LINK COMPLEXES OF SUBSPACE ARRANGEMENTS

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Abstract. Given a simplicial hyperplane arrangement $\mathcal{H}$ and a subspace arrangement $\mathcal{A}$ embedded in $\mathcal{H}$, we define a simplicial complex $\Delta_{\mathcal{A},\mathcal{H}}$ as the subdivision of the link of $\mathcal{A}$ induced by $\mathcal{H}$. In particular, this generalizes Steingrímsson's coloring complex of a graph.

We do the following:

(1) When $\mathcal{A}$ is a hyperplane arrangement, $\Delta_{\mathcal{A},\mathcal{H}}$ is shown to be shellable. As a special case, we answer affirmatively a question of Steingrímsson on coloring complexes.

(2) For $\mathcal{H}$ being a Coxeter arrangement of type A or B we obtain a close connection between the Hilbert series of the Stanley-Reisner ring of $\Delta_{\mathcal{A},\mathcal{H}}$ and the characteristic polynomial of $\mathcal{A}$. This extends results of Steingrímsson and provides an interpretation of chromatic polynomials of hypergraphs and signed graphs in terms of Hilbert polynomials.

1. Introduction

In [10], Steingrímsson introduced the *coloring complex* Δ_G. This is a simplicial complex associated with a graph G. The Hilbert polynomial of its Stanley-Reisner ring $k[\Delta_G]$ is closely related to the chromatic polynomial $P_G(x)$ in a way that is made precise in Section 5.

Answering a question of Steingrímsson, Jonsson [7] proved that Δ_G is a Cohen-Macaulay complex by showing that it is constructible. In particular, Δ_G being Cohen-Macaulay imposes restrictions on the Hilbert polynomial of $k[\Delta_G]$, hence on $P_G(x)$.

Since Δ_G is a Cohen-Macaulay complex, a natural question, asked already in [10], is whether it is shellable — a stronger property than constructibility.

In [10], Δ_G was defined in a combinatorially very explicit way. Another way to view Δ_G is, however, as a simplicial decomposition of the link (i.e. intersection with the unit sphere) of the *graphical hyperplane arrangement* associated with G. In this guise, Δ_G appeared in work of Herzog, Reiner and Welker [6]. Adopting this point of view, one may define a similar complex $\Delta_{\mathcal{A},\mathcal{H}}$ for any subspace arrangement $\mathcal{A}$, as long as it has an embedding in a simplicial hyperplane arrangement $\mathcal{H}$.

This paper has two goals. The first is addressed in Section 4 where we show that $\Delta_{\mathcal{A},\mathcal{H}}$ is shellable whenever $\mathcal{A}$ consists of hyperplanes. In particular, this proves that the coloring complexes are shellable.

The chromatic polynomial of G is essentially the characteristic polynomial of the corresponding graphical hyperplane arrangement. Bearing this in mind, one may hope to extend the aforementioned connection between the Hilbert polynomial of $k[\Delta_G]$ and $P_G(x)$ to more general complexes $\Delta_{\mathcal{A},\mathcal{H}}$. Achieved in Section 5, our

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second goal is to carry out this extension whenever $\mathcal{H}$ is a Coxeter arrangement of type A or B. When $\mathcal{A}$ consists of hyperplanes and $\mathcal{H}$ is of type A, Steingrímsson's result is recovered.

We define the complexes $\Delta_{\mathcal{A},\mathcal{H}}$ in Section 3 after reviewing some necessary background in the next section.

2. Preliminaries

2.1. **Subspace arrangements and characteristic polynomials.** By the term *subspace arrangement* we mean a finite collection $\mathcal{A} = \{A_1, \dots, A_t\}$ of linear subspaces, none of which contains another, of some ambient vector space. In our case, the ambient space will always be $\mathbb{R}^n$ for some n. To $\mathcal{A}$ we associate the *intersection lattice* $L_\mathcal{A}$ which consists of all intersections of subspaces in $\mathcal{A}$ ordered by reverse inclusion. (We emphasize the fact that $\mathcal{A}$ contains no strictly affine subspaces; in particular this implies that $L_\mathcal{A}$ is indeed a lattice.)

An important invariant of the arrangement $\mathcal{A}$ is its *characteristic polynomial*

$$\chi(\mathcal{A}; x) = \sum_{Y \in L_\mathcal{A}} \mu(\hat{0}, Y) x^{\dim(Y)},$$

where μ is the Möbius function of $L_\mathcal{A}$ and $\hat{0} = \mathbb{R}^n$ is the smallest element in $L_\mathcal{A}$.

