{F'(|\nabla u_2|)}{|\nabla u_2|}\nabla u_1 \nabla u_2] dx.\end{aligned}$$

The integrand in the first brackets is equal to $F^*(F'(|\nabla u_2|))$, where F^* is the Legendre transform of F, i.e.,

$$F^*(t) = \int_0^t g(s)ds,$$

where $g(s)$ is the inverse of the continuous and strictly increasing function $F'(s)$. What we used above is the so-called Young's formula

$$tF'(t) = F(t) + F^*(F'(t)).$$

Thus we have

$$\begin{aligned}\int_{\Omega_2} F(|\nabla u_1|) - F(|\nabla u_2|) &= \int_{\Omega_2\setminus\Omega_1} F^*(F'(|\nabla u_2|))dx + \\ &\int_{\Omega_1} F(|\nabla u_1|) - F(|\nabla u_2|) - \frac{F'(|\nabla u_2|)}{|\nabla u_2|}\nabla u_2 \nabla(u_1-u_2)dx.\end{aligned}$$

Now we are going to estimate the second integral. Let us consider the following function

$$\Phi(t) = F(|\nabla(u_2 + t(u_1-u_2))|).$$

From the convexity and monotonicity of F it follows that Φ is convex in t. So we can write

$$0 \le \Phi(1) - \Phi(0) - \Phi'(0) \le \Phi'(1) - \Phi'(0).$$

This gives us exactly the following

$$\begin{aligned}0 &\le F(|\nabla u_1|) - F(|\nabla u_2|) - \frac{F'(|\nabla u_2|)}{|\nabla u_2|}\nabla u_2 \nabla(u_1-u_2) \\ &\le \frac{F'(|\nabla u_1|)}{|\nabla u_1|}\nabla u_1 \nabla(u_1-u_2) - \frac{F'(|\nabla u_2|)}{|\nabla u_2|}\nabla u_2 \nabla(u_1-u_2)\end{aligned}$$

in Ω_1. Now we can continue as follows

$$0 \leq \int_{\Omega_1} F(|\nabla u_1|) - F(|\nabla u_2|) - \frac{F'(|\nabla u_2|)}{|\nabla u_2|}\nabla u_2 \nabla (u_1 - u_2)$$
$$\leq \int_{\Omega_1} \frac{F'(|\nabla u_1|)}{|\nabla u_1|}\nabla u_1 \nabla (u_1 - u_2) - \frac{F'(|\nabla u_2|)}{|\nabla u_2|}\nabla u_2 \nabla (u_1 - u_2) dx$$

and by using (4) we get that the last term equals

$$\int_{\Omega_1} \frac{F'(|\nabla u_1|)}{|\nabla u_1|}\nabla u_1 \nabla (u_1 - u_2) dx + \int_{\Omega_2 \setminus \Omega_1} \frac{F'(|\nabla u_2|)}{|\nabla u_2|}\nabla u_2 \nabla (u_1 - u_2) dx$$
$$= \int_{\partial\Omega_1 \setminus \partial\Omega_2} u_2 (H(|\nabla u_1|)\partial_\nu u_1 - H(|\nabla u_2|)\partial_\nu u_2) dH^{n-1}$$

Hence we have proved the following lemma.

Lemma 1. *If u_2 is the H-harmonic extension of u_1 from Ω_1 to Ω_2 then*

$$0 \leq \int_{\Omega_2} F(|\nabla u_1|) - F(|\nabla u_2|) dx - \int_{\Omega_2 \setminus \Omega_1} F^*(F'(|\nabla u_2|)) dx \leq \int_{\partial\Omega_1 \setminus \partial\Omega_2} u_2 [H(|\nabla u_1|)\partial_\nu u_1 - H(|\nabla u_2|)\partial_\nu u_2] dH^{n-1}. \tag{5}$$

Let us denote by

$$G(t) := F^*(F'(t)).$$

Note that G is continuous, monotone increasing on $[0, +\infty)$, $G(0) = 0$ and

$$G'(t) = tF''(t) \geq 0.$$

Remark 2. *From now on we will consider the classical case $F(t) = t^p$, $p > 1$. That means*

$$G(t) = (p-1)t^p. \tag{6}$$

3. The Hopf lemma for p-harmonic functions in domains with $C^{1,Dini}$ boundary

Here we prove that if the boundary of the domain is $C^{1,Dini}$ near some point y on the boundary and a p-harmonic function has its minimum at that point, then the gradient of the function is strictly positive. We was not able to find a reference for this, probably known, result. Anyway the proof presented here, which uses an elegant barrier construction, seems to be quite interesting.

