THE TWO-PHASE MEMBRANE PROBLEM – REGULARITY OF THE FREE BOUNDARIES IN HIGHER DIMENSIONS

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Abstract. For the two-phase membrane problem

$$\Delta u = \lambda_+ \chi_{\{u>0\}} \; - \; \lambda_- \chi_{\{u<0\}} \,,$$

where λ_+ and λ_- are positive Lipschitz functions, we prove in higher dimensions that the free boundary is in a neighborhood of each "branch point" the union of two C^1-graphs. The result is optimal in the sense that these graphs are in general not of class $C^{1,\mathrm{Dini}}$, as shown by a counter-example.

As application we obtain a stability result with respect to perturbations of the boundary data.

1. Introduction

1.1. **Background and main result.** In this paper we study the regularity of the obstacle-problem-like equation

$$\Delta u = \lambda_+ \chi_{\{u>0\}} \; - \; \lambda_- \chi_{\{u<0\}} \qquad \text{in } \Omega, \tag{1.1}$$

where $\lambda_+ > 0, \lambda_- > 0$ are Lipschitz functions and $\Omega \subset \mathbf{R}^n$ is a given domain. Physically the equation arises for example as the "two-phase membrane problem": consider an elastic membrane touching the planar phase boundary between two liquid/gaseous phases with densities $\rho_1 > \rho_2$ in a gravity field, for example water and air. If the density ρ_m of the membrane satisfies $\rho_1 > \rho_m > \rho_2$, then the membrane is being buoyed up in the phase with higher density and weighed down in the phase with lesser density, so the equilibrium state can be described by equation (1.1). In that case λ_+ is proportional to $\rho_1 - \rho_m$ and λ_- is proportional to $\rho_m - \rho_2$.
Properties of the solution, a Hausdorff dimension estimate of the free boundary etc. have been derived in [10] and in [9]. Moreover, in [5], the current authors gave a complete characterization of global two-phase solutions satisfying a quadratic

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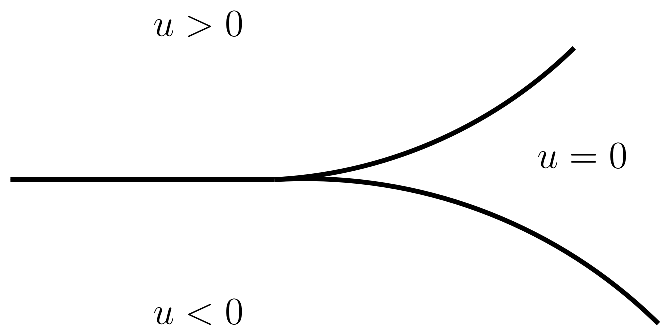

Figure 1. Example of a Branch Point

growth condition at a two-phase free boundary point and at infinity. It turned out that each global solution coincides after rotation with the one-dimensional solution $u(x) = \frac{\lambda_+}{2} \max(x_n, 0)^2 2 \; - \frac{\lambda_-}{2} \min(x_n, 0)^2$.

In [6, Theorem 4.1], it is proved that *in two dimensions* and for *constant coefficients* λ_+, λ_-, the free boundary is in a neighborhood of each branch point, i.e. a point in the set $\Omega \cap \partial\{u > 0\} \cap \partial\{u < 0\} \cap \{\nabla u = 0\}$, the union of (at most) two C^1-graphs. Note that the definition of "branch point" does not necessarily imply a bifurcation as that in Figure 1.

As application they also obtain the following stability result: If the free boundary contains no singular one-phase point for certain boundary data (B^0), then for boundary data (B) close to (B^0) the free boundary consists of C^1-arcs converging to those of (B) (cf. [6, Theorem 5.1]).

In higher dimensions an estimate for the $(n-1)$-dimensional Hausdorff measure of the free boundary has so far been the best known result (see [6]).

In the present paper we use another approach (related to that of [1]; see also [3]) to prove that *in higher dimensions and for non-constant coefficients* the free boundary is in a neighborhood of each branch point the union of (at most) two C^1-graphs (cf. Theorem 1.1). As application we obtain a stability result with respect to perturbations of the boundary data (see Theorem 5.1). Comparing the methods in this paper and in [6], the methods used here rely on a certain non-degeneracy of the nonlinearity, while the approach in [6] essentially requires two-dimensionality (for exceptions see [7] where the approach has been applied to a one-phase problem in higher dimensions) and reflection invariance of the nonlinearity. Apart from those restrictions both approaches can be generalized to a large class of nonlinear elliptic PDE operators.

We formulate the main result in this paper.

