EXISTENCE AND GEOMETRIC PROPERTIES OF SOLUTIONS OF A FREE BOUNDARY PROBLEM IN POTENTIAL THEORY

Björn Gustafsson
and
Henrik Shahgholian

Royal Institute of Technology
Department of Mathematics
S-100 44 Stockholm 70, Sweden

July 7, 1995

Abstract. Let $0 \leq g, h \in L^{\infty}(\mathbb{R}^N)(N \geq 2)$ be two given density functions, at least one of them bounded away from zero outside a compact set and g (Hölder) continuous. We prove that for any compactly supported positive measure μ which is sufficiently concentrated (e.g. has sufficiently high $(N-1)$-dimensional density) there exists a bounded open set $\Omega \subset \mathbb{R}^N$ such that the Newtonian potential of the measure $h\mathcal{L}^N\lfloor\Omega + g\mathcal{H}^{N-1}\lfloor\partial\Omega$ agrees with that of μ outside Ω. Some regularity of $\partial\Omega$ is obtained, as well as several results on the geometry of Ω. Example: if h and g are constant then, for any $x \in \partial\Omega$, the inward normal ray of $\partial\Omega$ at x (if it exists) intersects the closed convex hull of μ.

NOTATION

C	a generic constant , $(D \subset \mathbb{R}^N, N \geq 2)$
$H^1(D)$	$\{u \in L^2(D) : \nabla u \in L^2(D)\}$
$H^1_o(D)$	the closure of $C^\infty_o(D)$ in $H^1(D)$
$\mathcal{H}^s$	$s-$ dimensional Hausdorff measure
$\mathcal{L}^N$	$N-$ dimensional Lebesgue measure
$\mathbb{K}$	$\{u \in H^1(\mathbb{R}^N) : u \geq 0\}$
u^*	the radially symmetric decreasing rearrangement of u
$\mathbb{K}^*$	$\{u \in \mathbb{K} : u = u^*\}$
χ_D	the characteristic function of the set D
D^c	$\mathbb{R}^N \setminus D,$ the complement of D
$\overline{D}$	the closure of D
$B(x,\rho)$	$\{y \in \mathbb{R}^N : \|y-x\| < \rho\}$
$B_\rho(x)$	$B(x,\rho)$
$\\|v\\|$	$(\int v^2)^{1/2},\ L^2-$ norm of v
$\partial_{red}\Omega$	reduced boundary of Ω, see the discussion following Proposition 2.12
$\partial_{mes}\Omega$	measure theoretic boundary of Ω, see the discussion following Proposition 2.12
ω_N	area of the unit sphere in $\mathbb{R}^N$
μ,ν,σ	Radon measures
supp u	support of μ
U^μ	the Newtonian potential of μ
$\|D\|$	$N-$ dimensional volume of the set D
$\delta(x)$	distance function, see Corollary 2.6
δ_x	Dirac measure at $x \in \mathbb{R}^N$
$Q(\mu;h,g)$	quadrature domain, see $4.2-4.5$
$J_{f,g}(u)$	$\int_{\mathbb{R}^N}(\|\nabla u\|^2 - 2fu + g^2\chi_{\{u>0\}})dx$
$\fint_{\partial B_r} ud\mathcal{H}^{N-1}$	the average of u over $\partial B_r(x_0)$
$\mu\lfloor D$	the restriction of μ to the set D.

0. Introduction

This paper deals with a free boundary problem which arises in many areas of physics (free streamlines, jets, Hele-show flows, electromagnetic shaping, gravitational problems etc) but for which we take, as the title indicates, a somewhat potential theoretic point of view. From this point of view the problem can be stated as follows.

Let two nonnegative density functions, h and g, in $\mathbb{R}^N$ $(N \geq 2)$ be given. For any positive measure μ with compact support in $\mathbb{R}^N$ we ask for a bounded domain (or open set) Ω containing supp μ such that, outside Ω, the Newtonian potential U^μ of μ agrees with that of the measure

$$\nu = h\mathcal{L}^N\lfloor\Omega + g\mathcal{H}^{N-1}\lfloor\partial\Omega;$$

here $\mathcal{L}^N\lfloor\Omega$ denotes Lebesgue measure restricted to Ω and $\mathcal{H}^{N-1}\lfloor\partial\Omega$ denotes $(N-1)$-dimensional Hausdorff measure on $\partial\Omega$.

One may view this problem as a kind of balayage problem (or search for "quadrature domains"). In particular, if $h = 0$ then it is intimately connected with classical balayage (Poincaré sweeping) but with one major difference: we prescribe the density g of the swept out measure ν and ask for the domain Ω, whereas classically Ω is given and one asks for g.

In concrete terms our problem comes down to finding a domain Ω containing supp μ such that there exists a solution u of the overdetermined boundary value problem

$$-\Delta u = \mu - h \quad \text{in } \Omega, \tag{0.1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{0.2}$$

$$|\nabla u| = g \quad \text{on } \partial\Omega. \tag{0.3}$$

The relation with the previous formulation is that $u = U^\mu - U^\nu$ in $\mathbb{R}^N$ if u is extended by zero outside Ω.

The aim of the paper is, first, to prove existence of solutions of the problem when natural conditions are satisfied and, second, to study the geometry, and partly regularity, of solutions. Simple examples show that solutions cannot be expected to exist unless g is continuous, at least one of h and g is bounded away from zero outside a compact set and μ is "concentrated" enough, e.g. has a sufficiently high density with respect to $(N-1)$-dimensional Hausdorff measure on its support.

On the other hand, we prove, and this is our main result, that good solutions indeed exist when such conditions are fulfilled. By "good" we mean that the free boundary $\partial\Omega$ is reasonably regular. As to the geometry, we prove e.g. that if h and g are constant and if Ω is one of our constructed solutions, then for any $x \in \partial\Omega$ the inward normal ray of $\partial\Omega$ at x (if it exists) intersects the closed convex hull of μ. (This excludes e.g. domains with long fingers.)

We know of at least two general methods for proving existence of solutions of our problem. One is to first construct a kind of subsolution and then take the infimum of all supersolutions majorizing this. This idea goes back to A. Beurling [Beur] (for a related problem) and has recently been generalized and adapted to our problem by A. Henrot [Henrot]. In this way Henrot is able to find solutions of (0.1)–(0.3) with (0.3) holding in some weak sense. It is not proved in [Henrot] that $\partial\Omega$ is regular and that (0.3) holds in the sense we require it (e.g. that (0.5) below holds).

The other method, which is the one we use, goes back to K. Friedrichs [Friedr], or even to T. Carleman [Car], and was considerably developed and deepened by H.W.

Alt and L.A. Caffarelli [5]. Our work relies heavily on the methods and results in [5]. The method consists (in our case) of minimizing the functional

$$J(u) = \int_{\mathbb{R}^N} (|\nabla u|^2 - 2fu + g^2 \chi_{\{u>0\}}) dx \tag{0.4}$$

over all $0 \leq u \in H^1(\mathbb{R}^N)$. Here $f = \tilde{\mu} - h$, where $\tilde{\mu}$ is a mollified version of μ. If u minimizes (0.4) then $\Omega = \{u > 0\}$ solves our problem, provided supp $\mu \subset \Omega$. In [5] they have $f = 0$ but instead nonzero Dirichlet boundary conditions, working in a subdomain of $\mathbb{R}^N$. Using the method in this original form the second author obtained solutions of our problem in the case that μ is a finite sum of point masses [Shah94a].

Free boundary problems similar to (0.1)–(0.3) have been intensively studied for several decades now, and there is an enormous amount of literature. If $g = 0$ then (0.1)–(0.3) is equivalent to a variational inequality (of the same type as what occurs for the obstacle problem, the dam problem etc.) provided we moreover require that $u \geq 0$. General references here are [Ki-St], [F], [Rod] and special references, for our type of questions e.g. [Gust90], [Sak82], [Sak83], [Mar]. The emphasis of the paper is however on the case when $g > 0$ on at least part of $\mathbb{R}^N$. Here we may refer to [F] for an overview up to the year (1982) and (selection) [Friedr], [Beur], [2], [Shah92], [Shah94b], [LV2], [Serrin], [GNN], [Kaw] (uniqueness, symmetry, convexity) [Ki-Ni], [Caff(survey)], [5] (existence, regularity), [Henrot], [He-Pi], [Zol], [Shah94b] (quadrature surfaces).

In addition to the above literature there are papers treating the two-dimensional case with complex variable methods. We mention [Avci], [Sh-Ul], [Gust87], [Shap92].

The paper is organized as follows. In section 1 we show that the functional (0.4) is bounded from below and that its infimum is attained for at least one u. One ingredient in the proof (and in several later proofs) is a simple but useful rearrangement lemma (Lemma 1.1) which makes it possible to compare solutions with explicit solutions in a spherically symmetric case (Example 1.5) and in this way obtain estimates.

In section 2 we prove that minima (minimizers), or more generally local minima, of J solve (0.1)–(0.3) (with $\mu - h$ replaced by f) in an appropriate sense. Indeed, we show that any local minimum u is Lipschitz continuous in all $\mathbb{R}^N$ (assuming $f \in L^\infty(\mathbb{R}^N)$) and that

$$\Delta u + f\mathcal{L}^N \lfloor \Omega = g\mathcal{H}^{N-1} \lfloor \partial\Omega, \tag{0.5}$$

where $\Omega = \{u > 0\}$ (Theorem 2.13). Moreover, it is shown that $\partial\Omega$ is regular, at least at most points (e.g. $\partial_{red}\Omega$ is regular when $g > 0$).

Continuous functions $u \geq 0$ satisfying (0.5) are called weak solutions (for the problem of minimizing J). In section 3 we study questions of geometry of $\Omega = \{u > 0\}$ when u is a weak solution. We show e.g. (Corollary 3.8) that if g is constant and if Ω is convex and contains supp f_+ then $\{v > 0\} \subset \Omega$ for any other weak solution v. This partly generalizes a corresponding result in [Shah92]. We also prove (for local minima) the previously mentioned result on inward normal rays of $\partial\Omega$ (Corollary 3.11).

That (0.5) holds for local minima does not automatically mean that the original problem is solved: we also need to make sure that supp $\mu \subset \Omega$. This can be done if μ is sufficiently concentrated, and in section 4 we establish two results (Theorem 4.7 and 4.8) in this direction. These can be regarded as our main results.

It turns out that the original formulation of our problem is quite weak, indeed so weak that it (if $g > 0$) admits an abundance of "bad" solutions with irregular boundaries ("non-Smirnov" domains when $N = 2$). These can easily be ruled out by imposing additional conditions, but in principle there remains the question what is a really good formulation of our problem. The above matters are briefly discussed in Remark 4.2 and Example 4.3.

Acknowledgements. The authors are grateful to Harold S. Shapiro and Michel Zinsmeister for stimulating discussions on non-Smirnov domains. The first author has been partially supported by the Swedish Natural Science Research Council (NFR).

1. Existence of minimizers

Throughout this paper $N \geq 2$. Most of the paper (section 1–3) is devoted to studying a minimization problem. The data for this problem are two functions f and g in $\mathbb{R}^N$ satisfying Condition A below.

Condition A.

(A1) $f, g \in L^\infty(\mathbb{R}^N)$,

(A2) supp f_+ is compact ,

(A3) $g \geq 0$

(A4) at least one of
$f \leq$ const. < 0
$g \geq$ const. > 0
holds outside a compact set.

Let $\mathbb{K} = \{u \in H^1(\mathbb{R}^N) : \ u \geq 0\}$ and set

$$J(u) = J_{f,g}(u) = \int_{\mathbb{R}^N} (|\nabla u|^2 - 2fu + g^2 \chi_{\{u>0\}}) dx.$$

Then J is well defined on $\mathbb{K}$, taking values in $(-\infty, +\infty]$. We shall consider the problem:

$$\text{Minimize } J(u) \quad \text{for } u \in \mathbb{K}.$$

The following lemma turns out to be very useful. Similar results have previously been used by Friedman and Philips [F-P].

Lemma 1.1. *Let $J_k(u) = J_{f_k, g_k}(u)$, where $f_1 \leq f_2$, $g_1 \geq g_2$ and $k = 1, 2$. For $u_1, u_2 \in \mathbb{K}$ define $v = \min(u_1, u_2)$ and $w = \max(u_1, u_2)$. Then $v, w \in \mathbb{K}$ and*

$$J_1(v) + J_2(w) \leq J_1(u_1) + J_2(u_2).$$

In particular, if u_1 minimizes J_1 then $J_2(w) \leq J_2(u_2)$ and if u_2 minimizes J_2 then $J_1(v) \leq J_1(u_1)$. If u_k minimizes J_k for $k = 1, 2$ then v minimizes J_1 and w minimizes J_2.

Proof. In general, if $\Phi(t)$ is a nondecreasing function of $t \in \mathbb{R}$ and $h_1 \leq h_2$ then, as is easily seen,

$$\int (h_1 \Phi(u_1) + h_2 \Phi(u_2)) \leq \int (h_1 \Phi(v) + h_2 \Phi(w)).$$

Applying this with $h_j = f_j$, $\Phi(t) = t$ we find that

$$\int (f_1 u_1 + f_2 u_2) \leq \int (f_1 v + f_2 w),$$

and choosing $h_j = -g_j^2$, $\Phi(t) = 0$ for $t \leq 0$, $\Phi(t) = 1$ for $t > 0$ we find

$$\int (g_1^2 \chi_{\{u_1>0\}} + g_2^2 \chi_{\{u_2>0\}}) \geq \int (g_1^2 \chi_{\{v>0\}} + g_2^2 \chi_{\{w>0\}}).$$

Since also $\int (|\nabla u_1|^2 + |\nabla u_2|^2) = \int (|\nabla v|^2 + |\nabla w|^2)$ the proof is finished. □

In order to get comparison solutions we shall first prove the existence of solutions (minimizers) in a special case.

Lemma 1.2. *Let $f = a\chi_{B(0,R)} - b$ and $g = c\chi_{\mathbb{R}^N \setminus B(0,R_1)}$, where a, b, c, R and R_1 are nonnegative constants with $a > b$ and $b + c > 0$. Then J has at least one minimum (minimizer) u in $\mathbb{K}$. Any minimizing u is radially symmetric, radially nonincreasing and vanishes outside a compact set. Moreover the minima form a nested family and there is a largest minimum as well as a smallest one.*

Proof. For any u in $\mathbb{K}$ let u^* denote its radially symmetric decreasing rearrangement (for background see [Moss]). Then $u^* \in \mathbb{K}$ and

$$\int |\nabla u^*|^2 \leq \int |\nabla u|^2, \quad \int f u^* \geq \int f u, \quad \int g^2 \chi_{\{u^*>0\}} \leq \int g^2 \chi_{\{u>0\}};$$

where the first inequality follows from a classical theorem of Polya and Szegö (see [Moss, Theorem 4.1]) and the last two inequalities use the fact that f is nonincreasing and g is nondecreasing as functions of $r = |x|$. It follows that $J(u^*) \leq J(u)$ and hence that we only need to look for minima in $\mathbb{K}^* = \{u \in \mathbb{K} : u^* = u\}$. It should be observed also that $J(u^*) < J(u)$ unless $u^* = u$.

From now on we assume that $c > 0$, because if $c = 0$ then J is convex and it is well-known that there exists a unique minimizer u in $\mathbb{K}$. This u has compact support with radius of support $\rho = (\frac{a}{b})^{1/N} R$. (Note that $b > 0$ when $c = 0$.) Cf. Example 1.5 below. Thus the lemma holds if $c = 0$.

We first prove that J is bounded from below on $\mathbb{K}^*$ (and hence on $\mathbb{K}$). For u in $\mathbb{K}^*$ there is a unique ρ in $[0, \infty]$ depending on u such that $u(x) > 0$ for $|x| < \rho$ and $u(x) = 0$ for $|x| \geq \rho$. Set $\Omega = \{u > 0\} = B(0, \rho)$. Regarding f, g and u as functions of $r = |x|$ we have

$$\frac{1}{\omega_N} J(u) = \int_0^\rho (u'(r))^2 r^{N-1} dr - 2\int_0^\rho f u r^{N-1} dr + \frac{c^2}{N} \max(\rho^N - R_1^N, 0).$$

Since $J(u) = +\infty$ if $\rho = +\infty$ we need only to consider u with $\rho < \infty$. Set

$$\lambda = \left(\int_0^\rho (u'(r))^2 r^{N-1} dr \right)^{1/2} = \frac{1}{\sqrt{\omega_N}} \|\nabla u\|,$$

$$\phi(r) = \int_0^r f(s) s^{N-1} ds = \frac{a}{N} \min(r, R)^N - \frac{b}{N} r^N.$$

If $b \neq 0$ then there is an $r_0 > 0$ such that $\phi(r) \geq 0$ for $0 < r < r_0$, $\phi(r) \leq 0$ for $r > r_0$, and since $u' \leq 0$ we then get

$$\int_0^\rho f u r^{N-1} dr = -\int_0^\rho \phi u' dr \leq A\lambda, \tag{1.1}$$

where $A = (\int_0^{r_0} \phi^2 r^{1-N} dr)^{1/2}$ is a constant independent of u. Thus

$$\frac{1}{\omega_N} J(u) \geq \lambda^2 - 2A\lambda + \frac{c^2}{N} \max(\rho^N - R_1^N, 0) \geq -A^2. \tag{1.2}$$

If $b = 0$ then $0 \leq \phi(r) \leq$ const. $< \infty$ for $r > 0$ and (1.1), (1.2) hold with

$$A = A_\rho = \int_0^\rho \phi^2 r^{1-N} dr \leq \text{ const. } \int_0^\rho r^{1-N}.$$

Since $c > 0$ we still see from the second inequality in (1.2) that J is bounded from below (provided $N \geq 2$).

Thus J is always bounded from below. Let $\{u_n\} \subset \mathbb{K}^*$ be a minimizing sequence, ρ_n the radius of support of u_n and $\lambda_n^2 = \int_0^{\rho_n} (u_n')^2 r^{N-1} dr = \frac{1}{\sqrt{\omega_N}} \|\nabla u_n\|$. Then it follows from (1.2) that

$$\rho_n \leq \text{const.} < \infty, \tag{1.3}$$

$$\lambda_n \leq \text{const.} < \infty. \tag{1.4}$$

Now from (1.3) and (1.4) the existence of a minimum for J follows by standard arguments. In fact, (1.3) shows that we may work in $\mathbb{K} \cap H_o^1(B)$ for some fixed ball B and (1.4) then shows that the minimizing sequence $\{u_n\}$ is precompact in the w-$H_o^1(B)$ (i.e. $H_o^1(B)$ provided with the weak topology). As J is easily checked to be lower semicontinuous in w-$H_o^1(B)$ the existence of a minimum follows.

If $c > 0$ and a is not too large there may be several solutions u (cf. Example 1.5 below). However any solution is uniquely determined by its radius of support ρ (e.g. because u will satisfy $-\Delta u = f$ in $\Omega = \{|x| < \rho\}$, $u = 0$ on $\partial\Omega$, as will be proved later (Lemma 2.2) independently of the present proof), and a larger ρ will correspond to a larger solution u. Therefore the solutions form a nested family and it follows that there is a largest solution (note that any family of solutions at the same time is a minimizing sequence). □

Remark 1.3. When $N = 1$, $J(u)$ is not always bounded from below. Take e.g. $a = 2$, $b = 0$, $c = 1$, $R = 1$ and consider $u(x) = \frac{1}{2}(\rho - |x|)$ for $|x| < \rho$ and $u(x) = 0$ for $|x| > \rho$, where $\rho > 1$ is a parameter. Then $J(u) = 2 - \frac{3}{2}\rho$, which obviously goes to $-\infty$ as $\rho \to +\infty$.

When $N = 2$ $J(u)$ is not bounded from below if $b = c = 0$ (and $a > 0, R > 0$), while, as is seen from the proof, $J(u)$ is bounded from below when $N \geq 3$ even if $b = c = 0$. However the (unique) minimizer does not have compact support then.

It is also worth mentioning that the Condition A is not optimal. However g and f_- are not allowed to tend to zero too fast at infinity.

We now turn to the general case.

Theorem 1.4. *If f and g satisfy Condition A then J is bounded from below and its infimum is attained for at least one u in $\mathbb{K}$. All minimizers have support in a fixed compact set (which depends only on f and g) and the set of minimizers is compact in the weak topology of $H^1(\mathbb{R}^N)$.*

Proof. Let $\tilde{f} = a\chi_{B(0,R)} - b$, $\tilde{g} = c\chi_{\mathbb{R}^N \setminus B(0,R_1)}$ with $a, b, c, R, R_1 \geq 0$, $a > b$, $b + c > 0$ chosen so that $f \leq \tilde{f}$, $g \geq \tilde{g}$ and set $\tilde{J} = J_{\tilde{f},\tilde{g}}$. By Lemma 1.2 there is a largest minimizer $\tilde{u}$ in $\mathbb{K}$ of $\tilde{J}$. Clearly

$$J(u) \geq \tilde{J}(u) \qquad \text{for all } u \in \mathbb{K} \tag{1.5}$$

and also, by Lemma 1.1,

$$J(\min(u, \tilde{u})) \leq J(u). \tag{1.6}$$

Thus $J(u)$ decreases if u is replaced by $\min(u, \tilde{u})$. Choose an open ball B such that supp $\tilde{u} \subset B$. (1.5) together with Lemma 1.2 shows that J is bounded from below and (1.6) shows that if $\{u_n\}$ is a minimizing sequence then so is $\{\min(u_n, \tilde{u})\}$. Thus there exists a minimizing sequence $\{u_n\}$ with supp $u_n \subset B$. By Poincaré's lemma then $\|u_n\| \leq C\|\nabla u_n\|$, so that

$$J(u_n) \geq \|\nabla u_n\|^2 - 2\|f\chi_B\|\|u_n\| \geq \|\nabla u_n\|^2 - 2C\|\nabla u_n\| \geq -C^2.$$

Thus J is bounded from below, $\|\nabla u_n\| \leq C$ and the existence of a minimizer follows as in Lemma 1.2.

If now $u \in \mathbb{K}$ denotes any minimizer of J then Lemma 1.1 shows that $\max(u, \tilde{u}) \leq \tilde{u}$, since $\tilde{u}$ is the largest minimizer of $\tilde{J}$, hence that $u \leq \tilde{u}$. This shows that u has compact support in a fixed compact set. If $a > b + Nc/R$ and $R_1 = 0$ we in fact have supp $u \subset \overline{B(0,\rho)}$, where by Example 1.5 below ρ can be taken to be

$$\rho = \begin{cases} \left(\dfrac{a}{b}\right)^{1/N} R & \text{if } b \neq 0, \\ \left(\dfrac{aR^N}{Nc}\right)^{1/(N-1)} & \text{if } c \neq 0. \end{cases}$$

(If $a \leq b + Nc/R$ then $\rho = R$ works.) It moreover follows as above that $\|u\| + \|\nabla u\| \leq C < \infty$, C independent of u, and therefore that the set of minimizers is compact in w-$H^1(\mathbb{R}^N)$. $\square$

Example 1.5. Assume that f and g are radially symmetric with f nonincreasing and $g \geq 0$ nondecreasing as functions of $r = |x|$. As was noticed in the proof of Lemma 1.2 any minimum u in $\mathbb{K}$ of J is itself radially symmetric and nonincreasing as function of $r = |x|$, i.e. $u \in \mathbb{K}^*$. Moreover u has compact support. It will be proved later (without using the results of this example) that a necessary condition that a function $u \in \mathbb{K}^*$ is a minimum (or local minimum, Definition 2.1) is that it is a weak solution, i.e. u satisfies

$$\begin{aligned} -\Delta u &= f & &\text{in} \quad \Omega = \{u > 0\}, \\ u = 0 \quad |\nabla u| &= g, & &\text{on} \quad \partial\Omega, \end{aligned}$$

We shall now discuss weak solutions belonging to $\mathbb{K}^*$ and compare $J(u)$ for these. So let $u \in \mathbb{K}^*$ be a weak solution. Since f, g and u only depend on r and u is nonincreasing there is a unique ρ in $[0, \infty)$ such that $\Omega = B(0, \rho)$ and the equations above become

$$-u'' - \frac{N-1}{r}u' = f(r), \qquad 0 < r < \rho \tag{1.7}$$

$$u(\rho) = 0, \tag{1.8}$$

$$-u'(\rho) = g(\rho). \tag{1.9}$$

By (1.7) we have $(r^{N-1}u')' = -r^{N-1}f(r)$ and by (1.9)

$$r^{N-1}u'(r) = -\rho^{N-1}g(\rho) + \int_r^\rho s^{N-1}f(s)ds. \tag{1.10}$$

As $r \to 0$ we shall have $r^{N-1}u'(r) \to 0$ (otherwise we get a distributional contribution to Δu at the origin). Thus

$$\int_0^\rho s^{N-1}f(s)ds = \rho^{N-1}g(\rho). \tag{1.11}$$

This is a condition for ρ. Once ρ is determined u is obtained by integrating (1.10) and using (1.8). Explicitly

$$u(r) = \int_r^\rho t^{1-N}\left(\rho^{N-1}g(\rho) - \int_t^\rho s^{N-1}f(s)ds\right)dt,$$

for $0 < r < \rho$. Set

$$F(\rho) = \int_0^\rho s^{N-1}f(s)ds - \rho^{N-1}g(\rho),$$

for $\rho \geq 0$ so that (1.11) becomes $F(\rho) = 0$. Thus the weak solutions in $\mathbb{K}^*$ are in bijective correspondence to the zeros of F.

Let us now specialize to the case $f(r) = a\chi_{[0,R)} - b$ and $g(r) = c$, where $a, b, c, R \geq 0$ are constants with $a > b, b + c > 0, R > 0$. F becomes

$$F(\rho) = \begin{cases} \dfrac{a-b}{N}\rho^N - c\rho^{N-1} & 0 \leq \rho < R, \\ \dfrac{a}{N}R^N - \dfrac{b}{N}\rho^N - c\rho^{N-1} & \rho \geq R. \end{cases}$$

Note that $F'(\rho) \leq$ const. < 0 for $\rho \geq R$. It follows from the equations for a weak solution that $\int |\nabla u|^2 dx = \int fu dx$. Therefore

$$J(u) = \omega_N\left[\frac{c^2}{N}\rho^N - \int_0^\rho (u'(r))^2 r^{N-1}dr\right] \tag{1.12}$$

if u is the weak solution corresponding to ρ. By (1.10)

$$r^{N-1}u'(r) = -c\rho^{N-1} + \frac{a-b}{N}(\rho^N - r^N) \tag{1.13}$$

for $0 < r < \rho$ if $0 < \rho < R$, while

$$r^{N-1}u'(r) = -c\rho^{N-1} - \frac{b}{N}(\rho^N - r^N) + \frac{a}{N}(R^N - r^N)\chi_{[0,R)}(r) \tag{1.14}$$

for $0 < r < \rho$ if $\rho > R$. Inserting this into (1.12) gives

$$\frac{1}{\omega_N}J(u) = \frac{c^2}{N}\rho^N - \int_0^\rho \left[c\rho^{N-1} - \frac{a-b}{N}(\rho^N - r^N)\right]^2 r^{1-N}dr, \tag{1.15}$$

if $0 < \rho < R$ and

$$\begin{aligned} &\frac{1}{\omega_N}J(u) = \frac{c^2}{N}\rho^N \\ &- \int_0^\rho \left(-c\rho^{N-1} - \frac{b}{N}(\rho^N - r^N) + \frac{a}{N}(R^N - r^N)\chi_{[0,R)}\right)^2 r^{1-N}dr = \\ &\frac{c^2}{N}\rho^N - \int_0^R \left(\frac{a-b}{N}r^N\right)^2 r^{1-N}dr - \int_R^\rho \left(\frac{a}{N}R^N - \frac{b}{N}r^N\right)^2 r^{1-N}dr \leq \\ &\frac{c^2}{N}\rho^N - \frac{(a-b)^2}{N^2(N+2)}R^{N+2} - \frac{1}{N^2}\int_R^\rho (aR^N - br^N)^2 r^{1-N}dr, \end{aligned} \tag{1.16}$$

if $\rho > R$ (recall that $F(\rho) = 0$).

We shall now determine all zeros $\rho = \rho_n$ of F and compare $J(u_n)$ for the corresponding weak solutions $u_n \in \mathbb{K}^*$ $(n = 0, 1, ...)$. Observe first that $\rho = \rho_0 = 0$ is always a zero of F, corresponding to $u_0 = 0$ with $J(u_0) = 0$. Next we divide into cases.

Case 1: $c = 0$. Then J is convex and there is, besides ρ_0, exactly one more zero of F, namely $\rho_1 = \left(\frac{a}{b}\right)^{1/N} R > R$. It is easily seen from (1.16) that $J(u_1) < 0$. Thus u_1 is the only minimum of J and there are no other local minima.

Case 2: $c > 0$ and $b < a < b + \frac{Nc}{R}$. In this case $F(\rho) < 0$ for all $\rho > 0$. Hence $u_0 = 0$ is the only weak solution (in $\mathbb{K}^*$) and it is the global minimum of J. In particular $J(u) \geq 0$ for all $u \in \mathbb{K}$.

Case 3: $c > 0$ and $a = b + \frac{Nc}{R}$. Here $\rho_0 = 0$ and $\rho_1 = R$, are the zeros of F. Equation (1.16) gives that $J(u_1) > 0$. Thus u_0 is the only minimizer and $J \geq 0$.

Case 4: $c > 0$ and $a > b + \frac{Nc}{R}$. In this case $F(R) > 0$ and it follows that F has exactly three zeros: $\rho_0 = 0 < \rho_2 < R < \rho_1$. We have $\rho_2 = \frac{Nc}{a-b}$ and from (1.15) one finds that $J(u_2) > 0$ always. As to ρ_1, it is determined by

$$aR^N = b\rho_1^N + Nc\rho_1^{N-1}, \tag{1.17}$$

and $J(u_1)$ is then obtained by inserting this into (1.16). It is clear from (1.17) that when a increases from $b + \frac{Nc}{R}$ to $+\infty$ (with b, c and R kept fixed), then ρ_1 increases from R to $+\infty$. Moreover $J(u_1)$ at the same time decreases monotonically to $-\infty$ from its positive value when $a = b + \frac{Nc}{R}$. This can be seen e.g. by estimating the derivative $\frac{d}{da}J(u_1)$ or $\frac{d}{d\rho_1}J(u_1)$. It follows that there exists a critical value $a_0 > b + \frac{Nc}{R}$ such that we have the following three subcases.

Subcase 4a: $c > 0$ and $b + \frac{Nc}{R} < a < a_0$. Then $J(u_0) = 0$, $J(u_1) > 0$, $J(u_2) > 0$. Thus u_0 is the only minimizer and $J \geq 0$. However u_1 can be shown to be a local minimizer in this case. Indeed, it is not hard to see that u_1 is a local minimizer among other functions in $\mathbb{K}^*$, and when moving out from $\mathbb{K}^*$ (into $\mathbb{K} \setminus \mathbb{K}^*$) the functional J increases as was observed in the proof of Lemma 1.2.

Subcase 4b: $c > 0$ and $a = a_0$. Then $J(u_0) = J(u_1) = 0$, $J(u_2) > 0$. Thus we have two minima, and $J \geq 0$.

Subcase 4c: $c > 0$ and $a > a_0$. Then $J(u_0) = 0$, $J(u_1) < 0$, $J(u_2) > 0$ so that u_1 is the only minimizer. By the same argument as in subcase 4a, u_0 is a local minimizer.