Given a subspace $A \in \mathcal{A}$, we define two new arrangements, namely the *deletion*

$$\mathcal{A} \setminus A = \mathcal{A} \setminus \{A\}$$

and the *restriction*

$$\mathcal{A}/A = \max\{A \cap B | B \in \mathcal{A} \setminus A\},$$

where $\max \mathcal{S}$ denotes the collection of inclusion-maximal members of a set family $\mathcal{S}$. Another way to think of $\mathcal{A}/A$ is as the set of elements covering A in $L_\mathcal{A}$. In this way, we may extend the definition of $\mathcal{A}/A$ to arbitrary $A \in L_\mathcal{A}$. We consider $\mathcal{A} \setminus A$ to be an arrangement in $\mathbb{R}^n$, whereas $\mathcal{A}/A$ is an arrangement in A.

When $\mathcal{A}$ is a hyperplane arrangement, the next result is standard. We expect the general case to be known, too, although we have been unable to find it in the literature.

Theorem 2.1 (Deletion-Restriction)**.** *For a subspace arrangement $\mathcal{A}$ and any subspace $A \in \mathcal{A}$, we have*

$$\chi(\mathcal{A}; x) = \chi(\mathcal{A} \setminus A; x) - \chi(\mathcal{A}/A; x).$$

Proof. Choose $Y \in L_\mathcal{A}$. We claim that

$$\mu_\mathcal{A}(\hat{0}, Y) = \begin{cases} \mu_{\mathcal{A}\setminus A}(\hat{0}, Y) - \mu_\mathcal{A}(A, Y) & \text{if } Y \in L_{\mathcal{A}\setminus A}, \\ -\mu_\mathcal{A}(A, Y) & \text{otherwise,} \end{cases}$$

where $\mu_\mathcal{A}$ denotes the Möbius function of $L_\mathcal{A}$ which we think of as a function $L_\mathcal{A} \times L_\mathcal{A} \to \mathbb{Z}$ with $S \not\leq T \Rightarrow \mu_\mathcal{A}(S, T) = 0$ (and similarly for $\mathcal{A} \setminus A$).

The claim is true if $Y = \hat{0} = \mathbb{R}^n$, so assume it has been verified for all $Z < Y$ in $L_\mathcal{A}$. If $Y \in L_{\mathcal{A}\setminus A}$ we obtain

$$\begin{aligned} \mu_\mathcal{A}(\hat{0}, Y) &= -\sum_{\hat{0} \leq Z < Y} \mu_\mathcal{A}(\hat{0}, Z) = -\sum_{\substack{\hat{0} \leq Z < Y \\ Z \in L_{\mathcal{A}\setminus A}}} \mu_{\mathcal{A}\setminus A}(\hat{0}, Z) + \sum_{A \leq Z < Y} \mu_\mathcal{A}(A, Z) \\ &= \mu_{\mathcal{A}\setminus A}(\hat{0}, Y) - \mu_\mathcal{A}(A, Y), \end{aligned}$$

as desired. If, on the other hand, $Y \notin L_{\mathcal{A}\setminus A}$, then there is a unique largest element in $L_{\mathcal{A}\setminus A}$ which is below Y in $L_{\mathcal{A}}$, namely the join of all atoms (weakly) below Y except A; call this element W. If $W = \hat{0}$, then $Y = A$ and we are done. Otherwise,

$$\mu_{\mathcal{A}}(\hat{0}, Y) = -\sum_{\hat{0}\leq Z<Y} \mu_{\mathcal{A}}(\hat{0}, Z) = -\sum_{\substack{\hat{0}\leq Z\leq W\\ Z\in L_{\mathcal{A}\setminus A}}} \mu_{\mathcal{A}\setminus A}(\hat{0}, Z) + \sum_{A\leq Z<Y} \mu_{\mathcal{A}}(A, Z)$$
$$= \sum_{A\leq Z<Y} \mu_{\mathcal{A}}(A, Z) = -\mu_{\mathcal{A}}(A, Y),$$

establishing the claim.

We conclude that

$$\chi(\mathcal{A}; x) = \sum_{Y\in L_{\mathcal{A}\setminus A}} \mu_{\mathcal{A}\setminus A}(\hat{0}, Y)x^{\dim(Y)} - \sum_{Y\geq A} \mu_{\mathcal{A}}(A, Y)x^{\dim(Y)}.$$

Not every Y in the last sum belongs to $L_{\mathcal{A}/A}$ in general; the latter is join-generated by the elements covering A in $L_{\mathcal{A}}$. However, it follows from Rota's Crosscut theorem [8] that for every $Y \geq A$ in L_A,

$$\mu_{\mathcal{A}}(A, Y) = \begin{cases} \mu_{\mathcal{A}/A}(A, Y) & \text{if } Y \in L_{\mathcal{A}/A}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$\sum_{Y\geq A} \mu_{\mathcal{A}}(A, Y)x^{\dim(Y)} = \chi(\mathcal{A}/A; x),$$

and the theorem follows. □

Two (families of) hyperplane arrangements are of particular importance to us. The first is the *braid arrangement* $\mathcal{S}_n$. This is an arrangement whose ambient space is $\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 0\} \cong \mathbb{R}^{n-1}$. The $\binom{n}{2}$ hyperplanes in $\mathcal{S}_n$ are given by the equations $x_i = x_j$ for all $1 \leq i < j \leq n$.

