THE TOPOLOGY OF SPACES OF PHYLOGENETIC TREES WITH SYMMETRY

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Abstract. Natural Dowling analogues of the complex of phylogenetic trees are studied. Using discrete Morse theory, we find their homotopy types. In the process, the homotopy types of certain subposets of Dowling lattices are determined.

1. Introduction

Billera, Holmes and Vogtmann studied a space of phylogenetic trees in [2]. Partly, they were motivated by problems that arise in biology. For example, their space provides a notion of distance between phylogenetic trees. This is of relevance to the problem of finding the tree which most accurately incorporates observed data. One classical instance of this problem would be to find the evolutionary relationships within a collection of species given information about their genomes.

Topologically, (the link of the origin in) this space is a simplicial complex whose simplices are in bijection with certain leaf-labelled trees; the maximal trees correspond to phylogenetic ones in the sense of evolutionary biology. This complex happens to be homotopic (actually, homeomorphic [1]) to the order complex of the partition lattice.

A common way to generalise results on the partition lattice is to consider *Dowling analogues*. Given a finite group G and a natural number n, one can associate a geometric lattice Π_n^G. These lattices were introduced by Dowling [6], hence are called *Dowling lattices*. When G is trivial, the partition lattice on $n+1$ elements is recovered. In the case of the two-element group $\mathbb{Z}_2$, one obtains the intersection lattice of the arrangement of reflecting hyperplanes of the reflection group of type B_n (the hyperoctahedral group).

In this paper, we determine the homotopy types of natural Dowling analogues of the space of phylogenetic trees. They can be thought of as spaces of phylogenetic trees with symmetry given by a group G. In particular, if $G = \{\mathrm{id}\}$, the complex of Billera, Holmes and Vogtmann (actually, a cone over it) is recovered, whereas $G = \mathbb{Z}_2$ yields type B analogues.

Along the way, we compute the homotopy types of the order complexes of certain subposets of Dowling lattices that are of independent interest. Our methods involve discrete Morse theory as well as Quillen's fiber theorem and the Crosscut theorem.

Before we define and study our complexes of trees in Section 3, we review necessary material on combinatorial topology and Dowling lattices in the next section.

Key words and phrases. Phylogenetic trees, Dowling lattices, Discrete Morse theory.

Supported by the European Commission's IHRP Programme, grant HPRN-CT-2001-00272, "Algebraic Combinatorics in Europe".

Remark 1.1. After the first version of this paper appeared, some of our results have been extended and put in the context of nested set complexes by Delucchi [5].

2. Prerequisities

2.1. **The topology of posets.** To any poset P, we associate the *order complex* $\Delta(P)$. It is the simplicial complex whose simplices are the chains in P. When talking about topological properties of P, we have the corresponding properties of $\Delta(P)$ in mind. Here, and in what follows, no notational distinction will be made between an abstract simplicial complex and its geometric realisation. We now recall two important tools for computing the homotopy type of P.

Theorem 2.1 (Quillen's fiber theorem [9])**.** *Let P and Q be posets, and suppose we have an order-preserving map $f : P \to Q$ such that for every $q \in Q$, the subposet of P induced by*

$$\{p \in P \mid f(p) \leq q\}$$

is contractible. Then, P and Q are homotopy equivalent.

If a poset P has unique maximal and minimal elements, we denote them $\hat{1}$ and $\hat{0}$, respectively. The next result is a special case of the Crosscut theorem. It can be found e.g. in [3].

Theorem 2.2. *Suppose L is a finite lattice. Let Σ be the simplicial complex on the atoms of L consisting of those sets of atoms whose joins are not $\hat{1}$. Then, Σ and $L \setminus \{\hat{0}, \hat{1}\}$ are homotopy equivalent.*

2.2. **Discrete Morse theory.** We will need some elements of Forman's discrete Morse theory [7]. As has become fairly standard, we will deal with matchings rather than discrete Morse functions; this way of formulating things is due to Chari [4].