Let us take the function w to be the minimizer of the Dirichlet integral in $\{v \in H^1_0(\Omega) | v \equiv 1 \text{ on } K\}$, where K and Ω are convex domains with $C^{1,Dini}$ boundary and $K \Subset \Omega$. Thus we have $\Delta w = 0$ in $\Omega \setminus K$. From the Hopf lemma for harmonic functions (see [W]) we know that $\nabla w(x) \neq 0$, for any $x \in \Omega \setminus \overline{K}$. Now we will prove the existence of a smooth, convex function $f : [0,1] \to [0,1]$, $f(0) = 0$, $f(1) = 1$ such that

$$\Delta_p f(w) \geq 0$$

in $\Omega \backslash \overline{K}$ and $0 < f'(t) < +\infty$ for all $t \in [0,1]$. This will mean that the function $f(w)$ is a sub-solution for Δ_p and has non-vanishing gradient, thus it will work as a standard barrier function.

We have

$$\Delta_p f(w) = p|\nabla f(w)|^{p-2}\Delta f(w) + p(p-2)|\nabla f(w)|^{p-4}\Delta_\infty f(w),$$

where $\Delta_\infty v = \sum_{i,j} v_{ij} v_i v_j$ is the well known infinity Laplace operator. On the other hand

$$\nabla f(w) = f'(w)\nabla w,$$

$$\Delta f(w) = f'(w)\Delta w + f''(w)|\nabla w|^2 = f''(w)|\nabla w|^2,$$

$$\Delta_\infty f(w) = (f'(w))^3 \Delta_\infty w + (f'(w))^2 f''(w)|\nabla w|^4.$$

So we need to find a function f such that

$$\Delta_p f(w) = pf''(w)f'(w)^{p-2}|\nabla w|^p +$$
$$p(p-2)(f'(w)|\nabla w|)^{p-4}((f'(w))^3\Delta_\infty w + (f'(w))^2 f''(w)|\nabla w|^4) \geq 0,$$

or

$$\frac{f''(w)}{f'(w)} \geq \frac{2-p}{p-1}|\nabla w|^{-4}\Delta_\infty w. \tag{7}$$

We see that for $p \geq 2$ we can take $f(t) \equiv t$. This follows also from the monotonicity with respect to p of the p-potentials in convex rings proved in [MPS].

In case $1 < p < 2$ we continue as follows. We have from [W] that the the derivatives of w are continuous up to the boundary and do not vanish. Moreover we have bounds for the second derivatives of w near the boundary. Comming back to our case there exists a function $\zeta(t) \in L^1((0,1)) \cap C((0,1))$ such that $|\nabla w|^{-4}|\Delta_\infty w| \leq \zeta(w)$ in $\Omega \backslash \overline{K}$. Let us now integrate (7) in $w \in [t,1]$,

$$\int_t^1 \frac{f''(\tau)}{f'(\tau)} d\tau = \int_{f'(t)}^{f'(1)} \frac{ds}{s} \geq \frac{2-p}{p-1}\int_t^1 \zeta(\tau)d\tau.$$

Thus we can take for instance

$$f(t) = c\int_0^t \exp\left(-\frac{2-p}{p-1}\int_\tau^1 \zeta(s)ds\right)d\tau,$$

where the constant $c > 0$ is chosen to get $f(1) = 1$.

We have proved the following lemma.

Lemma 3. *Assume u is a p-harmonic function in the domain U. Further assume $y \in \partial U$, ∂U is locally $C^{1,Dini}$ near y and $u(x) \geq u(y)$ for all $x \in U$. Then $|\nabla u(y)|$ exists and does not vanish.*

Remark 4. *Note that the function $1 - f(w)$ is a super-solution for Δ_p in $\Omega\backslash\overline{K}$ and gives us bounds from above for the gradient of a p-harmonic function at the minimum point on the $C^{1,Dini}$ boundary.*

4. The free boundary condition

Let us recall the definition of the weak (viscosity) sub- and super solutions of free boundary relations, as it is defined in [ACKS].

Definition 5. *A surface S given by a graph of a continuous function $x_n = f(x_1, \dots, x_{n-1})$, defined on a open set $V \subset \mathbb{R}^{n-1}$, is a weak (viscosity) sub- (super-) solution of the relation*

$$\kappa(S) = g,$$

where g is a continuous function defined on S, if for every graph S_Q of quadratic polynomial $x_n = Q(x_1, \dots, x_{n-1})$ we have

$$\kappa(S_Q)(x_0) \geq g(x_0), \quad \textit{(respectively, } \leq\textit{)}$$

whenever x_0 is a local minimum (maximum) of $Q - f$. A surface S is called a weak solution, if it is both a weak sub- and super-solution.