Theorem 1.1. *Suppose that*

$$0 < \lambda_{\min} \le \inf_{B_1(0)} \min(\lambda_+, \lambda_-), \qquad \sup_{B_1(0)} \max(|\nabla\lambda_+|, |\nabla\lambda_-|) < +\infty$$

and that u is a weak solution of

$$\Delta u = \lambda_+ \chi_{\{u>0\}} - \lambda_- \chi_{\{u<0\}} \text{ in } B_1(0) ;$$

here $B_1(0)$ is the unit ball.
Then there are constants $\sigma > 0$ and $r_0 > 0$ such that

$$u(0) = 0 , \ |\nabla u(0)| \le \sigma , \ \operatorname{dist}(0, \{u > 0\}) \le \sigma \ \text{ and } \ \operatorname{dist}(0, \{u < 0\}) \le \sigma \tag{1.2}$$

imply $\partial\{u > 0\} \cap B_{r_0}(0)$ and $\partial\{u < 0\} \cap B_{r_0}(0)$ being C^1-surfaces. The constants σ, r_0 and the modulus of continuity of the normal vectors to these surfaces depend only on $\inf_{B_1(0)} \min(\lambda_+, \lambda_-)$, the Lipschitz norms of $\lambda_\pm$, the supremum norm of u and the space dimension n.
Moreover the C^1-regularity is optimal in the sense that the graphs are in general not of class $C^{1,\mathrm{Dini}}$.

The above "not of class $C^{1,\mathrm{Dini}}$" means that the normal of the free boundary may not be Dini continuous, i.e. if ω is the modulus of continuity of the normal vector then

$$\int_0^1 \omega(t) d(\log t) = \infty.$$

Corollary 1.2. *Suppose that*

$$0 < \lambda_{\min} \le \inf_{B_1(0)} \min(\lambda_+, \lambda_-), \qquad \sup_{B_1(0)} \max(|\nabla\lambda_+|, |\nabla\lambda_-|) < +\infty$$

and that u is a weak solution of

$$\Delta u = \lambda_+ \chi_{\{u>0\}} - \lambda_- \chi_{\{u<0\}} \text{ in } B_1(0) .$$

Then there is a constant $r_0 > 0$ such that if the origin is a branch point, then $\partial\{u > 0\} \cap B_{r_0}(0)$ and $\partial\{u < 0\} \cap B_{r_0}(0)$ are C^1-surfaces. The constant r_0 and the modulus of continuity of the normal vectors to these surfaces depend only on $\inf_{B_1(0)} \min(\lambda_+, \lambda_-)$, the Lipschitz norms of $\lambda_\pm$, the supremum norm of u and the space dimension n.

1.2. **Application in Optimal Control theory.** Before closing this introduction, we want to present yet another application of our problem which comes from optimal control theory. First, let us define for each $f \in L^\infty(\Omega)$ the solution v_f of the linear problem

$$\begin{cases} \Delta v_f = f & \text{in } \Omega, \\ \dfrac{\partial v_f}{\partial \nu} = \psi & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Here ν is the outward normal to $\partial\Omega$, ψ is a given function and f is a control function. Let

$$U_{ad} := \{f \in L^\infty(\Omega) :\ \text{esssup}_\Omega |f| \le 1, \quad \int_\Omega f = \int_{\partial\Omega} \psi \, d\mathcal{H}^{n-1}\}$$

be a set of admissible control functions and let us define

$$I(f) := \int_\Omega |\nabla v_f|^2 + |v_f| - \int_{\partial\Omega} \psi v_f \, d\mathcal{H}^{n-1}$$

for all $f \in U_{ad}$.
It is easy to calculate that

$$I(f) = \int_\Omega |v_f|(1 - f \text{sign } v_f) \ge 0 \, ;$$

here $I(v_f) = 0$ iff $f = \text{sign } v_f$, so $f = \text{sign } u$ minimizes the functional I if u is the solution of (1.1) with $\lambda_+ = \lambda_- = 1$.

2. Notation and Technical tools

Throughout this article $\mathbf{R}^n$ will be equipped with the Euclidean inner product $x \cdot y$ and the induced norm $|x|$. $B_r(x)$ will denote the open n-dimensional ball of center x, radius r and volume $r^n\,\omega_n$. When the center is not specified, it is assumed to be 0. $B'_r(x')$ will denote the open $n-1$-dimensional ball of center $x' \in \mathbf{R}^{n-1}$, radius r and volume $r^{n-1}\,\omega_{n-1}$.
We will use $\partial_e u = \nabla u \cdot e$ for the directional derivative.
When considering a set A, χ_A shall stand for the characteristic function of A, while ν shall typically denote the outward normal to a given boundary. Last, $\mathcal{H}^{n-1}$ is the usual $(n-1)$-dimensional Hausdorff measure.

Let $n \ge 2$, let Ω be a bounded open subset of $\mathbf{R}^n$ with Lipschitz boundary, let $\lambda_+ > 0$, $\lambda_- > 0$ be Lipschitz functions locally in Ω, and assume that $u_d \in W^{1,2}(\Omega)$. From [10] we know then that there exists a strong solution $u \in W^{2,2}(\Omega)$ of the equation $\Delta u = \lambda_+\,\chi_{\{u>0\}} - \lambda_-\,\chi_{\{u<0\}}$ in Ω, attaining the boundary data u_d in L^2 . The boundary condition may be replaced by other, more general boundary conditions.
A quadratic growth estimate near the set $\Omega \cap \{u = 0\} \cap \{\nabla u = 0\}$ had already been proved in [10] for more general coefficients λ_+ and λ_- , but local $W^{2,\infty}$- or $C^{1,1}$-regularity of the solution has been shown for the first time in [9]. See also [4]. So we know that

$$u \in W^{2,\infty}_{\text{loc}}(\Omega). \tag{2.1}$$

For the readers' convenience we also repeat one of our earlier results that will be referred to in the sequel. It concerns the classification of global solutions.