The braid arrangement is the set of reflecting hyperplanes of a Weyl group of type A. Considering type B instead, we find our second important family of arrangements. Explicitly, $\mathcal{B}_n$ is the arrangement of the n^2 hyperplanes in $\mathbb{R}^n$ that are given by the equations $x_i = \tau x_j$ for all $1 \leq i < j \leq n$, $\tau \in \{-1, 1\}$, and $x_i = 0$ for all $i \in [n] = \{1, \dots, n\}$.

2.2. **Stanley-Reisner rings and h-polynomials.** Let Δ be a simplicial complex on the vertex set $[n]$. Regarding the vertices as variables, we want to consider the ring of polynomials that live on Δ. To this end, for a field k, we define the *Stanley-Reisner ideal* $I_\Delta \subseteq k[x_1, \dots, x_n]$ by

$$I_\Delta = \langle \{x_{i_1} \dots x_{i_t} | \{i_1, \dots, i_t\} \notin \Delta\} \rangle.$$

The quotient ring

$$k[\Delta] = k[x_1, \dots, x_n]/I_\Delta$$

is the *Stanley-Reisner ring* of Δ, which is a graded algebra with the standard grading by degree. When speaking of algebraic properties, such as Cohen-Macaulayness, of Δ we have the corresponding properties of $k[\Delta]$ in mind.

Given a simplicial complex Δ of dimension $d - 1$, its *h-polynomial* is

$$h(\Delta; x) = \sum_{i=0}^{d} f_{i-1}(x - 1)^{d-i},$$

where f_i is the number of i-dimensional simplices in Δ (including $f_{-1} = 1$ if Δ is nonempty). One important feature of the h-polynomial is that it carries all information needed to compute the Hilbert series of $k[\Delta]$. Specifically,

$$\mathrm{Hilb}(k[\Delta]; x) = \frac{\overline{h}(\Delta; x)}{(1-x)^d},$$

where $\overline{h}$ denotes the reverse h-polynomial:

$$\overline{h}(\Delta; x) = x^d h\left(\Delta; \frac{1}{x}\right).$$

2.3. **Shellable complexes.** Suppose Δ is a *pure* simplicial complex, meaning that all facets (maximal simplices) have the same dimension $d-1$. A *shelling order* for Δ is a total ordering $F_1, \ldots, F_t$ of the facets of Δ such that $F_j \cap (\cup_{i<j} F_i)$ is pure of dimension $d-2$ for all $j = 2, \ldots, t$. We say that Δ is *shellable* if a shelling order for Δ exists.

One good reason to care about shellability is that it implies Cohen-Macaulayness.

3. The objects of study

Suppose $\mathcal{H}$ is a hyperplane arrangement in $\mathbb{R}^n$ such that $\cap\mathcal{H} = \{0\}$. Then, $\mathcal{H}$ determines a regular cell decomposition $\Delta_{\mathcal{H}}$ of the unit sphere S^{n-1}. In short, each point p on S^{n-1} has an associated sign vector in $\{0, -, +\}^{|\mathcal{H}|}$ recording for each hyperplane $h \in \mathcal{H}$ whether p is on, or on the negative, or on the positive side of h (for some choice of orientations of the hyperplanes). A cell in $\Delta_{\mathcal{H}}$ consists of the set of points with a common sign vector. The face poset of $\Delta_{\mathcal{H}}$ is the big face lattice of the corresponding oriented matroid, see [2].

If $\Delta_{\mathcal{H}}$ is a simplicial complex, then $\mathcal{H}$ is called *simplicial.* A prime example of a simplicial hyperplane arrangement is the collection of reflecting hyperplanes of a finite Coxeter group. In this case, $\Delta_{\mathcal{H}}$ coincides with the Coxeter complex.

From now on, let $\mathcal{H}$ be a simplicial hyperplane arrangement.

Consider an antichain $\mathcal{A}$ in $L_{\mathcal{H}}$. We say that the subspace arrangement $\mathcal{A}$ is *embedded* in $\mathcal{H}$. Observe that $\cup\mathcal{A} \cap S^{n-1}$, which is known as the *link* of $\mathcal{A}$, has the structure of a simplicial subcomplex of $\Delta_{\mathcal{H}}$. This subcomplex is the principal object of study in this paper. We denote it $\Delta_{\mathcal{A},\mathcal{H}}$.

Example 3.1. A graph $G = ([n], E)$ determines a *graphical hyperplane arrangement* $\widehat{G}$ in the $(n-1)$-dimensional subspace of $\mathbb{R}^n$ given by the equation $x_1 + \cdots + x_n = 0$. There is one hyperplane in $\widehat{G}$ for each edge in E; the hyperplane corresponding to the edge $\{i, j\}$ has the equation $x_i = x_j$.