Finally in this example we need (for later use) an estimate of a_0. We claim the following: if

$$a > \left(b + \frac{cN}{3R}\right) 3^N \tag{1.18}$$

then $J(u_1) < 0$ and

$$\rho_1 > 3R. \tag{1.19}$$

In particular $a_0 \leq (b + \frac{Nc}{3R})3^N$.

That $\rho_1 > 3R$ when (1.18) holds follows immediately from (1.17). Observe next that (1.17) also implies

$$\rho_1 \leq \left(\frac{aR}{Nc}\right)^{1/(N-1)} R \tag{1.20}$$

(and $\rho_1 \le (a/b)^N R$). Using (1.20) and (1.18) in (1.16) gives by a little computation that already the two first terms make $J(u_1)$ negative when $N \ge 3$. When $N = 2$ one also has to take the last term into account and the calculation becomes a little more tedious. (In the last term (the integral) one may replace ρ by $3R$ and b by $3^{-2}a$ according to (1.19) and (1.18).)

Subexample. If $N = 2$ and $b = 0$ then a_0 can easily be calculated to be $a_0 = \frac{2c}{R}\sqrt[4]{e}$.

As a corollary of Example 1.5 and Lemma 1.1 we have

Proposition 1.6. *Let f, g satisfy Condition A and set $a = \sup f$, $c = \inf g$ and let R be the radius of the smallest closed ball containing supp f_+. Then, if $aR \le Nc$, $J = J_{f,g} \ge 0$ and $u = 0$ is the only minimizer. There even exists a number $a_0 = a_0(N, R, c) > \frac{Nc}{R}$ such that the same conclusion holds whenever $a < a_0$.*

Proof. We have $f \le a\chi_{B_R} - b$, $g \ge c$ with $b = 0$. Then combine Example 1.5 with Lemma 1.1 □

Proposition 1.7. *Let f, g satisfy Condition A and let u be a minimizer of J. Assume that $u = 0$ on ∂B_R where B_R is a ball such that $R\sup_{B_R} f_+ \le N\inf_{B_R} g$. Then $u = 0$ in B_R.*

Proof. Set $v = u$ in B_R and $v = 0$ outside B_R. Clearly v minimizes $\tilde{J} = J_{\tilde{f},\tilde{g}}$ where $\tilde{f} = (f_+)\chi_{B_R}$, $\tilde{g} = g\chi_{B_R} + (\inf_{B_R} g)\chi_{\mathbb{R}^N\setminus B_R}$. Now apply Proposition 1.6 to $\tilde{J}$. □

Proposition 1.8. *If u and v are minima of J then also* $\min(u, v)$ *and* $\max(u, v)$ *are minima. Also, if $\{u_n\}$ are minima and $u_1 \le u_2 \le \ldots$ then $u = \sup u_n$ is a minimum. Similarly, if $u_1 \ge u_2 \ge \ldots$ then* $\inf u_n$ *is a minimum. Finally there is a largest minimizer of J, and also a smallest one.*

Proof. The first statement follows immediately from Lemma 1.1 and the second (and the third) from the compactness assertion of Theorem 1.4.

To prove the last assertion, first note that since $H^1(\mathbb{R}^N)$ is separable there is a finite or infinite sequence $\{v_n\}$ of minima which is dense in the set of all minima. Define $u_1 = v_1$ and, inductively for $n \ge 2$, $u_n = \sup(u_{n-1}, v_n)$, so that $u_1 \le u_2 \le \ldots$. As shown above $u = \sup u_n$ is also a minimizer and it is readily verified that $v \le u$ for every minimizer v. □

2. Local minima

In this section we deduce basic properties of minima, or more generally of local minima, of J. The data f and g will generally be assumed to satisfy Condition A. The main result of this section is Theorem 2.13, saying that any local minimum u solves the appropriate free boundary problem in a potential theoretically satisfactory sense, provided g is continuous. This means that the distributional Laplacian Δu can be expressed in terms of purely geometric quantities related to the open set $\Omega = \{u > 0\}$, more precisely that

$$\Delta u + f\mathcal{L}^N\lfloor\Omega = g\mathcal{H}^{N-1}\lfloor\partial\Omega.$$

Continuous functions $u \in \mathbb{K}$ satisfying this equation will be called *weak solutions* (Definition 3.1).

It should be told that this section is very much based on the methods and results of the pioneering paper [AC] (see also [ACF] and [F]). Many of our proofs are modifications of corresponding proofs in [AC].

Definition 2.1. A function $u \in \mathbb{K}$ is a local minimum of J if, for some $\epsilon > 0$, $J(v) \geq J(u)$ for every $v \in \mathbb{K}$ with

$$\int \left(|\nabla(v-u)|^2 + |\chi_{\{v>0\}} - \chi_{\{u>0\}}|\right) dx < \epsilon. \tag{2.1}$$

Lemma 2.2. *If u is a local minimum then*

$$\Delta u + f_+ \geq 0 \quad in \quad \mathbb{R}^N, \tag{2.2}$$

$$\Delta u + f = 0 \quad in \quad \Omega = \{u > 0\}, \tag{2.3}$$

$$\Delta u + f \leq 0 \quad in \quad \mathbb{R}^N \setminus supp\ g. \tag{2.4}$$

Remark. It follows from (2.2) that u has an upper semicontinuous representative, which is the one we will refer to in the sequel, and it will be proved later that this u actually is continuous. For the present proof of (2.3) Ω should strictly speaking be defined as the set of points $x \in \mathbb{R}^N$ such that there exists $0 \leq \phi \in C^\infty(\mathbb{R}^N)$ with $\phi(x) > 0$ and $u \geq \phi$ everywhere.

Proof. Take $0 \leq \phi \in C_o^\infty(\mathbb{R}^N)$ and define, for $\epsilon > 0$, $v_\epsilon = (u - \epsilon\phi)_+$. Then $v_\epsilon \in \mathbb{K}$, $0 \leq v_\epsilon \leq u$. Set $D_\epsilon = \{u \leq \epsilon\phi\} = \{v_\epsilon = 0\}$. Clearly $|D_\epsilon \cap \Omega| \to 0$ as $\epsilon \to 0$. Since $v_\epsilon - u = -u$ in D_ϵ, $v_\epsilon - u = -\epsilon\phi$ outside D_ϵ it follows that $v_\epsilon \to u$ in $H^1(\mathbb{R}^N)$ and that

$$\int g^2 |\chi_{\{v_\epsilon>0\}} - \chi_{\{u>0\}}| dx = \int_{\Omega\cap D_\epsilon} g^2 dx \quad \to 0.$$

Since u is a local minimum we conclude that $J(u) \leq J(v_\epsilon)$ for $\epsilon > 0$ small enough.

Next we estimate

$$\begin{aligned}
0 &\leq J(v_\epsilon) - J(u) \\
&= \int |\nabla v_\epsilon|^2 - \int |\nabla u|^2 - 2\int f(v_\epsilon - u) + \int g^2(\chi_{\{v_\epsilon>0\}} - \chi_{\{u>0\}}) \\
&= \int_{D_\epsilon^c} |\nabla(u-\epsilon\phi)|^2 - \int_{\mathbb{R}^N} |\nabla u|^2 + 2\epsilon\int_{D_\epsilon^c} f\phi + 2\int_{D_\epsilon} fu - \int_{\Omega\cap D_\epsilon} g^2 \\
&\leq -2\epsilon\int_{D_\epsilon^c} \nabla u\cdot\nabla\phi + \epsilon^2\int_{D_\epsilon^c} |\nabla\phi|^2 + 2\epsilon\int_{D_\epsilon^c} f_+\phi + 2\int_{D_\epsilon} f_+ u \\
&\leq 2\epsilon\left(\int_{\mathbb{R}^N} f_+\phi - \int_{\mathbb{R}^N} \nabla u\cdot\nabla\phi\right) + 2\epsilon\int_{D_\epsilon\cap\Omega} \nabla u\cdot\nabla\phi + \epsilon^2\int_{D_\epsilon^c} |\nabla\phi|^2.
\end{aligned}$$

Dividing both sides by ϵ and letting $\epsilon \to 0$ we obtain

$$0 \leq -\int \nabla u\cdot\nabla\phi + \int f_+\phi$$

for all $0 \le \phi \in C_o^\infty(\mathbb{R}^N)$, and hence that $\Delta u + f_+ \ge 0$ in $\mathbb{R}^N$.

If supp $\phi \subset \Omega$ we can take $v = u \pm \epsilon\phi \in \mathbb{K}$ for $\epsilon > 0$ small enough (and $0 \le \phi \in C_o^\infty(\mathbb{R}^N)$) and this readily gives that $\Delta u + f = 0$ in Ω.

Finally, taking $v_\epsilon = u + \epsilon\phi$ where supp $\phi \cap$ supp $g = \emptyset$ gives that $\Delta u + f \le 0$ in $\mathbb{R}^N \setminus$ supp g. □

Theorem 2.3. *Let u be a local minimum and assume $g^2 \in H^{1,1}(\mathbb{R}^N)$. Then*

$$\lim_{\epsilon \searrow 0} \int_{\partial\{u>\epsilon\}} (|\nabla u|^2 - g^2)\eta \cdot \nu d\mathcal{H}^{N-1} = 0$$

for every $\eta \in C_o^\infty(\mathbb{R}^N, \mathbb{R}^N)$. ($\nu$ denotes the outward normal vector of $\partial\{u > 0\}$.)

The proof is similar to that of Theorem 2.5 in [AC] and therefore omitted.

Lemma 2.4. *("Harnack") Assume $u \in H^1(B_r)$, $u \ge 0$ on ∂B_r ($B_r = B(0, r)$) and let $M \ge 0$.*

a) If $\Delta u \le M$ in B_r then

$$u(x) \ge r^N \frac{r - |x|}{(r + |x|)^{N-1}} \left[\frac{1}{r^2} \fint_{\partial B_r} u - \frac{2^{N-1} M}{N} \right],$$

for $x \in B_r$.

b) If $\Delta u \ge -M$ in B_r then

$$u(x) \le r^N \frac{r + |x|}{(r - |x|)^{N-1}} \left[\frac{1}{r^2} \fint_{\partial B_r} u + \frac{M}{2N} \right],$$

for $x \in B_r$.

c) If $|\Delta u| \le M$ in B_r then

$$r^{N-2} \frac{r - |x|}{(r + |x|)^{N-1}} u(0) - \frac{M r^2}{N} \le$$
$$u(x) \le r^{N-2} \frac{r + |x|}{(r - |x|)^{N-1}} u(0) + \frac{M r^N}{N} \frac{r + |x|}{(r - |x|)^{N-1}},$$

for $x \in B_r$.

d) If $|\Delta u| \le M$ in B_r, then

$$|\nabla u(0)| \le N \left[\frac{1}{r} \fint_{\partial B_r} u + \frac{M}{N+1} r \right].$$

Note that, by Lemma 2.2, b) is always applicable (with $M = \sup_{B_r} f_+$) if u is a local minimum, while a), c) and d) are applicable if $B_r \subset \Omega$.

The proof consists of straightforward applications of the Poisson formula combined with super- and subharmonicity properties of functions $u(x) \pm \frac{M}{2N}(r^2 - |x|^2)$. (Details are omitted.)

Lemma 2.5. *Suppose u is a local minimizer of J. Then there is an $r_0 > 0$ such that for any ball B_r with $0 < r < r_0$ we have*

$$\frac{1}{r}\fint_{\partial B_r} u > 2^N\left(\frac{r}{N}\sup_{B_r} f_- + \sup_{B_r} g\right) \qquad \Longrightarrow \qquad u > 0 \text{ and continuous in } B_r.$$

Note. The reason that r has to be small is simply that u is assumed only to be a local minimizer. For a global minimum the implication is true for all $r > 0$.

Proof. We may assume that B_r is centered at the origin. Define $v \in H^1(\mathbb{R}^N)$ by $v = u$ on $\mathbb{R}^N \setminus B_r$ (in particular on ∂B_r) and $-\Delta v = f$ in B_r. Note that v is continuous in B_r. Then, as in [AC, 3.2],

$$J(u) - J(v) \geq \int_{B_r} |\nabla(u-v)|^2 - \sup_{B_r} g^2 |\{u = 0\} \cap B_r|. \tag{2.5}$$

On the other hand a) of Lemma 2.4 (applied to v) shows that

$$v(x) \geq r^N \frac{r - |x|}{(r + |x|)^{N-1}}\left[\frac{1}{r^2}\fint_{\partial B_r} u - \frac{2^{N-1}M}{N}\right],$$

for $x \in B_r$. Here $M = \sup_{B_r} f_-$. Thus whenever

$$\frac{1}{r^2}\fint_{\partial B_r} u \geq \frac{2^N M}{N}, \tag{2.6}$$

we have

$$v(x) \geq \frac{r^N}{2}\frac{r - |x|}{(r + |x|)^{N-1}}\frac{1}{r^2}\fint_{\partial B_r} u \geq 2^{-N}(r - |x|)\frac{1}{r}\fint_{\partial B_r} u, \tag{2.7}$$

and in particular $v(x) > 0$ for $x \in B_r$. As in [Ac, 3.2] one derives from (2.7) the estimate

$$|\{u = 0\} \cap B_r| \left(\frac{1}{r}\fint_{\partial B_r} u\right)^2 \leq 2^{2N}\int_{B_r} |\nabla(u - v)|^2.$$

When (2.6) holds v is nonnegative by (2.7) so that $v \in \mathbb{K}$, and if $r > 0$ is small v is moreover close to u in the metric (2.1). Thus, since u is a local minimum, $J(u) \leq J(v)$, i.e. by (2.5)

$$\int_{B_r} |\nabla(u-v)|^2 \leq \sup_{B_r} g^2 |\{u = 0\} \cap B_r|.$$

Hence

$$|\{u = 0\} \cap B_r| \left(\frac{1}{r}\fint_{\partial B_r} u\right)^2 \leq 2^{2N}\sup_{B_r} g^2 |\{u = 0\} \cap B_r|, \tag{2.8}$$

whenever (2.6) holds.

This proves the lemma, for if

$$\frac{1}{r}\fint_{\partial B_r} u > 2^N\left(\frac{r}{N}\sup_{B_r} f_- + \sup_{B_r} g\right)$$

then (2.6) does hold, and (2.8) leads to a contradiction unless $|\{u = 0\} \cap B_r| = 0$. In the latter case we have $\int_{B_r} |\nabla(u - v)|^2 = 0$ and hence $u = v > 0$ in B_r as desired. □

Corollary 2.6. *Any local minimum u is Lipschitz continuous. Moreover near $\partial\Omega$ we have the estimates*

$$u(x) \leq 2^N \delta(x) \left(\sup_{B(x,2\delta(x))} g + M\delta(x) \right),$$

$$|\nabla u(x)| \leq N 2^N \left(\sup_{B(x,2\delta(x))} g + M\delta(x) \right), \tag{2.9}$$

where $M = \sup|f|$, $\Omega = \{u > 0\}$ and $\delta(x)$ denotes the distance from x to Ω^c. Thus $u(x) \leq C\delta(x)$ always, and if x approaches a point of $\partial\Omega$ where g vanishes we have a better estimate (e.g. $u(x) \leq C\delta^{1+\alpha}(x)$ if g is α-Hölder continuous).

For a proof see [AC, ??].

Remark 2.7. (On homogeneity) For $t > 0$ and $\varphi(x)$ any function of $x \in \mathbb{R}^N$, set $\varphi_t(x) = \varphi(x/t)$. Then a straightforward computation shows that for any real number α we have

$$J_{t^\alpha f_t, t^{\alpha+1} g_t}(t^{\alpha+2} u_t) = t^{N+2\alpha+2} J_{f,g}(u).$$

Lemma 2.8. *Let u be a local minimum. If $g \geq$ const. > 0 in an open set $D \subset \mathbb{R}^N$ then there is a constant $C > 0$ such that for any sufficiently small ball $B_r \subset D$ we have*

$$\frac{1}{r} \fint_{\partial B_r} u \leq C \qquad \Longrightarrow \qquad u = 0 \quad \text{in } B_{r/4}. \tag{2.10}$$

More precisely, C depends only on $\inf_{B_r} g$, $r \sup_{B_r} f_+$ and N and is positive whenever $\inf_{B_r} g > 0$ and $r \sup_{B_r} f_+$ is sufficiently small.

Remark. The lemma holds with $B_{\kappa r}$ in place of $B_{r/4}$ for any $0 < \kappa < 1$; C then also depends on κ.

Proof. For u a local minimizer of J b) of Lemma 2.4 always applies and gives, for some constants C_1 and C_2 only depending on N, that

$$u \leq C_1 \fint_{\partial B_r} u + C_2 r^2 \sup_{B_r} f_+ \tag{2.11}$$

in $B_{r/2}$. For notational convenience we assume that $B_r = B_r(0)$.

Set $m = \inf_{B_r} g$, $M = \sup_{B_r} f_+$ and define

$$J_r(v) = \int_{B_{r/2}} \left(|\nabla v|^2 - 2fv + g^2 \chi_{\{v>0\}} \right) dx,$$

$$\tilde{J}_r(v) = \int_{B_{r/2}} \left(|\nabla v|^2 - 2Mv + m^2 \chi_{\{v>0\}} \right) dx.$$

As in Lemma 1.1 we have

$$J_r(\min(u_1, u_2)) + \tilde{J}_r(\max(u_1, u_2)) \leq J_r(u_1) + \tilde{J}_r(u_2), \tag{2.12}$$

for any $u_1, u_2 \in H^1(B_{r/2})$.

Given a constant $\beta \geq 0$ consider the problem of minimizing $\tilde{J}_r(v)$ over $\{v \in H^1(B_{r/2}) : v \geq 0, v = \beta \text{ on } \partial B_{r/2}\}$. We claim that the largest minimizer v_β of $\tilde{J}_r$ vanishes on $B_{r/2}$ provided r and β are small enough. Clearly v_β is radially symmetric. Therefore the claim can be proved by comparing $\tilde{J}_r(w_n)$ for the various radially symmetric weak solutions $w_n = w_{\beta,n}$ for $\tilde{J}_r$, in a similar way as in Example 1.5.

We take $0 < r < 2Nm/M$. It then follows from Proposition 1.7 that if the largest minimizer v_β vanishes somewhere then there is some $0 < \rho < r/2$ such that $v_\beta = 0$ on $\overline{B}_\rho$, $v_\beta > 0$ in $B_{r/2} \setminus B_\rho$. Therefore it is enough to compare weak solutions w of the corresponding form, i.e. satisfying as functions of radius $|x|$ (cf. (1.7)–(1.9)),

$$\begin{aligned} w(|x|) &= 0 && 0 \leq |x| \leq \rho, \\ w'(\rho + 0) &= m && \\ (|x|^{N-1} w')' &= -|x|^{N-1} M && \rho < |x| < r/2, \\ w(r/2) &= \beta. && \end{aligned}$$

Note that by the third equation w' changes sign at most once. Therefore it is easy to see that the above system has at most three solutions, call them w_0, w_1 and w_2 ($w_n = w_{n,\beta}$). w_0 is the one corresponding to the largest value ρ_0 of ρ, with $w_0'(|x|) > 0$ for $\rho_0 < |x| < r/2$. w_1 is the solution obtained if w' changes sign. Thus $0 < \rho_1 < \rho_0$ and $w_1'(|x|) < 0$ for $|x|$ close to $r/2$. Finally w_2 is the uniquely determined weak solution which does not vanish at all.

Now consider what happens when $\beta \searrow 0$. Clearly $w_{0,\beta} \to w_{0,0} = 0$, e.g. in $H^1(B_{r/2})$ and $\rho_0 \to r/2$ (since $w'(\rho + 0) = m$) so that

$$\lim_{\beta \to 0} \tilde{J}_r(w_{0,\beta}) = \tilde{J}_r(0) = 0.$$

$w_{1,\beta}$ exists if and only if $M > 0$ and then $w_{1,\beta} \to w_{1,0} \neq 0$ and

$$\varliminf_{\beta \to 0} \tilde{J}_r(w_{1,\beta}) \geq \tilde{J}_r(w_{1,0}) > 0$$

where the last inequality follows from the fact that $v \equiv 0$ is the unique minimizer when $\beta = 0$ (by Proposition 1.6).

For the same reason we have $w_{2,\beta} \to w_{2,0}$ and

$$\varliminf_{\beta \to 0} \tilde{J}_r(w_{2,\beta}) \geq \tilde{J}_r(w_{2,0}) > 0$$

when $M > 0$. When $M = 0$ then $w_{2,0} \equiv 0$ so that $\tilde{J}_r(w_{2,0}) = 0$, but then $w_{2,\beta} \equiv \beta$ so that

$$\varliminf_{\beta \to 0} \tilde{J}_r(w_{2,\beta}) \geq m^2 |B_{r/2}| > 0.$$

From the above limits we conclude that $\tilde{J}(w_{0,\beta})$ is smaller than both $\tilde{J}(w_{1,\beta})$ and $\tilde{J}(w_{2,\beta})$ if β is small enough. Since clearly $w_{0,\beta}$ vanishes on $B_{r/4}$ if β is small this proves our claim: the largest minimizer $v_\beta(=w_{0,\beta})$ of $\tilde{J}_r$ vanishes on $B_{r/4}$ if $0<r<2Nm/M$ and, say, $0<\beta<\beta_0$.

Now β_0 depends on r, m and M. Indeed, it is easily seen (cf. Remark 2.7) that if r is scaled to tr ($t>0$) then the minimizer $v(x)$ of $\tilde{J}_r$ will be scaled to $tv(x/t)$ provided m, M and β, are scaled to, respectively, $m, M/t$ and $t\beta$. In other words, $\beta_0(tr,m,M/t)=t\beta_0(r,m,M)$ for $t>0$ or, with $t=1/r$,

$$\beta_0(r,m,M)=r\beta_0(1,m,rM). \tag{2.13}$$

For $M=0$ estimates for β_0 were computed in [AC, 2.6]. One has that $\beta_0(1,m,0)>0$ for $m>0$ and is an increasing function of m. Moreover, $\beta_0(1,m,M)$ is decreasing as a function of M and can be taken to depend continuously on (m,M) in a neighbourhood of $M=0$.

It follows from (2.11) and (2.13) that we can achieve

$$u\le\beta\le\beta_0(r,m,M) \tag{2.14}$$

on $\partial B_{r/2}$ by letting

$$C_1\frac{1}{r}⨍_{\partial B_r}u+C_2rM<\beta_0(1,m,rM).$$

Since $\beta_0(1,m,0)>0$ this shows that (2.14) holds if an estimate of the form (2.10) holds.

Now it only remains to show that (2.14) implies that $u=0$ in $B_{r/4}$. Let w denote the function which equals $\min(u,v)$ in $B_{r/2}$ and equals u outside $B_{r/2}$. When (2.14) holds then $w\in\mathbb{K}$, and if $r>0$ is small enough then w will be so close to u in the metric (2.1) that $J_r(u)\le J_r(w)$, u being a local minimizer of J. We also have $\tilde{J}_r(v)\le\tilde{J}_r(\max(u,v))$. But these two inequalities contradict (2.12) unless we have equality everywhere. Since v was the largest minimizer of $\tilde{J}_r$ it follows that $v=\max(u,v)$, i.e. that $u\le v$. Hence u vanishes in $B_{r/4}$. $\square$

Corollary 2.9. *If $g\ge$ const. >0 in a neighbourhood of a point $x_0\in\partial\Omega$ then*

$$u(x)\ge C\delta(x)$$

near x_0.

Proof. With C_1 the constant in (2.10) we have by a) of Lemma 2.4 and (2.10) (for $x\in\Omega$ close to x_0, $r=\delta(x)$, $B_r=B_r(x,r)$),

$$u(x)\ge ⨍_{\partial B_r}u-C_2r^2\ge C_1r-C_2r^2\ge Cr. \quad \square$$

Lemma 2.10. *Let u be a local minimum, $\Omega=\{u>0\}$ and assume that $g\ge$ const. >0 in a neighbourhood of a point $x_0\in\partial\Omega$. Then there are constants c_1 and c_2 such that*

$$0<c_1\le\frac{|B_r\cap\Omega|}{|B_r|}\le c_2<1$$

for small $r>0$ ($B_r=B(x_0,r)$).

For a proof see [AC, 2.7].

Remark 2.11. In addition to Lemma 2.8 the following lemma, due to Caffarelli, is useful:

Assume $0 \leq u \in H^1(B(0,R))$, $\Delta u \geq c > 0$ in $\Omega = \{u > 0\}$, $0 \in \overline{\Omega}$. Then, for any $0 < r < R$

$$\sup_{\partial B_r(0)} u \geq \frac{cr^2}{2N}.$$

(See [Caff1] for the simple proof). If u is a local minimum (or weak solution) for our problem, then this lemma shows that

$$\sup_{\partial B_r(x)} u \geq \frac{r^2}{2N} \inf_{B_r(x)} f_- \tag{2.15}$$

for any $x \in \overline{\Omega}$.

Proposition 2.12. *Any local minimizer u of J has compact support.*

Proof. By Lemma 2.2 u is subharmonic outside supp f_+. Therefore

$$u(x) \leq \frac{1}{|B_r(x)|} \int_{B_r(x)} u \leq \frac{1}{\sqrt{|B_r(x)|}} \left(\int_{B_r(x)} u^2 \right)^{1/2} \leq \frac{\|u\|}{\sqrt{|B_r|}}$$

for x a distance r away from supp f_+. Thus

$$u(x) \leq C|x|^{-N/2} \tag{2.16}$$

for $|x|$ large.

By Condition A either $g \geq c > 0$ or $f_- \geq c > 0$ (or both) far away. In the first case we conclude that $u(x) = 0$ for large $|x|$ by combining (2.16) with Lemma 2.8. In the second case the same conclusion follows from (2.16) combined with (2.15). □

In order to prepare for the main result in this section we need to recall a few facts about functions of bounded variation and sets of finite perimeter.[Gi],[EG, ch. 5] are good references for this.

Let $E \subset \mathbb{R}^N$ be a Lebesgue measurable set. The *measure theoretic boundary* $\partial_{mes}E$ of E is defined to be the set of points $x \in \mathbb{R}^N$ such that E and E^c have positive density at x. Thus $\partial_{mes}E \subset \partial E$ (the topological boundary). E is said to have *locally finite perimeter* if $\nabla \chi_E$ is a vector-valued Radon measure. This means that there exists a positive Radon measure $\mu = \mu_E$ in $\mathbb{R}^N$ and a μ-measurable function $\nu_E : \mathbb{R}^N \to S^{N-1} \cup \{0\}$ (the direction factor) such that $-\nabla \chi_E = \mu \lfloor \nu_E$, i.e. $\int_E \mathrm{div}\phi dx = \int \phi \cdot \nu_E d\mu$ for all $\phi \in C^1_c(\mathbb{R}^N, \mathbb{R}^N)$ (the left member being equal to $\langle -\nabla \chi_E, \phi \rangle$). The measure μ will occasionally be denoted $|\nabla \chi_E|$. It can be shown [EG, 5.11] that a measurable set E has locally finite perimeter if and only if $\mathcal{H}^{N-1}(K \cap \partial_{mes}E) < \infty$, for each compact set $K \subset \mathbb{R}^N$.

Assuming that E has locally finite perimeter the *reduced boundary* $\partial_{red}E$ of E can be defined as the set of points $x \in \mathbb{R}^N$ for which the density $\lim_{r\to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} \nu_E d\mu$ exists and has modulus one. It is convinient to work with that representative of ν_E which equals this limit on $\partial_{red}E$ and is zero elsewhere and ν_E then is the measure theoretic outward unit normal vector of E on $\partial_{red}E$. Clearly $\partial_{red}E \subset \partial_{mes}E$ and it is not hard to show that $\mathcal{H}^{N-1}(\partial_{mes}E \setminus \partial_{red}E) = 0$, ([EG, ch. 5.8]). A basic structure theorem says that $\mu = |\nabla \chi_E|$ actually agrees with $(N-1)$-dimensional Hausdorff measure restricted to $\partial_{red}E$: $|\nabla\chi_E| = \mathcal{H}^{N-1}\lfloor\partial_{red}E$. Thus also $\nabla\chi_E = -\nu_E\mathcal{H}^{N-1}\lfloor\partial_{red}E$.

All the above definitions and results carry over to the case with an open subset $G \subset \mathbb{R}^N$ in place of $\mathbb{R}^N$. One then speaks of sets having locally finite perimeter in G etc.

Theorem 2.13. *Assume that f, g satisfy Condition A, that g is continuous and set $G = \{x \in \mathbb{R}^N : g(x) > 0\}$. If $\partial G \neq \emptyset$ assume moreover that for some $0 < \alpha \leq 1$ g is α-Hölder continuous near ∂G and that $\mathcal{H}^{N-1+\alpha}(\partial G) = 0$. Then, if u is a local minimizer of $J = J_{f,g}$, then $\Omega = \{u > 0\}$ has locally finite perimeter in G,*

$$\mathcal{H}^{N-1}((\partial\Omega \setminus \partial_{red}\Omega) \cap G) = 0,$$

and

$$\Delta u + f\mathcal{L}^N\lfloor\Omega = g\mathcal{H}^{N-1}\lfloor\partial\Omega = g\mathcal{H}^{N-1}\lfloor\partial_{red}\Omega = g|\nabla\chi_\Omega|. \tag{2.17}$$

Here the right members shall be interpreted as zero outside G.

Remark 2.14. Ω need not have locally finite perimeter outside G. To see this, take e.g. $g = 0$, $f = a\chi_D - 1$ where D is a bounded domain such that ∂D has positive N-dimensional Lebesgue measure $|\partial D|$, and $1 < a < 1 + \frac{|\partial D|}{|D|}$ is a parameter. By (2.4), $\Delta u \leq 1 - a\chi_D < 0$ in D showing that $D \subset \Omega$. Also, $\overline{D} \subset \overline{\Omega}$. Next (2.17) yields $\Delta u = (1 - a\chi_D)\chi_\Omega = \chi_\Omega - a\chi_D$, by which $|\Omega| = a|D|$. Thus

$$|\Omega| < |D| + |\partial D| = |\overline{D}| \leq |\overline{\Omega}| = |\Omega| + |\partial\Omega|,$$

i.e. $\partial\Omega$ has even positive N-dimensional Lebesgue measure.

Corollary 2.15. *(to Lemma 2.10) With assumptions as in Theorem 2.13*

$$\partial\Omega \cap G = \partial_{mes}\Omega \cap G.$$

Remark. $\partial_{mes}\Omega$ may be strictly smaller than $\partial\Omega$ outside G. Indeed, in the case $g \equiv 0$ there are examples with $\partial\Omega$ having singular points (e.g. inward cusps and double points, when $N = 2$) at which Ω has density one.

For the proof of Theorem 2.13 we need the following observation.

Lemma 2.16. *Assume $u \geq 0$ is a continuous function such that Δu is a signed Radon measure. Then $\Delta u \geq 0$ on $\{u = 0\}$.*

The proof of Lemma 2.16 is quite straightforward and therefore omitted (cf. [AC, 4.2]).

Proof. (of Theorem 2.13) (2.2) shows that Δu is a Radon measure and (2.3) and Lemma 2.16 then show that $\Delta u + f\chi_\Omega = \lambda$, where λ is a positive Radon measure on $\partial\Omega$.