Theorem 2.1. *([5, Theorem 4.3]) Let $\lambda_+ > 0$, $\lambda_- > 0$ be constant, let u be a global solution in $\mathbf{R}^n$ such that $x_0 \in \partial\{u > 0\} \cap \partial\{u < 0\}$, $\nabla u(x_0) = 0$ for some $x_0 \in \mathbf{R}^n$ and $|D^2 u| \le C$ in $\mathbf{R}^n$. Then u is after a translation and rotation of the form*

$$u(x) = \frac{\lambda_+}{4} \max(x_n, 0)^2 - \frac{\lambda_-}{4} \min(x_n, 0)^2 .$$

3. Uniform flatness of the free boundary in the presence of both phases in a neighborhood

Uniform regularity of the free boundary close to branch points has been proved in [6] for the case of two space dimensions via an Aleksandrov reflection approach. Here we present another approach related to the approach in [1], and [3] that is based on a certain non-degeneracy of the equation. While the approach in [6] is not relying on non-degeneracy of the external force, the approach presented here has the advantages that it works in higher dimensions and for variable coefficients. We start out with a kind of directional-monotonicity property of solutions close to the one-dimensional solution

$$h := \frac{\lambda_+(0)}{2} \max(x_1, 0)^2 \; - \frac{\lambda_-(0)}{2} \min(x_1, 0)^2 \; . \tag{3.1}$$

Proposition 3.1. *Let $0 < \lambda_{\min} \le \inf_{B_1(0)} \min(\lambda_+, \lambda_-)$, h as in (3.1), and let $\varepsilon \in (0, 1)$. Then each solution u of (1.1) in $B_1(0)$ such that*

$$\mathrm{dist}_{C^1(B_1(0))}(u, h) \le \delta := \frac{\lambda_{\min}\varepsilon}{32n}$$

and

$$\sup_{B_1(0)} \max(|\nabla\lambda_+|, |\nabla\lambda_-|) \le \delta$$

satisfies $\varepsilon^{-1}\partial_e u - |u| \ge 0$ in $B_{1/2}(0)$ for every $e \in \partial B_1(0)$ such that $e_1 \ge \varepsilon$; here e_1 denotes the first component of the vector e.

Proof. First note that $\varepsilon^{-1}\partial_e h - |h| \ge 0$. It follows that

$$\varepsilon^{-1}\partial_e u - |u| \ge -2\delta\varepsilon^{-1} \tag{3.2}$$

provided that $\mathrm{dist}_{C^1(B_1(0))}(u, h) \le \delta$. Suppose now towards a contradiction that the statement is not true. Then there exist $\lambda_+, \lambda_- \in (\lambda_{\min}, +\infty), x^* \in B_{1/2}(0),\ e^*$, and a solution u of (1.1) in $B_1(0)$ such that $\mathrm{dist}_{C^1(B_1(0))}(u, h) \le \delta$,

$$\sup_{B_1(0)} \max(|\nabla\lambda_+|, |\nabla\lambda_-|) \le \delta,$$

$e_1^* \ge \varepsilon$ and $\varepsilon^{-1}\partial_{e^*} u(x^*) - |u(x^*)| < 0$. For the positive constant c to be defined later the functions $v := \varepsilon^{-1}\partial_{e^*} u - |u|$ and $w := \varepsilon^{-1}\partial_{e^*} u - |u| + c|x - x^*|^2$ satisfy then the following: as $\Delta|u| \ge \lambda_+\chi_{\{u>0\}} + \lambda_-\chi_{\{u<0\}}$ and $\nabla u \ne 0$ in $\{v < 0\} \cap \{u = 0\}$, we obtain in the set $D := B_1(0) \cap \{v < 0\}$ (cf. Figure 2),

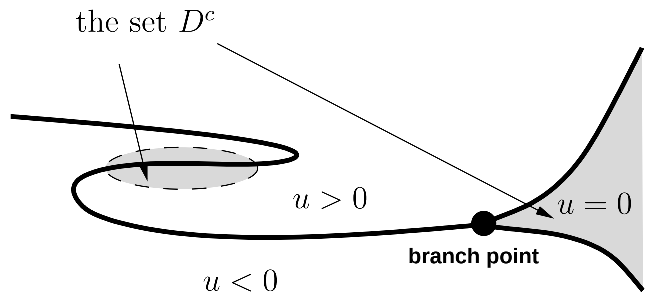

FIGURE 2. The set D

$$\Delta w \leq 2nc - \lambda_+ \chi_{\{u>0\}} - \lambda_- \chi_{\{u<0\}}$$
$$+\varepsilon^{-1}(\lambda_+ + \lambda_-)\nu \cdot e^* \mathcal{H}^{n-1} \lfloor (\{u = 0\} \cap \{\nabla u \neq 0\})$$
$$+\varepsilon^{-1}(\chi_{\{u>0\}} \partial_{e^*} \lambda_+ - \chi_{\{u<0\}} \partial_{e^*} \lambda_-)$$