The arrangement $\widehat{K_n}$ corresponding to the complete graph is nothing but the braid arrangement $\mathcal{S}_n$ which is simplicial. Any graph G thus determines a simplicial complex $\Delta_{\widehat{G},\mathcal{S}_n}$. It coincides with Steingrímsson's coloring complex of G which was denoted Δ_G in the Introduction. The complex $\Delta_{\widehat{G},\mathcal{S}_n}$ also appeared under the name $\Delta_{m,J}$ in [6].

We remark that the homotopy type of the link of $\mathcal{A}$, hence of $\Delta_{\mathcal{A},\mathcal{H}}$, can be computed in terms of the order complexes of lower intervals in $L_{\mathcal{A}}$ by a formula of Ziegler and Živaljević [13]. When $\mathcal{A}$ consists of hyperplanes we may simply note that $\Delta_{\mathcal{A},\mathcal{H}}$ is homotopy equivalent to the $(n-1)$-sphere with one point removed for each connected region in the complement $\mathbb{R}^n \setminus \cup\mathcal{A}$. Denoting by $R(\mathcal{A})$ the number of such regions, $\Delta_{\mathcal{A},\mathcal{H}}$ is thus homotopy equivalent to a wedge of $R(\mathcal{A})-1$ spheres of dimension $n-2$ in this case. For the arrangements $\widehat{G}$ of Example 3.1 it is not difficult to see that $R(\widehat{G})$ equals the number $\mathrm{AO}(G)$ of acyclic orientations of G. Thus, $\Delta_{\widehat{G},\mathcal{S}_n}$ has the homotopy type of a wedge of $AO(G)-1$ $(n-3)$-spheres ([6, 7]). In particular, the reduced Euler characteristic of $\Delta_{\widehat{G},\mathcal{S}_n}$ is $\pm(\mathrm{AO}(G)-1)$ ([10, Theorem 17]).

4. Shellability in the hyperplane case

Our goal in this section is to show that $\Delta_{\mathcal{A},\mathcal{H}}$ is shellable whenever $\mathcal{A}$ consists of hyperplanes. Applied to the complexes $\Delta_{\widehat{G},\mathcal{S}_n}$ of Example 3.1 this answers affirmatively a question of Steingrímsson [10] which was restated in [7]. The key tool is a particular class of shellings of $\Delta_{\mathcal{H}}$ determined by the *poset of regions* of $\mathcal{H}$ which we now define.

The complement $\mathbb{R}^n \setminus \cup\mathcal{H}$ is cut into disjoint open regions by $\mathcal{H}$. Restricting to the unit sphere, their closures are the facets of $\Delta_{\mathcal{H}}$. Let $\mathcal{F}=\mathcal{F}(\mathcal{H})$ be the set of such facets. For $R, R' \in \mathcal{F}$, say that $h \in \mathcal{H}$ *separates* R and R' if their respective interiors are on different sides of h.

Choose a *base region* $B \in \mathcal{F}$ arbitrarily. We have a distance function $\ell : \mathcal{F} \to \mathbb{N}$ which maps a region R to the number of hyperplanes in $\mathcal{H}$ which separate R and B. Now, for two regions $R, R' \in \mathcal{F}$, write $R \lhd R'$ iff R and R' are separated by exactly one hyperplane in $\mathcal{H}$ and $\ell(R) = \ell(R')-1$. The poset of regions $P_{\mathcal{H}}$ is the partial order on $\mathcal{F}$ whose covering relation is $\lhd$. It was first studied by Edelman [5].

From the point of view of this paper, the most important property of $P_{\mathcal{H}}$ is the following.

Theorem 4.1 (Theorem 4.3.3 in [2])**.** *Any linear extension of $P_{\mathcal{H}}$ is a shelling order for $\Delta_{\mathcal{H}}$.*

We are now ready to state and prove the main result of this section.

Theorem 4.2. *If $\mathcal{A}$ consists of hyperplanes, then $\Delta_{\mathcal{A},\mathcal{H}}$ is shellable.*

Proof. We proceed by induction over $|\mathcal{A}|$. When $\mathcal{A}=\{A\}$, we may apply Theorem 4.1 since $\Delta_{\mathcal{A},\mathcal{H}} = \Delta_{\mathcal{H}/A}$ in this case.

Now suppose $|\mathcal{A}| \geq 2$ and that we have a shelling order for $\Delta_{\mathcal{A}\setminus A,\mathcal{H}}$ for some $A \in \mathcal{A}$. We will append the remaining facets to this order.

The remaining facets are the facets of $\Delta_{\{A\},\mathcal{H}} = \Delta_{\mathcal{H}/A}$. They are divided into equivalence classes in the following way: F and G belong to the same class iff their interiors belong to the same connected component of $\mathbb{R}^n \setminus \cup(\mathcal{A}\setminus A)$ (or, equivalently, to the same connected component of $A \setminus \cup(\mathcal{A}/A)$). Observe that if F and G belong to different classes, then $F \cap G \in \Delta_{\mathcal{A}\setminus A,\mathcal{H}}$. Thus, it is enough to show that the facets in any equivalence class can be appended to the shelling order for $\Delta_{\mathcal{A}\setminus A,\mathcal{H}}$.