For any $x \in \mathbb{R}^N$, $|\nabla u|$ is integrable on $\partial B_r(x)$ for almost every $r > 0$ and for these r

$$|\int_{B_r(x)} \Delta u| \leq \int_{\partial B_r(x)} |\nabla u| d\mathcal{H}^{N-1} \leq Cr^{N-1} \sup_{\partial B_r(x)} |\nabla u|. \tag{2.18}$$

But $|\nabla u| \leq C$ by (2.9), hence (2.18) shows that Δu, and also λ, is absolutely continuous with respect to $\mathcal{H}^{N-1}$.

If $x \in \partial\Omega \setminus G$ then (2.9), (2.18) even yield that $|\int_{B_r(x)} \Delta u| \leq Cr^{N-1+\alpha}$ for $r > 0$ small, hence that Δu and λ are absolutely continuous with respect to $\mathcal{H}^{N-1+\alpha}$ on $\partial\Omega\setminus G$. From this it follows that $\lambda = 0$ on $\partial\Omega\setminus G$. Indeed, on $\partial\Omega\cap\partial G$ we have $\lambda = 0$ since by assumption $\mathcal{H}^{N-1+\alpha}(\partial G) = 0$. Outside $\overline{G}$ we have $\Delta u \in L^\infty$ by Lemma 2.2, and then standard arguments [Ki-St, II, Lemma A.4] show that $\lambda = \Delta u = 0$ a.e. on $\partial\Omega \setminus \overline{G}$.

By the above we see that $\Delta u + f\chi_\Omega = h\mathcal{H}^{N-1}\lfloor\partial\Omega$ for some Borel function $h \geq 0$ on $\partial\Omega \cap G$. It just remains to identify h with g i.e. to prove that

$$h(x) = g(x) \qquad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in \partial\Omega \cap G. \tag{2.19}$$

We shall merely give an outline of the proof of (2.19). The details are virtually the same as in [AC, 4.7–5.5].

It is enough to prove (2.19) for those $x \in \partial\Omega \cap G$ which satisfy $x \in \partial_{red}\Omega$,

$$\overline{\lim_{r\to 0}} \frac{\mathcal{H}^{N-1}(B(x,r) \cap \partial\Omega)}{\omega_{N-1}r^{N-1}} \leq 1, \tag{2.20}$$

$$\overline{\lim_{r\to 0}} \fint_{\partial\Omega\cap B(x,r)} |h - h(x)| d\mathcal{H}^{N-1} = 0 \tag{2.21}$$

since the remaining set has $\mathcal{H}^{N-1}$ measure zero (see [EG, Theorem 2, 2.3] for (2.20)). So fix such an $x \in \partial\Omega\cap G$. For simplicity of notation we assume that $x = 0$ and that $\nu_\Omega(0) = e_N = (0, ..., 0, 1)$.

Define the blow-up sequences $u_n(x) = nu(\frac{x}{n})$, $f_n(x) = \frac{1}{n}f(\frac{x}{n})$, $g_n(x) = g(\frac{x}{n})$, $h_n(x) = h(\frac{x}{n})$, $\Omega_n = \{u_n > 0\}$. Note that u, f and g are scaled in the right way according to Remark 2.7 (with $\alpha = -1$). Let $B = B(0,1)$, $B_r = B(0,r)$, $H = \{x_N < 0\}$.

By general properties of the reduced boundary [EG, 5.7.2]

$$|(\Omega_n \triangle H) \cap B| \to 0, \tag{2.22}$$

where $\triangle$ means the symmetric difference between the sets, and by our assumptions $f_n \to 0$ uniformly, and

$$\int_B |g(\frac{x}{n}) - g(0)| dx \to 0, \qquad \int_{\partial\Omega_n\cap B} |h(\frac{x}{n}) - h(0)| d\mathcal{H}^{N-1} \to 0$$

as $n \to \infty$.

As to u_n we know (Corollary 2.6 and 2.9) that $|\nabla u| \leq C$ and $u(x) \geq C\delta(x)$. Thus $|\nabla u_n| \leq C$, $u_n(x) \geq C\delta_n(x)$ where $\delta_n(x) = \text{dist}(x, \Omega_n^c) = n\delta(x/n)$. It follows that there exists a Lipschitz continuous limit function $u_0 \geq 0$ such that, for a subsequence,

$$u_n \to u_0 \qquad \text{uniformly in } B,$$
$$\nabla u_n \to \nabla u_0 \qquad w^* - L^\infty(B).$$

Setting $\Omega_0 = \{u_0 > 0\}$ it also follows, using nondegeneracy ($u_n \geq C\delta_n$) and that $|\Delta u_n| = |f_n| \leq C/n$ in Ω_n, that u_0 is harmonic in Ω_0, that $\Omega_n \cap B \to \Omega_0 \cap B$ in Hausdorff distance and in measure. By the last property combined with (2.22) $|\Omega_0 \triangle H| = 0$ and hence (since Ω_0 is open) $\Omega_0 \subset H$, $|H \setminus \Omega_0| = 0$.

Next one proves, and this is more technical [AC, 4.8], that actually $\Omega_0 = H$ (for this (2.20) has to be used) and that, due to (2.21),

$$u_0(x) = h(0)(-x_N)_+. \tag{2.23}$$

The final step consists of proving that u_0 is (global) minimum of

$$J_0(v) = \int_B (|\nabla v|^2 + g(0)^2 \chi_{\{v>0\}})$$

among all $0 \leq v \in H^1(B)$ with $v = u_0$ on ∂B ([AC, 5.4]). This is intuitively reasonable since by scaling (Remark 2.7) u_n is seen to be a minimum of

$$J_n(v) = \int_B (|\nabla v|^2 - 2f_n v + g_n^2 \chi_{\{v>0\}})$$

(among $0 \leq v \in H^1(B)$ with $v = u_n$ on ∂B) if n is large (recall also that $f_n \to 0$, $g_n \to g(0)$).

Now it follows from Theorem 2.3 (adapted to the unit ball B) that the function (2.23) can be a minimizer of J_0 only if $h(0) = g(0)$. This was the desired conclusion and the proof is finished. □

As to regularity of the free boundary $\partial\Omega$ we have

Theorem 2.17. *[AC], [Caff80] Assume that f and g satisfy Condition A and that u is a local minimum of J. Let $B_r = B_r(x_0)$ be a small ball.*

a) If g is Hölder continuous and satisfies $g \geq$ const. > 0 in B_r then for some $\alpha > 0$ $\partial_{red}\Omega$ is a $C^{1,\alpha}$ surface locally in B_r. If $N = 2$ then this even holds for $\partial\Omega$ (i.e. $\partial_{red}\Omega = \partial\Omega$ in B_r).

b) If $g = 0$, f is Hölder continuous and < 0 in B_r and if moreover Ω^c satisfies the minimal thickness condition of Caffarelli at x_0 (see [Caff80], [F]) then $\partial\Omega$ is a C^1 surface near x_0. This thickness condition is satisfied e.g. if Ω^c contains a nondegenerate cone in B_r with vertex at x_0.

c) If $g = 0$ and $f \geq 0$ in B_r then $\partial\Omega \cap B_r = \emptyset$.

Proof. a) is proved in [AC, 6–8] in the case $f = 0$. When $f \neq 0$ basically the same proof works. The modifications needed are listed in Appendix, section 5.

b) is proved in [Caff80].

As to c), (2.4) shows that $\Delta u \leq 0$ in B_r, hence either $u > 0$ in B_r or $u \equiv 0$ in B_r. □

Note. This theorem covers all cases except some limiting ones. However, in these limiting cases not much can be said in general. See e.g. Remark 2.14, where $g = 0$

and for any $x_0 \in \partial D \cap \partial\Omega$ f takes both positive and negative values in every neighbourhood of x_0. We may even redefine g to be any positive function outside Ω (e.g. $g(x) = \mathrm{dist}(x, \Omega)$, which is Lipschitz continuous) and we will still have the same irregular solution. See Remark 3.6.

As to higher regularity we just mention that if f and g are real analytic in B_r then, in Theorem 2.17, the conclusions $C^{1,\alpha}$ (in a)) and C^1 (in b)) can both be replaced by "real analytic" (see again [AC], [Caff80]). If f and g are real analytic and moreover $N = 2$ the regularity theory seems infact to be almost complete: If $g > 0$ in B_r then $\partial\Omega$ is real analytic by the above and if $g = 0$ and $f < 0$ in B_r then it is shown in [Sak91], [Sak94], that $\partial\Omega$ is analytic in B_r except possibly for a few types of singular points which may occur [Scha],[F]. These are certain types of inwards cusps, double points (including the case of a real analytic arc with Ω on both sides) and isolated points of $\partial\Omega$.

3. Geometry of local minima and weak solutions

In this section we derive some results on the geometry of $\Omega = \{u > 0\}$ when u is a local minimum. In some cases we really do not need the full strenght of u being a local minimum, just that u satisfies equation (2.17) in Theorem 2.13. We call such a function a weak solution. Our notion of weak solution is weaker than that of [AC].

Defintion 3.1. *Assume that f and g satisfy Condition A and that g moreover is continuous. Then by a weak solution for $J_{f,g}$ we mean a continuous function $u \geq 0$ with compact support satisfying*

$$\Delta u + f\mathcal{L}^N \lfloor \Omega = g\mathcal{H}^{N-1} \lfloor \partial\Omega \tag{3.1}$$

where $\Omega = \{u > 0\}$.

Remark 3.2.

a) $u = 0$ is always a weak solution.

b) A priori, $g\mathcal{H}^{N-1}\lfloor\partial\Omega$ is a (positive) Borel measure whereas the left member of (3.1) is a distribution. The equation (3.1) is to be interpreted as saying, first of all, that $g\mathcal{H}^{N-1}\lfloor\partial\Omega$ also is a distribution, hence a Radon measure, and, secondly, that equality holds in the sense of distributions. Thus it is a consequence of (3.1) that Δu is a (signed) Radon measure and also that Ω has locally finite perimeter in $G = \{g > 0\}$.

c) It follows, as in [AC, 4.2], that any weak solution u is in $\mathbb{K}$. Indeed, this readily follows from the estimate

$$\int_{u>\epsilon} |\nabla u|^2 = \int \nabla u \cdot \nabla(u-\epsilon)_+ = -\int \Delta u (u-\epsilon)_+ =$$
$$\int f(u-\epsilon)_+ = \int fu < \infty.$$

Also, by Theorem 2.13, if g satisfies the Hölder condition there (or $g > 0$ everywhere), any local minimum is a weak solution.

d) If some portion $\Gamma \subset G$ of $\partial\Omega$ bounds Ω from two sides (which is impossible for local minima by Lemma 2.10) then (3.1) is perhaps not the most natural definition: either u should be forced to have the normal derivative g in both directions from Γ (which would give $2g\mathcal{H}^{N-1}\lfloor\partial\Omega$ on Γ in (3.1)) or Γ should be neglected, which is accomplished by replacing the right member of (3.1) by $g\mathcal{H}^{N-1}\lfloor\partial_{mes}\Omega$ or $g\mathcal{H}^{N-1}\lfloor\partial_{red}\Omega$. However, for simplicity we shall stick to (3.1).

We begin with some miscellaneous comparison results for minima and local minima.

Proposition 3.3. *(Cf. [F-P]) Assume $f_1 \leq f_2$, $g_1 \geq g_2$, let $u_j \in \mathbb{K}$ be a minimizer of $J_j = J_{f_j,g_j}$ and let $\Omega_j = \{u_j > 0\}$ $(j = 1, 2)$.*
a) In each component D_1 of Ω_1 one of the following holds:

(1) $u_1 < u_2$ *in* D_1;
(2) $u_1 = u_2$ *and* $f_1 \equiv f_2$ *in* D_1;
(3) $u_1 > u_2$ *and* $f_1 \equiv f_2$ *in* D_1.

b) If $f_1 \leq 0$ in $\Omega_1 \cap \Omega_2$ then $J_1 \geq 0$ and $\Omega_1 \cap \Omega_2 = \emptyset$.
c) If $f_1 \leq 0$ in a component D_1 of Ω_1 then $D_1 \cap \Omega_2 = \emptyset$.
d) If $f_1 \leq 0$ in a component D_2 of Ω_2 then $\Omega_1 \cap D_2 = \emptyset$.

Proof. Let $v = \min(u_1, u_2)$ and $w = \max(u_1, u_2)$. Then, by Lemma 1.1, v minimizes J_1 and w minimizes J_2.

Next, to prove a), $\Delta w = -f_2$ in $\{w > 0\} = \Omega_1 \cup \Omega_2$ by Lemma 2.2. Thus $\Delta(w - u_1) = -f_2 + f_1 \leq 0$ in Ω_1 and $\Delta(w - u_2) = -f_2 + f_2 = 0$ in Ω_2. Moreover $w - u_j \geq 0$ $(j = 1, 2)$.

Let D_1 be a connected component of Ω_1. By the maximum principle, either $w - u_1 \equiv 0$ in D_1, in which case $f_1 = f_2$ there, or $w - u_1 > 0$ in D_1. In the latter case $u_1 < u_2$ in D_1, which is case (1) in the proposition. In the first case $u_1 \geq u_2$ in D_1, and it just remains to prove that either $u_1 > u_2$ or $u_1 \equiv u_2$ holds in all D_1.

Since $u_1 - u_2 = w - u_2 \geq 0$ is harmonic in Ω_2 we have, in each component of $D_1 \cap \Omega_2$, either $u_1 > u_2$ or $u_1 \equiv u_2$. Assume that $u_1 = u_2$ holds in one component D of $D_1 \cap \Omega_2$. Then $u_1 = u_2$ also on ∂D. It follows that $\partial D \cap D_1 = \emptyset$ (because if $x \in \partial D \cap D_1$ then $u_2(x) = u_1(x) > 0$ so that $x \in D_1 \cap \Omega_2$, contradicting $x \in \partial D$). Since $D \subset D_1$ and D_1 is connected this shows that $D = D_1$ i.e. $u_1 = u_2$ in all D_1 (case (2)).

If $u_1 = u_2$ on no component of $D_1 \cap \Omega_2$ then $u_1 > u_2$ in each component, and since trivially $u_1 > u_2$ in $D_1 \setminus \Omega_2$ we get $u_1 > u_2$ in all D_1 (case (3)). This completes the proof of a).

Since v minimizes J_1 we have $\Delta v + (f_1)_+ \geq 0$ in $\mathbb{R}^N$ by (2.2). Thus if D is an open set such that $f_1 \leq 0$ in D and $v = 0$ on ∂D then $v = 0$ in D by the maximum principle.

In b) the above is assumed to hold for $D = \Omega_1 \cap \Omega_2 = \{v > 0\}$. Thus $v = 0$ in D, hence $v = 0$ everywhere, $\Omega_1 \cap \Omega_2 = \emptyset$ and $J_1 \geq J_1(v) = 0$.

In c) we choose $D = D_1$ ($v = 0$ on ∂D_1 since $u_1 = 0$ on $\partial D_1 \subset \partial\Omega_1$) and we conclude that $v = 0$ in D_1, i.e. $D_1 \cap \Omega_2 = \emptyset$. d) is proved similarly. □

Corollary 3.4. *Let $u \in \mathbb{K}$ be a minimizer of $J_{f,g}$ and let $f_1 = (1 - \chi_\Omega)f_+ - f_-$ where $\Omega = \{u > 0\}$. Then $J_{f_1,g} \geq 0$ and $\Omega_1 \cap \Omega = \emptyset$ where $\Omega_1 = \{u_1 > 0\}$ for any minimizer u_1 of $J_{f_1,g}$ ($u_1 = 0$ is a minimizer).*

Proof. Apply b) of Proposition 3.3 with $f_2 = f$ and f_1 as in the statement. Note that $f_1 \leq 0$ in Ω ($\Omega = \Omega_2$). □

Corollary 3.5. *Let $u \in \mathbb{K}$ be a minimzer of $J_{f,g}$, let Ω_1 be a component of $\Omega = \{u > 0\}$ and set $f_1 = \chi_{\Omega_1} f_+ - f_-$, $u_1 = u\chi_{\Omega_1}$, $\Omega_2 = \Omega \setminus \Omega_1$. Then u_1 minimizes $J_1 = J_{f_1,g}$ and for any minimizer v of J_1 we have $\{v > 0\} \cap \Omega_2 = \emptyset$.*

Proof. Apply d) of Proposition 3.3 with $f_2 = f$ and f_1 as in the statement (of the corollary). Note that Ω_2 is a union of components of Ω and that $f_1 \le 0$ in Ω_2. It follows that $\{v > 0\} \cap \Omega_2 = \emptyset$ for any minimizer v of J_1, and hence also that u_1 minimizes J_1 (for otherwise J_1 could be made smaler by changing u in Ω_1 to a minimizer of J_1). □

Note that if $f \ge 0$ then Corollary 3.4 roughly says that if minimization of J does not produce a domain Ω covering f then another minimization, for the uncovered part, does not help. Similarly, Corollary 3.5 says (when $f \ge 0$) that if Ω turns out to be disconnected, then separate minimizations for the parts of f in each component of Ω always produces domains which do not meet each other.

Remark 3.6. Assume that $u \in \mathbb{K}$ is a (local) minimizer of $J = J_{f,g}$ and let $\tilde{f} = f$, $\tilde{g} \le g$ in $\Omega = \{u > 0\}$, $\tilde{f} \le f$, $\tilde{g} \ge g$ outside Ω. Then u is a (local) minimum also for $\tilde{J} = J_{\tilde{f},\tilde{g}}$. Indeed, one immediately finds that $\tilde{J}(u) - J(u) \le \tilde{J}(v) - J(v)$, and hence $\tilde{J}(u) \le \tilde{J}(v)$, for every $v \in \mathbb{K}$ (close to u).

One conclusion from this observation is that if g is not continuous then a local minimizer u cannot be expected to satisfy the equation (3.1) for a weak solution (because g can be replaced by any larger function on $\partial\Omega$ (or on $\mathbb{R}^N \setminus \Omega$) and u will still be a local minimizer). Cf. [AC, 5.9].

Theorem 3.7. *Assume that $u_j \ge 0$ ($j = 1, 2$) are weak solutions for $J_{f,g}$, $\Omega_j = \{u_j > 0\}$ and that $f \le 0$ outside Ω_2 (or simply that $\int_{\Omega_1 \setminus \Omega_2} f \le 0$). Then*

$$\int_{\partial(\Omega_1 \cup \Omega_2)} g d\mathcal{H}^{N-1} \le \int_{\partial\Omega_2} g d\mathcal{H}^{N-1}.$$

Proof. We have

$$\partial(\Omega_1 \cup \Omega_2) = (\partial\Omega_1 \setminus \Omega_2) \cup (\partial\Omega_2 \setminus \overline{\Omega}_1), \qquad \partial\Omega_2 = (\partial\Omega_2 \cap \overline{\Omega}_1) \cup (\partial\Omega_2 \setminus \overline{\Omega}_1),$$

where the unions are disjoint. Thus it is enough (and necessary) to prove that

$$\int_{\partial\Omega_1 \setminus \Omega_2} g d\mathcal{H}^{N-1} \le \int_{\partial\Omega_2 \cap \overline{\Omega}_1} g d\mathcal{H}^{N-1}.$$

Set $u = \inf(u_1, u_2)$. Then $u \ge 0$, u is continuous and $\Omega_1 \cap \Omega_2 = \{u > 0\}$. Since $u_j \in H^1(\mathbb{R}^N)$ (Remark 3.2) also $u \in H^1(\mathbb{R}^N)$. Since $-\Delta u_j = f$ in Ω_j we have

$$-\Delta u \ge f \qquad \text{in } \Omega_1 \cap \Omega_2. \tag{3.9}$$

In particular, Δu is a Radon measure in $\Omega_1 \cap \Omega_2$.

Now we claim that Δu actually is a Radon measure in all $\mathbb{R}^N$. It is not hard to show (cf. [AC, 4.2]) that this is the case if and only if Δu has finite total mass in $\Omega_1 \cap \Omega_2$, i.e.

$$-\int_{u>\epsilon} \Delta u \le C < \infty, \tag{3.10}$$

for all $\epsilon > 0$. Here the left member can also be written

$$-\int_{u>\epsilon} \Delta(u-\epsilon)_+ = \int_{u\leq\epsilon} \Delta(u-\epsilon)_+.$$

Since $\Delta(u-\epsilon)_+$ is a Radon measure with compact support in $\Omega_1 \cap \Omega_2$ Lemma 2.16 can be applied to $(u-\epsilon)_+$ and $(u_j-\epsilon)_+ - (u-\epsilon)_+$ showing that

$$0 \leq \Delta(u-\epsilon)_+ \leq \Delta(u_j-\epsilon)_+ \qquad \text{in } \{u_j \leq \epsilon\}.$$

From this (3.10) easily follows using the fact that Δu_j have finite total masses.

Next we apply Lemma 2.16 to u, $u_1 - u$ and $u_2 - u$. This gives

$$0 \leq \Delta u \leq \Delta u_1 \quad \text{on } \partial\Omega_1, \qquad \text{and} \qquad 0 \leq \Delta u \leq \Delta u_2 \quad \text{on } \partial\Omega_2.$$

Combining with (3.9) we obtain

$$\int_{\Omega_1\cap\Omega_2} f \leq -\int_{\Omega_1\cap\Omega_2} \Delta u = \int_{(\Omega_1\cap\Omega_2)^c} \Delta u = \int_{\partial(\Omega_1\cap\Omega_2)} \Delta u \leq$$
$$\int_{\partial\Omega_1\cap\Omega_2} \Delta u + \int_{\partial\Omega_2\cap\overline{\Omega}_1} \Delta u \leq \int_{\partial\Omega_1\cap\Omega_2} \Delta u_1 + \int_{\partial\Omega_2\cap\overline{\Omega}_1} \Delta u_2$$

and hence

$$\int_{\partial\Omega_2\cap\overline{\Omega}_1} g d\mathcal{H}^{N-1} = \int_{\partial\Omega_2\cap\overline{\Omega}_1} \Delta u_2 \geq \int_{\Omega_1\cap\Omega_2} f - \int_{\partial\Omega_1\cap\Omega_2} \Delta u_1 =$$
$$\int_{\Omega_1\cap\Omega_2} f + \int_{\partial\Omega_1\setminus\Omega_2} \Delta u_1 + \int_{\Omega_1} \Delta u_1 = \int_{\Omega_1\cap\Omega_2} f + \int_{\partial\Omega_1\setminus\Omega_2} g d\mathcal{H}^{N-1} -$$
$$\int_{\Omega_1} f = -\int_{\Omega_1\setminus\Omega_2} f + \int_{\partial\Omega_1\setminus\Omega_2} g d\mathcal{H}^{N-1} \geq \int_{\partial\Omega_1\setminus\Omega_2} g d\mathcal{H}^{N-1}$$

as required. □

Corollary 3.8. *Assume $u_j \geq 0$ $(j = 1, 2)$ are weak solutions with $g =$ const. > 0, that $f \leq 0$ outside Ω_2 and that Ω_2 is convex. Then $\Omega_1 \subset \Omega_2$ $(\Omega_j = \{u_j > 0\})$.*

Proof. Let $P : \mathbb{R}^N \to \overline{\Omega}_2$ be the projection, taking $x \in \mathbb{R}^N$ onto the closest point $P(x)$ on the compact convex set $\overline{\Omega}_2$. Then

$$|P(x) - P(y)| \leq |x - y|, \tag{3.11}$$
$$P(\partial(\Omega_1 \cup \Omega_2)) = \partial\Omega_2, \tag{3.12}$$

as is easily seen. But (3.11) implies [E-G,Theorem 1, p. 75] that P shrinks Hausdorff measure, in particular

$$\mathcal{H}^{N-1}(P(\partial(\Omega_1 \cup \Omega_2))) \leq \mathcal{H}^{N-1}(\partial(\Omega_1 \cup \Omega_2)).$$

Thus, by (3.12)

$$\mathcal{H}^{N-1}(\partial\Omega_2) \leq \mathcal{H}^{N-1}(\partial(\Omega_1 \cup \Omega_2)). \tag{3.13}$$

If $\Omega_1 \not\subset \Omega_2$ then $\Omega_1 \setminus \overline{\Omega}_2 \neq \emptyset$ (since Ω_2 is convex), and it is easy to see that the inequality (3.13) must be strict in this case. But this contradicts Theorem 3.7. Thus $\Omega_1 \subset \Omega_2$. $\square$

Corollary 3.8 partly generalizes [Shah1, Theorem 2.6], where the same conclusion was obtained assuming some regularity of $\partial\Omega$ but without any positivity assumption on u. Other results related to convexity can be found in [Beur], [Acker81] and [Kaw].

Next we shall use some reflection methods to obtain a result on monotonicity or convexity along lines. The method is related to the "moving plane method" which has previously been used in similar problems in [Serrin], [GNN], [B-N], [Shah94b], [Gu-Sa]. Important points in our approach are that we do not require any regularity of the solutions u and that we are able to work with local minima (not only global minima).

For a fixed unit vector $a \in \mathbb{R}^N$ and for $\lambda \in \mathbb{R}$ set

$$T_\lambda = T_{a,\lambda} := \{x \cdot a = \lambda\}, \qquad T_\lambda^- := \{x \cdot a < \lambda\}, \qquad T_\lambda^+ := \{x \cdot a > \lambda\}.$$

For $x \in \mathbb{R}^N$ let x^λ denote the reflected point with respect to T_λ and for φ a function set $\varphi^\lambda(x) = \varphi(x^\lambda)$. If $\Omega \subset \mathbb{R}^N$ we define

$$\begin{aligned} \Omega_\lambda &= \Omega \cap T_\lambda^+ = \text{ the cap cut off by } T_\lambda, \\ \tilde{\Omega}_\lambda &= \{x^\lambda : x \in \Omega_\lambda\} = \text{ the reflection of } \Omega_\lambda \text{ in } T_\lambda. \end{aligned}$$

Theorem 3.9. *Assume that f and g satisfy Condition A and moreover that for some unit vector $a \in \mathbb{R}^N$ and some $\lambda_0 \in \mathbb{R}^N$ we have*

$$f \leq f^\lambda, \qquad g \geq g^\lambda \quad \text{in } T_\lambda^+ \tag{3.14}$$

for all $\lambda \geq \lambda_0$. Then for any local minimum u of J the following hold.

$$u < u^\lambda \quad \text{in } \Omega_\lambda \quad \text{for all } \lambda > \lambda_0, \tag{3.15}$$

$$\tilde{\Omega}_\lambda \subset \Omega \quad \text{for all } \lambda \geq \lambda_0, \tag{3.16}$$

$$a \cdot \nabla u < 0 \quad \text{in } \Omega_{\lambda_0}. \tag{3.17}$$

Note. (3.14) holding for all $\lambda \geq \lambda_0$ is equivalent to that

$$\begin{aligned} f &\leq f^{\lambda_0}, & a \cdot \nabla f &\leq 0, \\ g &\geq g^{\lambda_0}, & a \cdot \nabla g &\geq 0 \end{aligned}$$

hold on $T_{\lambda_0}^+$ (in the sense of distributions).

Proof. Define

$$v^\lambda = \begin{cases} \min(u, u^\lambda) & \text{in } T_\lambda^+, \\ \max(u, u^\lambda) & \text{in } T_\lambda^-, \end{cases}$$

$$I(\varphi) = \int_{T_\lambda^+} (|\nabla\varphi|^2 - 2f\varphi + g^2\chi_{\{\varphi>0\}})dx,$$

$$I_\lambda(\varphi) = \int_{T_\lambda^+} (|\nabla\varphi|^2 - 2f^\lambda\varphi + (g^\lambda)^2\chi_{\{\varphi>0\}})dx.$$

Then Lemma 1.1 (with $\mathbb{R}^N$ replaced by T_λ^+) shows that

$$\begin{aligned} J(v^\lambda) &= I(\min(u,u^\lambda)) + I_\lambda(\max(u,u^\lambda)) \\ &\le I(u) + I_\lambda(u^\lambda) = J(u) \end{aligned} \tag{3.18}$$

for all $\lambda \ge \lambda_0$.

On the other hand $J(v^\lambda) \ge J(u)$ whenever v^λ is close enough to u in the metric (2.1), since u is a local minimum. Thus

$$J(v^\lambda) = J(u) \tag{3.19}$$

for all values of $\lambda \ge \lambda_0$ such that v^λ is close to u.

Now for $\lambda \ge \lambda_0$ so large that $\Omega \subset T_\lambda^-$ we have

$$v^\lambda = u, \tag{3.20}$$

i.e. $u \le u^\lambda$ in T_λ^+. Note that (3.20) implies

$$\tilde{\Omega}_\lambda \subset \Omega. \tag{3.21}$$

We shall prove that (3.20) holds for all $\lambda \ge \lambda_0$. For this it is enough to prove that if for some $\lambda_1 > \lambda_0$ (3.20) holds for all $\lambda \ge \lambda_1$ then it also hold for all λ in a full neighbourhood of λ_1. Note that the set of values of λ for which (3.20) holds is a closed set.

By Lemma 1.1

$$J(\min(u,v^\lambda)) + J(\max(u,v^\lambda)) \le J(u) + J(v^\lambda)$$

and if v^λ is close to u then also $\min(u,v^\lambda)$ and $\max(u,v^\lambda)$ are close to u. Thus by (3.19) also $\min(u,v^\lambda)$ and $\max(u,v^\lambda)$ are local minima when v^λ is close to u. In particular, by Lemma 2.2

$$-\Delta \max(u,v^\lambda) = f \qquad \text{in } \Omega \tag{3.22}$$

(note that $\max(u,v^\lambda) > 0$ in Ω).

Set

$$\varphi = \max(u,v^\lambda) - u = \begin{cases} 0 & \text{in } T_\lambda^+ \cup T_\lambda \\ (u^\lambda - u)_+ & \text{in } T_\lambda^-. \end{cases}$$

Then (3.20) is equivalent to $\varphi = 0$ in $\mathbb{R}^N$. Clearly we have

$$\varphi = 0 \qquad \text{in } \mathbb{R}^N \setminus \tilde{\Omega}_\lambda \tag{3.23}$$

and, when (3.22) holds,

$$\Delta\varphi = 0 \qquad \text{in } \Omega. \tag{3.24}$$

Thus by the maximum principle (3.21) implies $\varphi = 0$, i.e. (3.20) (when (3.22) holds).

If Ω is connected then the above readily shows what we want, namely that (3.20) holds for all $\lambda \geq \lambda_0$. Indeed assume that (3.20), and hence (3.21), holds for all $\lambda \geq \lambda_1 > \lambda_0$. Then, for any λ in some small neighbourhood of λ_1 we have (3.22) and hence (3.24). If $\Omega \subset T_\lambda^-$ (for such λ) then obviously (3.21), and hence (3.20), holds. If $\Omega \cap T_\lambda \neq \emptyset$ then (3.23) implies that $\varphi = 0$ in an open subset of Ω. Therefore, by (3.24) $\varphi = 0$ in all Ω and hence (3.20), (3.21) hold. Finally note that, by (3.21), $|\Omega_{\lambda_1}| \leq \frac{1}{2}|\Omega|$. Therefore, the remaining case, namely that $\Omega \subset T_\lambda^+$ cannot occur for λ close to λ_1.

Thus (3.20), (3.21) hold for all $\lambda \geq \lambda_0$ provided Ω is connected. If Ω is not connected a similar reasoning can be applied to each component (we omit the details) and the same conclusion is obtained.

We have now proved (3.16) and that $u \leq u^\lambda$ in T_λ^+ for all $\lambda \geq \lambda_0$. This readily implies that $a \cdot \nabla u \leq 0$ in Ω_{λ_0} (note that $u \in C^1(\Omega)$).