Lemma 6. *Let u be the minimizer of (1). Then Γ^* is a weak (viscosity) solution of the free boundary relation*

$$G(|\nabla u|) = (p-1)|\nabla u|^p = \kappa(\Gamma^*). \tag{8}$$

Moreover on $\overline{\Omega}_+ \cap \partial B_R$ we have pointwise the inequality

$$G(|\nabla u|) = (p-1)|\nabla u|^p \geq \kappa(\partial B_R). \tag{9}$$

Proof. Assume Γ^* is given by a graph of a continuous function $x_n = f(x_1, \dots, x_{n-1})$, $f(0) = |\nabla f(0)| = 0$ and the outward (with respect to Ω_u) normal at 0 is $(0, \dots, 0, 1)$. Further assume that that the graph S of the quadratic polynomial $x_n = Q(x_1, \dots, x_{n-1})$ touches Γ^* from inside (below). Let us consider the set $\Omega_t := \Omega_u \cup (\{x_n < Q(x_1, \dots, x_{n-1}) + t\} \cap B_{r_0})$, $r_0 > 0$ small, and the minimizer u_t of (3) over $\{v \in H^1_0(\Omega_t) | v \equiv 1 \text{ in } K\}$. Note that u_t is the H-harmonic extension of u from Ω_u to Ω_t.

Since u is a (local) minimizer we have that for small t

$$I(u) - I(u_t) \leq 0.$$

Taking $d_t(x) := dist(x, S_t)$, where $S_t := \{x_n = Q(x_1, \dots, x_{n-1}) + t\}$, and $V_t := \Omega_t \backslash \Omega_u$ we get that

$$\int F(|\nabla u|)dx - \int F(|\nabla u_t|)dx \leq H^{n-1}(S_t \backslash \Omega_u) - H^{n-1}(\Gamma \cap \Omega_t) \leq$$
$$- \int_{\partial V_t} \partial_\nu d_t dH^{n-1} = - \int_{V_t} \Delta d_t dx,$$

where in the second inequality we use the fact that $\partial_\nu d_t \le 1$. On the other hand from the first inequality in Lemma 1

$$\int_{V_t} G(|\nabla u_t|)dx \le \int F(|\nabla u|)dx - \int F(|\nabla u_t|)dx,$$

thus

$$|V_t|^{-1}\int_{V_t} G(|\nabla u_t|)dx \le -|V_t|^{-1}\int_{V_t} \Delta d_t dx.$$

Letting $t \to 0$ we obtain

$$G(|\nabla u(0)|) \le \kappa(S_Q)(0),$$

thus Γ^* is a weak sub-solution. Here we used the continuity of the gradient, the proof follows from Lemma 7 and Remark 4.

To obtain that Γ^* is a weak super-solution we take a quadratic polynomial Q, which touches Γ^* at 0 locally from outside and consider the set $\Omega_{-t} := \Omega_u \backslash (B_{r_0} \cap \{x_n > Q(x_1, \dots, x_{n-1}) - t\})$. Analogously to the previous case we assume u_{-t} is the minimizer of (3) over $\{v \in H^1_0(\Omega_{-t}) | v \equiv 1 \text{ in } K\}$ and take $V_{-t} = \Omega_u \backslash \Omega_{-t}$ and d_{-t} and S_{-t} as above. Similarly to the previous case we have

$$\int F(|\nabla u|)dx - \int F(|\nabla u_{-t}|)dx \le H^{n-1}(S_{-t}\cap\Omega_u) - H^{n-1}(\Gamma\backslash\Omega_{-t}) \le \\ -\int_{\partial V_{-t}} \partial_\nu d_{-t} dH^{n-1} = -\int_{V_{-t}} \Delta d_{-t} dx.$$

Since now u is the H-harmonic extension of u_{-t} we have to use the second inequality in Lemma 1

$$-\int_{V_{-t}} G(|\nabla u|)dx - \int_{S_{-t}\cap\Omega_u} u[H(|\nabla u_{-t}|)\partial_\nu u_{-t} - H(|\nabla u|)\partial_\nu u] dH^{n-1} \\ \le \int F(|\nabla u|)dx - \int F(|\nabla u_{-t}|).$$

We thus obtain

$$|V_{-t}|^{-1}\int_{V_{-t}} G(|\nabla u|)dx + \\ |V_{-t}|^{-1}\int_{S_{-t}\cap\Omega_u} u[H(|\nabla u_{-t}|)\partial_\nu u_{-t} - H(|\nabla u|)\partial_\nu u] dH^{n-1} \ge \\ |V_{-t}|^{-1}\int_{V_{-t}} \Delta d_{-t} dx \to_{t\to 0} \kappa(S_Q)(0).$$