Without loss of generality, consider the class which contains the maximal element in $P_{\mathcal{H}/A}$, i.e. the region opposite to the base region. Call this class C. If $F \in C$ and $G \notin C$ for $F, G \in P_{\mathcal{H}/A}$, then some hyperplane in $\mathcal{A}/A \subseteq \mathcal{H}/A$ separates F from G, and G is on the positive side of this hyperplane. Thus, $F \not\leq G$. This shows that C is an order filter in $P_{\mathcal{H}/A}$. According to Theorem 4.1, $\Delta_{\mathcal{H}/A}$ has a shelling order which ends with the facets in C. Now observe that $(\cup C) \cap (\cup(P_{\mathcal{H}/A} \setminus C)) = (\cup C) \cap \Delta_{\mathcal{A}\setminus A,\mathcal{H}}$.

The facets in C may therefore be appended in this order to the shelling order for $\Delta_{\mathcal{A}\setminus A,\mathcal{H}}$. $\square$

5. The h-polynomial of $\Delta_{\mathcal{A},\mathcal{H}}$

For brevity we write $h(\mathcal{A},\mathcal{H};x)$ meaning $h(\Delta_{\mathcal{A},\mathcal{H}};x)$ and similarly for $\overline{h}$. The following result of Steingrímsson serves as a motivating example for this section:

Theorem 5.1 (Theorem 13 in [10]). *Recall the complex $\Delta_{\widehat{G},\mathcal{S}_n}$ defined in Example 3.1. We have*

$$\frac{x\overline{h}(\widehat{G},\mathcal{S}_n;x)}{(1-x)^n} = \sum_{m\geq 0} (m^n - P_G(m))\, x^m,$$

where P_G is the chromatic polynomial of G.

This theorem is interesting because of the connection between the left hand side and the Hilbert series of the Stanley-Reisner ring $k[\Delta_{\widehat{G},\mathcal{S}_n}]$. In [3], Brenti began a systematic study of which polynomials arise as Hilbert polynomials of standard graded algebras. A question left open in [3], and later answered affirmatively by Almkvist [1], was whether chromatic polynomials of graphs have this property. Theorem 5.1 implies something similar, namely that $(m+1)^n - P_G(m+1)$ is the Hilbert polynomial (in m) of a standard graded algebra; for details, see Corollary 5.7 below.

It is well-known that $P_G(x) = x\chi(\widehat{G};x)$; one way to prove it is to compare Theorem 2.1 with the standard deletion-contraction recurrence for P_G. The identity suggests the possibility of extending Theorem 5.1 to other complexes $\Delta_{\mathcal{A},\mathcal{H}}$. This turns out to be possible at least if $\mathcal{H}\in\{\mathcal{S}_n,\mathcal{B}_n\}$ and is the topic of this section.

Given a subspace T of $\mathbb{R}^n$, let $d(T)$ denote its dimension. For a subspace arrangement $\mathcal{T}$, we also write

$$d(\mathcal{T}) = \max_{T\in\mathcal{T}} d(T).$$

Lemma 5.2. *Let $A\in\mathcal{A}$. Then,*

$$\begin{aligned} h(\mathcal{A},\mathcal{H};x) = &\quad (x-1)^{d(\mathcal{A})-d(\mathcal{A}\setminus A)} h(\mathcal{A}\setminus A,\mathcal{H};x)\\ &+ (x-1)^{d(\mathcal{A})-d(A)} h(\{A\},\mathcal{H};x)\\ &- (x-1)^{d(\mathcal{A})-d(\mathcal{A}/A)} h(\mathcal{A}/A,\mathcal{H}/A;x). \end{aligned}$$

Proof. Each simplex in $\Delta_{\mathcal{A},\mathcal{H}}$ belongs to $\Delta_{\mathcal{A}\setminus A,\mathcal{H}}$ or to $\Delta_{\{A\},\mathcal{H}}$ or to both. Also, observe that $\Delta_{\mathcal{A}\setminus A,\mathcal{H}}\cap\Delta_{\{A\},\mathcal{H}} = \Delta_{\mathcal{A}/A,\mathcal{H}/A}$. Denoting by $f_i(\mathcal{S},\mathcal{T})$ the number of i-dimensional simplices in $\Delta_{\mathcal{S},\mathcal{T}}$, we thus obtain for all i

$$f_i(\mathcal{A},\mathcal{H}) = f_i(\mathcal{A}\setminus A,\mathcal{H}) + f_i(\{A\},\mathcal{H}) - f_i(\mathcal{A}/A,\mathcal{H}/A).$$

The lemma now follows from the fact that $\dim(\Delta_{\mathcal{S},\mathcal{T}}) = d(\mathcal{S})-1$. $\square$

We may use Lemma 5.2 to recursively compute $h(\mathcal{A},\mathcal{H};x)$. As it turns out, this recursion is particularly useful when $\mathcal{H}\in\{\mathcal{S}_n,\mathcal{B}_n\}$. The reason is given by the following two lemmata.