Let Δ be a regular CW complex. The *face poset* $\mathcal{F} = \mathcal{F}(\Delta)$ is the poset of cells in Δ (including the empty cell) ordered by inclusion. A *matching* on Δ is an involution $M : Q \to Q$, for some $Q \subseteq \mathcal{F}$, such that for all $q \in Q$, either $M(q) \lhd q$ or $M(q) \rhd q$, where $\lhd$ denotes the covering relation in $\mathcal{F}$. Informally, M is just a matching of the Hasse diagram of $\mathcal{F}(\Delta)$ in the graph-theoretical sense. The *critical cells* of M are the unmatched cells, i.e. the elements in $\mathcal{F} \setminus Q$.

A matching M on Δ is *acyclic* if for every sequence

$$q_0 \lhd M(q_0) \rhd q_1 \lhd M(q_1) \rhd \cdots \lhd M(q_{t-1}) \rhd q_t$$

with $q_0 \neq q_1$ it holds that $q_0 \neq q_t$. This condition can be interpreted as follows. In the Hasse diagram of $\mathcal{F}$, direct the edges that are part of the matching upwards and direct the others downwards. Then, M is acyclic iff this directed graph is acyclic.

Theorem 2.3 (Forman [7])**.** *Suppose M is an acyclic matching on Δ. Let C be the set of critical cells. If every cell in C has dimension d, then Δ is homotopy equivalent to a wedge of $|C|$ spheres of dimension d.*

The following result can be derived from [7]. See [8] for a direct proof.

Theorem 2.4. *Suppose M is an acyclic matching on Δ. Let C be the critical cells. If C is a subcomplex of Δ, then C and Δ are homotopy equivalent.*

2.3. **Dowling lattices.** Let G be a finite group and n a positive integer. We have a natural G-action on the set $([n] \times G) \cup \{0\}$, where $[n] = \{1, \dots, n\}$, by $g0 = 0$ and $g(i,h) = (i, gh)$; we extend this action in the natural way to subsets and partitions of $([n] \times G) \cup \{0\}$. A partition $\sigma = \sigma_1 | \dots | \sigma_t$ of $([n] \times G) \cup \{0\}$ is called *G-invariant* if $g\sigma = \sigma$ for all $g \in G$. If $g\sigma_i \neq \sigma_i$ for all $g \in G \setminus \{\mathrm{id}\}$, then the block σ_i is called *simple*. Note that if σ is G-invariant and σ_i is the block containing 0, then $g\sigma_i = \sigma_i$ for all $g \in G$.

Definition 2.5. *Ordering partitions by refinement, the* Dowling lattice Π_n^G *is the lattice of all G-invariant partitions σ of $([n] \times G) \cup \{0\}$ in which all blocks that do not contain 0 are simple. We call the block containing 0 the* zero block *of σ.*

Notice that $\Pi_n^{\{\mathrm{id}\}} \cong \Pi_{n+1}$, where Π_{n+1} is the familiar partition lattice on $n+1$ elements. Another well-known special case is $\Pi_n^{\mathbb{Z}_2}$, which is the *signed partition lattice*, i.e. the intersection lattice of the arrangement of reflecting hyperplanes in the Weyl group of type B_n. Dowling lattices were first introduced in [6].

The G-orbit of any non-zero block in $\sigma \in \Pi_n^G$ contains $|G|$ representatives. We sometimes use the term *non-zero block orbit* when referring to such an orbit. Similar terminology is not employed for the zero block, since it is alone in its orbit.

3. The complex of G-symmetric phylogenetic n-trees

An *(n,G)-tree* is a rooted tree with $n|G|$ leaves, each with a unique label from the set $[n] \times G$. The group G acts on the set of (n,G)-trees via the natural action on the leaf labels: $g(i,h) = (i, gh)$ for $g, h \in G$, $i \in [n]$.

Definition 3.1. *An (n,G)-tree T is a G-*symmetric phylogenetic n-tree *if the following three conditions are met:*

(i) *Every vertex except the root and the leaves has degree at least 3.*
(ii) *For all $g \in G$, $gT = T$, i.e. T is invariant under the G-action.*
(iii) *For any $i \in [n]$ and $g \neq h \in G$, the unique shortest path between the leaf labelled (i,g) and the one labelled (i,h) passes through the root.*

We denote the set of G-symmetric phylogenetic n-trees by $\mathcal{T}_n^G$.