Plugging in $H(t) = pt^{p-2}$ we see that to complete the proof it is enough to check that

$$|V_{-t}|^{-1}\int_{S_{-t}\cap\Omega_u} u[p|\nabla u_{-t}|^{p-2}\partial_\nu u_{-t} - p|\nabla u|^{p-2}\partial_\nu u] dH^{n-1} \to_{t\to 0} 0.$$

Let us take $S^\epsilon_{-t} := \{x \in S_{-t} | dist(x, \Gamma) > \epsilon\sqrt{t}\}$. Then since u and u_{-t} are Lipschitz functions ([Maz])

$$\Big| \int_{(S_{-t}\cap\Omega_u)\setminus S^\epsilon_{-t}} u[p|\nabla u_{-t}|^{p-2}\partial_\nu u_{-t} - p|\nabla u|^{p-2}\partial_\nu u] dH^{n-1}\Big| < C\epsilon|V_{-t}|,$$

and we need to show that

$$|p|\nabla u_{-t}|^{p-2}\partial_\nu u_{-t} - p|\nabla u|^{p-2}\partial_\nu u| < c(\epsilon, t)$$

on $S^\epsilon_{-t} \cap \Omega_u$, where $c(\epsilon, t) \to_{t\to 0} 0$ for every fixed ϵ. This fact follows from the next lemma, which is the complete analogue of Lemma 6.3 from [ACKS]. □

Lemma 7.

$$\partial_\nu u_{-t}(-te_n) \to \partial_\nu u(0) \ \ as \ \ t \to 0.$$

Proof. The proof follows the same lines as that of the proof of Lemma 6.3 from [ACKS].

From the linear growth near the boundary we have that

$$u(x) = \alpha x_n^- + o(|x|), \ \ u_{-t}(x) = \beta(t)(x_n + t)^- + o(|x|)$$

Observe that the Lipschitz continuity, Remark 4 and weak maximum principle for p-harmonic functions give that for some constant $C > 0$

$$|u - u_{-t}| \le Ct, \ \text{ in } \ \Omega_u.$$

If we now consider the blow-up limits

$$w_t(x) = \frac{u(tx)}{t}, \ \ v_t(x) = \frac{u_{-t}(tx)}{t},$$

then as $t \to 0$, at least for a subsequence, $w_t \to w_0 = \alpha x_n^-$, $v_t \to v_0 = \beta_0(x_n + 1)^-$ and $|w_0 - v - 0| \le C$ in $\mathbb{R}^n$. Hence $\alpha = \beta_0$. □

Corollary 8. *Due to the the Hopf lemma for p-harmonic functions the gradient of the solution u does not vanish on the "good" part of the free boundary and we can use the hodograph transformation. The non-vanishing of the gradient and its Hölder continuity up to the boundary (see* [Li]*) allows us to use the classical results from the theory of viscosity solutions of elliptic equations with Hölder continuous coefficients (see* [CC]*), which give that* Γ^* *is locally* $C^{2,1/2}$ *and thus, by bootstrapping argument analytic. See also Corollary 6.4 in* [ACKS].

Remark 9. *The weak (viscosity) equality (8) we proved is true pointwise on* Γ^**.*

5. A concavity result

From now on we denote by $\kappa(\partial U)$ the interior mean curvature of the $C^{1,1}$ part of the boundary of a domain U (in viscosity sense) as follows.

Assume $0 \in \partial U$ and the interior normal $\nu_{\partial U}(0)$ shows in the direction of the e-axis. We take

$$\kappa(\partial U)(0) := \inf_{\mathcal{A}\in\mathfrak{A}} \kappa(S_{\mathcal{A}})(0),$$

where $S_{\mathcal{A}} = \{(x,e)|e = \langle \mathcal{A}x, x\rangle\}$ and $\mathfrak{A}$ is the set of all symmetric matrices $\mathcal{A}$ such that the set $S_{\mathcal{A}}$ (the graph of a quadratic polynomial) locally touches ∂U from inside.

Let us consider the convex hull $\mathrm{cov}(U)$ of a (non-convex) set U with C^2 boundary. Note that then $\mathrm{cov}(U)$ has a $C^{1,1}$ boundary. For notational reasons let us assume $U \subset \mathbb{R}^{n+1} = \{(x,e)|x \in \mathbb{R}^n, u \in \mathbb{R}\}$.

The following lemma will be useful.