Lemma 5.3. *We have*

$$\frac{x\overline{h}(\Delta_{\mathcal{S}_n};x)}{(1-x)^{n+1}} = \sum_{m\geq 0} m^n x^m$$

and

$$\frac{\overline{h}(\Delta_{\mathcal{B}_n};x)}{(1-x)^{n+1}} = \sum_{m\geq 0}(2m+1)^n x^m.$$

Proof. The complexes $\Delta_{\mathcal{S}_n}$ and $\Delta_{\mathcal{B}_n}$ coincide with the Coxeter complexes of types A_{n-1} and B_n, respectively. For the h-polynomials this implies that $x\overline{h}(\Delta_{\mathcal{S}_n};x) = A_n(x)$ and $\overline{h}(\Delta_{\mathcal{B}_n};x) = B_n(x)$, where A_n is the nth Eulerian polynomial and B_n is the nth B-Eulerian polynomial, see [4]. The assertions are well-known properties of these polynomials [4, Theorem 3.4.ii]. □

Lemma 5.4.

(i) *For any subspace* $A \in L_{\mathcal{S}_n}$*, we have*

$$\frac{x\overline{h}(\{A\},\mathcal{S}_n;x)}{(1-x)^{d(A)+2}} = \sum_{m\geq 0} m^{d(A)+1}x^m.$$

(ii) *For any subspace* $\mathcal{A} \in L_{\mathcal{B}_n}$*, we have*

$$\frac{\overline{h}(\{A\},\mathcal{B}_n;x)}{(1-x)^{d(A)+1}} = \sum_{m\geq 0}(2m+1)^{d(A)}x^m.$$

Proof. A key property of $\mathcal{S}_n$ ($\mathcal{B}_n$), which is readily checked, is that its restriction to any subspace in the intersection lattice is again a type A (B) hyperplane arrangement. Thus, $\Delta_{\{A\},\mathcal{S}_n} = \Delta_{\mathcal{S}_n/A} \cong \Delta_{\mathcal{S}_{d(A)+1}}$ ($\Delta_{\{A\},\mathcal{B}_n} = \Delta_{\mathcal{B}_n/A} \cong \Delta_{\mathcal{B}_{d(A)}}$). The assertions now follow from Lemma 5.3 □

The leading term of $\chi(\mathcal{A};x)$ is always x^n, where n is the dimension of the ambient space. It is convenient to introduce the *tail* $T(\mathcal{A};x) = x^n - \chi(\mathcal{A};x)$.

When $\mathcal{A}$ consists of hyperplanes, the following result coincides with Theorem 5.1.

Theorem 5.5. *Suppose* $\mathcal{A}$ *is a subspace arrangement embedded in* $\mathcal{S}_n$*. Then,*

$$\frac{x\overline{h}(\mathcal{A},\mathcal{S}_n;x)}{(1-x)^{d(\mathcal{A})+2}} = \sum_{m\geq 0} mT(\mathcal{A};m)x^m.$$

Proof. We proceed by induction over $|\mathcal{A}|$, noting that $|\mathcal{A}\setminus A| < |\mathcal{A}|$ and $|\mathcal{A}/A| < |\mathcal{A}|$ for every $A \in \mathcal{A}$. If $|\mathcal{A}| = 1$, we have $\chi(\mathcal{A};m) = m^{n-1} - m^{d(\mathcal{A})}$, so that $T(\mathcal{A};m) = m^{d(\mathcal{A})}$, and the theorem follows from part (i) of Lemma 5.4.

Now suppose $|\mathcal{A}| \geq 2$ and pick a subspace $A \in \mathcal{A}$. Using Lemma 5.2 and the induction hypothesis, we obtain