where $\nu = \frac{\nabla u}{|\nabla u|}$. As

$$\nu \cdot e^* \leq 0 \text{ on } \{u = 0\} \cap \{v < 0\} = \{u = 0\} \cap \{\partial_{e^*} u < 0\} ,$$

we obtain by the definition of δ that w is superharmonic in D provided that c has been chosen accordingly, say $c := \lambda_{\min}/(4n)$. It follows that the negative infimum of w is attained on

$$\partial D \subset \partial B_1(0) \cup (B_1(0) \cap \partial\{v < 0\}) \; .$$

Consequently it is attained on $\partial B_1(0)$, say at the point $\bar{x} \in \partial B_1(0)$. Since $|x^* - \bar{x}| \geq 1/2$, we obtain that

$$\varepsilon^{-1} \partial_{e^*} u(\bar{x}) - |u(\bar{x})| = v(\bar{x}) = w(\bar{x}) - c|x^* - \bar{x}|^2 < -c/4 = -\lambda_{\min}/(16n) \; .$$

But this contradicts (3.2) in view of $\delta = \frac{\lambda_{\min}\varepsilon}{32n}$.

Lemma 3.2. *Let u be a solution of (1.1) in $B_1(0)$. Then, given $\delta > 0$, there are constants $r_\delta > 0, \sigma_\delta > 0$ (depending only on $\inf_{B_1(0)} \min(\lambda_+, \lambda_-)$, the Lipschitz norms of $\lambda_\pm$, the supremum norm of u and the space dimension n) such that the following holds:*
If $r \in (0, r_\delta]$, $u(y) = 0$, $|\nabla u(y)| \leq \sigma_\delta r$, $\operatorname{dist}(y, \{u > 0\}) \leq \sigma_\delta r$ and $\operatorname{dist}(y, \{u < 0\}) \leq \sigma_\delta r$ for some $y \in B_{1/2}(0)$ then in $B_r(y)$, the solution $u(y + \cdot)$ is δr^2-close to a rotated version $\tilde{h}$ of the one-dimensional solution h defined in (3.1), more precisely

$$r^{-2} \sup_{B_r(0)} |u(y + \cdot) - \tilde{h}| + r^{-1} \sup_{B_r(0)} |\nabla u(y + \cdot) - \nabla \tilde{h}| \leq \delta.$$

Proof. Suppose towards a contradiction that the statement of the lemma fails. Then for some $\delta > 0$ there exist $\sigma_j \to 0, r_j \to 0,\ y^j \to y^0 \in \overline{B_{1/2}}$, a sequence u_j of solutions such that $y^j \in B_{1/2}(0),\ u_j(y^j) = 0,\ |\nabla u_j(y^j)| \le \sigma_j r_j,\ \operatorname{dist}(y^j, \{u_j > 0\}) \le \sigma_j r_j,\ \operatorname{dist}(y^j, \{u_j < 0\}) \le \sigma_j r_j$ and

$$r_j^{-2} \sup_{B_1(0)} |u_j(y^j + r_j\cdot) - \tilde{h}(r_j\cdot)| \ +\ r_j^{-1} \sup_{B_1(0)} |\nabla u_j(y^j + r_j\cdot) - \nabla\tilde{h}(r_j\cdot)| > \delta$$

for all possible rotations $\tilde{h}$ of h.

We may define

$$U_j(x) := \frac{u_j(r_j x + y^j)}{r_j^2}$$

and arrive at

$$\|U_j - \tilde{h}\|_{C^1(B_1)} > \delta,$$

for all possible rotations $\tilde{h}$ of h.
Observe that U_j is a solution of (1.1) in B_1 with respect to the scaled coefficients $\lambda_+(r_j x + y^j)$ and $\lambda_-(r_j x + y^j)$. Since $U_j(0) = 0,\ |\nabla U_j(0)| \le \sigma_j, \operatorname{dist}(0, \{U_j > 0\}) \le \sigma_j, \operatorname{dist}(0, \{U_j < 0\}) \le \sigma_j$ and the second derivatives of U_j are uniformly bounded, we obtain by standard compactness arguments a global limit solution U_0 of (1.1) in $\mathbf{R}^n$ with respect to $\lambda_+(y^0)$ and $\lambda_-(y^0)$ which satisfies $0 \in \partial\{U_0 > 0\} \cap \partial\{U_0 < 0\} \cap \{\nabla U_0 = 0\}$ and preserves the above property, i.e.

$$\|U_0 - \tilde{h}\|_{C^1(B_1)} \ge \delta$$

for all possible rotations $\tilde{h}$ of h. This is a contradiction to Theorem 2.1. □

4. Proof of the main theorem

The theorem is proven in several simple steps, using mainly Proposition 3.1, and Lemma 3.2. Note that the proof can be simplified substantially in the case that we are dealing not with a whole class of solutions but a single solution.

Part I: In this first part we prove uniform C^1-regularity.