Next $\Delta(u^\lambda - u) = f - f^\lambda \leq 0$ in Ω_λ. On $\partial\Omega_\lambda \cap T_\lambda^+$, $u^\lambda - u = u^\lambda \geq 0$ and on $\partial\Omega_\lambda \cap T_\lambda$ we have $u^\lambda - u = 0$. Moreover, when $\lambda > \lambda_0$ then u^λ must be strictly positive somewhere on $\partial\Omega_\lambda \cap T_\lambda^+$ (or even on $\partial D \cap T_\lambda^+$ for any component D of Ω_λ) because λ can be decreased further with (3.21) still holding. Therefore it follows from the minimum principle for superharmonic functions that $u^\lambda - u > 0$ in $\Omega_\lambda \cap T_\lambda^+$ when $\lambda > \lambda_0$. It also readily follows that $a \cdot \nabla u < 0$ in Ω_{λ_0}. The proof is finished. □

Corollary 3.10. *Let u, f and g be as in Theorem 3.9 and assume moreover f and g are symmetric in T_{λ_0}. Then u is symmetric in T_{λ_0}.*

Corollary 3.11. *Assume that f and g satisfy Condition A and that moreover both f and g are constant outside some compact convex set K (then necessarily $\operatorname{supp} f_+ \subset K$). Let $\Omega = \{u > 0\}$ where u is a local minimum for J. Then for any $x \in \partial_{red}\Omega \setminus K$ the inward normal ray $N_x = \{-t\nu_\Omega(x) : t > 0\}$ of $\partial\Omega$ at x intersects K. Moreover, $\partial\Omega \setminus K$ is Lipschitz.*

Proof. If for $x \in \partial_{red}\Omega \setminus K$ we have $N_x \cap K = \emptyset$ then one can find $a \in \mathbb{R}^N$ and $\lambda_0 \in \mathbb{R}$ such that $K \subset T_{a,\lambda_0}^-$, $N_x \subset T_{a,\lambda_0}^+$. The first inclusion implies that the assumption of Theorem 3.9 are satisfied while the second inclusion implies that the conclusions do not hold (e.g. $\partial\Omega \cap T_{a,\lambda_0}^+$ is not a graph near x). This contradiction proves the first statement of the corollary. The second statement follows easily by varying a and λ_0 such that $K \subset T_{a,\lambda_0}^-$. □

Theorem 3.12. *Assume that f, g satisfy Condition A and that u is a local minimizer of $J_{f,g}$. Assume moreover that*

$$f(x/t) \leq tf(x) \qquad \text{and} \qquad g(x/t) \geq g(x)$$

for all $0 < t < 1$ (and all $x \in \mathbb{R}^N$). Then

$$tu(x/t) \leq u(x)$$

for all $0 < t < 1$. In particular $\Omega = \{u > 0\}$ is starshaped with respect to the origin.

More generally the same conclusion holds with the above inequalities replaced by, respectively

$$t^\alpha f(x/t) \leq f(x), \qquad t^{\alpha+1} g(x/t) \geq g(x) \qquad \text{and} \qquad t^{\alpha+2} u(x/t) \leq u(x)$$

for any (fixed) real number α.

Proof. Fix $\alpha \in \mathbb{R}^N$ and set $\varphi_t(x) = \varphi(x/t)$ for any function φ. It follows from Remark 2.7 that $t^{\alpha+2}u_t$ is a local minimizer of $J_t = J_{t^\alpha f_t, t^{\alpha+1}g_t}$.

Since $t^\alpha f_t \le f$, $t^{\alpha+1}g_t \ge g$ Lemma 1.1 therefore shows that $w_t = \max(t^{\alpha+2}u_t, u)$ is a local minimizer of J and that $J(w_t) = J(u)$, provided $t \le 1$ is close enough to 1. Clearly $w_t = u$ for $t = 1$. Now similar arguments as those in the proof of Theorem 3.9 show that actually $w_t = u$ for all $0 < t \le 1$. Thus $t^{\alpha+2}u_t \le u$ $(0 < t \le 1)$, and this readily shows that Ω is starshaped. □

Theorem 3.13. *Assume that f^ϵ, f, g^ϵ, g satisfy Condition A. Let u^ϵ, u be the largest minimizers of $J^\epsilon = J_{f^\epsilon, g^\epsilon}$ and $J = J_{f,g}$ respectively, and let $\Omega^\epsilon = \{u^\epsilon > 0\}$, $\Omega = \{u > 0\}$. Assume also that*

$$f_- + g \ge \text{ const. } > 0 \tag{3.25}$$

outside Ω. Then if

$$f^\epsilon \searrow f \qquad \text{and} \qquad g^\epsilon \nearrow g \qquad \text{a.e.}$$

(or in the sense of distributions) as $\epsilon \searrow 0$ we have

$$u^\epsilon \searrow u \qquad \text{uniformly and in } w\text{-}H^1(\mathbb{R}^N), \tag{3.26}$$

$$\Omega^\epsilon \searrow \Omega \qquad \text{with respect to Hausdorff distance.} \tag{3.27}$$

Note. Condition (3.25) is needed only for (3.27)

Proof. By Lemma 1.1 u^ϵ decreases (pointwise) with ϵ. Thus $v = \lim_{\epsilon\to 0} u^\epsilon = \inf_{\epsilon>0} u^\epsilon$ exists. As in the proof of Lemma 1.2 one has $||\nabla u^\epsilon|| \le C < \infty$. Hence $u^\epsilon \to v$ weakly in $H^1(\mathbb{R}^N)$ (and strongly in $L^2(\mathbb{R}^N)$). It is now easy to check that $J(v) \le \underline{\lim}\, J^\epsilon(u^\epsilon)$. Since $J^\epsilon(u^\epsilon) \le J^\epsilon(u) \le J(u)$ (also, $\lim J^\epsilon(u) = J(u)$) it follows that $v \in \mathbb{K}$ minimizes J. But $v \ge u$ since $u \le u^\epsilon$ for $\epsilon > 0$. Thus $v = u$ since u was the largest minimizer. Thus $u^\epsilon \searrow u$, and the convergence is uniform since u^ϵ and u are continuous. This proves (3.26).

Clearly Ω^ϵ decreases with ϵ and $\Omega \subset \cap_{\epsilon>0}\Omega^\epsilon$. In order to prove (3.27) it is enough to prove the following: for any ball $B_r = B_r(x)$ with $B_{2r} \cap \Omega = \emptyset$ we have $B_r \cap \Omega^\epsilon = \emptyset$ for $\epsilon > 0$ small enough.

So assume $B_{2r} \cap \Omega = \emptyset$. Then $u^\epsilon \searrow 0$ uniformly in B_{2r}. Assume now that $B_r \cap \Omega^\epsilon \neq \emptyset$ for some $\epsilon > 0$. By combining Corollary 2.9 and Remark 2.11 we have, if (3.25) holds in B_{2r},

$$\sup_{B_{2r}} u^\epsilon \ge \sup_{\partial B_r(y)\cap\Omega^\epsilon} u^\epsilon \ge C_1 r + C_2 r^2$$

for $y \in B_r \cap \Omega^\epsilon$, where $C_1, C_2 \ge 0$, $C_1 + C_2 > 0$. This contradicts the uniform convergence of u^ϵ if ϵ is small enough, proving (3.27). □

Theorem 3.14. *Assume that f^ϵ, f, g^ϵ, g satisfy Condition A. Let u^ϵ, u be the smallest minimizers of $J^\epsilon = J_{f^\epsilon,g^\epsilon}$ and $J = J_{f,g}$ respectively, and let $\Omega^\epsilon = \{u^\epsilon > 0\}$, $\Omega = \{u > 0\}$. Then if*

$$f^\epsilon \nearrow f \qquad \text{and} \qquad g^\epsilon \searrow g \qquad a.e.$$

(or in the sense of distributions) as $\epsilon \searrow 0$ we have

$$\begin{aligned} u^\epsilon &\nearrow u && \text{uniformly and in } w\text{-}H^1(\mathbb{R}^N),\\ \Omega^\epsilon &\nearrow \Omega && \text{with respect to Hausdorff distance.} \end{aligned}$$

The proof is similar to (and somewhat simpler than) that of Theorem 3.13 and therefore omitted.

Example 3.15. Take $f = f_a = a\chi_{B_R} - b$, $g = c$ as in Example 1.5, where now $R > 0$, $b \geq 0$, $c > 0$ are kept fixed and $a > 0$ is regarded as a parameter. Then, as we saw in Example 1.5, there is a critical value a_0 with $b + Nc/R < a_0 \leq (b + Nc/3R)3^N$ such that for $a < a_0$ $u = u_0 \equiv 0$ is the unique minimizer of $J_a = J_{f_a,g}$, for $a = a_0$ there are two minimizers, $u_0 = 0$ and $u_1 \not\equiv 0$, say, while for $a > a_0$ there is again a unique minimizer $u_1 \not\equiv 0$ (depending on a).

For $a \geq a_0$ the set $\Omega_a = \{u_1 > 0\}$ is a ball whose radius $\rho = \rho(a) > R$ (given by equation (1.17)) increases with a.

Thus we see that the largest solution is continuous from the left with respect to a (i.e. it depends continuously on a, on the intervals $0 < a < a_0$ and $a_0 \leq a < \infty$) while the smallest solution is continuous from the right. This is in accordance with Theorem 3.13 and 3.14, and it also shows that one cannot expect to have more than the semicontinuities stated. See also Example 4.4.

4. Quadrature domains and balayage

Let $0 \leq g, h \in L^\infty(\mathbb{R}^N)$ be given density functions. In this section we shall study the following type of balayage problem. Given a positive Radon measure μ with compact support find a bounded open set Ω containing supp μ such that μ is "graviequivalent" to the measure

$$\nu = h\mathcal{L}^N\lfloor\Omega + g\mathcal{H}^{N-1}\lfloor\partial\Omega, \tag{4.1}$$

in the sense that $U^\nu = U^\mu$ in $\mathbb{R}^N \setminus \Omega$. Here, if σ is any (positive) Borel measure, U^σ denotes its Newtonian potential, i.e.

$$U^\sigma(x) = \int E(x - y)d\sigma(y), \qquad (x \in \mathbb{R}^N)$$

where $E(x) = (\omega_N/N)|x|^{2-N}$ $(N \geq 3)$, and $E(x) = \frac{1}{2\pi}\log|x|$ $(N = 2)$ so that $-\Delta U^\sigma = \sigma$.

When a measure μ is graviequivalent to a measure ν associated with a domain Ω, as in (4.1), and Ω contains supp μ (or at least $\mu(\Omega^c) = 0$) then the word "quadrature domain" for Ω is sometimes used [Sak82], [Shap92]. The reason for this terminology is indicated after Remark 4.2 below. In this paper we shall use the following definition of a quadrature domain.

Definition 4.1. Let h, g and μ be given as above. Then Ω is a *quadrature domain* for μ (and for the given densities g and h) if Ω is a bounded open set in $\mathbb{R}^N$ such that

$$\text{supp}\ \mu \subset \Omega, \tag{4.2}$$

$$U^\nu = U^\mu \qquad \text{on } \mathbb{R}^N \setminus \Omega, \tag{4.3}$$

where

$$\nu = h\mathcal{L}^N \lfloor \Omega + g\mathcal{H}^{N-1} \lfloor \partial\Omega. \tag{4.4}$$

We then write

$$\Omega \in Q(\mu) \quad \text{or} \quad \Omega \in Q(\mu; h, g).$$

Remark 4.2.

a) As in Remark 3.2 b) it follows that if $\Omega \in Q(\mu; h, g)$ then Ω has finite perimeter in any open set in which $g \geq$ const. > 0.

b) Suppose that (4.2) holds. Then, by definition, $\Omega \in Q(\mu; h, g)$ if and only if the "quadrature identity"

$$\int_\Omega \varphi d\mu = \int_\Omega \varphi h dx + \int_{\partial\Omega} \varphi g d\mathcal{H}^{N-1} \tag{4.5}$$

holds for a certain class of harmonic functions φ in Ω, namely for all linear combinations of the functions $\varphi(x) = E(x - y)$, with $y \in \Omega^c$. By an approximation argument, (4.5) then also holds for every harmonic φ in Ω which can be extended to a smooth function in a neighbourhood of $\overline{\Omega}$.

c) Our definition of quadrature domain is quite weak e.g. in the sense that the identity (4.5) is required to hold only for a rather small class of harmonic functions φ. Indeed, as is explained in Example 4.3 below, our definition allows for a large class of nonsmooth members in $Q(\mu; h, g)$ when $g > 0$. (When $g = 0$ the situation is much better.)

Therefore we wish to point out conceivable ways of strengthening the requirements. In addition to (4.2), (4.3) one could ask e.g. to have

$$U^\nu \leq U^\mu \qquad \text{in } \mathbb{R}^N, \tag{4.6}$$

$$|\nabla U^\nu| \leq \text{ const. } < \infty. \tag{4.7}$$

Since these inequalities look a little *ad hoc* we have preferred not to put them into the definitions, but they do have some good properties: (4.7) rules out the type of nonsmooth domains occurring in Example 4.3 (relevant when $g > 0$) and (4.6) implies uniqueness (up to nullsets) of quadrature domains when $g = 0$ [Sak82], [Gust90]. Moreover, both (4.6) and (4.7) hold for the quadrature domains we construct in Theorem 4.7, 4.8.

Quadrature domains have been extensively studied in the case $h = 1, g = 0$ [Davis], [Ah-Sh], [Sak82], [Gust90], [Shap92] and also (to a smaller extent) when $h = 0, g = 1$ [Shah94b] [Gust87], [Henrot], [Sh-Ul], [Avci], [LV1], [LV2]. If e.g. $h = 1, g = 0$ and μ is a finite sum of point masses then the identity (4.5) gives a very simple way of computing the integral $\int_\Omega \varphi dx$ for φ harmonic in Ω. This explains the terminology. Let us now give a couple of examples, primarily for the case $h = 0, g = 1$.

Example 4.3. Let $\mu = \delta_0$ be the point mass at the origin and let $h \equiv b$, $g \equiv c$ be constant. Then the ball $\Omega = B(0, R)$ with $R > 0$ chosen so that $b\omega_N R^N + cN\omega_N R^{N-1} = 1$ is in $Q(\delta_0; b, c)$. If $b > 0$, $c = 0$ this Ω is the unique element in $Q(\delta_0; b, 0)$ (see [Kuran], [ASZ], etc).

If $b = 0$, $c > 0$, Ω is still unique among domains with smooth boundary [Shah92]. Indeed, it is even shown in [LV2] that Ω is unique among domains in $Q(\delta_0; 0, c)$ satisfying in addition (4.7) and $\mathcal{H}^{N-1}(\partial\Omega \setminus \partial_{mes}\Omega) = 0$ ((4.6) is automatically satisfied). However, without these additional assumptions there turns out to exist also a quite large family of domains in $Q(\delta_0; 0, c)$ with rather "pathological" boundaries. In two dimensions these are the famous non-Smirnov domains first found by Keldysh and Lavrentiev and later (in a more constructive way) by Duren, Shapiro and Shields [DSS], [Shap66] (see also [Duren]). In higher dimension such domains were recently constructed by Lewis and Vogel [LV1].

It should be told that these non-smooth domains Ω are not extremely pathological. E.g. they are images of the unit ball under Hölder class homeomorphisms $\mathbb{R}^N \to \mathbb{R}^N$ (which when $N = 2$ even can be taken to be quasiconformal). They satisfy

$$c \int_{\partial\Omega} \varphi d\mathcal{H}^{N-1} = \varphi(0) \tag{4.8}$$

for every φ harmonic in Ω and continuous on $\overline{\Omega}$. Also, it follows from our Corollary 3.8 that $\Omega \subset B(0, R) \in Q(\delta_0; 0, c)$.

Finally we mention that, when $N = 2$, we have uniqueness of finitely connected domains satisfying (4.8) if the test class of functions φ is enlarged to the appropriate Hardy (or "Smirnov") space. See [Avci, Theorem 2.1], [Gust87, Remark 3.4].

Example 4.4. We cite the following interesting example due to Henrot [Henrot]. Let $N = 2$, $g = 1$, $h = 0$, $\mu = a(\delta_{(-1,0)} + \delta_{(1,0)})$ where $a > 0$.

(i) If $0 < a < 2\pi$ then $\Omega_0 = B((-1, 0), a/2\pi) \cup B((1, 0), a/2\pi)$ is a disconnected element in $Q(\mu)$.

(ii) If $4.60... < a < 2\pi$ then there moreover exist two connected domains, Ω_1 and Ω_2, in $Q(\mu)$. We have $\Omega_0 \subset \Omega_1 \subset \Omega_2$, and for $a = 4.60...$ $\Omega_1 = \Omega_2$.

(iii) As a increases towards 2π Ω_0 expands and Ω_1 shrinks. For $a = 2\pi$ $\Omega_0 = \Omega_1$. For $a > 2\pi$ Ω_0 and Ω_1 do not exist.

(iv) As $a > 4.60...$ increases towards $+\infty$ Ω_2 expands all the time, and from $a = 5.65...$ on it is convex.

The above is proved by conformal mapping. For more details, see [Henrot]. This example illustrates in a beautiful way several of our (and also Henrot's) results, e.g. Corollary 3.8 and Theorems 3.13 and 3.14. As to the different behavior (shrinking, expanding etc) of $\Omega_0, \Omega_1, \Omega_2$ as functions of the parameter $a > 0$, there is a classification of weak solutions (into "hyperbolic", "elliptic" and "parabolic") based on such properties due to Beurling [Beur].

The existence of two different simply connected and smoothly bounded quadrature domains for a measure as simple as the above μ is particularly interesting. In the case $g = 0$, $h = 1$ there is no such example known for any μ.

Example 4.5. Let $N = 2$, $g = 1$, $h = 0$ and let $\mu = a\mathcal{H}^1\lfloor I$ where $a > 0$ and I is the closed line segment from $(-1,0)$ to $(1,0)$. If $\Omega \in Q(\mu;0,1)$ then $I \subset \Omega$ by (4.2), which implies that $\mathcal{H}^1(\partial\Omega) > 4$. On the other hand (4.3) implies $\int d\nu = \int d\mu$ and hence $\mathcal{H}^1(\partial\Omega) = \int d\nu = 2a$. Thus we see that a *necessary* condition for the existence of a quadrature domain for μ is that $a > 2$.

Now to relate quadrature domains with our minimization problem, assume $\mu \in L^\infty(\mathbb{R}^N)$ (i.e. μ is absolutely continuous with a bounded density function, also denoted μ) and set $f = \mu - h$. Let $u \geq 0$ be a weak solution for $J_{f,g}$ so that

$$\Delta u + f\mathcal{L}^N\lfloor\Omega = g\mathcal{H}^{N-1}\lfloor\partial\Omega,$$

where $\Omega = \{u > 0\}$. This identity can also be written

$$\mu + \Delta u = \nu,$$

where

$$\nu = h\mathcal{L}^N\lfloor\Omega + g\mathcal{H}^{N-1}\lfloor\partial\Omega + \mu\lfloor\Omega^c.$$

Clearly, $u = U^\mu - U^\nu$. Thus we see that

$$\Omega \in Q(\mu;h,g) \quad \Longleftrightarrow \quad \text{supp } \mu \subset \Omega. \tag{4.9}$$

When μ is a more general measure (not in L^∞), e.g. a sum of point masses, then our minimization problem does not make sense, but one can still pass between quadrature domains and the minimization problem by mollifying.

Lemma 4.6. *Let $0 \leq \psi \in L^\infty(\mathbb{R}^N)$ be radially symmetric, non-increasing as a function of $|x|$, have compact support and satisfy $\int \psi dx = 1$. Then, for μ a positive measure with compact support,*

$$\Omega \in Q(\mu * \psi; h, g) \Longrightarrow \Omega \in Q(\mu; h, g).$$

Moreover, if (4.6) holds for $\mu * \psi$ it holds also for μ.

Proof. By the supermeanvalue property for superharmonic functions

$$U^{\mu*\psi} \leq U^\mu \qquad \text{everywhere}$$

and by the ordinary meanvalue property (for harmonic functions)

$$U^{\mu*\psi} = U^\mu \qquad \text{outside supp } (\mu * \psi).$$

Note that supp $\mu \subset$ supp $(\mu * \psi)$. Thus the assertions of the lemma follows directly from Definition 4.1. □

From (4.9) and Lemma 4.6 it is clear that in order to construct quadrature domains for general positive measures μ using the minimization problem one just has to make sure that supp $(\mu * \psi) \subset \Omega = \{u > 0\}$ for a suitable mollifier ψ, where u is a (local) minimizer. This is the way two of our main results, Theorem 4.7 and 4.8 below, are proved. Since $u \geq 0$ and u is Lipschitz continuous (Corollary 2.6) the quadrature domains constructed will automatically satisfy (4.6) and (4.7).

In these theorems $b, c \geq 0$ are constants with $b + c > 0$, $g, h \in L^\infty(\mathbb{R}^N)$ are density functions satisfying

$$0 \leq h \leq b,$$
$$0 \leq g \leq c.$$

Moreover, at least one of h and g is assumed to be $\geq$ const. > 0 outside a compact set, and g is assumed to be continuous and to satisfy the Hölder condition in Theorem 2.13 (unless $g > 0$ everywhere).

Theorem 4.7. *Let μ be a positive measure which is concentrated to a ball $B_R = B(x_0, R)$ to the extent that*

$$\mu(B_R^c) = 0, \tag{4.10}$$

$$\mu(B_R) > \left(b + \frac{Nc}{3R}\right) 6^N |B_R|. \tag{4.11}$$

Then, for any h, g as above there exists $\Omega \in Q(\mu; h, g)$, which moreover satisfies (4.6), (4.7) and $B_{3R} \subset \Omega$.

Proof. For $\rho > 0$ set

$$\psi_\rho = \frac{1}{|B_\rho(0)|} \chi_{B_\rho(0)}. \tag{4.12}$$

Then (4.10), (4.11) imply that

$$\mu * \psi_{2R} > \left(b + \frac{Nc}{3R}\right) 3^N \qquad \text{in } \overline{B}_R$$
$$\mu * \psi_{2R} = 0 \qquad \text{outside } B_{3R}.$$

Setting $\tilde{f} = \mu * \psi_{2R} - h$ $(\in L^\infty(\mathbb{R}^N))$ we may choose $a > (b + \frac{Nc}{3R})3^N$ so that $\tilde{f} \geq a\chi_{B_R} - b$. Note that a satisfies (1.18) of Example 1.5.

Let w denote the largest and unique minimizer of $J_{a\chi_{B_R} - b, c}$ and $\tilde{u}$ the largest minimizer of $\tilde{J} = J_{\tilde{f}, g}$. Then $0 \leq w \leq \tilde{u}$ by Proposition 3.3 and $\overline{B}_{3R} \subset \{w > 0\}$ by Example 1.5 (subcase 4c). Now denote $\Omega = \{\tilde{u} > 0\}$. Then supp $(\mu * \psi_{2R}) \subset \overline{B}_{3R} \subset \{w > 0\} \subset \Omega$ and it follows that $\Omega \in Q(\mu * \psi_{2R}; h, g)$. Thus $\Omega \in Q(\mu; h, g)$ by Lemma 4.6. $\square$

Theorem 4.8. *With b, c, h, g as in Theorem 4.7 there exists a constant $C = C(N, b, c)$ such that if*

$$\sup_{r>0} \frac{\mu(B_r(x))}{r^{N-1}} \geq C \qquad \textit{for every } x \in \textit{supp } \mu \tag{4.13}$$

(for μ a positive measure with compact support) then there exists $\Omega \in Q(\mu; h, g)$, which moreover satisfies (4.6), (4.7). If $c = 0$ the condition (4.13) can even be replaced by

$$\sup_{r>0} \frac{\mu(B_r(x))}{r^{N}} \geq C \qquad \textit{for every } x \in \textit{supp } \mu \tag{4.14}$$

(for another $C = C(N, b)$).

Proof. Take (if $c > 0$) any

$$C > \sup_{0<R<1} R^{1-N} \left(b + \frac{Nc}{3R}\right) 6^N |B_R|.$$

Then, if (4.13) holds with this C there exists for each $x \in \text{supp}\ \mu$ a radius $R_x > 0$ such that

$$\mu(B(x, R_x)) > \left(b + \frac{Nc}{3R_x}\right) 6^N |B(x, R_x)|. \tag{4.15}$$

Since supp μ is compact we can select finitely many $x_1, ..., x_m \in \text{supp}\ \mu$ such that

$$\text{supp}\ \mu \subset B(x_1, R_1) \cup ... \cup B(x_m, R_m),$$

where $R_j = R_{x_j}$. Now we can choose $\epsilon > 0$ small enough so that

$$\mu(B(x_j, R_j)) > \left(b + \frac{Nc}{3(R_j + \epsilon)}\right) 6^N |B_{R_j+\epsilon}|$$

for $j = 1, ..., m$. With ψ_ρ as in (4.12) this means that

$$(\mu * \psi_\epsilon)(B(x_j, R_j + \epsilon)) > \left(b + \frac{Nc}{3(R_j + \epsilon)}\right) 6^N |B_{R_j+\epsilon}|.$$

Now minimize $J_{f,g}$ with $f = \mu * \psi_\epsilon - h$ and set $\Omega = \{u > 0\}$ where u is a minimizer. Then as in Theorem 4.7 $B(x_j, 3(R_j+\epsilon)) \subset \Omega$, $j = 1, ..., m$. In particular supp $(\mu * \psi_\epsilon) \subset \Omega$ and it follows that $\Omega \in Q(\mu * \psi_\epsilon; h, g)$ and $\Omega \in Q(\mu; h, g)$. This proves the theorem if $c > 0$. If $c = 0$ then one also gets (4.15) if (4.14) holds with $C > b|B_1|$, and the rest of the proof is unchanged. □

Comments. Clearly (4.13) is satisfied if μ is a finite sum of point masses or if μ is supported by a finite system of manifolds of dimension $\leq N-1$ and has a sufficiently high density on them (for dimension $s < N - 1$ it is enough that the H^s density is bounded away from zero).

Moreover it is clear from Example 4.5 that an assumption of the sort (4.13) really is necessary for the existence of a quadrature domain. However, the constant C obtained in the proof is probably far from the best possible. Indeed, Example 4.5 indicates that if $N = 2$, $h = 0$, $g = 1$ then any $C > 1$ should work in (4.13), whereas our proof needs $C > 24\pi$ in this case. If $c = 0$ then, according to [Mar], [Sak, unpublished], Theorem 4.7 holds with 6^N in (4.11) replaced by 2^N, which is the best constant. This also gives the best constant in (4.14), namely $C(N, b) > 2^N|B_1|b$.

Aside from interesting cases of nonuniqueness of quadrature domains, as in Example 4.4, there is sometimes, for nonconstant g, a kind of trivial nonuniqueness: if $\mu \geq 0$ is any measure, take $h = 0$, $g = |\nabla U^\mu|$ (outside supp μ at least). Then $\Omega_t = \{x \in \mathbb{R}^N : U^\mu(x) > t\}$ is in $Q(\mu; h, g)$ for any $t \in \mathbb{R}$ such that Ω_t contains supp μ and is bounded. Cf. discussions in [Beur]. The function $u = U^\mu - U^\nu$ in this case simply is $(U^\mu - t)_+$.

Finally note that when Ω is a quadrature domain obtained from a local minimizer of J then the regularity results of section 2 and the geometric results of section 3 apply.

References

A. Acker, *Heat flow inequalities with applications to heat flow and optimization problems*, SIAM J. Math. Anal. **8** (1977), 604–618.

A. Acker, *Interior free boundary problem for the Laplace equation*, Arch. Rat. Mech. Anal **75** (1981), 157–168.
D. Aharonov, H.S. Shapiro, *Domains on which analytic functions satisfy quadrature identities*, J. Analyse Math. **30** (1976), 39–73.
D. Aharonov, M.M. Schiffer, L. Zalcman, *Potato kugel*, Israel J. Math. **40** (1981), 331–339.
H.W. Alt, L.A. Caffarelli, *Existence and regularity for a minimum problem with free boundary*, J. Reine Angew. Math. **325** (1981), 434–448.
H.W. Alt, L.A. Caffarelli, A. Friedman, *Variational problems with two phases and their free boundaries*, Trans. Amer. Math. Soc. **282 nr. 2** (1984), 431–461.
Y. Avci, *'Quadrature Identities and the Schwarz Function'*, Doctoral dissertation, Stanford University (1977).
H. Berestycki, L. Nirenberg, *On the method of moving planes and the sliding method*, Bol. da Soc. Brasileira de Mathematica **22** (1991), 1–37.
A. Beurling, *On free-boundary problems fro the Laplace equation*, Sem. on Analytic Functions, Inst. for Advanced Study Princeton **1** (1957), 248–263.
L.A. Caffarelli, *Compactness methods in free boundary problems*, Partial Differential Equations **5** (1980), 427–448.
L. A. Caffarelli, *Free boundary problems, a survey*, ??, 31– 61.
T. Carleman, *Über ein Minimumproblem der mathematischen Physik*, Math. Z. **1** (1918), 208–212.
P. Davis, *The Schwarz Function and Its Application*, Carus Math. Monographs, Math. Assoc. of America, 1974.
P. Duren, *Theory of H^p Spaces*, Academic Press, New York, 1970.
P. Duren, H.S. Shapiro, A. Shield, *Singular measures and domains not of Smirnov type*, Duke Math. J. **33** (1966), 247–254.
L.C. Evans, R.F. Gariepy, *Measure Theory and Fine Property of Functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton (Florida), 1992.
A. Friedman, *Variational Principles and Free Boundary Problems*, Wiley, New York, 1982.
A. Friedman, D. Philips, *The free boundary problem of a semilinear elliptic equation*, Trans. Amer. Math. Soc. **282** (1984), 153–182.
K. Friedrichs, *Über ein Minimumproblem für Potentialströmungen mit freiem Rand*, Math. Ann. **109** (1934), 60–82.
G. Gidas, W. Ni and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Commun. Math. Phys. **68** (1979), 209–243.
E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*, Birkhäuser, Boston, 1984.
B. Gustafsson, *Application of half-order differentials on Riemann surfaces to quadrature identities for arc-length*, J. Analyse Math. **49** (1987), 54–88.
B. Gustafsson, *On quadrature domains and an inverse problem in potential theory*, J. Analyse Math. **55** (1990), 172–216.
B. Gustafsson, M. Sakai, *Properties of some balayage operators, with applications to quadrature domains and moving boundary problems*, Nonlinear Anal. **22** (1994), 1221–1245.
A. Henrot, *Subsolutions and supersolutions in free boundary problems*, Ark. Mat. **32** (1994), 79–98.
A. Henrot, M. Pierre, *About existence of equilibria in electromagnetic casting*, Quarterly of Applied Mathematics **XLIXX** (1991), 563–575.
B. Kawohl, *On Rearrangements and Convexity of Level Sets in PDE*, vol. 1150, Lecture Notes in Math. Springer Verlag, Berlin-Heidelberg, 1985.
D. Kinderlehrer and L. Nirenberg, *Regularity in free boundary problems*, Ann. Scoula Norm. Sup. Pisa Cl. Sci. **4** (1977), 373–391.
D. Kinderlehrer, G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, New York, 1980.
Ü. Kuran, *On the mean-value property of harmonic functions*, Bull. London Math. Soc. **4** (1972), 311–312.
J.L. Lewis, A. Vogel, *On Pseudospheres*, Rev. Mat. Iberoamericana **7** (1991), 25–54.
J.L. Lewis, A. Vogel, *On some almost everywhere symmetry theorems*, Progress in nonlinear differential equations (Lloyd N. G. et al., eds), Nonlinear diffusion equations and their equilibrium states,, Birkhäuser, Boston, 347–374.
A. S. Margulis, *The moving boundary problem of potential theory*, preprint (1993).
J. Mossino, *Inégalités Isopérimétriques et Applications en Physique*, Herman, Paris, 1984.