$$\begin{aligned}
\frac{x^{d(\mathcal{A})+1} h\left(\mathcal{A}, \mathcal{S}_n; \frac{1}{x}\right)}{(1-x)^{d(\mathcal{A})+2}} &= \left(\frac{1-x}{x}\right)^{d(\mathcal{A})-d(\mathcal{A}\setminus A)} \frac{x^{d(\mathcal{A})+1} h\left(\mathcal{A}\setminus A, \mathcal{S}_n; \frac{1}{x}\right)}{(1-x)^{d(\mathcal{A})+2}} \\
&\quad + \left(\frac{1-x}{x}\right)^{d(\mathcal{A})-d(A)} \frac{x^{d(\mathcal{A})+1} h\left(\{A\}, \mathcal{S}_n; \frac{1}{x}\right)}{(1-x)^{d(\mathcal{A})+2}} \\
&\quad - \left(\frac{1-x}{x}\right)^{d(\mathcal{A})-d(\mathcal{A}/A)} \frac{x^{d(\mathcal{A})+1} h\left(\mathcal{A}/A, \mathcal{S}_n/A; \frac{1}{x}\right)}{(1-x)^{d(\mathcal{A})+2}} \\
&= \sum_{m\geq 0} m(m^{n-1} - \chi(\mathcal{A}\setminus A; m))x^m \\
&\quad + \sum_{m\geq 0} m(m^{n-1} - (m^{n-1} - m^{d(A)}))x^m \\
&\quad - \sum_{m\geq 0} m(m^{d(A)} - \chi(\mathcal{A}/A; m))x^m \\
&= \sum_{m\geq 0} m(m^{n-1} - \chi(\mathcal{A}; m))x^m,
\end{aligned}$$

where the last equality follows from Deletion-Restriction.

For completeness, we should also check the uninteresting case $|\mathcal{A}| = 0$ which is not covered by the above arguments. Here, $\overline{h}(\emptyset, \mathcal{S}_n; x) = 0$ and $T(\emptyset; x) = 0$, and the assertion holds. □

Employing part (ii) of Lemma 5.4 instead of part (i), and keeping track of the fact that $\mathcal{B}_n$ is an arrangement in $\mathbb{R}^n$, whereas $\mathcal{S}_n$ sits in $\mathbb{R}^{n-1}$, the proof of Theorem 5.5 is easily adjusted to a proof of the next result.

Theorem 5.6. *Suppose $\mathcal{A}$ is a subspace arrangement embedded in $\mathcal{B}_n$. Then,*

$$\frac{\overline{h}(\mathcal{A}, \mathcal{B}_n; x)}{(1-x)^{d(\mathcal{A})+1}} = \sum_{m\geq 0} T(\mathcal{A}; 2m+1)x^m.$$

For subspace arrangements covered by Theorem 5.5 or Theorem 5.6, we may now draw the promised algebraic conclusions. To this end, for a simplicial complex Γ and a subcomplex $\Gamma' \subseteq \Gamma$, let $\mathcal{J}_{\Gamma',\Gamma}$ be the ideal in the Stanley-Reisner ring $k[\Gamma]$ generated by the (equivalence classes of) monomials corresponding to simplices in Γ that do not belong to Γ'.

Corollary 5.7. *Suppose $\mathcal{A}$ is a subspace arrangement embedded in $\mathcal{S}_n$. Let Γ denote the double cone over $\Delta_{\mathcal{S}_n}$, and write Γ' for the double cone over $\Delta_{\mathcal{A},\mathcal{S}_n}$ with the same cone points. The following holds:*

(i) *The Hilbert polynomial of $k[\Gamma']$ is $F(k[\Gamma']; m) = (m+1)T(\mathcal{A}; m+1)$.*
(ii) *The Hilbert polynomial of $\mathcal{J}_{\Gamma',\Gamma}$ is $F(\mathcal{J}_{\Gamma',\Gamma}; m) = (m+1)\chi(\mathcal{A}; m+1)$.*

Proof. The dimension of Γ' is $d(\mathcal{A}) + 1$. Taking a cone over a simplicial complex does not affect the $\overline{h}$-polynomial. Thus,

$$\mathrm{Hilb}(k[\Gamma']; x) = \frac{\overline{h}(\mathcal{A}, \mathcal{S}_n; x)}{(1-x)^{d(\mathcal{A})+2}} = \frac{1}{x} \sum_{m\geq 0} mT(\mathcal{A}; m)x^m,$$

where the second equality follows from Theorem 5.5. This proves (i).

For (ii), we use that

$$k[\Gamma'] \cong k[\Gamma]/\mathcal{J}_{\Gamma',\Gamma}.$$

For the Hilbert series, this implies

$$\mathrm{Hilb}(k[\Gamma'];x) = \mathrm{Hilb}(k[\Gamma];x) - \mathrm{Hilb}(\mathcal{J}_{\Gamma',\Gamma};x).$$

From part (i) and the fact that

$$\mathrm{Hilb}(k[\Gamma]) = \frac{\overline{h}(\Delta_{\mathcal{S}_n};x)}{(1-x)^{n+1}} = \frac{1}{x}\sum_{m\geq 0} m^n x^m,$$

we conclude

$$\mathrm{Hilb}(\mathcal{J}_{\Gamma',\Gamma};x) = \frac{1}{x}\sum_{m\geq 0} m^n x^m - \frac{1}{x}\sum_{m\geq 0} mT(\mathcal{A};m)x^m = \frac{1}{x}\sum_{m\geq 0} m\chi(\mathcal{A};m)x^m.$$

□

The situation for $\mathcal{B}_n$ is analogous, although we use cones instead of double cones. This is a manifestation of the fact that $\mathcal{B}_n$ and $\mathcal{S}_n$ differ by one in dimension.