Step 1 (Directional monotonicity): Given $\varepsilon > 0$, there are $\sigma_\varepsilon > 0$ and $r_\varepsilon > 0$ (depending only on the parameters of the statement) such that $2\varepsilon^{-1} r_\varepsilon \partial_e u - |u| \ge 0$ in $B_{r_\varepsilon/2}(y)$. The inequality holds for every $y \in B_{1/2}(0)$ satisfying $u(y) = 0, |\nabla u(y)| \le \sigma_\varepsilon r_\varepsilon, \operatorname{dist}(y, \{u > 0\}) \le \sigma_\varepsilon r_\varepsilon$ and $\operatorname{dist}(y, \{u < 0\}) \le \sigma_\varepsilon r_\varepsilon$, for some unit vector $\nu_\varepsilon(y)$ and for every $e \in \partial B_1$ satisfying $e \cdot \nu_\varepsilon(y) \ge \frac{\varepsilon}{2}$. In particular, for $\varepsilon = 1$, the solution u is by condition (1.2) with $\sigma = \sigma_1 r_1$ non-decreasing in $B_{r_1/2}(0)$ for every $e \in \partial B_1(0)$ such that $e \cdot \nu_\varepsilon(0) \ge \frac{1}{2}$.

Proof: By Lemma 3.2 there are $\sigma_\varepsilon > 0$ and $r_\varepsilon > 0$ as above such that the scaled function $u_{r_\varepsilon}(x) = u(y + r_\varepsilon x)/r_\varepsilon^2$ is $\delta := \varepsilon\frac{\lambda_{\min}}{64n}$-close in $C^1(B_1(0))$ to a rotated version $\tilde{h}$ of h in B_1. Since u_{r_ε} solves (1.1) with respect to $\lambda_+(r_\varepsilon \cdot + y)$ and $\lambda_-(r_\varepsilon \cdot + y)$, and since $\max(|\nabla(\lambda_+(r_\varepsilon \cdot + y))|, |\nabla(\lambda_-(r_\varepsilon \cdot + y))|) \le C_1 r_\varepsilon$, we may choose $r_\varepsilon < \delta/C_1$ in order to apply Proposition 3.1 to u_{r_ε} in B_1 and to conclude that for some unit vector $\nu_\varepsilon(y)$, $2\varepsilon^{-1}\partial_e u_{r_\varepsilon} - |u_{r_\varepsilon}| \ge 0$ in $B_{1/2}(0)$ for every $e \in \partial B_1(0)$ such that

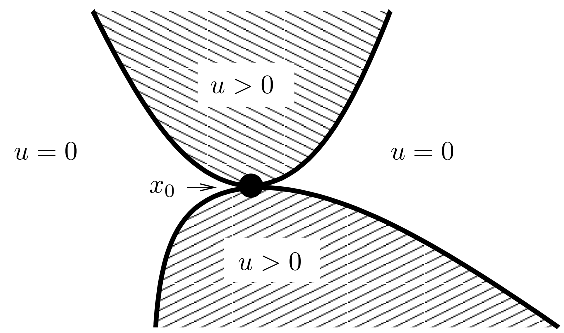

FIGURE 3. Example of a Singular One-Phase Free Boundary Point

$e \cdot \nu_\varepsilon(y) \geq \varepsilon/2$. Scaling back we obtain the statement of Step 1.

Step 2 (Lipschitz continuity): $\partial\{u > 0\} \cap B_{r_1/2}(0)$ and $\partial\{u < 0\} \cap B_{r_1/2}(0)$ are Lipschitz graphs in the direction of $\nu_\varepsilon(0)$ with Lipschitz norms less than 1. Moreover, for each $\varepsilon \in (0,1)$ and $y \in \{u = 0\} \cap B_{1/2}$ satisfying $|\nabla u(y)| \leq \sigma_\varepsilon r_\varepsilon$, $\text{dist}(y, \{u > 0\}) \leq \sigma_\varepsilon r_\varepsilon$ and $\text{dist}(y, \{u < 0\}) \leq \sigma_\varepsilon r_\varepsilon$, the free boundaries $\partial\{u > 0\} \cap B_{r_{\varepsilon/2}}(y)$ and $\partial\{u < 0\} \cap B_{r_{\varepsilon/2}}(y)$ are Lipschitz graphs (in the direction of $\nu_\varepsilon(y)$) with Lipschitz norms not greater than ε.

Proof: This follows from the monotonicity obtained in Step 1.

Step 3 (Existence of a tangent plane at points $y \in \partial\{u > 0\} \cap \partial\{u < 0\} \cap B_{1/2}(0)$ **satisfying** $|\nabla u(y)| = 0$**):** The Lipschitz graphs of Step 2 are both differentiable at the point y, and the two tangent planes at y coincide.

Proof: This follows from Step 2 by letting ε tend to zero.

Step 4 (One-phase points are regular): If $y \in B_{r_1/2}(0)$ is a free boundary point and the solution u is non-negative or non-positive in $B_\delta(y)$, then the free boundary is the graph of a $C^{1,\alpha}$-function in $B_{c_1\delta}(y)$, where c_1 and the $C^{1,\alpha}$-norm depend only on the parameters in the statement. Consequently, in $B_{r_1/2}(0)$, there exist no singular one-phase free boundary points as in Figure 3.

Proof: By Step 2, the sets $\{u > 0\} \cap B_{r_1/2}(0)$ and $\{u < 0\} \cap B_{r_1/2}(0)$ are sub/supergraphs of Lipschitz continuous functions. Therefore $\{u = 0\} \cap B_\delta(y)$ satisfies the thickness condition required for [1, Theorem 7] and the statement follows.