J.F. Rodrigues, *Obstacle Problems in Mathematical Physics*, North-Holland, Amsterdam, 1987.
M. Sakai, *Lecture Notes in Math.*, vol. 934, Springer -Verlag, Berlin-Heidelberg, 1982.
M. Sakai, *Application of variational inequalities to the existence theorem on quadrature domains*, Trans. Amer. Math. Soc. **276** (1983), 267–279.
M. Sakai, *Regularity of boundary having a Schwarz function*, Acta Math. **166** (19991), 263–297.
M. Sakai, *Regularity of boundaries of quadrature domains in two dimensions,*, J. Math. Anal **24** (1994), 341–364.
M. Sakai, Private Communicatin.
D.G. Schaeffer, *Some examples of singularities in a free boundary*, Ann. Scu. Norm. Sup. Pisa 4 **4** (1977), 131–144.
J. Serrin, *A symmetry problem in potential theory*, Arch. Rat. Mech. Anal. **43** (1971), 304–318.
H. Shahgholian, *A characterization of the sphere in terms of single-layer potentials*, Proc. Amer. Math. Soc. **115** (1992), no. 4, 1167–1168.
H. Shahgholian, *Existence of quadrature surfaces for positive measures with finite support*, Potential . Anal. **3** (1994), 245–255.
H. Shahgholian, *Quadrature surfaces as free boundaries*, Ark. Mat. (1994).
H. H.S. Shapiro, *Remarks concerning domains of Smirnov type*, Michigan Math. J. **13** (1966), 341–348.
H. S. Shapiro, *The Schwarz Function and its Generalization to Higher Dimensions*, Wiley, New York, University of Arkansas lecture Notes, 1992.
H.S. Shapiro, C. Ullemar, *Conformal mappings satisfying certain extremal properties, and associated quadrature identities*, research bulletin TRITA-MATH-1981-6, Royal Inst. of Technology, Stockholm.
J.P. Zolésio, *Weak Shape formulation of free boundary problems*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **21** (1994), 11–44.

1. Existence of minimizers

Throughout this paper $N \geq 2$. Most of the paper (section 1–3) is devoted to studying a minimization problem. The data for this problem are two functions f and g in $\mathbb{R}^N$ satisfying Condition A below.

Condition A.

(A1) $f, g \in L^\infty(\mathbb{R}^N)$,

(A2) supp f_+ is compact ,

(A3) $g \geq 0$

(A4) at least one of
$f \leq$ const. < 0
$g \geq$ const. > 0
holds outside a compact set.

Let $\mathbb{K} = \{u \in H^1(\mathbb{R}^N) : \ u \geq 0\}$ and set

$$J(u) = J_{f,g}(u) = \int_{\mathbb{R}^N} (|\nabla u|^2 - 2fu + g^2 \chi_{\{u>0\}}) dx.$$

Then J is well defined on $\mathbb{K}$, taking values in $(-\infty, +\infty]$. We shall consider the problem:

$$\text{Minimize } J(u) \quad \text{for } u \in \mathbb{K}.$$

The following lemma turns out to be very useful. Similar results have previously been used by Friedman and Philips [F-P].

Lemma 1.1. *Let $J_k(u) = J_{f_k, g_k}(u)$, where $f_1 \leq f_2$, $g_1 \geq g_2$ and $k = 1, 2$. For $u_1, u_2 \in \mathbb{K}$ define $v = \min(u_1, u_2)$ and $w = \max(u_1, u_2)$. Then $v, w \in \mathbb{K}$ and*

$$J_1(v) + J_2(w) \leq J_1(u_1) + J_2(u_2).$$

In particular, if u_1 minimizes J_1 then $J_2(w) \le J_2(u_2)$ and if u_2 minimizes J_2 then $J_1(v) \le J_1(u_1)$. If u_k minimizes J_k for $k = 1, 2$ then v minimizes J_1 and w minimizes J_2.

Proof. In general, if $\Phi(t)$ is a nondecreasing function of $t \in \mathbb{R}$ and $h_1 \le h_2$ then, as is easily seen,

$$\int (h_1\Phi(u_1) + h_2\Phi(u_2)) \le \int (h_1\Phi(v) + h_2\Phi(w)).$$

Applying this with $h_j = f_j$, $\Phi(t) = t$ we find that

$$\int (f_1u_1 + f_2u_2) \le \int (f_1v + f_2w),$$

and choosing $h_j = -g_j^2$, $\Phi(t) = 0$ for $t \le 0$, $\Phi(t) = 1$ for $t > 0$ we find

$$\int (g_1^2\chi_{\{u_1>0\}} + g_2^2\chi_{\{u_2>0\}}) \ge \int (g_1^2\chi_{\{v>0\}} + g_2^2\chi_{\{w>0\}}).$$

Since also $\int(|\nabla u_1|^2 + |\nabla u_2|^2) = \int(|\nabla v|^2 + |\nabla w|^2)$ the proof is finished. □

In order to get comparison solutions we shall first prove the existence of solutions (minimizers) in a special case.

Lemma 1.2. *Let $f = a\chi_{B(0,R)} - b$ and $g = c\chi_{\mathbb{R}^N \setminus B(0,R_1)}$, where a, b, c, R and R_1 are nonnegative constants with $a > b$ and $b + c > 0$. Then J has at least one minimum (minimizer) u in $\mathbb{K}$. Any minimizing u is radially symmetric, radially nonincreasing and vanishes outside a compact set. Moreover the minima form a nested family and there is a largest minimum as well as a smallest one.*

Proof. For any u in $\mathbb{K}$ let u^* denote its radially symmetric decreasing rearrangement (for background see [Moss]). Then $u^* \in \mathbb{K}$ and

$$\int |\nabla u^*|^2 \le \int |\nabla u|^2, \quad \int fu^* \ge \int fu, \quad \int g^2\chi_{\{u^*>0\}} \le \int g^2\chi_{\{u>0\}};$$

where the first inequality follows from a classical theorem of Polya and Szegö (see [Moss, Theorem 4.1]) and the last two inequalities use the fact that f is nonincreasing and g is nondecreasing as functions of $r = |x|$. It follows that $J(u^*) \le J(u)$ and hence that we only need to look for minima in $\mathbb{K}^* = \{u \in \mathbb{K} : u^* = u\}$. It should be observed also that $J(u^*) < J(u)$ unless $u^* = u$.

From now on we assume that $c > 0$, because if $c = 0$ then J is convex and it is well-known that there exists a unique minimizer u in $\mathbb{K}$. This u has compact support with radius of support $\rho = (\frac{a}{b})^{1/N}R$. (Note that $b > 0$ when $c = 0$.) Cf. Example 1.5 below. Thus the lemma holds if $c = 0$.

We first prove that J is bounded from below on $\mathbb{K}^*$ (and hence on $\mathbb{K}$). For u in $\mathbb{K}^*$ there is a unique ρ in $[0, \infty]$ depending on u such that $u(x) > 0$ for $|x| < \rho$ and $u(x) = 0$ for $|x| \ge \rho$. Set $\Omega = \{u > 0\} = B(0, \rho)$. Regarding f, g and u as functions of $r = |x|$ we have

$$\frac{1}{\omega_N}J(u) = \int_0^\rho (u'(r))^2 r^{N-1}dr - 2\int_0^\rho fur^{N-1}dr + \frac{c^2}{N}\max(\rho^N - R_1^N, 0).$$

Since $J(u) = +\infty$ if $\rho = +\infty$ we need only to consider u with $\rho < \infty$. Set

$$\lambda = \left(\int_0^\rho (u'(r))^2 r^{N-1}dr\right)^{1/2} = \frac{1}{\sqrt{\omega_N}}\|\nabla u\|,$$
$$\phi(r) = \int_0^r f(s)s^{N-1}ds = \frac{a}{N}\min(r, R)^N - \frac{b}{N}r^N.$$

If $b \ne 0$ then there is an $r_0 > 0$ such that $\phi(r) \ge 0$ for $0 < r < r_0$, $\phi(r) \le 0$ for $r > r_0$, and since $u' \le 0$ we then get

$$\int_0^\rho fur^{N-1}dr = -\int_0^\rho \phi u'dr \le A\lambda, \tag{1.1}$$

where $A = (\int_0^{r_0} \phi^2 r^{1-N} dr)^{1/2}$ is a constant independent of u. Thus

$$\frac{1}{\omega_N} J(u) \geq \lambda^2 - 2A\lambda + \frac{c^2}{N} \max(\rho^N - R_1^N, 0) \geq -A^2. \tag{1.2}$$

If $b = 0$ then $0 \leq \phi(r) \leq$ const. $< \infty$ for $r > 0$ and (1.1), (1.2) hold with

$$A = A_\rho = \int_0^\rho \phi^2 r^{1-N} dr \leq \text{ const. } \int_0^\rho r^{1-N}.$$

Since $c > 0$ we still see from the second inequality in (1.2) that J is bounded from below (provided $N \geq 2$).

Thus J is always bounded from below. Let $\{u_n\} \subset \mathbb{K}^*$ be a minimizing sequence, ρ_n the radius of support of u_n and $\lambda_n^2 = \int_0^{\rho_n} (u_n')^2 r^{N-1} dr = \frac{1}{\sqrt{\omega_N}} \|\nabla u_n\|$. Then it follows from (1.2) that

$$\rho_n \leq \text{const.} < \infty, \tag{1.3}$$

$$\lambda_n \leq \text{const.} < \infty. \tag{1.4}$$

Now from (1.3) and (1.4) the existence of a minimum for J follows by standard arguments. In fact, (1.3) shows that we may work in $\mathbb{K} \cap H_o^1(B)$ for some fixed ball B and (1.4) then shows that the minimizing sequence $\{u_n\}$ is precompact in the w-$H_o^1(B)$ (i.e. $H_o^1(B)$ provided with the weak topology). As J is easily checked to be lower semicontinuous in w-$H_o^1(B)$ the existence of a minimum follows.

If $c > 0$ and a is not too large there may be several solutions u (cf. Example 1.5 below). However any solution is uniquely determined by its radius of support ρ (e.g. because u will satisfy $-\Delta u = f$ in $\Omega = \{|x| < \rho\}$, $u = 0$ on $\partial\Omega$, as will be proved later (Lemma 2.2) independently of the present proof), and a larger ρ will correspond to a larger solution u. Therefore the solutions form a nested family and it follows that there is a largest solution (note that any family of solutions at the same time is a minimizing sequence). ☐

Remark 1.3. When $N = 1$, $J(u)$ is not always bounded from below. Take e.g. $a = 2$, $b = 0$, $c = 1$, $R = 1$ and consider $u(x) = \frac{1}{2}(\rho - |x|)$ for $|x| < \rho$ and $u(x) = 0$ for $|x| > \rho$, where $\rho > 1$ is a parameter. Then $J(u) = 2 - \frac{3}{2}\rho$, which obviously goes to $-\infty$ as $\rho \to +\infty$.

When $N = 2$ $J(u)$ is not bounded from below if $b = c = 0$ (and $a > 0, R > 0$), while, as is seen from the proof, $J(u)$ is bounded from below when $N \geq 3$ even if $b = c = 0$. However the (unique) minimizer does not have compact support then.

It is also worth to mention that the Condition A is not optimal. However g and f_- are not allowed to tend to zero too fast at infinity.

We now turn to the general case.

Theorem 1.4. *If f and g satisfy Condition A then J is bounded from below and its infimum is attained for at least one u in $\mathbb{K}$. All minimizers have support in a fixed compact set (which depends only on f and g) and the set of minimizers is compact in the weak topology of $H^1(\mathbb{R}^N)$.*

Proof. Let $\tilde{f} = a\chi_{B(0,R)} - b$, $\tilde{g} = c\chi_{\mathbb{R}^N \setminus B(0,R_1)}$ with $a, b, c, R, R_1 \geq 0$, $a > b$, $b + c > 0$ chosen so that $f \leq \tilde{f}$, $g \geq \tilde{g}$ and set $\tilde{J} = J_{\tilde{f},\tilde{g}}$. By Lemma 1.2 there is a largest minimizer $\tilde{u}$ in $\mathbb{K}$ of $\tilde{J}$. Clearly

$$J(u) \geq \tilde{J}(u) \qquad \text{for all } u \in \mathbb{K} \tag{1.5}$$

and also, by Lemma 1.1,

$$J(\min(u, \tilde{u})) \leq J(u). \tag{1.6}$$

Thus $J(u)$ decreases if u is replaced by $\min(u, \tilde{u})$. Choose an open ball B such that supp $\tilde{u} \subset B$. (1.5) together with Lemma 1.2 shows that J is bounded from below and (1.6) shows that if $\{u_n\}$ is a minimizing sequence then so is $\{\min(u_n, \tilde{u})\}$. Thus there exists a minimizing sequence $\{u_n\}$ with supp $u_n \subset B$. By Poincaré's lemma then $\|u_n\| \leq C\|\nabla u_n\|$, so that

$$J(u_n) \geq \|\nabla u_n\|^2 - 2\|f\chi_B\|\|u_n\| \geq \|\nabla u_n\|^2 - 2C\|\nabla u_n\| \geq -C^2.$$

Thus J is bounded from below, $\|\nabla u_n\| \leq C$ and the existence of a minimizer follows as in Lemma 1.2.

If now $u \in \mathbb{K}$ denotes any minimizer of J then Lemma 1.1 shows that $\max(u, \tilde{u}) \leq \tilde{u}$, since $\tilde{u}$ is the largest minimizer of $\tilde{J}$, hence that $u \leq \tilde{u}$. This shows that u has compact support in a fixed compact set. If $a > b + Nc/R$ and $R_1 = 0$ we in fact have supp $u \subset \overline{B(0,\rho)}$, where by Example 1.5 below ρ can be taken to be

$$\rho = \begin{cases} \left(\frac{a}{b}\right)^{1/N} R & \text{if } b \neq 0, \\ \left(\frac{aR^N}{Nc}\right)^{1/(N-1)} & \text{if } c \neq 0. \end{cases}$$

(If $a \leq b + Nc/R$ then $\rho = R$ works.) It moreover follows as above that $\|u\| + \|\nabla u\| \leq C < \infty$, C independent of u, and therefore that the set of minimizers is compact in w-$H^1(\mathbb{R}^N)$. $\square$

Example 1.5. Assume that f and g are radially symmetric with f nonincreasing and $g \geq 0$ nondecreasing as functions of $r = |x|$. As was noticed in the proof of Lemma 1.2 any minimum u in $\mathbb{K}$ of J is itself radially symmetric and nonincreasing as function of $r = |x|$, i.e. $u \in \mathbb{K}^*$. Moreover u has compact support. It will be proved later (without using the results of this example) that a necessary condition that a function $u \in \mathbb{K}^*$ is a minimum (or local minimum, Definition 2.1) is that it is a weak solution, i.e. u satisfies

$$\begin{aligned} -\Delta u = f \qquad & \text{in} \quad \Omega = \{u > 0\}, \\ |\nabla u| = g, \quad u = 0 \qquad & \text{on} \quad \partial\Omega, \end{aligned}$$

We shall now discuss weak solutions belonging to $\mathbb{K}^*$ and compare $J(u)$ for these. So let $u \in \mathbb{K}^*$ be a weak solution. Since f, g and u only depend on r and u is nonincreasing there is a unique ρ in $[0, \infty)$ such that $\Omega = B(0, \rho)$ and the equations above become

$$-u'' - \frac{N-1}{r} u' = f(r), \qquad 0 < r < \rho \tag{1.7}$$

$$u(\rho) = 0, \tag{1.8}$$

$$-u'(\rho) = g(\rho). \tag{1.9}$$

(1.7) gives $(r^{N-1}u')' = -r^{N-1}f(r)$ and by (1.9)

$$r^{N-1}u'(r) = -\rho^{N-1}g(\rho) + \int_r^\rho s^{N-1}f(s)ds. \tag{1.10}$$

As $r \to 0$ we shall have $r^{N-1}u'(r) \to 0$ (otherwise we get a distributional contribution to Δu at the origin). Thus

$$\int_0^\rho s^{N-1}f(s)ds = \rho^{N-1}g(\rho). \tag{1.11}$$

This is a condition for ρ. Once ρ is determined u is obtained by integrating (1.10) and using (1.8). Explicitly

$$u(r) = \int_r^\rho t^{1-N}\left(\rho^{N-1}g(\rho) - \int_t^\rho s^{N-1}f(s)ds\right) dt,$$

for $0 < r < \rho$. Set

$$F(\rho) = \int_0^\rho s^{N-1}f(s)ds - \rho^{N-1}g(\rho),$$

for $\rho \geq 0$ so that (1.11) becomes $F(\rho) = 0$. Thus the weak solutions in $\mathbb{K}^*$ are in bijective correspondence to the zeros of F.

Let us now specialize to the case $f(r) = a\chi_{[0,R)} - b$ and $g(r) = c$, where $a, b, c, R \geq 0$ are constants with $a > b, b + c > 0, R > 0$. F becomes

$$F(\rho) = \begin{cases} \frac{a-b}{N}\rho^N - c\rho^{N-1} & 0 \leq \rho < R, \\ \frac{a}{N}R^N - \frac{b}{N}\rho^N - c\rho^{N-1} & \rho \geq R. \end{cases}$$

Note that $F'(\rho) \leq$ const. < 0 for $\rho \geq R$. It follows from the equations for a weak solution that $\int |\nabla u|^2 dx = \int f u dx$. Therefore

$$J(u) = \omega_N \left[\frac{c^2}{N}\rho^N - \int_0^\rho (u'(r))^2 r^{N-1} dr \right] \tag{1.12}$$

if u is the weak solution corresponding to ρ. By (1.10)

$$r^{N-1}u'(r) = -c\rho^{N-1} + \frac{a-b}{N}(\rho^N - r^N) \tag{1.13}$$

for $0 < r < \rho$ if $0 < \rho < R$, while

$$r^{N-1}u'(r) = -c\rho^{N-1} - \frac{b}{N}(\rho^N - r^N) + \frac{a}{N}(R^N - r^N)\chi_{[0,R)}(r) \tag{1.14}$$

for $0 < r < \rho$ if $\rho > R$. Inserting this into (1.12) gives

$$\frac{1}{\omega_N} J(u) = \frac{c^2}{N}\rho^N - \int_0^\rho \left[c\rho^{N-1} - \frac{a-b}{N}(\rho^N - r^N) \right]^2 r^{1-N} dr, \tag{1.15}$$

if $0 < \rho < R$ and

$$\begin{aligned} &\frac{1}{\omega_N} J(u) = \frac{c^2}{N}\rho^N \\ &- \int_0^\rho \left(-c\rho^{N-1} - \frac{b}{N}(\rho^N - r^N) + \frac{a}{N}(R^N - r^N)\chi_{[0,R)} \right)^2 r^{1-N} dr = \\ &\frac{c^2}{N}\rho^N - \int_0^R \left(\frac{a-b}{N} r^N \right)^2 r^{1-N} dr - \int_R^\rho \left(\frac{a}{N}R^N - \frac{b}{N} r^N \right)^2 r^{1-N} dr \leq \\ &\frac{c^2}{N}\rho^N - \frac{(a-b)^2}{N^2(N+2)} R^{N+2} - \frac{1}{N^2}\int_R^\rho (aR^N - br^N)^2 r^{1-N} dr, \end{aligned} \tag{1.16}$$

if $\rho > R$ (recall that $F(\rho) = 0$).

We shall now determine all zeros $\rho = \rho_n$ of F and compare $J(u_n)$ for the corresponding weak solutions $u_n \in \mathbb{K}^*$ $(n = 0, 1, ...)$. Observe first that $\rho = \rho_0 = 0$ is always a zero of F, corresponding to $u_0 = 0$ with $J(u_0) = 0$. Next we divide into cases.

Case 1: $c = 0$. Then J is convex and there is, besides ρ_0, exactly one more zero of F, namely $\rho_1 = \left(\frac{a}{b}\right)^{1/N} R > R$. It is easily seen from (1.16) that $J(u_1) < 0$. Thus u_1 is the only minimum of J and there are no other local minima.

Case 2: $c > 0$ and $b < a < b + \frac{Nc}{R}$. In this case $F(\rho) < 0$ for all $\rho > 0$. Hence $u_0 = 0$ is the only weak solution (in $\mathbb{K}^*$) and it is the global minimum of J. In particular $J(u) \geq 0$ for all $u \in \mathbb{K}$.

Case 3: $c > 0$ and $a = b + \frac{Nc}{R}$. Here $\rho_0 = 0$ and $\rho_1 = R$, are the zeros of F. Equation (1.16) gives that $J(u_1) > 0$. Thus u_0 is the only minimizer and $J \geq 0$.

Case 4: $c > 0$ and $a > b + \frac{Nc}{R}$. In this case $F(R) > 0$ and it follows that F has exactly three zeros: $\rho_0 = 0 < \rho_2 < R < \rho_1$. We have $\rho_2 = \frac{Nc}{a-b}$ and from (1.15) one finds that $J(u_2) > 0$ always. As to ρ_1, it is determined by

$$aR^N = b\rho_1^N + Nc\rho_1^{N-1}, \tag{1.17}$$

and $J(u_1)$ is then obtained by inserting this into (1.16). It is clear from (1.17) that when a increases from $b + \frac{Nc}{R}$ to $+\infty$ (with b, c and R kept fixed), then ρ_1 increases from R to $+\infty$. Moreover $J(u_1)$ at the same time decreases monotonically to $-\infty$ from its positive value when $a = b + \frac{Nc}{R}$. This can be seen e.g. by estimating the derivative $\frac{d}{da}J(u_1)$ or $\frac{d}{d\rho_1}J(u_1)$. It follows that there exists a critical value $a_0 > b + \frac{Nc}{R}$ such that we have the following three subcases.

Subcase 4a: $c > 0$ and $b + \frac{Nc}{R} < a < a_0$. Then $J(u_0) = 0$, $J(u_1) > 0$, $J(u_2) > 0$. Thus u_0 is the only minimizer and $J \geq 0$. However u_1 can be shown to be a local minimizer in this case. Indeed, it is not hard to see that u_1 is a local minimizer among other functions in $\mathbb{K}^*$, and when moving out from $\mathbb{K}^*$ (into $\mathbb{K} \setminus \mathbb{K}^*$) the functional J increases as was observed in the proof of Lemma 1.2.

Subcase 4b: $c > 0$ and $a = a_0$. Then $J(u_0) = J(u_1) = 0$, $J(u_2) > 0$. Thus we have two minima, and $J \geq 0$.

Subcase 4c: $c > 0$ and $a > a_0$. Then $J(u_0) = 0$, $J(u_1) < 0$, $J(u_2) > 0$ so that u_1 is the only minimizer. By the same argument as in subcase 4a, u_0 is a local minimizer.

Finally in this example we need (for later use) an estimate of a_0. We claim the following: if

$$a > \left(b + \frac{cN}{3R}\right) 3^N \tag{1.18}$$

then $J(u_1) < 0$ and

$$\rho_1 > 3R. \tag{1.19}$$

In particular $a_0 \leq (b + \frac{Nc}{3R})3^N$.

That $\rho_1 > 3R$ when (1.18) holds follows immediately from (1.17). Observe next that (1.17) also implies

$$\rho_1 \leq \left(\frac{aR}{Nc}\right)^{1/(N-1)} R \tag{1.20}$$

(and $\rho_1 \leq (a/b)^N R$). Using (1.20) and (1.18) in (1.16) gives by a little computation that already the two first terms make $J(u_1)$ negative when $N \geq 3$. When $N = 2$ one also has to take the last term into account and the calculation becomes a little more tedious. (In the last term (the integral) one may replace ρ by $3R$ and b by $3^{-2}a$ according to (1.19) and (1.18).)

Subexample. If $N = 2$ and $b = 0$ then a_0 can easily be calculated to be $a_0 = \frac{2c}{R}\sqrt[4]{e}$.

As a corollary of Example 1.5 and Lemma 1.1 we have

Proposition 1.6. *Let f, g satisfy Condition A and set $a = \sup f$, $c = \inf g$ and let R be the radius of the smallest closed ball containing supp f_+. Then, if $aR \leq Nc$, $J = J_{f,g} \geq 0$ and $u = 0$ is the only minimizer. There even exists a number $a_0 = a_0(N, R, c) > \frac{Nc}{R}$ such that the same conclusion holds whenever $a < a_0$.*

Proof. We have $f \leq a\chi_{B_R} - b$, $g \geq c$ with $b = 0$. Then combine Example 1.5 with Lemma 1.1 □

Proposition 1.7. *Let f, g satisfy Condition A and let u be a minimizer of J. Assume that $u = 0$ on ∂B_R where B_R is a ball such that $R \sup_{B_R} f_+ \leq N \inf_{B_R} g$. Then $u = 0$ in B_R.*

Proof. Set $v = u$ in B_R and $v = 0$ outside B_R. Clearly v minimizes $\tilde{J} = J_{\tilde{f},\tilde{g}}$ where $\tilde{f} = (f_+)\chi_{B_R}$, $\tilde{g} = g\chi_{B_R} + (\inf_{B_R} g)\chi_{\mathbb{R}^N \setminus B_R}$. Now apply Proposition 1.6 to $\tilde{J}$. □

Proposition 1.8. *If u and v are minima of J then also $\min(u, v)$ and $\max(u, v)$ are minima. Also, if $\{u_n\}$ are minima and $u_1 \leq u_2 \leq ...$ then $u = \sup u_n$ is a minimum. Similarly, if $u_1 \geq u_2 \geq ...$ then $\inf u_n$ is a minimum. Finally there is a largest minimizer of J, and also a smallest one.*

Proof. The first statement follows immediately from Lemma 1.1 and the second (and the third) from the compactness assertion of Theorem 1.4.

To prove the last assertion, first note that since $H^1(\mathbb{R}^N)$ is separable there is a finite or infinite sequence $\{v_n\}$ of minima which is dense in the set of all minima. Define $u_1 = v_1$ and, inductively for $n \geq 2$, $u_n = \sup(u_{n-1}, v_n)$, so that $u_1 \leq u_2 \leq ....$ As shown above $u = \sup u_n$ is also a minimizer and it is readily verified that $v \leq u$ for every minimizer v. □

2. Local minima

In this section we deduce basic properties of minima, or more generally of local minima, of J. The data f and g will generally be assumed to satisfy Condition A. The main result of this section is Theorem 2.13, saying that any local minimum u solves the appropriate free boundary problem in a potential theoretically satisfactory sense, provided g is continuous. This means that the distributional Laplacian Δu can be expressed in terms of purely geometric quantities related to the open set $\Omega = \{u > 0\}$, more precisely that

$$\Delta u + f\mathcal{L}^N \lfloor \Omega = g\mathcal{H}^{N-1}\lfloor \partial\Omega.$$

Continuous functions $u \in \mathbb{K}$ satisfying this equation will be called *weak solutions* (Definition 3.1).

It should be told that this section is very much based on the methods and results of the pioneering paper [AC] (see also [ACF] and [F]). Many of our proofs are modifications of corresponding proofs in [AC].

Definition 2.1. A function $u \in \mathbb{K}$ is a local minimum of J if, for some $\epsilon > 0$, $J(v) \geq J(u)$ for every $v \in \mathbb{K}$ with

$$\int \left(|\nabla(v-u)|^2 + |\chi_{\{v>0\}} - \chi_{\{u>0\}}|\right) dx < \epsilon. \tag{2.1}$$

Lemma 2.2. *If u is a local minimum then*

$$\Delta u + f_+ \geq 0 \quad in \quad \mathbb{R}^N, \tag{2.2}$$

$$\Delta u + f = 0 \quad in \quad \Omega = \{u > 0\}, \tag{2.3}$$

$$\Delta u + f \leq 0 \quad in \quad \mathbb{R}^N \setminus supp\ g. \tag{2.4}$$

Remark. It follows from (2.2) that u has an upper semicontinuous representative, which is the one we will refer to in the sequel, and it will be proved later that this u actually is continuous. For the present proof of (2.3) Ω should strictly speaking be defined as the set of points $x \in \mathbb{R}^N$ such that there exists $0 \leq \phi \in C^\infty(\mathbb{R}^N)$ with $\phi(x) > 0$ and $u \geq \phi$ everywhere.

Proof. Take $0 \leq \phi \in C_o^\infty(\mathbb{R}^N)$ and define, for $\epsilon > 0$, $v_\epsilon = (u - \epsilon\phi)_+$. Then $v_\epsilon \in \mathbb{K}$, $0 \leq v_\epsilon \leq u$. Set $D_\epsilon = \{u \leq \epsilon\phi\} = \{v_\epsilon = 0\}$. Clearly $|D_\epsilon \cap \Omega| \to 0$ as $\epsilon \to 0$. Since $v_\epsilon - u = -u$ in D_ϵ, $v_\epsilon - u = -\epsilon\phi$ outside D_ϵ it follows that $v_\epsilon \to u$ in $H^1(\mathbb{R}^N)$ and that

$$\int g^2 |\chi_{\{v_\epsilon>0\}} - \chi_{\{u>0\}}| dx = \int_{\Omega \cap D_\epsilon} g^2 dx \quad \to 0.$$

Since u is a local minimum we conclude that $J(u) \leq J(v_\epsilon)$ for $\epsilon > 0$ small enough.

Next we estimate

$$\begin{aligned} 0 &\leq J(v_\epsilon) - J(u) \\ &= \int |\nabla v_\epsilon|^2 - \int |\nabla u|^2 - 2\int f(v_\epsilon - u) + \int g^2(\chi_{\{v_\epsilon>0\}} - \chi_{\{u>0\}}) \\ &= \int_{D_\epsilon^c} |\nabla(u - \epsilon\phi)|^2 - \int_{\mathbb{R}^N} |\nabla u|^2 + 2\epsilon \int_{D_\epsilon^c} f\phi + 2\int_{D_\epsilon} fu - \int_{\Omega \cap D_\epsilon} g^2 \\ &\leq -2\epsilon \int_{D_\epsilon^c} \nabla u \cdot \nabla\phi + \epsilon^2 \int_{D_\epsilon^c} |\nabla\phi|^2 + 2\epsilon \int_{D_\epsilon^c} f_+\phi + 2\int_{D_\epsilon} f_+ u \\ &\leq 2\epsilon \left(\int_{\mathbb{R}^N} f_+\phi - \int_{\mathbb{R}^N} \nabla u \cdot \nabla\phi\right) + 2\epsilon \int_{D_\epsilon \cap \Omega} \nabla u \cdot \nabla\phi + \epsilon^2 \int_{D_\epsilon^c} |\nabla\phi|^2. \end{aligned}$$

Dividing both sides by ϵ and letting $\epsilon \to 0$ we obtain

$$0 \leq -\int \nabla u \cdot \nabla\phi + \int f_+\phi$$

for all $0 \leq \phi \in C_o^\infty(\mathbb{R}^N)$, and hence that $\Delta u + f_+ \geq 0$ in $\mathbb{R}^N$.

If supp $\phi \subset \Omega$ we can take $v = u \pm \epsilon\phi \in \mathbb{K}$ for $\epsilon > 0$ small enough (and $0 \leq \phi \in C_o^\infty(\mathbb{R}^N)$) and this readily gives that $\Delta u + f = 0$ in Ω.