Corollary 5.8. *Suppose $\mathcal{A}$ is a subspace arrangement embedded in $\mathcal{B}_n$. Let Γ denote the cone over $\Delta_{\mathcal{B}_n}$, and write Γ' for the cone over $\Delta_{\mathcal{A},\mathcal{B}_n}$ with the same cone point. Then, the following holds:*

(i) *The Hilbert polynomial of $k[\Gamma']$ is $F(k[\Gamma'];m) = T(\mathcal{A};2m+1)$.*
(ii) *The Hilbert polynomial of $\mathcal{J}_{\Gamma',\Gamma}$ is $F(\mathcal{J}_{\Gamma',\Gamma};m) = \chi(\mathcal{A};2m+1)$.*

Proof. Proceeding as in the proof of Corollary 5.7, using Theorem 5.6 instead of Theorem 5.5, we prove (i) by observing

$$\mathrm{Hilb}(k[\Gamma'];x) = \frac{\overline{h}(\mathcal{A},\mathcal{B}_n;x)}{(1-x)^{d(\mathcal{A})+1}} = \sum_{m\geq 0} T(\mathcal{A};2m+1)x^m.$$

For (ii), note that

$$\mathrm{Hilb}(k[\Gamma];x) = \frac{\overline{h}(\Delta_{\mathcal{B}_n};x)}{(1-x)^{n+1}} = \sum_{m\geq 0}(2m+1)^n x^m.$$

Thus,

$$\mathrm{Hilb}(\mathcal{J}_{\Gamma',\Gamma};x) = \sum_{m\geq 0}(2m+1)^n x^m - \sum_{m\geq 0} T(\mathcal{A};2m+1)x^m = \sum_{m\geq 0}\chi(\mathcal{A};2m+1)x^m.$$

□

Any hypergraph (without inclusions among edges) G on n vertices corresponds to a subspace arrangement $\widehat{G}$ embeddable in $\mathcal{S}_n$. The construction is virtually the same as in Example 3.1; with the hyperedge $\{i_1,\dots,i_t\}$ is associated the subspace given by $x_{i_1} = \cdots = x_{i_t}$. As for ordinary graphs (the hyperplane case), we have $x\chi(\widehat{G};x) = P_G(x)$, cf. [9, Theorem 3.4]. In this way, Corollary 5.7 allows us to interpret chromatic polynomials of hypergraphs in terms of Hilbert polynomials. For ordinary graphs, this is the content of Steingrímsson's [10, Corollary 10].

Corollary 5.8, too, has an impact on chromatic polynomials. Any *signed graph* (in the sense of Zaslavsky [11]) G on n vertices corresponds to a hyperplane arrangement $\widehat{G} \subseteq \mathcal{B}_n$, and vice versa. A signed graph G has a chromatic polynomial $P_G(x)$, and $P_G(x) = \chi(\widehat{G}; x)$ [12].

References

[1] G. Almkvist, The chromatic polynomial is a Hilbert polynomial, preprint 1998.

[2] A. Björner, M. Las Vergnas, B. Sturmfels, N. White and G. M. Ziegler, *Oriented matroids. 2nd ed.*, Encyclopedia of Mathematics and its Applications **46**, Cambridge Univ. Press, Cambridge, 1999.

[3] F. Brenti, Hilbert polynomials in combinatorics, *J. Algebraic Combin.* **7** (1998), 127–156.

[4] F. Brenti, q-Eulerian polynomials arising from Coxeter groups, *European J. Combin.* **15** (1994), 417–441.

[5] P. H. Edelman, A partial order on the regions of R^n dissected by hyperplanes, *Trans. Amer. Math. Soc.* **283** (1984), 617–631.

[6] J. Herzog, V. Reiner and V. Welker, The Koszul property in affine semigroup rings, *Pacific J. Math.* **186** (1998), 39–65.

[7] J. Jonsson, The topology of the coloring complex, *J. Algebraic Combin.* **21** (2005), 311–329.

[8] G.-C. Rota, On the foundations of combinatorial theory. I. Theory of Möbius functions, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **2** (1964), 340–368.

[9] R. P. Stanley, Graph colorings and related symmetric functions: ideas and applications: a description of results, interesting applications, & notable open problems, *Discrete Math.* **193** (1998), 267–286.

[10] E. Steingrímsson, The coloring ideal and coloring complex of a graph, *J. Algebraic Combin.* **14** (2001), 73–84.

[11] T. Zaslavsky, The geometry of root systems and signed graphs, *Amer. Math. Monthly* **88** (1981), 88–105.

[12] T. Zaslavsky, Signed graph coloring, *Discrete Math.* **39** (1982), 215–228.

[13] G. M. Ziegler and R. T. Živaljević, Homotopy types of subspace arrangements via diagrams of spaces, *Math. Ann.* **295** (1993), 527–548.

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