The edge set of $T \in \mathcal{T}_n^G$ is partitioned into G-orbits. Condition (iii) of Definition 3.1 implies that no edge is fixed by any other group element than the identity. Thus, the orbit of any edge has cardinality $|G|$. In particular, $|G|$ is the smallest possible degree of the root.

An *inner edge* in T is an edge which is not incident to a leaf. Its G-orbit is called an *inner orbit*. Suppose $\mathfrak{o}$ is an inner orbit of some $T \in \mathcal{T}_n^G$. If we remove every edge in $\mathfrak{o}$ from T, the remaining connected components induce a partition of the leaves of T. Adding 0 to the block corresponding to the component which contains the root (which possibly contains no leaf), this is an element $\pi(\mathfrak{o}) \in \Pi_n^G$. It is easily seen that $\pi(\mathfrak{o})$ has exactly one non-zero block orbit, and this orbit cannot consist of singletons. In fact, every such partition arises in this way. Thus, to every $T \in \mathcal{T}_n^G$, we may associate the set of partitions

$$P(T) = \{\pi(\mathfrak{o}) \mid \mathfrak{o} \text{ is an inner orbit of } T\} \subseteq \Pi_n^G.$$

As an example, suppose T is the left one of the two maximal trees in Figure 1. Then T has two inner orbits, and $P(T) = \{0|1\underline{2}3|\underline{1}2\underline{3}, 03\underline{3}|1\underline{2}|\underline{1}2\} \subseteq \Pi_3^{\mathbb{Z}_2}$.

Hence, we have a map $P : \mathcal{T}_n^G \to \mathcal{P}(\Pi_n^G)$, where $\mathcal{P}(\Pi_n^G)$ denotes the Boolean lattice of subsets of Π_n^G.

Remark 3.2. The bipartition of a phylogenetic tree obtained by removing one of its edges is often called a *split* by the bioinformatics community. Thus, $P(T)$ may be seen as a set of Dowling analogues of splits. It is well-known that a phylogenetic tree is determined by its set of splits. This extends to the Dowling situation according to Proposition 3.4 below.

Observe that the simultaneous contraction of every edge in an inner orbit turns T into another tree $T' \in \mathcal{T}_n^G$. We say that T' was obtained from T by an *inner orbit contraction*. Analogously, we say that T is constructed by an *inner orbit extension* of T'. This is the covering relation of a partial order on $\mathcal{T}_n^G$:

Definition 3.3. *Given T, $T' \in \mathcal{T}_n^G$, we write $T' \leq T$ iff T' can be obtained from T by a sequence of inner orbit contractions.*

A part of this poset when $n = 3$ and G is the two-element group is shown in Figure 1.

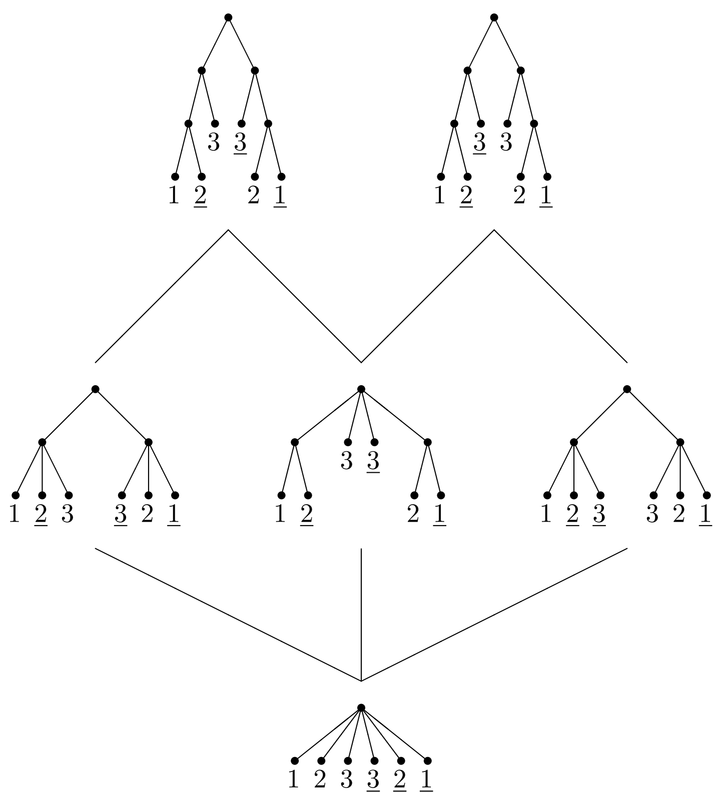

FIGURE 1. The order ideal in the poset $\mathcal{T}_3^{\mathbb{Z}_2}$ generated by two of the twelve maximal elements. To simplify notation, we write i for the leaf label (i, id) and $\underline{i}$ for the label (i, g), where g is the non-identity in $\mathbb{Z}_2$.