Finally, taking $v_\epsilon = u + \epsilon\phi$ where supp $\phi \cap$supp $g = \emptyset$ gives that $\Delta u + f \leq 0$ in $\mathbb{R}^N \setminus$supp g. □

Theorem 2.3. *Let u be a local minimum and assume $g^2 \in H^{1,1}(\mathbb{R}^N)$. Then*

$$\lim_{\epsilon \searrow 0} \int_{\partial\{u>\epsilon\}} (|\nabla u|^2 - g^2)\eta \cdot \nu d\mathcal{H}^{N-1} = 0$$

for every $\eta \in C_o^\infty(\mathbb{R}^N, \mathbb{R}^N)$. ($\nu$ denotes the outward normal vector of $\partial\{u > 0\}$.)

The proof is similar to that of Theorem 2.5 in [AC] and therefore omitted.

Lemma 2.4. *("Harnack") Assume $u \in H^1(B_r)$, $u \geq 0$ on ∂B_r ($B_r = B(0,r)$) and let $M \geq 0$.*
a) If $\Delta u \leq M$ in B_r then

$$u(x) \geq r^N \frac{r - |x|}{(r + |x|)^{N-1}} \left[\frac{1}{r^2} \fint_{\partial B_r} u - \frac{2^{N-1} M}{N} \right],$$

for $x \in B_r$.
b) If $\Delta u \geq -M$ in B_r then

$$u(x) \leq r^N \frac{r + |x|}{(r - |x|)^{N-1}} \left[\frac{1}{r^2} \fint_{\partial B_r} u + \frac{M}{2N} \right],$$

for $x \in B_r$.
c) If $|\Delta u| \leq M$ in B_r then

$$r^{N-2} \frac{r - |x|}{(r + |x|)^{N-1}} u(0) - \frac{M r^2}{N} \leq$$
$$u(x) \leq r^{N-2} \frac{r + |x|}{(r - |x|)^{N-1}} u(0) + \frac{M r^N}{N} \frac{r + |x|}{(r - |x|)^{N-1}},$$

for $x \in B_r$.
d) If $|\Delta u| \leq M$ in B_r, then

$$|\nabla u(0)| \leq N \left[\frac{1}{r} \fint_{\partial B_r} u + \frac{M}{N+1} r \right].$$

Note that, by Lemma 2.2, b) is always applicable (with $M = \sup_{B_r} f_+$) if u is a local minimum, while a), c) and d) are applicable if $B_r \subset \Omega$.

The proof is standard and therefore omitted. However, one has to work with $u + \varphi$, where $\varphi(x) = \frac{M}{2N}(r^2 - |x|^2)$ so that $u + \varphi$ becomes a superharmonic function.

Lemma 2.5. *Suppose u is a local minimizer of J. Then there is an $r_0 > 0$ such that for any ball B_r with $0 < r < r_0$ we have*

$$\frac{1}{r} \fint_{\partial B_r} u > 2^N \left(\frac{r}{N} \sup_{B_r} f_- + \sup_{B_r} g \right) \qquad \Longrightarrow \qquad u > 0 \text{ and continuous in } B_r.$$

Note. The reason that r has to be small is simply that u is assumed only to be a local minimizer. For a global minimum the implication is true for all $r > 0$.

Proof. We may assume that B_r is centered at the origin. Define $v \in H^1(\mathbb{R}^N)$ by $v = u$ on $\mathbb{R}^N \setminus B_r$ (in particular on ∂B_r) and $-\Delta v = f$ in B_r. Note that v is continuous in B_r. Then as in [AC, 3.2]

$$J(u) - J(v) \geq \int_{B_r} |\nabla(u - v)|^2 - \sup_{B_r} g^2 |\{u = 0\} \cap B_r|. \tag{2.5}$$

On the other hand a) of Lemma 2.4 (applied to v) shows that

$$v(x) \geq r^N \frac{r - |x|}{(r + |x|)^{N-1}} \left[\frac{1}{r^2} \fint_{\partial B_r} u - \frac{2^{N-1} M}{N} \right],$$

for $x \in B_r$. Here $M = \sup_{B_r} f_-$. Thus whenever

$$\frac{1}{r^2} \fint_{\partial B_r} u \geq \frac{2^N M}{N}, \tag{2.6}$$

we have

$$v(x) \geq \frac{r^N}{2} \frac{r - |x|}{(r + |x|)^{N-1}} \frac{1}{r^2} \fint_{\partial B_r} u \geq 2^{-N}(r - |x|) \frac{1}{r} \fint_{\partial B_r} u, \tag{2.7}$$

and in particular $v(x) > 0$ for $x \in B_r$.

Now as in [AC, 3.2] one obtains

$$|\{u = 0\} \cap B_r| \left(\frac{1}{r} \fint_{\partial B_r} u \right)^2 \leq 2^{2N} \int_{B_r} |\nabla(u - v)|^2$$

provided (2.6) holds.

When (2.6) holds v is nonnegative by (2.7) so that $v \in \mathbb{K}$, and if $r > 0$ is small v is moreover close to u in the metric (2.1). Thus, since u is a local minimum, $J(u) \leq J(v)$, i.e. by (2.5)

$$\int_{B_r} |\nabla(u - v)|^2 \leq \sup_{B_r} g^2 |\{u = 0\} \cap B_r|.$$

Hence

$$|\{u = 0\} \cap B_r| \left(\frac{1}{r} \fint_{\partial B_r} u \right)^2 \leq 2^{2N} \sup_{B_r} g^2 |\{u = 0\} \cap B_r|, \tag{2.8}$$

whenever (2.6) holds.

This proves the lemma, for if

$$\frac{1}{r} \fint_{\partial B_r} u > 2^N \left(\frac{r}{N} \sup_{B_r} f_- + \sup_{B_r} g \right)$$

then (2.6) does hold, and (2.8) leads to a contradiction unless $|\{u = 0\} \cap B_r| = 0$. In the latter case we have $\int_{B_r} |\nabla(u - v)|^2 = 0$ and hence $u = v > 0$ in B_r as desired. □

Corollary 2.6. *Any local minimum u is Lipschitz continuous. Moreover near $\partial\Omega$ we have the estimates*

$$u(x) \leq 2^N \delta(x) \left(\sup_{B(x, 2\delta(x))} g + M\delta(x) \right),$$

$$|\nabla u(x)| \leq N 2^N \left(\sup_{B(x, 2\delta(x))} g + M\delta(x) \right), \tag{2.9}$$

where $M = \sup |f|$, $\Omega = \{u > 0\}$ and $\delta(x)$ denotes the distance from x to Ω^c. Thus $u(x) \leq C\delta(x)$ always, and if x approaches a point of $\partial\Omega$ where g vanishes we have a better estimate (e.g. $u(x) \leq C\delta^{1+\alpha}(x)$ if g is α-Hölder continuous).

For a proof see [AC, ??].

Remark 2.7. (On homogeneity) For $t > 0$ and $\varphi(x)$ any function of $x \in \mathbb{R}^N$, set $\varphi_t(x) = \varphi(x/t)$. Then a straightforward computation shows that for any real number α we have

$$J_{t^\alpha f_t, t^{\alpha+1} g_t}(t^{\alpha+2} u_t) = t^{N+2\alpha+2} J_{f,g}(u).$$

Lemma 2.8. *Let u be a local minimum. If $g \geq$ const. > 0 in an open set $D \subset \mathbb{R}^N$ then there is a constant $C > 0$ such that for any sufficiently small ball $B_r \subset D$ we have*

$$\frac{1}{r}\fint_{\partial B_r} u \leq C \qquad \Longrightarrow \qquad u = 0 \quad \text{in } B_{r/4}. \tag{2.10}$$

More precisely, C depends only on $\inf_{B_r} g$, $r \sup_{B_r} f_+$ and N and is positive whenever $\inf_{B_r} g > 0$ and $r \sup_{B_r} f_+$ is sufficiently small.

Remark. The lemma holds with $B_{\kappa r}$ in place of $B_{r/4}$ for any $0 < \kappa < 1$; C then also depends on κ.

Proof. For u a local minimizer of J b) of Lemma 2.4 always applies and gives, for some constants C_1 and C_2 only depending on N, that

$$u \leq C_1 \fint_{\partial B_r} u + C_2 r^2 \sup_{B_r} f_+ \tag{2.11}$$

in $B_{r/2}$. For notational convenience we assume that $B_r = B_r(0)$.

Set $m = \inf_{B_r} g$, $M = \sup_{B_r} f_+$ and define

$$J_r(v) = \int_{B_{r/2}} \left(|\nabla v|^2 - 2fv + g^2 \chi_{\{v>0\}}\right) dx,$$
$$\tilde{J}_r(v) = \int_{B_{r/2}} \left(|\nabla v|^2 - 2Mv + m^2 \chi_{\{v>0\}}\right) dx.$$

As in Lemma 1.1 we have

$$J_r(\min(u_1, u_2)) + \tilde{J}_r(\max(u_1, u_2)) \leq J_r(u_1) + \tilde{J}_r(u_2), \tag{2.12}$$

for any $u_1, u_2 \in H^1(B_{r/2})$.

Given a constant $\beta \geq 0$ consider the problem of minimizing $\tilde{J}_r(v)$ over $\{v \in H^1(B_{r/2}) : v \geq 0, v = \beta \text{ on } \partial B_{r/2}\}$. We claim that the largest minimizer v_β of $\tilde{J}_r$ vanishes on $B_{r/2}$ provided r and β are small enough. Clearly v_β is radially symmetric. Therefore the claim can be proved by comparing $\tilde{J}_r(w_n)$ for the various radially symmetric weak solutions $w_n = w_{\beta,n}$ for $\tilde{J}_r$, in a similar way as in Example 1.5.

We take $0 < r < 2Nm/M$. It then follows from Proposition 1.7 that if the largest minimizer v_β vanishes somewhere then there is some $0 < \rho < r/2$ such that $v_\beta = 0$ on $\overline{B}_\rho$, $v_\beta > 0$ in $B_{r/2} \setminus B_\rho$. Therefore it is enough to compare weak solutions w of the corresponding form, i.e. satisfying as functions of radius $|x|$ (cf. (1.7)–(1.9)),

$$\begin{aligned} w(|x|) &= 0 && 0 \leq |x| \leq \rho, \\ w'(\rho + 0) &= m && \\ (|x|^{N-1} w')' &= -|x|^{N-1} M && \rho < |x| < r/2, \\ w(r/2) &= \beta. && \end{aligned}$$

Note that by the third equation w' changes sign at most once. Therefore it is easy to see that the above system has at most three solutions, call them w_0, w_1 and w_2 ($w_n = w_{n,\beta}$). w_0 is the one corresponding to the largest value ρ_0 of ρ, with $w_0'(|x|) > 0$ for $\rho_0 < |x| < r/2$. w_1 is the solution obtained if w' changes sign. Thus $0 < \rho_1 < \rho_0$ and $w_1'(|x|) < 0$ for $|x|$ close to $r/2$. Finally w_2 is the uniquely determined weak solution which does not vanish at all.

Now consider what happens when $\beta \searrow 0$. Clearly $w_{0,\beta} \to w_{0,0} = 0$, e.g. in $H^1(B_{r/2})$ and $\rho_0 \to r/2$ (since $w'(\rho + 0) = m$) so that

$$\lim_{\beta \to 0} \tilde{J}_r(w_{0,\beta}) = \tilde{J}_r(0) = 0.$$

$w_{1,\beta}$ exists if and only if $M > 0$ and then $w_{1,\beta} \to w_{1,0} \neq 0$ and

$$\varliminf_{\beta \to 0} \tilde{J}_r(w_{1,\beta}) \geq \tilde{J}_r(w_{1,0}) > 0$$

where the last inequality follows from the fact that $v \equiv 0$ is the unique minimizer when $\beta = 0$ (by Proposition 1.6).

For the same reason we have $w_{2,\beta} \to w_{2,0}$ and

$$\varliminf_{\beta \to 0} \tilde{J}_r(w_{2,\beta}) \geq \tilde{J}_r(w_{2,0}) > 0$$

when $M > 0$. When $M = 0$ then $w_{2,0} \equiv 0$ so that $\tilde{J}_r(w_{2,0}) = 0$, but then $w_{2,\beta} \equiv \beta$ so that

$$\varliminf_{\beta \to 0} \tilde{J}_r(w_{2,\beta}) \geq m^2 |B_{r/2}| > 0.$$

From the above limits we conclude that $\tilde{J}(w_{0,\beta})$ is smaller than both $\tilde{J}(w_{1,\beta})$ and $\tilde{J}(w_{2,\beta})$ if β is small enough. Since clearly $w_{0,\beta}$ vanishes on $B_{r/4}$ if β is small this proves our claim: the largest minimizer $v_\beta (= w_{0,\beta})$ of $\tilde{J}_r$ vanishes on $B_{r/4}$ if $0 < r < 2Nm/M$ and, say, $0 < \beta < \beta_0$.

Now β_0 depends on r, m and M. Indeed, it is easily seen (cf. Remark 2.7) that if r is scaled to tr $(t > 0)$ then the minimizer $v(x)$ of $\tilde{J}_r$ will be scaled to $tv(x/t)$ provided m, M and β, are scaled to, respectively, $m, M/t$ and $t\beta$. In other words, $\beta_0(tr, m, M/t) = t\beta_0(r, m, M)$ for $t > 0$ or, with $t = 1/r$,

$$\beta_0(r, m, M) = r\beta_0(1, m, rM). \tag{2.13}$$

For $M = 0$ estimates for β_0 were computed in [AC, 2.6]. One has that $\beta_0(1, m, 0) > 0$ for $m > 0$ and is an increasing function of m. Moreover, $\beta_0(1, m, M)$ is decreasing as a function of M and can be taken to depend continuously on (m, M) in a neighbourhood of $M = 0$.

It follows from (2.11) and (2.13) that we can achieve

$$u \leq \beta \leq \beta_0(r, m, M) \tag{2.14}$$

on $\partial B_{r/2}$ by letting

$$C_1 \frac{1}{r} \fint_{\partial B_r} u + C_2 r M < \beta_0(1, m, rM).$$

Since $\beta_0(1, m, 0) > 0$ this shows that (2.14) holds if an estimate of the form (2.10) holds.

Now it only remains to show that (2.14) implies that $u = 0$ in $B_{r/4}$. Let w denote the function which equals $\min(u, v)$ in $B_{r/2}$ and equals u outside $B_{r/2}$. When (2.14) holds then $w \in \mathbb{K}$, and if $r > 0$ is small enough then w will be so close to u in the metric (2.1) that $J_r(u) \leq J_r(w)$, u being a local minimizer of J. We also have $\tilde{J}_r(v) \leq \tilde{J}_r(\max(u, v))$. But these two inequalities contradict (2.12) unless we have equality everywhere. Since v was the largest minimizer of $\tilde{J}_r$ it follows that $v = \max(u, v)$, i.e. that $u \leq v$. Hence u vanishes in $B_{r/4}$. □

Corollary 2.9. *If $g \geq$ const. > 0 in a neighbourhood of a point $x_0 \in \partial\Omega$ then*

$$u(x) \geq C\delta(x)$$

near x_0.

Proof. With C_1 the constant in (2.10) we have by a) of Lemma 2.4 and (2.10) (for $x \in \Omega$ close to x_0, $r = \delta(x)$, $B_r = B_r(x, r)$),

$$u(x) \geq \fint_{\partial B_r} u - C_2 r^2 \geq C_1 r - C_2 r^2 \geq Cr. \quad \square$$

Lemma 2.10. *Let u be a local minimum, $\Omega = \{u > 0\}$ and assume that $g \geq$ const. > 0 in a neighbourhood of a point $x_0 \in \partial\Omega$. Then there are constants c_1 and c_2 such that*

$$0 < c_1 \leq \frac{|B_r \cap \Omega|}{|B_r|} \leq c_2 < 1$$

for small $r > 0$ $(B_r = B(x_0, r))$.

For a proof see [AC, 2.7].

Remark 2.11. In addition to Lemma 2.8 the following lemma, due to Caffarelli, is useful:
Assume $0 \le u \in H^1(B(0,R))$, $\Delta u \ge c > 0$ in $\Omega = \{u > 0\}$, $0 \in \overline{\Omega}$. Then, for any $0 < r < R$

$$\sup_{\partial B_r(0)} u \ge \frac{cr^2}{2N}.$$

(See [Caff1] for the simple proof). If u is a local minimum (or weak solution) for our problem, then this lemma shows that

$$\sup_{\partial B_r(x)} u \ge \frac{r^2}{2N} \inf_{B_r(x)} f_- \tag{2.15}$$

for any $x \in \overline{\Omega}$.

Proposition 2.12. *Any local minimizer u of J has compact support.*

Proof. By Lemma 2.2 u is subharmonic outside supp f_+. Therefore

$$u(x) \le \frac{1}{|B_r(x)|} \int_{B_r(x)} u \le \frac{1}{\sqrt{|B_r(x)|}} \left(\int_{B_r(x)} u^2 \right)^{1/2} \le \frac{\|u\|}{\sqrt{|B_r|}}$$

for x a distance r away from supp f_+. Thus

$$u(x) \le C|x|^{-N/2} \tag{2.16}$$

for $|x|$ large.

By Condition A either $g \ge c > 0$ or $f_- \ge c > 0$ (or both) far away. In the first case we conclude that $u(x) = 0$ for large $|x|$ by combining (2.16) with Lemma 2.8. In the second case the same conclusion follows from (2.16) combined with (2.15). □

In order to prepare for the main result in this section we need to recall a few facts about functions of bounded variation and sets of finite perimeter.[Gi],[EG, ch. 5] are good references for this.

Let $E \subset \mathbb{R}^N$ be a Lebesgue measurable set. The *measure theoretic boundary* $\partial_{mes}E$ of E is defined to be the set of points $x \in \mathbb{R}^N$ such that E and E^c have positive density at x. Thus $\partial_{mes}E \subset \partial E$ (the topological boundary). E is said to have *locally finite perimeter* if $\nabla\chi_E$ is a vector-valued Radon measure. This means that there exists a positive Radon measure $\mu = \mu_E$ in $\mathbb{R}^N$ and a μ-measurable function $\nu_E : \mathbb{R}^N \to S^{N-1} \cup \{0\}$ (the direction factor) such that $-\nabla\chi_E = \mu\lfloor\nu_E$, i.e. $\int_E \text{div}\phi dx = \int \phi \cdot \nu_E d\mu$ for all $\phi \in C_c^1(\mathbb{R}^N, \mathbb{R}^N)$ (the left member being equal to $\langle -\nabla\chi_E, \phi\rangle$). The measure μ will occasionally be denoted $|\nabla\chi_E|$. It can be shown [EG, 5.11] that a measurable set E has locally finite perimeter if and only if $\mathcal{H}^{N-1}(K \cap \partial_{mes}E) < \infty$, for each compact set $K \subset \mathbb{R}^N$.

Assuming that E has locally finite perimeter the *reduced boundary* $\partial_{red}E$ of E can be defined as the set of points $x \in \mathbb{R}^N$ for which the density $\lim_{r\to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} \nu_E d\mu$ exists and has modulus one. It is convinient to work with that representative of ν_E which equals this limit on $\partial_{red}E$ and is zero elsewhere and ν_E then is the measure theoretic outward unit normal vector of E on $\partial_{red}E$. Clearly $\partial_{red}E \subset \partial_{mes}E$ and it is not hard to show that $\mathcal{H}^{N-1}(\partial_{mes}E \setminus \partial_{red}E) = 0$, ([EG, ch. 5.8]). A basic structure theorem says that $\mu = |\nabla\chi_E|$ actually agrees with $(N-1)$-dimensional Hausdorff measure restricted to $\partial_{red}E$: $|\nabla\chi_E| = \mathcal{H}^{N-1}\lfloor\partial_{red}E$. Thus also $\nabla\chi_E = -\nu_E\mathcal{H}^{N-1}\lfloor\partial_{red}E$.

All the above definitions and results carry over to the case with an open subset $G \subset \mathbb{R}^N$ in place of $\mathbb{R}^N$. One then speaks of sets having locally finite perimeter in G etc.

Theorem 2.13. *Assume that f, g satisfy Condition A, that g is continuous and set $G = \{x \in \mathbb{R}^N : g(x) > 0\}$. If $\partial G \ne \emptyset$ assume moreover that for some $0 < \alpha \le 1$ g is α-Hölder continuous near ∂G and that $\mathcal{H}^{N-1+\alpha}(\partial G) = 0$. Then, if u is a local minimizer of $J = J_{f,g}$, then $\Omega = \{u > 0\}$ has locally finite perimeter in G,*

$$\mathcal{H}^{N-1}((\partial\Omega \setminus \partial_{red}\Omega) \cap G) = 0,$$

and

$$\Delta u + f\mathcal{L}^N\lfloor\Omega = g\mathcal{H}^{N-1}\lfloor\partial\Omega = g\mathcal{H}^{N-1}\lfloor\partial_{red}\Omega = g|\nabla\chi_\Omega|. \tag{2.17}$$

Here the right members shall be interpreted as zero outside G.

Remark 2.14. Ω need not have locally finite perimeter outside G. To see this, take e.g. $g = 0$, $f = a\chi_D - 1$ where D is a bounded domain such that ∂D has positive N-dimensional Lebesgue measure $|\partial D|$, and $1 < a < 1+\frac{|\partial D|}{|D|}$ is a parameter. By (2.4), $\Delta u \leq 1-a\chi_D < 0$ in D showing that $D \subset \Omega$. Also, $\overline{D} \subset \overline{\Omega}$. Next (2.17) yields $\Delta u = (1-a\chi_D)\chi_\Omega = \chi_\Omega - a\chi_D$, by which $|\Omega| = a|D|$. Thus

$$|\Omega| < |D| + |\partial D| = |\overline{D}| \leq |\overline{\Omega}| = |\Omega| + |\partial\Omega|,$$

i.e. $\partial\Omega$ has even positive N-dimensional Lebesgue measure.

Corollary 2.15. *(to Lemma 2.10) With assumptions as in Theorem 2.13*

$$\partial\Omega \cap G = \partial_{mes}\Omega \cap G.$$

Remark. $\partial_{mes}\Omega$ may be strictly smaller than $\partial\Omega$ outside G. Indeed, in the case $g \equiv 0$ there are examples with $\partial\Omega$ having singular points (e.g. inward cusps and double points, when $N = 2$) at which Ω has density one.

For the proof of Theorem 2.13 we need the following observation.

Lemma 2.16. *Assume $u \geq 0$ is a continuous function such that Δu is a signed Radon measure. Then $\Delta u \geq 0$ on $\{u = 0\}$.*

The proof of Lemma 2.16 is quite straightforward and therefore omitted (cf. [AC, 4.2]).

Proof. (of Theorem 2.13) (2.2) shows that Δu is a Radon measure and (2.3) and Lemma 2.16 then show that $\Delta u + f\chi_\Omega = \lambda$, where λ is a positive Radon measure on $\partial\Omega$.

For any $x \in \mathbb{R}^N$, $|\nabla u|$ is integrable on $\partial B_r(x)$ for almost every $r > 0$ and for these r

$$|\int_{B_r(x)} \Delta u| \leq \int_{\partial B_r(x)} |\nabla u| d\mathcal{H}^{N-1} \leq Cr^{N-1} \sup_{\partial B_r(x)} |\nabla u|. \tag{2.18}$$

But $|\nabla u| \leq C$ by (2.9), hence (2.18) shows that Δu, and also λ, is absolutely continuous with respect to $\mathcal{H}^{N-1}$.

If $x \in \partial\Omega \setminus G$ then (2.9), (2.18) even yield that $|\int_{B_r(x)} \Delta u| \leq Cr^{N-1+\alpha}$ for $r > 0$ small, hence that Δu and λ are absolutely continuous with respect to $\mathcal{H}^{N-1+\alpha}$ on $\partial\Omega \setminus G$. From this it follows that $\lambda = 0$ on $\partial\Omega \setminus G$. Indeed, on $\partial\Omega \cap \partial G$ we have $\lambda = 0$ since by assumption $\mathcal{H}^{N-1+\alpha}(\partial G) = 0$. Outside $\overline{G}$ we have $\Delta u \in L^\infty$ by Lemma 2.2, and then standard arguments [Ki-St, II, Lemma A.4] show that $\lambda = \Delta u = 0$ a.e. on $\partial\Omega \setminus \overline{G}$.

By the above we see that $\Delta u + f\chi_\Omega = h\mathcal{H}^{N-1}\lfloor\partial\Omega$ for some Borel function $h \geq 0$ on $\partial\Omega \cap G$. It just remains to identify h with g i.e. to prove that

$$h(x) = g(x) \qquad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in \partial\Omega \cap G. \tag{2.19}$$

We shall merely give an outline of the proof of (2.19). The details are virtually the same as in [AC, 4.7–5.5].

It is enough to prove (2.19) for those $x \in \partial\Omega \cap G$ which satisfy $x \in \partial_{red}\Omega$,

$$\overline{\lim_{r\to 0}} \frac{\mathcal{H}^{N-1}(B(x,r) \cap \partial\Omega)}{\omega_{N-1}r^{N-1}} \leq 1, \tag{2.20}$$

$$\overline{\lim_{r\to 0}} \fint_{\partial\Omega\cap B(x,r)} |h - h(x)| d\mathcal{H}^{N-1} = 0 \tag{2.21}$$

since the remaining set has $\mathcal{H}^{N-1}$ measure zero (see [EG, Theorem 2, 2.3] for (2.20)). So fix such an $x \in \partial\Omega\cap G$. For simplicity of notation we assume that $x = 0$ and that $\nu_\Omega(0) = e_N = (0, ..., 0, 1)$.

Define the blow-up sequences $u_n(x) = nu(\frac{x}{n})$, $f_n(x) = \frac{1}{n}f(\frac{x}{n})$, $g_n(x) = g(\frac{x}{n})$, $h_n(x) = h(\frac{x}{n})$, $\Omega_n = \{u_n > 0\}$. Note that u, f and g are scaled in the right way according to Remark 2.7 (with $\alpha = -1$). Let $B = B(0,1)$, $B_r = B(0,r)$, $H = \{x_N < 0\}$.

By general properties of the reduced boundary [EG, 5.7.2]

$$|(\Omega_n \triangle H) \cap B| \to 0, \tag{2.22}$$

where $\triangle$ means the symmetric difference between the sets, and by our assumptions $f_n \to 0$ uniformly, and

$$\int_B |g(\frac{x}{n}) - g(0)|dx \to 0, \qquad \int_{\partial\Omega_n \cap B} |h(\frac{x}{n}) - h(0)|d\mathcal{H}^{N-1} \to 0$$

as $n \to \infty$.

As to u_n we know (Corollary 2.6 and 2.9) that $|\nabla u| \le C$ and $u(x) \ge C\delta(x)$. Thus $|\nabla u_n| \le C$, $u_n(x) \ge C\delta_n(x)$ where $\delta_n(x) = \mathrm{dist}(x, \Omega_n^c) = n\delta(x/n)$. It follows that there exists a Lipschitz continuous limit function $u_0 \ge 0$ such that, for a subsequence,

$$\begin{aligned} u_n \to u_0 &\qquad \text{uniformly in } B, \\ \nabla u_n \to \nabla u_0 &\qquad w^* - L^\infty(B). \end{aligned}$$

Setting $\Omega_0 = \{u_0 > 0\}$ it also follows, using nondegeneracy ($u_n \ge C\delta_n$) and that $|\Delta u_n| = |f_n| \le C/n$ in Ω_n, that u_0 is harmonic in Ω_0, that $\Omega_n \cap B \to \Omega_0 \cap B$ in Hausdorff distance and in measure. By the last property combined with (2.22) $|\Omega_0 \triangle H| = 0$ and hence (since Ω_0 is open) $\Omega_0 \subset H$, $|H \setminus \Omega_0| = 0$.

Next one proves, and this is more technical [AC, 4.8], that actually $\Omega_0 = H$ (for this (2.20) has to be used) and that, due to (2.21),

$$u_0(x) = h(0)(-x_N)_+. \tag{2.23}$$

The final step consists of proving that u_0 is (global) minimum of

$$J_0(v) = \int_B (|\nabla v|^2 + g(0)^2 \chi_{\{v>0\}})$$

among all $0 \le v \in H^1(B)$ with $v = u_0$ on ∂B ([AC, 5.4]). This is intuitively reasonable since by scaling (Remark 2.7) u_n is seen to be a minimum of

$$J_n(v) = \int_B (|\nabla v|^2 - 2f_n v + g_n^2 \chi_{\{v>0\}})$$

(among $0 \le v \in H^1(B)$ with $v = u_n$ on ∂B) if n is large (recall also that $f_n \to 0$, $g_n \to g(0)$).

Now it follows from Theorem 2.3 (adapted to the unit ball B) that the function (2.23) can be a minimizer of J_0 only if $h(0) = g(0)$. This was the desired conclusion and the proof is finished. □

As to regularity of the free boundary $\partial\Omega$ we have

Theorem 2.17. *[AC], [Caff80] Assume that f and g satisfy Condition A and that u is a local minimum of J. Let $B_r = B_r(x_0)$ be a small ball.*

a) If g is Hölder continuous and satisfies $g \ge$ const. > 0 in B_r then for some $\alpha > 0$ $\partial_{red}\Omega$ is a $C^{1,\alpha}$ surface locally in B_r. If $N = 2$ then this even holds for $\partial\Omega$ (i.e. $\partial_{red}\Omega = \partial\Omega$ in B_r).

b) If $g = 0$, f is Hölder continuous and < 0 in B_r and if moreover Ω^c satisfies the minimal thickness condition of Caffarelli at x_0 (see [Caff80], [F]) then $\partial\Omega$ is a C^1 surface near x_0. This thickness condition is satisfied e.g. if Ω^c contains a nondegenerate cone in B_r with vertex at x_0.

c) If $g = 0$ and $f \ge 0$ in B_r then $\partial\Omega \cap B_r = \emptyset$.

Proof. a) is proved in [AC, 6–8] in the case $f = 0$. When $f \ne 0$ basically the same proof works. The modifications needed are listed in Appendix, section 5.

b) is proved in [Caff80].

As to c), (2.4) shows that $\Delta u \le 0$ in B_r, hence either $u > 0$ in B_r or $u \equiv 0$ in B_r. □

Note. This theorem covers all cases except some limiting ones. However, in these limiting cases not much can be said in general. See e.g. Remark 2.14, where $g = 0$ and for any $x_0 \in \partial D \cap \partial\Omega$ f takes both positive and negative values in every neighbourhood of x_0. We may even redefine g to be any positive function outside Ω (e.g. $g(x) = \mathrm{dist}(x, \Omega)$, which is Lipschitz continuous) and we will still have the same irregular solution. See Remark 3.6.

As to higher regularity we just mention that if f and g are real analytic in B_r then, in Theorem 2.17, the conclusions $C^{1,\alpha}$ (in a)) and C^1 (in b)) can both be replaced by "real analytic" (see again [AC], [Caff80]). If f and g are real analytic and moreover $N = 2$ the regularity theory seems infact to be almost complete: If $g > 0$ in B_r then $\partial\Omega$ is real analytic by the above and if $g = 0$ and $f < 0$ in B_r then it is shown in [Sak91], [Sak94], that $\partial\Omega$ is analytic in B_r except possibly for a few types of singular points which may occur [Scha],[F]. These are certain types of inwards cusps, double points (including the case of a real analytic arc with Ω on both sides) and isolated points of $\partial\Omega$.