Lemma 10. *The function $\kappa(\partial cov(U))(x)$ is upper semi-continuous on $\partial cov(U)$.*

Assume we have a point $x_0 \in \partial\mathrm{cov}(U)\backslash\partial U$ then there are points $y_0, z_0 \in \partial\mathrm{cov}(U) \cap \partial U$ such that y_0, x_0 and z_0 lay on a line.

Lemma 11. *The function*

$$\frac{1}{\kappa(\partial cov(U))}(x)$$

is concave on (y_0, z_0). Moreover if $\kappa(x) = 0$ for some $x \in (y_0, z_0)$ then $\kappa(x) = 0$ for all $x \in (y_0, z_0)$.

Proof. We need to show that

$$\frac{1}{\kappa(\partial\mathrm{cov}(U))}\Big(\frac{x^1+x^2}{2}\Big) \geq \frac{1}{2}\Big(\frac{1}{\kappa(\partial\mathrm{cov}(U))}(x^1) + \frac{1}{\kappa(\partial\mathrm{cov}(U))}(x^2)\Big)$$

for all $x^1, x^2 \in (y_0, z_0)$. Without loss of generality we can assume $x^1 = (-1, 0, \ldots, 0)$ and $x^2 = (1, 0, \ldots, 0)$. Further assume that graphs of quadratic polynomials

$$u = \langle \mathcal{A}_1(x - x^1), (x - x^1)\rangle \;\text{ and }\; u = \langle \mathcal{A}_2(x - x^2), (x - x^2)\rangle$$

given by positive symmetric matrices $\mathcal{A}_1$ and $\mathcal{A}_2$ locally touch the boundary $\partial\mathrm{cov}(U)$ from inside and $0 < \mathrm{Tr}\mathcal{A}_i - \kappa(x^i) < \epsilon$ for $i = 1, 2$.

Since x^1, x^2 lie on the x_1 axis we can assume that for $i = 1, 2$

$$\mathcal{A}_i = \begin{pmatrix} a_i & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & \mathcal{B}_i & \\ 0 & & & \end{pmatrix},$$

where $\mathcal{B}_i$ are positive symmetric matrices and $0 < a_i < \epsilon$.

Let us now consider the sets

$$\{(-1, x', e)|e > \langle \mathcal{B}_1 x', x'\rangle\} \;\text{ and }\; \{(1, x', e)|e > \langle \mathcal{B}_2 x', x'\rangle\},$$

which touch the boundary of $\mathrm{cov}(U)$ from inside locally at the points x^1 and x^2 respectively. Here $x' = (x_2, \ldots, x_n)$. We will now "calculate" the intersection of the convex hull of this two sets with the plane

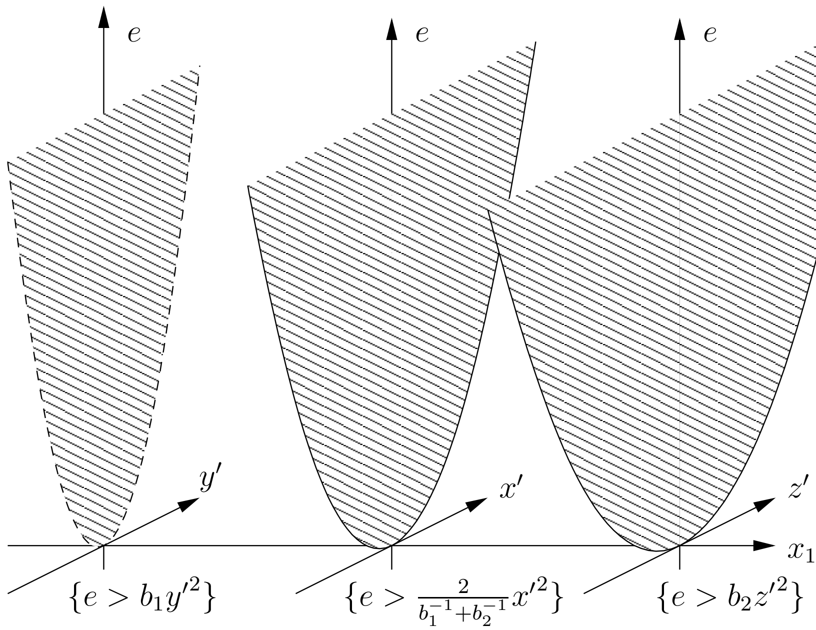

FIGURE 1. Convex hull of two parabolas

$\{x|x_1 = 0\}$. This will locally touch the boundary $\partial\mathrm{cov}(U)$ from inside and give us the desired estimate on the mean curvature. The intersection of the convex hull of these two sets with the mentioned plane is $\{(0, x', e)|e > u(x')\}$, where

$$u(x') = \inf_{y'+z'=2x'} \frac{1}{2} \left(\langle \mathcal{B}_1 y', y'\rangle + \langle \mathcal{B}_2 z', z'\rangle\right). \tag{10}$$