Step 5 (Existence of normals in $B_{r_1/2}(0)$**):** $\partial\{u > 0\} \cap B_{r_1/2}(0)$ and $\partial\{u < 0\} \cap B_{r_1/2}(0)$ are graphs of differentiable functions.

Proof: Let $y \in B_{r_1/2}(0)$ be a free boundary point. We have to prove existence of a tangent plane at y.

First, if y is a one-phase point, i.e. if the solution u is non-negative or non-positive in $B_\delta(y)$, then the statement holds at y by the result of Step 4. Second, if $|\nabla u(y)| \neq 0$, the statement holds by the implicit function theorem. Last, if $|\nabla u(y)| = 0$ and y is the limit point of both phases $\{u > 0\}$ and $\{u < 0\}$, then Step 3 applies.

Step 6 (Equicontinuity of the normals): It remains to prove that the normals are equicontinuous on $B_{r_1/2}(0) \cap \partial\{u > 0\}$ and on $B_{r_1/2}(0) \cap \partial\{u < 0\}$ for u in the class of solutions specified in the statement of the main theorem.

Proof: By Step 2 we know already that the Lipschitz norms of $\partial\{u > 0\} \cap B_{r_1/2}(0)$ and $\partial\{u < 0\} \cap B_{r_1/2}(0)$ are less than 1. We prove that the normals are equicontinuous on $B_{r_1/2}(0) \cap \partial\{u > 0\}$.

We may assume that $\nu(0)$ points in the direction of the x_1-axis and that $x_1 = f(x_2, \ldots, x_n)$ is the representation of $\partial\{u > 0\} \cap B_{r_1/2}(0)$. Besides we have $|\nabla f(x')| < 1$ for $x = (x_1, x') \in \partial\{u > 0\} \cap B_{r_1/2}(0)$. We claim that for $\varepsilon > 0$ there is $\delta_\varepsilon > 0$ depending only on the parameters in the statement such that for any pair of free boundary points $y^1, y^2 \in \partial\{u > 0\} \cap B_{r_1/2}(0)$,

$$|y^1 - y^2| \le \delta_\varepsilon \quad \Rightarrow \quad |\nu(y^1) - \nu(y^2)| \le 2\varepsilon. \tag{4.1}$$

In what follows let $\rho_\varepsilon := \sigma_\varepsilon r_\varepsilon/2$.

Suppose first that u is non-negative in $B_{\rho_\varepsilon}(y^1)$. Here we may as in Step 4 apply [1, Theorem 7] to the scaled function $w(x) := u(y^1 + \rho_\varepsilon x)/\rho_\varepsilon^2$; since the $C^{1,\alpha}$-norm of the free boundary normal of w is on $B_{c_2} \cap \partial\{w > 0\}$ bounded by a constant C_3, where $c_2 > 0$ and $C_3 < +\infty$ depend only on the parameters in the statement, we may choose

$$\delta_\varepsilon := \min\Big(\frac{\varepsilon^{\frac{1}{\alpha}}}{C_3^{\frac{1}{\alpha}}}, c_2\Big)\rho_\varepsilon$$

to obtain (4.1).

Next, suppose that u changes its sign in $B_{\rho_\varepsilon}(y^1)$. If there is a point $y \in B_{\rho_\varepsilon}(y^1) \cap \partial\{u > 0\}$ such that $|\nabla u(y)| \le \rho_\varepsilon$ then we are in the situation of Step 1. By Step 2 the free boundary $\partial\{u > 0\} \cap B_{r_\varepsilon/2}(y)$ is Lipschitz with Lipschitz norm not greater than ε. Hence (4.1) follows in this case with $\delta_\varepsilon := r_\varepsilon/2$.

Last, if $|\nabla u| \ge \rho_\varepsilon$ for all points $y \in B_{\rho_\varepsilon}(y^1) \cap \partial\{u > 0\}$, we proceed as follows: from the equation $u(f(x'), x') = 0$ we infer that $\nabla' u + \partial_1 u \nabla' f = 0$ on $\partial\{u > 0\} \cap B_{r_1/2}(0)$. Hence we obtain

$$|\nabla f(y^1) - \nabla f(y^2)| \le 4M\rho_\varepsilon^{-1}|y^1 - y^2|,$$

where $M = \sup_{B_{1/2}(0)} |D^2 u|$. In particular we may choose

$$\delta_\varepsilon := \frac{\varepsilon}{4M}\rho_\varepsilon$$

to arrive at (4.1).

Note that the above equicontinuity result could – in view of the non-Dini property shown below – not be inferred from higher regularity!

Part II: Let us now prove the second part of the theorem. Namely the sharpness, and non-$C^{1,\mathrm{Dini}}$ property.

Proof. Note that in [8], a similar counter-example has been constructed for the case of the classical obstacle problem.