3. Geometry of local minima and weak solutions

In this section we derive some results on the geometry of $\Omega = \{u > 0\}$ when u is a local minimum. In some cases we really do not need the full strenght of u being a local minimum, just that u satisfies equation (2.17) in Theorem 2.13. We call such a function a weak solution. Our notion of weak solution is weaker than that of [AC].

Defintion 3.1. *Assume that f and g satisfy Condition A and that g moreover is continuous. Then by a weak solution for $J_{f,g}$ we mean a continuous function $u \geq 0$ with compact support satisfying*

$$\Delta u + f\mathcal{L}^N \lfloor \Omega = g\mathcal{H}^{N-1} \lfloor \partial\Omega \tag{3.1}$$

where $\Omega = \{u > 0\}$.

Remark 3.2.

a) $u = 0$ is always a weak solution.

b) A priori, $g\mathcal{H}^{N-1}\lfloor\partial\Omega$ is a (positive) Borel measure whereas the left member of (3.1) is a distribution. The equation (3.1) is to be interpreted as saying, first of all, that $g\mathcal{H}^{N-1}\lfloor\partial\Omega$ also is a distribution, hence a Radon measure, and, secondly, that equality holds in the sense of distributions. Thus it is a consequence of (3.1) that Δu is a (signed) Radon measure and also that Ω has locally finite perimeter in $G = \{g > 0\}$.

c) It follows, as in [AC, 4.2], that any weak solution u is in $\mathbb{K}$. Indeed, this readily follows from the estimate

$$\int_{u>\epsilon} |\nabla u|^2 = \int \nabla u \cdot \nabla(u-\epsilon)_+ = -\int \Delta u(u-\epsilon)_+ =$$
$$\int f(u-\epsilon)_+ = \int fu < \infty.$$

Also, by Theorem 2.13, if g satisfies the Hölder condition there (or $g > 0$ everywhere), any local minimum is a weak solution.

d) If some portion $\Gamma \subset G$ of $\partial\Omega$ bounds Ω from two sides (which is impossible for local minima by Lemma 2.10) then (3.1) is perhaps not the most natural definition: either u should be forced to have the normal derivative g in both directions from Γ (which would give $2g\mathcal{H}^{N-1}\lfloor\partial\Omega$ on Γ in (3.1)) or Γ should be neglected, which is accomplished by replacing the right member of (3.1) by $g\mathcal{H}^{N-1}\lfloor\partial_{mes}\Omega$ or $g\mathcal{H}^{N-1}\lfloor\partial_{red}\Omega$. However, for simplicity we shall stick to (3.1).

We begin with some miscellaneous comparison results for minima and local minima.

Proposition 3.3. *(Cf. [F-P]) Assume $f_1 \leq f_2$, $g_1 \geq g_2$, let $u_j \in \mathbb{K}$ be a minimizer of $J_j = J_{f_j,g_j}$ and let $\Omega_j = \{u_j > 0\}$ $(j = 1, 2)$.*

a) In each component D_1 of Ω_1 one of the following holds:

(1) $u_1 < u_2$ *in* D_1*;*
(2) $u_1 = u_2$ *and* $f_1 \equiv f_2$ *in* D_1*;*
(3) $u_1 > u_2$ *and* $f_1 \equiv f_2$ *in* D_1*.*

b) If $f_1 \le 0$ in $\Omega_1 \cap \Omega_2$ then $J_1 \ge 0$ and $\Omega_1 \cap \Omega_2 = \emptyset$.
c) If $f_1 \le 0$ in a component D_1 of Ω_1 then $D_1 \cap \Omega_2 = \emptyset$.
d) If $f_1 \le 0$ in a component D_2 of Ω_2 then $\Omega_1 \cap D_2 = \emptyset$.

Proof. Let $v = \min(u_1, u_2)$ and $w = \max(u_1, u_2)$. Then, by Lemma 1.1, v minimizes J_1 and w minimizes J_2.

Next, to prove a), $\Delta w = -f_2$ in $\{w > 0\} = \Omega_1 \cup \Omega_2$ by Lemma 2.2. Thus $\Delta(w - u_1) = -f_2 + f_1 \le 0$ in Ω_1 and $\Delta(w - u_2) = -f_2 + f_2 = 0$ in Ω_2. Moreover $w - u_j \ge 0$ $(j = 1, 2)$.

Let D_1 be a connected component of Ω_1. By the maximum principle, either $w - u_1 \equiv 0$ in D_1, in which case $f_1 = f_2$ there, or $w - u_1 > 0$ in D_1. In the latter case $u_1 < u_2$ in D_1, which is case (1) in the proposition. In the first case $u_1 \ge u_2$ in D_1, and it just remains to prove that either $u_1 > u_2$ or $u_1 \equiv u_2$ holds in all D_1.

Since $u_1 - u_2 = w - u_2 \ge 0$ is harmonic in Ω_2 we have, in each component of $D_1 \cap \Omega_2$, either $u_1 > u_2$ or $u_1 \equiv u_2$. Assume that $u_1 = u_2$ holds in one component D of $D_1 \cap \Omega_2$. Then $u_1 = u_2$ also on ∂D. It follows that $\partial D \cap D_1 = \emptyset$ (because if $x \in \partial D \cap D_1$ then $u_2(x) = u_1(x) > 0$ so that $x \in D_1 \cap \Omega_2$, contradicting $x \in \partial D$). Since $D \subset D_1$ and D_1 is connected this shows that $D = D_1$ i.e. $u_1 = u_2$ in all D_1 (case (2)).

If $u_1 = u_2$ on no component of $D_1 \cap \Omega_2$ then $u_1 > u_2$ in each component, and since trivially $u_1 > u_2$ in $D_1 \setminus \Omega_2$ we get $u_1 > u_2$ in all D_1 (case (3)). This completes the proof of a).

Since v minimizes J_1 we have $\Delta v + (f_1)_+ \ge 0$ in $\mathbb{R}^N$ by (2.2). Thus if D is an open set such that $f_1 \le 0$ in D and $v = 0$ on ∂D then $v = 0$ in D by the maximum principle.

In b) the above is assumed to hold for $D = \Omega_1 \cap \Omega_2 = \{v > 0\}$. Thus $v = 0$ in D, hence $v = 0$ everywhere, $\Omega_1 \cap \Omega_2 = \emptyset$ and $J_1 \ge J_1(v) = 0$.

In c) we choose $D = D_1$ ($v = 0$ on ∂D_1 since $u_1 = 0$ on $\partial D_1 \subset \partial \Omega_1$) and we conclude that $v = 0$ in D_1, i.e. $D_1 \cap \Omega_2 = \emptyset$. d) is proved similarly. □

Corollary 3.4. *Let $u \in \mathbb{K}$ be a minimizer of $J_{f,g}$ and let $f_1 = (1 - \chi_\Omega) f_+ - f_-$ where $\Omega = \{u > 0\}$. Then $J_{f_1,g} \ge 0$ and $\Omega_1 \cap \Omega = \emptyset$ where $\Omega_1 = \{u_1 > 0\}$ for any minimizer u_1 of $J_{f_1,g}$ ($u_1 = 0$ is a minimizer).*

Proof. Apply b) of Proposition 3.3 with $f_2 = f$ and f_1 as in the statement. Note that $f_1 \le 0$ in Ω $(\Omega = \Omega_2)$. □

Corollary 3.5. *Let $u \in \mathbb{K}$ be a minimzer of $J_{f,g}$, let Ω_1 be a component of $\Omega = \{u > 0\}$ and set $f_1 = \chi_{\Omega_1} f_+ - f_-$, $u_1 = u\chi_{\Omega_1}$, $\Omega_2 = \Omega \setminus \Omega_1$. Then u_1 minimizes $J_1 = J_{f_1,g}$ and for any minimizer v of J_1 we have $\{v > 0\} \cap \Omega_2 = \emptyset$.*

Proof. Apply d) of Proposition 3.3 with $f_2 = f$ and f_1 as in the statement (of the corollary). Note that Ω_2 is a union of components of Ω and that $f_1 \le 0$ in Ω_2. It follows that $\{v > 0\} \cap \Omega_2 = \emptyset$ for any minimizer v of J_1, and hence also that u_1 minimizes J_1 (for otherwise J_1 could be made smaler by changing u in Ω_1 to a minimizer of J_1). □

Note that if $f \ge 0$ then Corollary 3.4 roughly says that if minimization of J does not produce a domain Ω covering f then another minimization, for the uncovered part, does not help. Similarly, Corollary 3.5 says (when $f \ge 0$) that if Ω turns out to be disconnected, then separate minimizations for the parts of f in each component of Ω always produces domains which do not meet each other.

Remark 3.6. Assume that $u \in \mathbb{K}$ is a (local) minimizer of $J = J_{f,g}$ and let $\tilde{f} = f$, $\tilde{g} \le g$ in $\Omega = \{u > 0\}$, $\tilde{f} \le f$, $\tilde{g} \ge g$ outside Ω. Then u is a (local) minimum also for $\tilde{J} = J_{\tilde{f},\tilde{g}}$. Indeed, one immediately finds that $\tilde{J}(u) - J(u) \le \tilde{J}(v) - J(v)$, and hence $\tilde{J}(u) \le \tilde{J}(v)$, for every $v \in \mathbb{K}$ (close to u).

One conclusion from this observation is that if g is not continuous then a local minimizer u cannot be expected to satisfy the equation (3.1) for a weak solution (because g can be replaced by any larger function on $\partial\Omega$ (or on $\mathbb{R}^N \setminus \Omega$) and u will still be a local minimizer). Cf. [AC, 5.9].

Theorem 3.7. *Assume that $u_j \ge 0$ ($j = 1, 2$) are weak solutions for $J_{f,g}$, $\Omega_j = \{u_j > 0\}$ and that $f \le 0$ outside Ω_2 (or simply that $\int_{\Omega_1 \setminus \Omega_2} f \le 0$). Then*

$$\int_{\partial(\Omega_1 \cup \Omega_2)} g d\mathcal{H}^{N-1} \le \int_{\partial \Omega_2} g d\mathcal{H}^{N-1}.$$

Proof. We have

$$\partial(\Omega_1 \cup \Omega_2) = (\partial\Omega_1 \setminus \Omega_2) \cup (\partial\Omega_2 \setminus \overline{\Omega}_1), \qquad \partial\Omega_2 = (\partial\Omega_2 \cap \overline{\Omega}_1) \cup (\partial\Omega_2 \setminus \overline{\Omega}_1),$$

where the unions are disjoint. Thus it is enough (and necessary) to prove that

$$\int_{\partial\Omega_1\setminus\Omega_2} g d\mathcal{H}^{N-1} \le \int_{\partial\Omega_2\cap\overline{\Omega}_1} g d\mathcal{H}^{N-1}.$$

Set $u = \inf(u_1, u_2)$. Then $u \ge 0$, u is continuous and $\Omega_1 \cap \Omega_2 = \{u > 0\}$. Since $u_j \in H^1(\mathbb{R}^N)$ (Remark 3.2) also $u \in H^1(\mathbb{R}^N)$. Since $-\Delta u_j = f$ in Ω_j we have

$$-\Delta u \ge f \qquad \text{in } \Omega_1 \cap \Omega_2. \tag{3.9}$$

In particular, Δu is a Radon measure in $\Omega_1 \cap \Omega_2$.

Now we claim that Δu actually is a Radon measure in all $\mathbb{R}^N$. It is not hard to show (cf. [AC, 4.2]) that this is the case if and only if Δu has finite total mass in $\Omega_1 \cap \Omega_2$, i.e.

$$-\int_{u>\epsilon} \Delta u \le C < \infty, \tag{3.10}$$

for all $\epsilon > 0$. Here the left member can also be written

$$-\int_{u>\epsilon} \Delta(u-\epsilon)_+ = \int_{u\le\epsilon} \Delta(u-\epsilon)_+.$$

Since $\Delta(u-\epsilon)_+$ is a Radon measure with compact support in $\Omega_1 \cap \Omega_2$ Lemma 2.16 can be applied to $(u-\epsilon)_+$ and $(u_j-\epsilon)_+ - (u-\epsilon)_+$ showing that

$$0 \le \Delta(u-\epsilon)_+ \le \Delta(u_j-\epsilon)_+ \qquad \text{in } \{u_j \le \epsilon\}.$$

From this (3.10) easily follows using the fact that Δu_j have finite total masses.

Next we apply Lemma 2.16 to u, $u_1 - u$ and $u_2 - u$. This gives

$$0 \le \Delta u \le \Delta u_1 \quad \text{on } \partial\Omega_1, \qquad \text{and} \qquad 0 \le \Delta u \le \Delta u_2 \quad \text{on } \partial\Omega_2.$$

Combining with (3.9) we obtain

$$\int_{\Omega_1\cap\Omega_2} f \le -\int_{\Omega_1\cap\Omega_2} \Delta u = \int_{(\Omega_1\cap\Omega_2)^c} \Delta u = \int_{\partial(\Omega_1\cap\Omega_2)} \Delta u \le$$
$$\int_{\partial\Omega_1\cap\Omega_2} \Delta u + \int_{\partial\Omega_2\cap\overline{\Omega}_1} \Delta u \le \int_{\partial\Omega_1\cap\Omega_2} \Delta u_1 + \int_{\partial\Omega_2\cap\overline{\Omega}_1} \Delta u_2$$

and hence

$$\int_{\partial\Omega_2\cap\overline{\Omega}_1} g d\mathcal{H}^{N-1} = \int_{\partial\Omega_2\cap\overline{\Omega}_1} \Delta u_2 \ge \int_{\Omega_1\cap\Omega_2} f - \int_{\partial\Omega_1\cap\Omega_2} \Delta u_1 =$$
$$\int_{\Omega_1\cap\Omega_2} f + \int_{\partial\Omega_1\setminus\Omega_2} \Delta u_1 + \int_{\Omega_1} \Delta u_1 = \int_{\Omega_1\cap\Omega_2} f + \int_{\partial\Omega_1\setminus\Omega_2} g d\mathcal{H}^{N-1} -$$
$$\int_{\Omega_1} f = -\int_{\Omega_1\setminus\Omega_2} f + \int_{\partial\Omega_1\setminus\Omega_2} g d\mathcal{H}^{N-1} \ge \int_{\partial\Omega_1\setminus\Omega_2} g d\mathcal{H}^{N-1}$$

as required. □

Corollary 3.8. *Assume $u_j \ge 0$ $(j = 1, 2)$ are weak solutions with $g =$ const. > 0, that $f \le 0$ outside Ω_2 and that Ω_2 is convex. Then $\Omega_1 \subset \Omega_2$ $(\Omega_j = \{u_j > 0\})$.*

Proof. Let $P : \mathbb{R}^N \to \overline{\Omega}_2$ be the projection, taking $x \in \mathbb{R}^N$ onto the closest point $P(x)$ on the compact convex set $\overline{\Omega}_2$. Then

$$|P(x) - P(y)| \le |x - y|, \tag{3.11}$$
$$P(\partial(\Omega_1 \cup \Omega_2)) = \partial\Omega_2, \tag{3.12}$$

as is easily seen. But (3.11) implies [E-G,Theorem 1, p. 75] that P shrinks Hausdorff measure, in particular

$$\mathcal{H}^{N-1}(P(\partial(\Omega_1 \cup \Omega_2))) \le \mathcal{H}^{N-1}(\partial(\Omega_1 \cup \Omega_2)).$$

Thus, by (3.12)

$$\mathcal{H}^{N-1}(\partial\Omega_2) \le \mathcal{H}^{N-1}(\partial(\Omega_1 \cup \Omega_2)). \tag{3.13}$$

If $\Omega_1 \not\subset \Omega_2$ then $\Omega_1 \setminus \overline{\Omega}_2 \ne \emptyset$ (since Ω_2 is convex), and it is easy to see that the inequality (3.13) must be strict in this case. But this contradicts Theorem 3.7. Thus $\Omega_1 \subset \Omega_2$. □

Corollary 3.8 partly generalizes [Shah1, Theorem 2.6], where the same conclusion was obtained assuming some regularity of $\partial\Omega$ but without any positivity assumption on u. Other results related to convexity can be found in [Beur], [Acker81] and [Kaw].

Next we shall use some reflection methods to obtain a result on monotonicity or convexity along lines. The method is related to the "moving plane method" which has previously been used in similar problems in [Serrin], [GNN], [B-N], [Shah94b], [Gu-Sa]. Important points in our approach are that we do not require any regularity of the solutions u and that we are able to work with local minima (not only global minima).

For a fixed unit vector $a \in \mathbb{R}^N$ and for $\lambda \in \mathbb{R}$ set

$$T_\lambda = T_{a,\lambda} := \{x \cdot a = \lambda\}, \qquad T_\lambda^- := \{x \cdot a < \lambda\}, \qquad T_\lambda^+ := \{x \cdot a > \lambda\}.$$

For $x \in \mathbb{R}^N$ let x^λ denote the reflected point with respect to T_λ and for φ a function set $\varphi^\lambda(x) = \varphi(x^\lambda)$. If $\Omega \subset \mathbb{R}^N$ we define

$$\begin{aligned} \Omega_\lambda &= \Omega \cap T_\lambda^+ = \text{ the cap cut off by } T_\lambda, \\ \tilde{\Omega}_\lambda &= \{x^\lambda : x \in \Omega_\lambda\} = \text{ the reflection of } \Omega_\lambda \text{ in } T_\lambda. \end{aligned}$$

Theorem 3.9. *Assume that f and g satisfy Condition A and moreover that for some unit vector $a \in \mathbb{R}^N$ and some $\lambda_0 \in \mathbb{R}^N$ we have*

$$f \le f^\lambda, \qquad g \ge g^\lambda \quad \text{in } T_\lambda^+ \tag{3.14}$$

for all $\lambda \ge \lambda_0$. Then for any local minimum u of J the following hold.

$$u < u^\lambda \qquad \text{in } \Omega_\lambda \qquad \text{for all } \lambda > \lambda_0, \tag{3.15}$$

$$\tilde{\Omega}_\lambda \subset \Omega \qquad \text{for all } \lambda \ge \lambda_0, \tag{3.16}$$

$$a \cdot \nabla u < 0 \qquad \text{in } \Omega_{\lambda_0}. \tag{3.17}$$

Note. (3.14) holding for all $\lambda \ge \lambda_0$ is equivalent to that

$$\begin{aligned} f &\le f^{\lambda_0}, & a \cdot \nabla f &\le 0, \\ g &\ge g^{\lambda_0}, & a \cdot \nabla g &\ge 0 \end{aligned}$$

hold on $T_{\lambda_0}^+$ (in the sense of distributions).

Proof. Define

$$v^\lambda = \begin{cases} \min(u, u^\lambda) & \text{in } T_\lambda^+, \\ \max(u, u^\lambda) & \text{in } T_\lambda^-, \end{cases}$$

$$I(\varphi) = \int_{T_\lambda^+} (|\nabla\varphi|^2 - 2f\varphi + g^2\chi_{\{\varphi>0\}})dx,$$

$$I_\lambda(\varphi) = \int_{T_\lambda^+} (|\nabla\varphi|^2 - 2f^\lambda\varphi + (g^\lambda)^2\chi_{\{\varphi>0\}})dx.$$

Then Lemma 1.1 (with $\mathbb{R}^N$ replaced by T_λ^+) shows that

$$J(v^\lambda) = I(\min(u, u^\lambda)) + I_\lambda(\max(u, u^\lambda)) \leq I(u) + I_\lambda(u^\lambda) = J(u) \tag{3.18}$$

for all $\lambda \geq \lambda_0$.

On the other hand $J(v^\lambda) \geq J(u)$ whenever v^λ is close enough to u in the metric (2.1), since u is a local minimum. Thus

$$J(v^\lambda) = J(u) \tag{3.19}$$

for all values of $\lambda \geq \lambda_0$ such that v^λ is close to u.

Now for $\lambda \geq \lambda_0$ so large that $\Omega \subset T_\lambda^-$ we have

$$v^\lambda = u, \tag{3.20}$$

i.e. $u \leq u^\lambda$ in T_λ^+. Note that (3.20) implies

$$\tilde{\Omega}_\lambda \subset \Omega. \tag{3.21}$$

We shall prove that (3.20) holds for all $\lambda \geq \lambda_0$. For this it is enough to prove that if for some $\lambda_1 > \lambda_0$ (3.20) holds for all $\lambda \geq \lambda_1$ then it also hold for all λ in a full neighbourhood of λ_1. Note that the set of values of λ for which (3.20) holds is a closed set.

By Lemma 1.1

$$J(\min(u, v^\lambda)) + J(\max(u, v^\lambda)) \leq J(u) + J(v^\lambda)$$

and if v^λ is close to u then also $\min(u, v^\lambda)$ and $\max(u, v^\lambda)$ are close to u. Thus by (3.19) also $\min(u, v^\lambda)$ and $\max(u, v^\lambda)$ are local minima when v^λ is close to u. In particular, by Lemma 2.2

$$-\Delta \max(u, v^\lambda) = f \qquad \text{in } \Omega \tag{3.22}$$

(note that $\max(u, v^\lambda) > 0$ in Ω).

Set

$$\varphi = \max(u, v^\lambda) - u = \begin{cases} 0 & \text{in } T_\lambda^+ \cup T_\lambda \\ (u^\lambda - u)_+ & \text{in } T_\lambda^-. \end{cases}$$

Then (3.20) is equivalent to $\varphi = 0$ in $\mathbb{R}^N$. Clearly we have

$$\varphi = 0 \qquad \text{in } \mathbb{R}^N \setminus \tilde{\Omega}_\lambda \tag{3.23}$$

and, when (3.22) holds,

$$\Delta\varphi = 0 \qquad \text{in } \Omega. \tag{3.24}$$

Thus by the maximum principle (3.21) implies $\varphi = 0$, i.e. (3.20) (when (3.22) holds).

If Ω is connected then the above readily shows what we want, namely that (3.20) holds for all $\lambda \geq \lambda_0$. Indeed assume that (3.20), and hence (3.21), holds for all $\lambda \geq \lambda_1 > \lambda_0$. Then, for any λ in some small neighbourhood of λ_1 we have (3.22) and hence (3.24). If $\Omega \subset T_\lambda^-$ (for such λ) then obviously (3.21), and hence (3.20), holds. If $\Omega \cap T_\lambda \neq \emptyset$ then (3.23) implies that $\varphi = 0$ in an open subset of Ω. Therefore, by (3.24) $\varphi = 0$ in all Ω and hence (3.20), (3.21) hold. Finally note that, by (3.21), $|\Omega_{\lambda_1}| \leq \frac{1}{2}|\Omega|$. Therefore, the remaining case, namely that $\Omega \subset T_\lambda^+$ cannot occur for λ close to λ_1.

Thus (3.20), (3.21) hold for all $\lambda \geq \lambda_0$ provided Ω is connected. If Ω is not connected a similar reasoning can be applied to each component (we omit the details) and the same conclusion is obtained.

We have now proved (3.16) and that $u \leq u^\lambda$ in T_λ^+ for all $\lambda \geq \lambda_0$. This readily implies that $a \cdot \nabla u \leq 0$ in Ω_{λ_0} (note that $u \in C^1(\Omega)$).

Next $\Delta(u^\lambda - u) = f - f^\lambda \leq 0$ in Ω_λ. On $\partial\Omega_\lambda \cap T_\lambda^+$, $u^\lambda - u = u^\lambda \geq 0$ and on $\partial\Omega_\lambda \cap T_\lambda$ we have $u^\lambda - u = 0$. Moreover, when $\lambda > \lambda_0$ then u^λ must be strictly positive somewhere on $\partial\Omega_\lambda \cap T_\lambda^+$ (or even on $\partial D \cap T_\lambda^+$ for any component D of Ω_λ) because λ can be decreased further with (3.21) still holding. Therefore it follows from the minimum principle for superharmonic functions that $u^\lambda - u > 0$ in $\Omega_\lambda \cap T_\lambda^+$ when $\lambda > \lambda_0$. It also readily follows that $a \cdot \nabla u < 0$ in Ω_{λ_0}. The proof is finished. □

Corollary 3.10. *Let u, f and g be as in Theorem 3.9 and assume moreover f and g are symmetric in T_{λ_0}. Then u is symmetric in T_{λ_0}.*

Corollary 3.11. *Assume that f and g satisfy Condition A and that moreover both f and g are constant outside some compact convex set K (then necessarily supp $f_+ \subset K$). Let $\Omega = \{u > 0\}$ where u is a local minimum for J. Then for any $x \in \partial_{red}\Omega \setminus K$ the inward normal ray $N_x = \{-t\nu_\Omega(x) : t > 0\}$ of $\partial\Omega$ at x intersects K. Moreover, $\partial\Omega \setminus K$ is Lipschitz.*

Proof. If for $x \in \partial_{red}\Omega \setminus K$ we have $N_x \cap K = \emptyset$ then one can find $a \in \mathbb{R}^N$ and $\lambda_0 \in \mathbb{R}$ such that $K \subset T^-_{a,\lambda_0}$, $N_x \subset T^+_{a,\lambda_0}$. The first inclusion implies that the assumption of Theorem 3.9 are satisfied while the second inclusion implies that the conclusions do not hold (e.g. $\partial\Omega \cap T^+_{a,\lambda_0}$ is not a graph near x). This contradiction proves the first statement of the corollary. The second statement follows easily by varying a and λ_0 such that $K \subset T^-_{a,\lambda_0}$. $\square$

Theorem 3.12. *Assume that f, g satisfy Condition A and that u is a local minimizer of $J_{f,g}$. Assume moreover that*

$$f(x/t) \le tf(x) \qquad \text{and} \qquad g(x/t) \ge g(x)$$

for all $0 < t < 1$ (and all $x \in \mathbb{R}^N$). Then

$$tu(x/t) \le u(x)$$

for all $0 < t < 1$. In particular $\Omega = \{u > 0\}$ is starshaped with respect to the origin.

More generally the same conclusion holds with the above inequalities replaced by, respectively

$$t^\alpha f(x/t) \le f(x), \qquad t^{\alpha+1}g(x/t) \ge g(x) \qquad \text{and} \qquad t^{\alpha+2}u(x/t) \le u(x)$$

for any (fixed) real number α.

Proof. Fix $\alpha \in \mathbb{R}^N$ and set $\varphi_t(x) = \varphi(x/t)$ for any function φ. It follows from Remark 2.7 that $t^{\alpha+2}u_t$ is a local minimizer of $J_t = J_{t^\alpha f_t, t^{\alpha+1}g_t}$.

Since $t^\alpha f_t \le f$, $t^{\alpha+1}g_t \ge g$ Lemma 1.1 therefore shows that $w_t = \max(t^{\alpha+2}u_t, u)$ is a local minimizer of J and that $J(w_t) = J(u)$, provided $t \le 1$ is close enough to 1. Clearly $w_t = u$ for $t = 1$. Now similar arguments as those in the proof of Theorem 3.9 show that actually $w_t = u$ for all $0 < t \le 1$. Thus $t^{\alpha+2}u_t \le u$ $(0 < t \le 1)$, and this readily shows that Ω is starshaped. $\square$

Theorem 3.13. *Assume that f^ϵ, f, g^ϵ, g satisfy Condition A. Let u^ϵ, u be the largest minimizers of $J^\epsilon = J_{f^\epsilon, g^\epsilon}$ and $J = J_{f,g}$ respectively, and let $\Omega^\epsilon = \{u^\epsilon > 0\}$, $\Omega = \{u > 0\}$. Assume also that*

$$f_- + g \ge \text{ const. } > 0 \tag{3.25}$$

outside Ω. Then if

$$f^\epsilon \searrow f \qquad \text{and} \qquad g^\epsilon \nearrow g \qquad a.e.$$

(or in the sense of distributions) as $\epsilon \searrow 0$ we have

$$u^\epsilon \searrow u \qquad \text{uniformly and in } w\text{-}H^1(\mathbb{R}^N), \tag{3.26}$$

$$\Omega^\epsilon \searrow \Omega \qquad \text{with respect to Hausdorff distance.} \tag{3.27}$$

Note. Condition (3.25) is needed only for (3.27)

Proof. By Lemma 1.1 u^ϵ decreases (pointwise) with ϵ. Thus $v = \lim_{\epsilon\to 0} u^\epsilon = \inf_{\epsilon>0} u^\epsilon$ exists. As in the proof of Lemma 1.2 one has $||\nabla u^\epsilon|| \le C < \infty$. Hence $u^\epsilon \to v$ weakly in $H^1(\mathbb{R}^N)$ (and strongly in $L^2(\mathbb{R}^N)$). It is now easy to check that $J(v) \le \underline{\lim}\, J^\epsilon(u^\epsilon)$. Since $J^\epsilon(u^\epsilon) \le J^\epsilon(u) \le J(u)$ (also, $\lim J^\epsilon(u) = J(u)$) it follows that $v \in \mathbb{K}$ minimizes J. But $v \ge u$ since $u \le u^\epsilon$ for $\epsilon > 0$. Thus $v = u$ since u was the largest minimizer. Thus $u^\epsilon \searrow u$, and the convergence is uniform since u^ϵ and u are continuous. This proves (3.26).

Clearly Ω^ϵ decreases with ϵ and $\Omega \subset \cap_{\epsilon>0}\Omega^\epsilon$. In order to prove (3.27) it is enough to prove the following: for any ball $B_r = B_r(x)$ with $B_{2r} \cap \Omega = \emptyset$ we have $B_r \cap \Omega^\epsilon = \emptyset$ for $\epsilon > 0$ small enough.

So assume $B_{2r} \cap \Omega = \emptyset$. Then $u^\epsilon \searrow 0$ uniformly in B_{2r}. Assume now that $B_r \cap \Omega^\epsilon \ne \emptyset$ for some $\epsilon > 0$. By combining Corollary 2.9 and Remark 2.11 we have, if (3.25) holds in B_{2r},

$$\sup_{B_{2r}} u^\epsilon \ge \sup_{\partial B_r(y)\cap\Omega^\epsilon} u^\epsilon \ge C_1 r + C_2 r^2$$

for $y \in B_r \cap \Omega^\epsilon$, where $C_1, C_2 \ge 0$, $C_1 + C_2 > 0$. This contradicts the uniform convergence of u^ϵ if ϵ is small enough, proving (3.27). $\square$

Theorem 3.14. *Assume that f^ϵ, f, g^ϵ, g satisfy Condition A. Let u^ϵ, u be the smallest minimizers of $J^\epsilon = J_{f^\epsilon, g^\epsilon}$ and $J = J_{f,g}$ respectively, and let $\Omega^\epsilon = \{u^\epsilon > 0\}$, $\Omega = \{u > 0\}$. Then if*

$$f^\epsilon \nearrow f \qquad \text{and} \qquad g^\epsilon \searrow g \qquad a.e.$$

(or in the sense of distributions) as $\epsilon \searrow 0$ we have

$$\begin{aligned} u^\epsilon &\nearrow u && \text{uniformly and in } w\text{-}H^1(\mathbb{R}^N), \\ \Omega^\epsilon &\nearrow \Omega && \text{with respect to Hausdorff distance.} \end{aligned}$$

The proof is similar to (and somewhat simpler than) that of Theorem 3.13 and therefore omitted.

Example 3.15. Take $f = f_a = a\chi_{B_R} - b$, $g = c$ as in Example 1.5, where now $R > 0$, $b \geq 0$, $c > 0$ are kept fixed and $a > 0$ is regarded as a parameter. Then, as we saw in Example 1.5, there is a critical value a_0 with $b + Nc/R < a_0 \leq (b + Nc/3R)3^N$ such that for $a < a_0$ $u = u_0 \equiv 0$ is the unique minimizer of $J_a = J_{f_a,g}$, for $a = a_0$ there are two minimizers, $u_0 = 0$ and $u_1 \not\equiv 0$, say, while for $a > a_0$ there is again a unique minimizer $u_1 \not\equiv 0$ (depending on a).