We are going to calculate explicitly the expression on the right hand side. So for each x' we are looking for the minimum of the following function

$$w_{x'}(y') = \frac{1}{2}(\langle \mathcal{B}_1 y', y'\rangle + \langle \mathcal{B}_2(2x' - y'), 2x' - y'\rangle).$$

After differentiation in y' and some (simple) calculations we get that the infimum in (10) is attained at the values

$$y' = 2(\mathcal{B}_1 + \mathcal{B}_2)^{-1}\mathcal{B}_2 x'$$

and

$$z' = 2x' - y' = 2(\mathcal{B}_1 + \mathcal{B}_2)^{-1}\mathcal{B}_1 x'.$$

Substituting now the values of y' and z' into (10) and using the identity

$$\mathcal{B}_1(\mathcal{B}_1 + \mathcal{B}_2)^{-1}\mathcal{B}_2 = (\mathcal{B}_1^{-1} + \mathcal{B}_2^{-1})^{-1}$$

we get

$$u(x') = 2\langle(\mathcal{B}_1^{-1} + \mathcal{B}_2^{-1})^{-1}x', x'\rangle.$$

Note that the invertibility of $\mathcal{B}_1 + \mathcal{B}_2$ and $\mathcal{B}_1^{-1} + \mathcal{B}_2^{-1}$ follows from the strict positivity of all eigenvalues of $\mathcal{B}_1, \mathcal{B}_2$. In three dimensions, when matrices $\mathcal{B}_1, \mathcal{B}_2$ are given by positive numbers b_1, b_2, this interesting result is illustrated in Figure 1.

The proof now follows from the inequalities bellow:

$$\frac{2}{\kappa(\partial \mathrm{cov}(U))}\Big(\frac{x^1+x^2}{2}\Big) \geq \frac{1}{\mathrm{Tr}(\mathcal{B}_1^{-1}+\mathcal{B}_2^{-1})^{-1}} \geq \frac{1}{\mathrm{Tr}\mathcal{B}_1}+\frac{1}{\mathrm{Tr}\mathcal{B}_2} \geq \frac{1}{\kappa(\partial \mathrm{cov}(U))(x^1)+\epsilon}+\frac{1}{\kappa(\partial \mathrm{cov}(U))(x^2)+\epsilon}. \tag{11}$$

Note that $\epsilon > 0$ is arbitrary small and we have the first and the third inequalities in (11) by the construction of $\mathcal{B}_1, \mathcal{B}_2$ and from the properties of the convex hull. The second inequality seems to be classical, but we could not find a reference for that: the proof is given in the Appendix. □

6. Convexity of the free boundary

In the proof of the key Lemma 13 we will use the following lemma from [LS]. Let $K \subset U$ be a compact convex set, U be open and non-convex and $\mathrm{cov}(U)$ be the convex hull of U. Further assume that the function u minimizes the functional (3) over the set $\{v \in H_0^1(\mathrm{cov}(U)) | v \equiv 1 \text{ on } K\}$ and that the segment $[y_0, z_0] \subset \partial \mathrm{cov}(U)$. Then the following lemma is true.

Lemma 12. *The function*

$$\frac{1}{|\nabla u|}(x)$$

is convex on (y_0, z_0).

This is due to the fact (see [L]) that the level sets of a p-harmonic potential in a convex ring are convex.

The following lemma is the key to the proof of the main result.

Lemma 13. *Let u be a (local) minimizer of (2) and denote by $cov(\Omega)$ the convex hull of Ω_u. Assume u^c be the minimizer of*

$$\int_{cov(\Omega)\setminus K} |\nabla u(x)|^p dx \tag{12}$$

over the set $\{v \in H_0^1(cov(\Omega)) | v \equiv 1 \text{ on } K\}$.

Then $\partial cov(\Omega)$ is a solution of the (pointwise) free boundary inequality

$$\frac{1}{p-1}\left(\frac{1}{|\nabla u^c(x)|}\right)^p \geq \kappa(\partial cov(\Omega)), \tag{13}$$

where κ is the interior mean curvature.

Remark 14. *Since at all points $x \in \partial cov(\Omega) \cap \partial \Omega_u$ we have a supporting plane at Ω_u and since the free boundary is an almost minimal* *surface (the blow-up is a plane or a minimal cone, see* [Maz]*) we get that*

$$\partial cov(\Omega) \cap \partial\Omega_u \subset \Gamma^*.$$

Proof. We get the desired inequality on $\partial\mathrm{cov}(\Omega) \cap \partial\Omega_u$ from the maximum principle and Lemma 6.