Lemma 4.1. *If $v \in W^{1,2}(\Omega)$ is a solution of the one-phase obstacle problem*

$$\Delta v = \chi_{\{v>0\}} \text{ in } \Omega$$

such that $v = 0$ on $\Sigma \subset \partial\Omega$, then for any $B_r(x^0) \subset \mathbf{R}^n$ satisfying $B_r(x^0) \cap (\partial\Omega - \Sigma) = \emptyset$,

$$\sup_{\Omega\cap B_r(x^0)} v \le r^2/(8n) \Rightarrow v \equiv 0 \text{ in } \Omega \cap B_{r/2}(x^0) \; .$$

Proof. Comparison of v in $\Omega\cap B_{r/2}(y)$ to $w^y(x) = |x-y|^2/(2n)$ for $y \in B_{r/2}(x^0)\cap\Omega$.

Let now $\zeta \in C^\infty(\mathbf{R})$ be such that $\zeta = 0$ in $[-1/2, +\infty)$, $\zeta = 1/16$ in $(-\infty, -1]$ and ζ is strictly decreasing in $(-1, -1/2)$. Moreover define for $M \in [0, 1]$ the function u_M as the solution of the one-phase obstacle problem

$$\Delta u_M = \chi_{\{u_M>0\}} \text{ in } Q := \{x \in \mathbf{R}^2 : x_1 \in (0, 1), x_2 \in (-1, 0)\} \; ,$$

$$u_M(x_1, x_2) = M\zeta(x_2) \text{ on } \{x_1 = 0\} \cap \partial Q,$$

$$u_M(x_1, x_2) = M/2 \text{ on } \{x_1 = 1\} \cap \partial Q,$$

$$\partial_2 u_M = 0 \text{ on } (\{x_2 = -1\} \cup \{x_2 = 0\}) \cap \partial Q \; .$$

For $M = 1$ we may compare u_M to the function $x_1^2/2$ to deduce that

$$u_1 > 0 \text{ in } Q \; .$$

For $M = 0$, clearly $u_0 \equiv 0$.
On the other hand, as $\partial_2 u_M$ is harmonic in the set $Q\cap\{\partial_2 u_M > 0\}$ and non-positive on $\partial(Q \cap \{\partial_2 u_M > 0\})$, we obtain from the maximum principle that $\partial_2 u_M \le 0$ in Q. Thus the free boundary of u_M is a graph of the x_1-variable.
Suppose now towards a contradiction that $\{0\} \times (-1/4, 0) \subset \partial\{u_M = 0\}^\circ$ for all $M \in (0, 1)$. Then, as $M \to 1$, we obtain $u_1 = |\nabla u_1| = 0$ on $\{0\}\times[-1/4, 0]$, implying by the fact that $u_1 > 0$ in Q and by the Cauchy-Kovalevskaya theorem (applied repeatedly to $w = u_1 - x_1^2/2$) that $u_1 \equiv x_1^2/2$ in Q; this is a contradiction in view of the boundary data of u_1.
From the continuous dependence of u_M on the boundary data as well as Lemma 4.1 we infer therefore the existence of an $M_0 \in (0, 1)$ as well as $\bar x = (\bar x_1, \bar x_2) \in (\{0\} \times [-1/4, 0]) \cap \partial\{u_{M_0} = 0\}^\circ \cap \partial\{u_{M_0} > 0\}$ (cf. Figure 4). Note that Hopf's principle, applied at the line segment $\{0\} \times (-1/2, \bar x_2)$, yields $\nabla u_{M_0} \ne 0$ on $\{0\} \times (-1/2, \bar x_2)$.
Now we may extend u_{M_0} by odd reflection at the line $\{x_1 = 0\}$ to a solution u of (1.1) in an open neighborhood of $\bar x$; here $\lambda_+ = \lambda_- = 1$. The point $\bar x$ is a branch point, so we may apply [6] or the main theorem of the present paper to obtain that the free boundary is the union of two C^1-graphs in a neighborhood of $\bar x$.

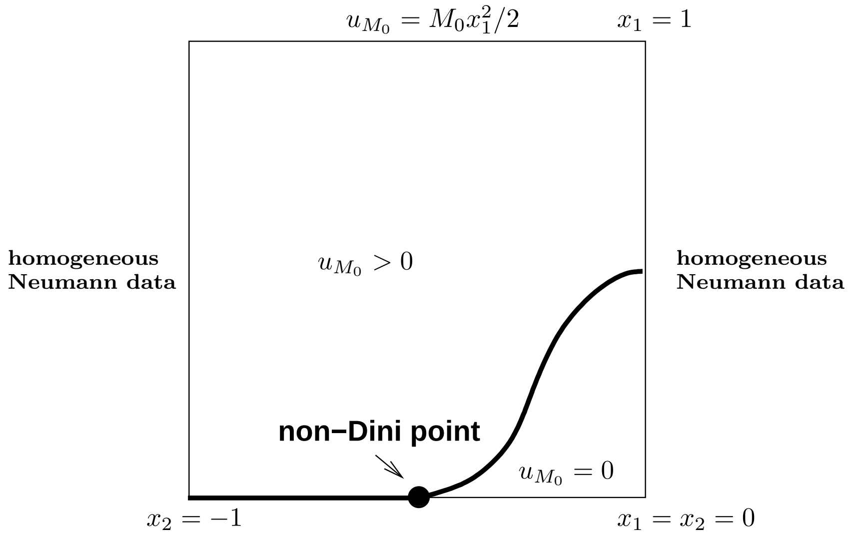

FIGURE 4. A counter-example to $C^{1,\text{Dini}}$-regularity

Suppose now towards a contradiction that $\partial\{u > 0\}$ is of class $C^{1,\text{Dini}}$ in a neighborhood of $\bar{x}$. Then by [11, Theorem 2.5], the Hopf principle holds at $\bar{x}$ and tells us that

$$\liminf_{x_1 \to 0} \frac{\partial_2 u_{M_0}(x_1, \bar{x}_2)}{x_1} \ < \ 0 \ .$$