For $a \geq a_0$ the set $\Omega_a = \{u_1 > 0\}$ is a ball whose radius $\rho = \rho(a) > R$ (given by equation (1.17)) increases with a.

Thus we see that the largest solution is continuous from the left with respect to a (i.e. it depends continuously on a, on the intervals $0 < a < a_0$ and $a_0 \leq a < \infty$) while the smallest solution is continuous from the right. This is in accordance with Theorem 3.13 and 3.14, and it also shows that one cannot expect to have more than the semicontinuities stated. See also Example 4.4.

4. Quadrature domains and balayage

Let $0 \leq g, h \in L^\infty(\mathbb{R}^N)$ be given density functions. In this section we shall study the following type of balayage problem. Given a positive Radon measure μ with compact support find a bounded open set Ω containing supp μ such that μ is "graviequivalent" to the measure

$$\nu = h\mathcal{L}^N\lfloor\Omega + g\mathcal{H}^{N-1}\lfloor\partial\Omega, \tag{4.1}$$

in the sense that $U^\nu = U^\mu$ in $\mathbb{R}^N \setminus \Omega$. Here, if σ is any (positive) Borel measure, U^σ denotes its Newtonian potential, i.e.

$$U^\sigma(x) = \int E(x-y)d\sigma(y), \qquad (x \in \mathbb{R}^N)$$

where $E(x) = (\omega_N/N)|x|^{2-N}$ $(N \geq 3)$, and $E(x) = \frac{1}{2\pi}\log|x|$ $(N = 2)$ so that $-\Delta U^\sigma = \sigma$.

When a measure μ is graviequivalent to a measure ν associated with a domain Ω, as in (4.1), and Ω contains supp μ (or at least $\mu(\Omega^c) = 0$) then the word "quadrature domain" for Ω is sometimes used [Sak82], [Shap92]. The reason for this terminology is indicated after Remark 4.2 below. In this paper we shall use the following definition of a quadrature domain.

Definition 4.1. Let h, g and μ be given as above. Then Ω is a *quadrature domain* for μ (and for the given densities g and h) if Ω is a bounded open set in $\mathbb{R}^N$ such that

$$\text{supp } \mu \subset \Omega, \tag{4.2}$$

$$U^\nu = U^\mu \qquad \text{on } \mathbb{R}^N \setminus \Omega, \tag{4.3}$$

where

$$\nu = h\mathcal{L}^N\lfloor\Omega + g\mathcal{H}^{N-1}\lfloor\partial\Omega. \tag{4.4}$$

We then write

$$\Omega \in Q(\mu) \quad \text{or} \quad \Omega \in Q(\mu; h, g).$$

Remark 4.2.

a) As in Remark 3.2 b) it follows that if $\Omega \in Q(\mu; h, g)$ then Ω has finite perimeter in any open set in which $g \geq$ const. > 0.

b) Suppose that (4.2) holds. Then, by definition, $\Omega \in Q(\mu; h, g)$ if and only if the "quadrature identity"

$$\int_{\Omega} \varphi d\mu = \int_{\Omega} \varphi h dx + \int_{\partial\Omega} \varphi g d\mathcal{H}^{N-1} \tag{4.5}$$

holds for a certain class of harmonic functions φ in Ω, namely for all linear combinations of the functions $\varphi(x) = E(x-y)$, with $y \in \Omega^c$. By an approximation argument, (4.5) then also holds for every harmonic φ in Ω which can be extended to a smooth function in a neighbourhood of $\overline{\Omega}$.

c) Our definition of quadrature domain is quite weak e.g. in the sense that the identity (4.5) is required to hold only for a rather small class of harmonic functions φ. Indeed, as is explained in Example 4.3 below, our definition allows for a large class of nonsmooth members in $Q(\mu; h, g)$ when $g > 0$. (When $g = 0$ the situation is much better.)

Therefore we wish to point out conceivable ways of strengthening the requirements. In addition to (4.2), (4.3) one could ask e.g. to have

$$U^{\nu} \leq U^{\mu} \qquad \text{in } \mathbb{R}^N, \tag{4.6}$$

$$|\nabla U^{\nu}| \leq \text{ const. } < \infty. \tag{4.7}$$

Since these inequalities look a little *ad hoc* we have preferred not to put them into the definitions, but they do have some good properties: (4.7) rules out the type of nonsmooth domains occurring in Example 4.3 (relevant when $g > 0$) and (4.6) implies uniqueness (up to nullsets) of quadrature domains when $g = 0$ [Sak82], [Gust90]. Moreover, both (4.6) and (4.7) hold for the quadrature domains we construct in Theorem 4.7, 4.8.

Quadrature domains have been extensively studied in the case $h = 1, g = 0$ [Davis], [Ah-Sh], [Sak82], [Gust90], [Shap92] and also (to a smaller extent) when $h = 0, g = 1$ [Shah94b] [Gust87], [Henrot], [Sh-Ul], [Avci], [LV1], [LV2]. If e.g. $h = 1, g = 0$ and μ is a finite sum of point masses then the identity (4.5) gives a very simple way of computing the integral $\int_{\Omega} \varphi dx$ for φ harmonic in Ω. This explains the terminology. Let us now give a couple of examples, primarily for the case $h = 0, g = 1$.

Example 4.3. Let $\mu = \delta_0$ be the point mass at the origin and let $h \equiv b$, $g \equiv c$ be constant. Then the ball $\Omega = B(0, R)$ with $R > 0$ chosen so that $b\omega_N R^N + cN\omega_N R^{N-1} = 1$ is in $Q(\delta_0; b, c)$. If $b > 0$, $c = 0$ this Ω is the unique element in $Q(\delta_0; b, 0)$ (see [Kuran], [ASZ], etc).

If $b = 0$, $c > 0$, Ω is still unique among domains with smooth boundary [Shah92]. Indeed, it is even shown in [LV2] that Ω is unique among domains in $Q(\delta_0; 0, c)$ satisfying in addition (4.7) and $\mathcal{H}^{N-1}(\partial\Omega \setminus \partial_{mes}\Omega) = 0$ ((4.6) is automatically satisfied). However, without these additional assumptions there turns out to exist also a quite large family of domains in $Q(\delta_0; 0, c)$ with rather "pathological" boundaries. In two dimensions these are the famous non-Smirnov domains first found by Keldysh and Lavrentiev and later (in a more constructive way) by Duren, Shapiro and Shields [DSS], [Shap66] (see also [Duren]). In higher dimension such domains were recently constructed by Lewis and Vogel [LV1].

It should be told that these non-smooth domains Ω are not extremely pathological. E.g. they are images of the unit ball under Hölder class homeomorphisms $\mathbb{R}^N \to \mathbb{R}^N$ (which when $N = 2$ even can be taken to be quasiconformal). They satisfy

$$c \int_{\partial\Omega} \varphi d\mathcal{H}^{N-1} = \varphi(0) \tag{4.8}$$

for every φ harmonic in Ω and continuous on $\overline{\Omega}$. Also, it follows from our Corollary 3.8 that $\Omega \subset B(0, R) \in Q(\delta_0; 0, c)$.

Finally we mention that, when $N = 2$, we have uniqueness of finitely connected domains satisfying (4.8) if the test class of functions φ is enlarged to the appropriate Hardy (or "Smirnov") space. See [Avci, Theorem 2.1], [Gust87, Remark 3.4].

Example 4.4. We cite the following interesting example due to Henrot [Henrot]. Let $N = 2$, $g = 1$, $h = 0$, $\mu = a(\delta_{(-1,0)} + \delta_{(1,0)})$ where $a > 0$.

(i) If $0 < a < 2\pi$ then $\Omega_0 = B((-1,0), a/2\pi) \cup B((1,0), a/2\pi)$ is a disconnected element in $Q(\mu)$.

(ii) If $4.60... < a < 2\pi$ then there moreover exist two connected domains, Ω_1 and Ω_2, in $Q(\mu)$. We have $\Omega_0 \subset \Omega_1 \subset \Omega_2$, and for $a = 4.60...$ $\Omega_1 = \Omega_2$.

(iii) As a increases towards 2π Ω_0 expands and Ω_1 shrinks. For $a = 2\pi$ $\Omega_0 = \Omega_1$. For $a > 2\pi$ Ω_0 and Ω_1 do not exist.

(iv) As $a > 4.60...$ increases towards $+\infty$ Ω_2 expands all the time, and from $a = 5.65...$ on it is convex.

The above is proved by conformal mapping. For more details, see [Henrot]. This example illustrates in a beautiful way several of our (and also Henrot's) results, e.g. Corollary 3.8 and Theorems 3.13 and 3.14. As to the different behavior (shrinking, expanding etc) of $\Omega_0, \Omega_1, \Omega_2$ as functions of the parameter $a > 0$, there is a classification of weak solutions (into "hyperbolic", "elliptic" and "parabolic") based on such properties due to Beurling [Beur].

The existence of two different simply connected and smoothly bounded quadrature domains for a measure as simple as the above μ is particularly interesting. In the case $g = 0$, $h = 1$ there is no such example known for any μ.

Example 4.5. Let $N = 2$, $g = 1$, $h = 0$ and let $\mu = a\mathcal{H}^1\lfloor I$ where $a > 0$ and I is the closed line segment from $(-1,0)$ to $(1,0)$. If $\Omega \in Q(\mu; 0, 1)$ then $I \subset \Omega$ by (4.2), which implies that $\mathcal{H}^1(\partial\Omega) > 4$. On the other hand (4.3) implies $\int d\nu = \int d\mu$ and hence $\mathcal{H}^1(\partial\Omega) = \int d\nu = 2a$. Thus we see that a *necessary* condition for the existence of a quadrature domain for μ is that $a > 2$.

Now to relate quadrature domains with our minimization problem, assume $\mu \in L^\infty(\mathbb{R}^N)$ (i.e. μ is absolutely continuous with a bounded density function, also denoted μ) and set $f = \mu - h$. Let $u \geq 0$ be a weak solution for $J_{f,g}$ so that

$$\Delta u + f\mathcal{L}^N\lfloor\Omega = g\mathcal{H}^{N-1}\lfloor\partial\Omega,$$

where $\Omega = \{u > 0\}$. This identity can also be written

$$\mu + \Delta u = \nu,$$

where

$$\nu = h\mathcal{L}^N\lfloor\Omega + g\mathcal{H}^{N-1}\lfloor\partial\Omega + \mu\lfloor\Omega^c.$$

Clearly, $u = U^\mu - U^\nu$. Thus we see that

$$\Omega \in Q(\mu; h, g) \qquad \Longleftrightarrow \qquad \text{supp } \mu \subset \Omega. \tag{4.9}$$

When μ is a more general measure (not in L^∞), e.g. a sum of point masses, then our minimization problem does not make sense, but one can still pass between quadrature domains and the minimization problem by mollifying.

Lemma 4.6. *Let $0 \leq \psi \in L^\infty(\mathbb{R}^N)$ be radially symmetric, non-increasing as a function of $|x|$, have compact support and satisfy $\int \psi dx = 1$. Then, for μ a positive measure with compact support,*

$$\Omega \in Q(\mu * \psi; h, g) \Longrightarrow \Omega \in Q(\mu; h, g).$$

Moreover, if (4.6) holds for $\mu * \psi$ it holds also for μ.

Proof. By the supermeanvalue property for superharmonic functions

$$U^{\mu*\psi} \leq U^\mu \qquad \text{everywhere}$$

and by the ordinary meanvalue property (for harmonic functions)

$$U^{\mu*\psi} = U^\mu \qquad \text{outside supp } (\mu * \psi).$$

Note that supp $\mu \subset$ supp $(\mu*\psi)$. Thus the assertions of the lemma follows directly from Definition 4.1. □

From (4.9) and Lemma 4.6 it is clear that in order to construct quadrature domains for general positive measures μ using the minimization problem one just has to make sure that supp $(\mu*\psi) \subset$

$\Omega = \{u > 0\}$ for a suitable mollifier ψ, where u is a (local) minimizer. This is the way two of our main results, Theorem 4.7 and 4.8 below, are proved. Since $u \geq 0$ and u is Lipschitz continuous (Corollary 2.6) the quadrature domains constructed will automatically satisfy (4.6) and (4.7).

In these theorems $b, c \geq 0$ are constants with $b + c > 0$, $g, h \in L^\infty(\mathbb{R}^N)$ are density functions satisfying

$$0 \leq h \leq b,$$
$$0 \leq g \leq c.$$

Moreover, at least one of h and g is assumed to be $\geq$ const. > 0 outside a compact set, and g is assumed to be continuous and to satisfy the Hölder condition in Theorem 2.13 (unless $g > 0$ everywhere).

Theorem 4.7. *Let μ be a positive measure which is concentrated to a ball $B_R = B(x_0, R)$ to the extent that*

$$\mu(B_R^c) = 0, \tag{4.10}$$

$$\mu(B_R) > \left(b + \frac{Nc}{3R}\right) 6^N |B_R|. \tag{4.11}$$

Then, for any h, g as above there exists $\Omega \in Q(\mu; h, g)$, which moreover satisfies (4.6), (4.7) and $B_{3R} \subset \Omega$.

Proof. For $\rho > 0$ set

$$\psi_\rho = \frac{1}{|B_\rho(0)|} \chi_{B_\rho(0)}. \tag{4.12}$$

Then (4.10), (4.11) imply that

$$\mu * \psi_{2R} > \left(b + \frac{Nc}{3R}\right) 3^N \qquad \text{in } \overline{B}_R$$
$$\mu * \psi_{2R} = 0 \qquad \text{outside } B_{3R}.$$

Setting $\tilde{f} = \mu * \psi_{2R} - h$ ($\in L^\infty(\mathbb{R}^N)$) we may choose $a > (b + \frac{Nc}{3R}) 3^N$ so that $\tilde{f} \geq a\chi_{B_R} - b$. Note that a satisfies (1.18) of Example 1.5.

Let w denote the largest and unique minimizer of $J_{a\chi_{B_R} - b, c}$ and $\tilde{u}$ the largest minimizer of $\tilde{J} = J_{\tilde{f}, g}$. Then $0 \leq w \leq \tilde{u}$ by Proposition 3.3 and $\overline{B}_{3R} \subset \{w > 0\}$ by Example 1.5 (subcase 4c). Now denote $\Omega = \{\tilde{u} > 0\}$. Then supp $(\mu * \psi_{2R}) \subset \overline{B}_{3R} \subset \{w > 0\} \subset \Omega$ and it follows that $\Omega \in Q(\mu * \psi_{2R}; h, g)$. Thus $\Omega \in Q(\mu; h, g)$ by Lemma 4.6. □

Theorem 4.8. *With b, c, h, g as in Theorem 4.7 there exists a constant $C = C(N, b, c)$ such that if*

$$\sup_{r>0} \frac{\mu(B_r(x))}{r^{N-1}} \geq C \qquad \text{for every } x \in \text{supp } \mu \tag{4.13}$$

(for μ a positive measure with compact support) then there exists $\Omega \in Q(\mu; h, g)$, which moreover satisfies (4.6), (4.7). If $c = 0$ the condition (4.13) can even be replaced by

$$\sup_{r>0} \frac{\mu(B_r(x))}{r^N} \geq C \qquad \text{for every } x \in \text{supp } \mu \tag{4.14}$$

(for another $C = C(N, b)$).

Proof. Take (if $c > 0$) any

$$C > \sup_{0<R<1} R^{1-N} \left(b + \frac{Nc}{3R}\right) 6^N |B_R|.$$

Then, if (4.13) holds with this C there exists for each $x \in \text{supp } \mu$ a radius $R_x > 0$ such that

$$\mu(B(x, R_x)) > \left(b + \frac{Nc}{3R_x}\right) 6^N |B(x, R_x)|. \tag{4.15}$$

Since supp μ is compact we can select finitely many $x_1, ..., x_m \in \text{supp } \mu$ such that

$$\text{supp } \mu \subset B(x_1, R_1) \cup ... \cup B(x_m, R_m),$$

where $R_j = R_{x_j}$. Now we can choose $\epsilon > 0$ small enough so that

$$\mu(B(x_j, R_j)) > \left(b + \frac{Nc}{3(R_j + \epsilon)}\right) 6^N |B_{R_j+\epsilon}|$$

for $j = 1, ..., m$. With ψ_ρ as in (4.12) this means that

$$(\mu * \psi_\epsilon)(B(x_j, R_j + \epsilon)) > \left(b + \frac{Nc}{3(R_j + \epsilon)}\right) 6^N |B_{R_j+\epsilon}|.$$

Now minimize $J_{f,g}$ with $f = \mu * \psi_\epsilon - h$ and set $\Omega = \{u > 0\}$ where u is a minimizer. Then as in Theorem 4.7 $B(x_j, 3(R_j + \epsilon)) \subset \Omega$, $j = 1, ..., m$. In particular supp $(\mu * \psi_\epsilon) \subset \Omega$ and it follows that $\Omega \in Q(\mu * \psi_\epsilon; h, g)$ and $\Omega \in Q(\mu; h, g)$. This proves the theorem if $c > 0$. If $c = 0$ then one also gets (4.15) if (4.14) holds with $C > b|B_1|$, and the rest of the proof is unchanged. □

Comments. Clearly (4.13) is satisfied if μ is a finite sum of point masses or if μ is supported by a finite system of manifolds of dimension $\leq N - 1$ and has a sufficiently high density on them (for dimension $s < N - 1$ it is enough that the H^s density is bounded away from zero).

Moreover it is clear from Example 4.5 that an assumption of the sort (4.13) really is necessary for the existence of a quadrature domain. However, the constant C obtained in the proof is probably far from the best possible. Indeed, Example 4.5 indicates that if $N = 2$, $h = 0$, $g = 1$ then any $C > 1$ should work in (4.13), whereas our proof needs $C > 24\pi$ in this case. If $c = 0$ then, according to [Mar], [Sak, unpublished], Theorem 4.7 holds with 6^N in (4.11) replaced by 2^N, which is the best constant. This also gives the best constant in (4.14), namely $C(N, b) > 2^N |B_1| b$.

Aside from interesting cases of nonuniqueness of quadrature domains, as in Example 4.4, there is sometimes, for nonconstant g, a kind of trivial nonuniqueness: if $\mu \geq 0$ is any measure, take $h = 0$, $g = |\nabla U^\mu|$ (outside supp μ at least). Then $\Omega_t = \{x \in \mathbb{R}^N : U^\mu(x) > t\}$ is in $Q(\mu; h, g)$ for any $t \in \mathbb{R}$ such that Ω_t contains supp μ and is bounded. Cf. discussions in [Beur]. The function $u = U^\mu - U^\nu$ in this case simply is $(U^\mu - t)_+$.

Finally note that when Ω is a quadrature domain obtained from a local minimizer of J then the regularity results of section 2 and the geometric results of section 3 apply.

5. Appendix

In order to prove a) of Theorem 2.16 we need to reprove (modify) some of the lemmas and theorems in [AC].

In the case of [AC] the weak solution u is harmonic in $\{u > 0\}$ whereas in our case $-\Delta u = f$. However, when we "blow-up" the solution u (call it u_ρ) then $-\Delta u_\rho = \rho f$ and as $\rho \to 0$ the functions u_ρ converge to a harmonic function.

In most cases the changes are just a remark. It is our objective here to provide the reader with a step-to-step remark on necessary changes in the proofs of the results of [AC, 6–8].

We will adopt notations from [AC] and we only give changes in the proofs and not the statements of the results.

Lemmas and theorems that need changes are as follows: 6.1–6.3, 7.2, 7.5, 7.6, 7.10. The remaining results in [AC, 6–8] go through without changes.

6.1. The function v in the proof is $v = u + x_N$ and in our case

$$|\nabla v| \leq C_N \delta + \left(\frac{r}{2\kappa}\right)^2 \frac{\kappa}{r} |f| \leq 2C_N \delta,$$

provided $r < 4\kappa\delta C_N / M$. Here $M = \sup |f|$. Hence we need to adjust r and the choice of r depends on C_N, δ, κ, and M.

6.2. Here we get an additional term

$$\int_{B_r\cap\{u=\epsilon\}} |\nabla u| d\mathcal{H}^{N-1} \leq C\int_{\partial B_r} d\mathcal{H}^{N-1} + Mr^N \leq Cr^{N-1},$$

for r small. Here again $C = C(N, M)$ and $M = \sup|f|$.

6.3. Step 1,2,3 and 5 need no changes. In step 4 let h_ϵ be defined as $-\Delta h_\epsilon = f$ in $B_r \setminus \Lambda_\epsilon$ and $h_\epsilon = 0$ on $\partial B_r \cup \Lambda_\epsilon$. Then at the end of step 4 we'll have

$$u_\epsilon - h_\epsilon \leq u \leq cr = crw_\epsilon \qquad \text{on } \partial B_r$$

and we get $u_\epsilon - h_\epsilon \leq crw_\epsilon$ in B_r. Hence we arrive at

$$\begin{aligned}&\lim_{\epsilon\searrow 0}\sup \lambda_{w_\epsilon}(B_{r/2}) \geq \frac{c}{r}\lim_{\epsilon\searrow 0}\sup \lambda_{(u_\epsilon - h_\epsilon)}(B_{r/2}) \geq \\ &\frac{c}{r}\lambda_u(B_{r/2}) - \frac{c}{r}\lambda_h(B_{r/2}) \geq \frac{c}{r}r^{N-1} - \frac{c}{r}r^N \geq cr^{N-2},\end{aligned}$$

for r small enough.

7.2. v should be such that $-\Delta v = M \geq f$ in D. Then the estimate $\partial_{-\nu}v(z) \leq 1 + C_N\sigma$ still holds and we also have $u \leq v$. Hence we get $1-\sigma \leq l \leq 1 + C\sigma$ as in [AC] and their proof works all along. However, since our v is not harmonic but superharmonic we need to take a much smaller ball $B_r(\xi)$ ($r = 1/10$ in [AC]) where r depends on M and should be smaller for larger M. We need this in particular for the use of Harnack inequlity. Indeed we have

$$(v-u)(\xi) \leq C((v-u)(x_\xi) + r^2) \leq C(c\sigma + r^2) \leq C_N\sigma,$$

if r is small ($r^2 < \sigma$).

7.5. The inequality at the end of page 135 in [AC] involves an additional term in our case, namely

$$\frac{M}{1-\tau}|\{u_k > 0\} \cap Z^+| \tag{5.1}$$

where $Z^+ = Z^+(\sigma_k g)$. Observe that g here is not the same as our g.

We have to take into consideration that while "blowing-up" our solution M will change and we get $|\Delta u_k| = |\rho_k f| \leq \rho_k M$. Hence in 5.1 we should replace M by $M\rho_k$. We will thus arrive, as in [AC], at

$$C\sigma_k^2 \leq \tau_k \mathcal{H}^{N-1}(Z^0 \cap \{u_k > 0\}) + \rho_k C(N, M)|\{u_k > 0\} \cap Z^+|.$$

If we assume $\tau_k, \rho_k = o(\sigma_k)$ then this is a contradiction, as in [AC].

We should be careful here, since our f and g are different from those of [AC].

7.6. Again our solution will give an additional term and we obtain

$$\int_{B^k_{1/2}\cap\partial_{red}\{u_k>0\}} Q_k G^k_h \leq -\int_{B^k_{1/2}} \nabla u_k \cdot \nabla G^k_h + \rho_k\int_{B^k_{1/2}} MG^k_h,$$

where we assume $|\Delta u_k| \leq M$ in B_{ρ_k}. Now the "blow-up" solutions $u_k(\rho_k x)/\rho_k$ (which again is called u_k in [AC]) give $|\Delta u_k| \leq \rho_k M$. This justifies the above inequlity.

Now proceeding as in the proof of [AC] we only have to prove that

$$\frac{\rho_k M}{\sigma_k(1-\tau_k)}\int_{B^k_{1/2}} G^k_h \qquad \to \quad 0,$$

which is true if $\rho_k = o(\sigma_k)$.

7.10. Since in our case $U = \max(|\nabla u_\rho| - \sup_{B_{3r_1}} Q, 0)$ is, in general, not subharmonic but involves a defect of magnitude $\rho^2 M$ under the action of Δ, we have to take a superharmonic function V_ρ with $V_\rho = \tau$ on ∂B_{2r_1}, $V_\rho = 0$ on ∂B and $-\Delta V_\rho = \rho^2 M$ in $B_{2r_1} \setminus B$. This gives for ρ small $V \leq (1 - C(N,\rho))\tau$ in B_{r_1}. This is the only point that needs attention.

References

A. Acker, *Heat flow inequalities with applications to heat flow and optimization problems*, SIAM J. Math. Anal. **8** (1977), 604–618.

A. Acker, *Interior free boundary problem for the Laplace equation*, Arch. Rat. Mech. Anal **75** (1981), 157–168.

D. Aharonov, H.S. Shapiro, *Domains on which analytic functions satisfy quadrature identities*, J. Analyse Math. **30** (1976), 39–73.

D. Aharonov, M.M. Schiffer, L. Zalcman, *Potato kugel*, Israel J. Math. **40** (1981), 331–339.

H.W. Alt, L.A. Caffarelli, *Existence and regularity for a minimum problem with free boundary*, J. Reine Angew. Math. **325** (1981), 434–448.

H.W. Alt, L.A. Caffarelli, A. Friedman, *Variational problems with two phases and their free boundaries*, Trans. Amer. Math. Soc. **282 nr. 2** (1984), 431–461.

Y. Avci, *'Quadrature Identities and the Schwarz Function'*, Doctoral dissertation, Stanford University (1977).

H. Berestycki, L. Nirenberg, *On the method of moving planes and the sliding method*, Bol. da Soc. Brasileira de Mathematica **22** (1991), 1–37.

[be] L. Bers, An approximation theorem, J. Analyse Math. **14** (1965), 1–4.

A. Beurling, *On free-boundary problems fro the Laplace equation*, Sem. on Analytic Functions, Inst. for Advanced Study Princeton **1** (1957), 248–263.

L.A. Caffarelli, *Compactness methods in free boundary problems*, Partial Differential Equations **5** (1980), 427–448.

L. A. Caffarelli, *Free boundary problems, a survey*, ??, 31– 61.

T. Carleman, *Über ein Minimumproblem der mathematischen Physik*, Math. Z. **1** (1918), 208–212.

P. Davis, *The Schwarz Function and Its Application*, Carus Math. Monographs, Math. Assoc. of America, 1974.

P. Duren, *Theory of H^p Spaces*, Academic Press, New York, 1970.

P. Duren, H.S. Shapiro, A. Shield, *Singular measures and domains not of Smirnov type*, Duke Math. J. **33** (1966), 247–254.

L.C. Evans, R.F. Gariepy, *Measure Theory and Fine Property of Functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton (Florida), 1992.

A. Friedman, *Variational Principles and Free Boundary Problems*, Wiley, New York, 1982.

A. Friedman, D. Philips, *The free boundary problem of a semilinear elliptic equation*, Trans. Amer. Math. Soc. **282** (1984), 153–182.

K. Friedrichs, *Über ein Minimumproblem für Potentialströmungen mit freiem Rand*, Math. Ann. **109** (1934), 60–82.

G. Gidas, W. Ni and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Commun. Math. Phys. **68** (1979), 209–243.

E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*, Birkhäuser, Boston, 1984.

B. Gustafsson, *Application of half-order differentials on Riemann surfaces to quadrature identities for arc-length*, J. Analyse Math. **49** (1987), 54–88.

B. Gustafsson, *On quadrature domains and an inverse problem in potential theory*, J. Analyse Math. **55** (1990), 172–216.

B. Gustafsson, M. Sakai, *Properties of some balayage operators, with applications to quadrature domains and moving boundary problems*, Nonlinear Anal. **22** (1994), 1221–1245.

A. Henrot, *Subsolutions and supersolutions in free boundary problems*, Ark. Mat. **32** (1994), 79–98.

A. Henrot, M. Pierre, *About existence of equilibria in electromagnetic casting*, Quarterly of Applied Mathematics **XLIXX** (1991), 563–575.

B. Kawohl, *On Rearrangements and Convexity of Level Sets in PDE*, vol. 1150, Lecture Notes in Math. Springer Verlag, Berlin-Heidelberg, 1985.

D. Kinderlehrer and L. Nirenberg, *Regularity in free boundary problems*, Ann. Scoula Norm. Sup. Pisa Cl. Sci. **4** (1977), 373–391.

D. Kinderlehrer, G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, New York, 1980.

Ü. Kuran, *On the mean-value property of harmonic functions*, Bull. London Math. Soc. **4** (1972), 311–312.

J.L. Lewis, A. Vogel, *On Pseudospheres*, Rev. Mat. Iberoamericana **7** (1991), 25–54.

J.L. Lewis, A. Vogel, *On some almost everywhere symmetry theorems*, Progress in nonlinear differential equations (Lloyd N. G. et al., eds), Nonlinear diffusion equations and their equilibrium states,, Birkhäuser, Boston, 347–374.

A. S. Margulis, *The moving boundary problem of potential theory*, preprint (1993).

J. Mossino, *Inégalités Isopérimétriques et Applications en Physique*, Herman, Paris, 1984.

J.F. Rodrigues, *Obstacle Problems in Mathematical Physics*, North-Holland, Amsterdam, 1987.

M. Sakai, *Lecture Notes in Math.*, vol. 934, Springer -Verlag, Berlin-Heidelberg, 1982.

M. Sakai, *Application of variational inequalities to the existence theorem on quadrature domains*, Trans. Amer. Math. Soc. **276** (1983), 267–279.

M. Sakai, *Regularity of boundary having a Schwarz function*, Acta Math. **166** (19991), 263–297.

M. Sakai, *Regularity of boundaries of quadrature domains in two dimensions,*, J. Math. Anal **24** (1994), 341–364.

M. Sakai, Private Communicatin.

D.G. Schaeffer, *Some examples of singularities in a free boundary*, Ann. Scu. Norm. Sup. Pisa 4 **4** (1977), 131–144.

J. Serrin, *A symmetry problem in potential theory*, Arch. Rat. Mech. Anal. **43** (1971), 304–318.

[shah] H. Shahgholian, On quadrature domains and the Schwarz potential, J. Math. Anal. vol. 171, No. 1, (1992), 61–78.

H. Shahgholian, *A characterization of the sphere in terms of single-layer potentials*, Proc. Amer. Math. Soc. **115** (1992), no. 4, 1167–1168.

H. Shahgholian, *Existence of quadrature surfaces for positive measures with finite support*, Potential . Anal. **3** (1994), 245–255.

H. Shahgholian, *Quadrature surfaces as free boundaries*, Ark. Mat. 32 (1994), 475–492.

H. H.S. Shapiro, *Remarks concerning domains of Smirnov type*, Michigan Math. J. **13** (1966), 341–348.

H. S. Shapiro, *The Schwarz Function and its Generalization to Higher Dimensions*, Wiley, New York, University of Arkansas lecture Notes, 1992.

[shap1] H.S. Shapiro, Unbounded quadrature domains, in Proc. of Special Year in Complex Analysis at U. of Maryland 1985, Lecture Notes in Math. 1275, Springer-Verlag, 1987, pp. 287–331.

H.S. Shapiro, C. Ullemar, *Conformal mappings satisfying certain extremal properties, and associated quadrature identities*, research bulletin TRITA-MATH-1981-6, Royal Inst. of Technology, Stockholm.

J.P. Zolésio, *Weak Shape formulation of free boundary problems*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **21** (1994), 11–44.