Assume now that $x_0 \in \partial\mathrm{cov}(\Omega)\backslash\partial\Omega_u$. From the definition of the convex hull follows that we can always write $x_0 = \sum_{k=1}^m \alpha_k y_k$, $y_k \in \partial\mathrm{cov}(\Omega) \cap \partial\Omega_u$, $\alpha_k > 0$, $\sum_{k=1}^m \alpha_k = 1$, $2 \leq m \leq n$.

We proceed by induction in m. Assume there exist two points $y_1, y_2 \in \partial\mathrm{cov}(\Omega) \cap \partial\Omega_u$ such that y_1, x_0 and y_2 lay on one line.

We need to show that

$$\frac{1}{p-1}\left(\frac{1}{|\nabla u^c(x)|}\right)^p - \frac{1}{\kappa(\partial\mathrm{cov}(\Omega))(x_0)} \leq 0. \tag{14}$$

We know that $\frac{1}{|\nabla u^c(x)|}$ and thus $\left(\frac{1}{|\nabla u^c(x)|}\right)^p$ is convex on $[y_1, y_2]$. Since (14) is true at the points y_1 and y_2 the proof follows from the concavity of $\frac{1}{\kappa(\partial\mathrm{cov}(\Omega))(x)}$ on the line segment (y_1, y_2) and its lower semi-continuity (Lemma 10).

The induction step $m \Rightarrow m+1$ finishes the proof. $\square$

Theorem. *If K is convex and u is a minimizer of (2) then Ω_u is also convex.*

Proof. Assume Ω_u is not convex. Let us take u^c and $\mathrm{cov}(\Omega_u)$ as in Lemma 13 and assume $0 \in \mathrm{int}K$. Further take $u^c_r(x) := u^c(rx)$, $\mathrm{cov}(\Omega_u^r) = r^{-1}\mathrm{cov}(\Omega_u)$ and $0 < r_0 := \inf\{r > 0 | \mathrm{cov}(\Omega_u^r) \subset \Omega_u\} < 1$. Assume $\partial\mathrm{cov}(\Omega_u^{r_0})$ touches $\partial\Omega_u$ at the point $\tilde{x}$. First note that as in Remark 14 we have that $\partial\Omega_u$ is smooth near $\tilde{x}$ and that $\tilde{x}$ is not on ∂B_R. We have now

$$\begin{aligned}\kappa(\partial\mathrm{cov}(\Omega_u^{r_0}))(\tilde{x}) \leq r_0^{p-1}(p-1)|\nabla u^c_{r_0}|^p(\tilde{x}) \leq \\ r_0^{p-1}(p-1)|\nabla u|^p(\tilde{x}) = r_0^{p-1}\kappa(\partial\Omega_u)(\tilde{x}),\end{aligned} \tag{15}$$

where the first inequality follows from Lemma 13, the second one from the comparison principle and the third one from Remark 9. On the other hand from the definition of r_0 we get that $\kappa(\partial\mathrm{cov}(\Omega_u^{r_0}))(\tilde{x}) \geq \kappa(\partial\Omega_u)(\tilde{x})$ and $r_0 < 1$, a contradiction. $\square$

Corollary 15. *The free boundary is an analytic surface.*

Corollary 16. *Using the same method as in the proof of the theorem one can easily prove the uniqueness of the minimizer by a contradiction argument. Note that in the two-phase (interior) case (see* [ACKS]*) we do not have uniqueness.*

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Appendix

Let us for simplicity re-write the second inequality in (11) in the form

$$\frac{1}{\mathrm{Tr}(\mathcal{B}_1+\mathcal{B}_2)^{-1}} \geq \frac{1}{\mathrm{Tr}\mathcal{B}_1^{-1}} + \frac{1}{\mathrm{Tr}\mathcal{B}_2^{-1}}, \tag{16}$$

and denote by $\lambda(\mathcal{C}) = (\lambda_1(\mathcal{C}), \dots, \lambda_d(\mathcal{C}))$ the eigenvalues of a $d \times d$ (positive) symmetric matrix $\mathcal{C}$ with $\lambda_1(\mathcal{C}) \geq \lambda_2(\mathcal{C}) \geq \dots \geq \lambda_d(\mathcal{C})$. We need to prove

$$\mathrm{Tr}(\mathcal{B}_1+\mathcal{B}_2)^{-1} = \sum_{k=1}^{d} \frac{1}{\lambda_k(\mathcal{B}_1+\mathcal{B}_2)} \leq \tag{17}$$

$$\left(\frac{1}{\sum_{k=1}^{d}\lambda_k(\mathcal{B}_1)^{-1}} + \frac{1}{\sum_{k=1}^{d}\lambda_k(\mathcal{B}_2)^{-1}}\right)^{-1} = \left(\frac{1}{\mathrm{Tr}\mathcal{B}_1^{-1}} + \frac{1}{\mathrm{Tr}\mathcal{B}_2^{-1}}\right)^{-1}$$