But that contradicts Lemma 3.2 which, applied to the solution u at $y = \bar{x}$, shows that

$$\liminf_{x_1 \to 0} \frac{\partial_2 u_{M_0}(x_1, \bar{x}_2)}{x_1} \ = \ 0 \ .$$

Consequently $\partial\{u > 0\}$ and $\partial\{u < 0\}$ are not of class $C^{1,\text{Dini}}$.

5. Stability of the free boundary

Theorem 5.1. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain and assume that for given Dirichlet data $u_d \in W^{1,2}(\Omega)$, the free boundary does not contain any one-phase singular free boundary point (cf. Figure 3; for a characterization of one-phase singular free boundary points see* [6, Lemma 2.3]*).*
Then for $K \subset\subset \Omega$ and $\tilde{u}_d \in W^{1,2}(\Omega)$ satisfying $\sup_{\partial\Omega} |u_d - \tilde{u}_d| < \delta_K$, there is $\omega > 0$ such that the free boundary is for every $y \in K$ in $B_\omega(y)$ the union of (at most) two C^1-graphs converging in C^1 to those of the solution with respect to boundary data u_d as $\sup_{\partial\Omega} |u_d - \tilde{u}_d| \to 0$.

Proof. The proof is very similar to that of [6, Theorem 5.1], where uniform convergence of the free boundaries in two dimensions has been shown.
First, by [6, Theorem 3.1], for $\sup_{\partial\Omega} |u_d - \tilde{u}_d|$ small the free boundaries of the solution with respect to $\tilde{u}_d$ cannot contain any one-phase singular free boundary point.

Note that in the case of variable coefficients λ_+, λ_-, we have to replace the use of the monotonicity formula by Alt-Caffarelli-Friedman in the proof of [6, Theorem 3.1] by the use of the monotonicity formula by Caffarelli-Jerison-Kenig ([2]).
Let u and $\tilde{u}$ be the solutions with respect to u_d and $\tilde{u}_d$, respectively. By the comparison principle, $\sup_\Omega |u - \tilde{u}| \to 0$ as $\sup_{\partial\Omega} |u_d - \tilde{u}_d| \to 0$. Consequently, $\tilde{u} \to u$ in $C^{1,\beta}_{\text{loc}}(\Omega)$ as $\sup_{\partial\Omega} |u_d - \tilde{u}_d| \to 0$. But then – provided that $B_\omega(y) \cap \{\tilde{u} = 0\}$ is non-empty – one of the following three cases applies:
1. The assumptions of Theorem 1.1 are satisfied, and the free boundary of $\tilde{u}$ is in $B_\omega(y)$ the union of two C^1-graphs whose normals are equicontinuous.
2. $|\nabla\tilde{u}| \geq \sigma/2$ in $B_\omega(y)$. We infer from the implicit function theorem that the 0-level set of $\tilde{u}$ is in $B_\omega(y)$ a C^1-graph whose normal is equicontinuous.
3. The solution $\tilde{u}$ has in $B_\omega(y)$ a sign and the thickness condition required for [1, Theorem 7] holds, implying that the free boundary of $\tilde{u}$ is in $B_\omega(y)$ a C^1-graph whose normal is equicontinuous.
In all three cases, fixing $z \in \Omega \cap (\partial\{u > 0\} \cup \partial\{u < 0\})$ and translating and rotating once, we obtain $r_0 > 0$ such that $\partial\{\tilde{u} > 0\} \cup \partial\{\tilde{u} < 0\}$ is for $\sup_{\partial\Omega} |u_d - \tilde{u}_d| < \delta_K$ in $B_{r_0}(0)$ the union of the graphs of the C^1-functions $\tilde{g}^+$ and $\tilde{g}^-$ in the direction of a fixed unit vector e; moreover, the class of functions $\tilde{g}^+, \tilde{g}^-$ is precompact in C^1.
In order to identify the limit, suppose now towards a contradiction that

$$\sup_{B'_{r_0/2}} |\tilde{g}^+ - g^+| \geq c_1 > 0 \quad \text{or} \quad \sup_{B'_{r_0/2}} |\tilde{g}^- - g^-| \geq c_1 > 0$$

for some sequence $\tilde{u}_d \to u_d$. Then the fact that u and $\tilde{u}$ are near free boundary points close to monotone one-dimensional solutions with superquadratic growth ([6, Theorem 3.1]) implies that

$$\sup_{B_{r_0/2}} |\tilde{u} - u| \geq c_2 > 0$$

for the same sequence, and we obtain a contradiction.

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