The proof consists of two steps:

Step 1:

$$\sum_{k=1}^{d} \frac{1}{\lambda_k(\mathcal{B}_1+\mathcal{B}_2)} \leq \sum_{k=1}^{d} \frac{1}{\lambda_k(\mathcal{B}_1)+\lambda_k(\mathcal{B}_2)}.$$

Step 2:

$$\sum_{k=1}^{d} \frac{1}{\lambda_k+\mu_k} \leq \left(\frac{1}{\sum_{k=1}^{d}\lambda_k^{-1}} + \frac{1}{\sum_{k=1}^{d}\mu_k^{-1}}\right)^{-1}, \tag{18}$$

where $\lambda_k > 0$, $\mu_k > 0$, $k = 1, \dots, d$.

Proof of the Step 1. We use now the following notions from the theory of majorisation (see [S], [MO]). Let $\mathbf{a} = (a_1, \dots, a_d)$ and $\mathbf{b} = (b_1, \dots, b_d)$ be vectors in $\mathbb{R}^d$ and denote by $a_{[1]} \geq \dots \geq a_{[d]}$ and $b_{[1]} \geq \dots \geq b_{[d]}$ the elements of the vectors $\mathbf{a}$ and $\mathbf{b}$ in the decreasing order. We say that $\mathbf{b}$ majorizes $\mathbf{a}$ and write

$$\mathbf{a} \prec \mathbf{b} \tag{19}$$

if

$$\sum_{k=1}^{d'} a_{[k]} \leq \sum_{k=1}^{d'} b_{[k]}$$

for $1 \leq d' \leq d$ and equality holds if $d' = d$. Note that if Φ is a convex function, then from (19) it follows

$$\sum_{k=1}^{d} \Phi(a_k) \leq \sum_{k=1}^{d} \Phi(b_k),$$

(see [MO] p. 108). The proof of the Step 1 now follows from the convexity of $\Phi(t) = t^{-1}$ for $t > 0$ and the following fact proved in [Fan] (see also [MO] p. 241)

$$\lambda(\mathcal{B}_1+\mathcal{B}_2) \prec \lambda(\mathcal{B}_1)+\lambda(\mathcal{B}_2).$$

Proof of the Step 2. The proof of (18) is done by induction. The key step is the proof in $d=2$ case. We need to prove that

$$\frac{1}{\lambda_1+\mu_1}+\frac{1}{\lambda_2+\mu_2} \le \left(\frac{1}{\lambda_1^{-1}+\lambda_2^{-1}}+\frac{1}{\mu_1^{-1}+\mu_2^{-1}}\right)^{-1}. \tag{20}$$

Without loss of generality we can assume that $\lambda_1+\mu_1+\lambda_2+\mu_2=1$. If we now denote by

$$\begin{aligned}\alpha &:= \lambda_1+\lambda_2,\\ \beta &:= \lambda_1+\mu_2,\\ \gamma &:= \lambda_1+\mu_1,\end{aligned}$$

then (20) is equivalent to

$$\alpha(1-\alpha+\gamma-\beta)(1-\gamma-\alpha+\beta)+(1-\alpha)(\alpha+\gamma+\beta-1)(\alpha-\gamma-\beta+1) \\ \le 4\alpha(1-\alpha)\gamma(1-\gamma).$$

Observe that for fixed α and γ the expression on the right hand side is the maximum over β of the one on the left hand side.

Now we can easyly see how the induction step $d \Rightarrow d+1$ works.

$$\sum_{k=1}^{d+1}\frac{1}{\lambda_k+\mu_k} \le \left(\frac{1}{\sum_{k=1}^{d}\lambda_k^{-1}}+\frac{1}{\sum_{k=1}^{d}\mu_k^{-1}}\right)^{-1}+\frac{1}{\lambda_{d+1}+\mu_{d+1}} \le \\ \left(\frac{1}{\sum_{k=1}^{d+1}\lambda_k^{-1}}+\frac{1}{\sum_{k=1}^{d+1}\mu_k^{-1}}\right)^{-1},$$

where in the second inequality we use the case $d=2$ proved above.

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