Obviously, $T < T' \Rightarrow P(T) \subset P(T')$, i.e. P is order-preserving. Observe that the image of P is an order ideal in $\mathcal{P}(\Pi_n^G)$.

Proposition 3.4. *The map* $P : \mathcal{T}_n^G \to \mathcal{P}(\Pi_n^G)$ *is injective.*

Proof. We prove that $T \neq T' \Rightarrow P(T) \neq P(T')$ for all T, $T' \in \mathcal{T}_n^G$ by induction over $|P(T)|$, the assertion being true for $P(T) = \emptyset$.

Suppose, in order to get a contradiction, that $T \neq T'$ and $P(T) = P(T')$. Then there exist inner orbits $\mathfrak{o}$ and $\mathfrak{o}'$ in T and T', respectively, such that $P(T) \setminus \{\pi(\mathfrak{o})\} = P(T') \setminus \{\pi(\mathfrak{o}')\}$; in particular, $\pi(\mathfrak{o}) = \pi(\mathfrak{o}')$. By the induction assumption, there is a unique $U \in \mathcal{T}_n^G$ such that $P(U) = P(T) \setminus \{\pi(\mathfrak{o})\} = P(T') \setminus \{\pi(\mathfrak{o}')\}$. Thus, from U we obtain T and T' by inner orbit expansions corresponding to $\mathfrak{o}$ and $\mathfrak{o}'$, respectively. Since different expansions in the same tree, by construction, give rise to different partitions, this implies $\pi(\mathfrak{o}) \neq \pi(\mathfrak{o}')$, a contradiction. □

A simplicial complex is called *pure* if its maximal simplices are equidimensional.

Corollary 3.5. *The partial order* $\leq$ *turns* $\mathcal{T}_n^G$ *into the face poset of a pure simplicial complex of dimension* $n-2$.

Proof. It is easily seen that the inverse of P (which is defined on the image of P according to Proposition 3.4) is order-preserving. Thus, as a poset, $\mathcal{T}_n^G$ is isomorphic to the image of P. The latter is an order ideal in a Boolean lattice, i.e. the face poset of a simplicial complex. Furthermore, this ideal is generated by subsets of cardinality $n-1$. □

By abuse of notation, we will use $\mathcal{T}_n^G$ to denote both this complex and its face poset. An example with $n = 3$, $G = \mathbb{Z}_2$ is shown in Figure 2.

Remark 3.6. Observe that the degree of the root in a maximal tree in $\mathcal{T}_n^G$ is $|G|$. Thus, when $G = \{\mathrm{id}\}$, $\mathcal{T}_n^G$ is a cone. Under P, its apex is mapped to the unique partition in $\Pi_n^{\{\mathrm{id}\}}$ consisting of one non-zero and a singleton zero block. The link of this apex in $\mathcal{T}_n^{\{\mathrm{id}\}}$ coincides with the link of the origin in the space of phylogenetic trees as defined by Billera, Holmes and Vogtmann in [2].

Remark 3.7. The type B complex $\mathcal{T}_n^{\mathbb{Z}_2}$ is studied from a combinatorial point of view in Zehnpfund's Diploma thesis [10].

The vertices in the complex $\mathcal{T}_n^G$ may be identified with the set of all inner orbits of trees in $\mathcal{T}_n^G$; we let V_n^G denote this set. Given $\mathfrak{o} \in V_n^G$, $T \in \mathcal{T}_n^G$, we define $T - \mathfrak{o}$ to be the unique tree in $\mathcal{T}_n^G$ satisfying $P(T - \mathfrak{o}) = P(T) \setminus \{\pi(\mathfrak{o})\}$. Proposition 3.4 shows that this is well-defined. Similarly, $T + \mathfrak{o}$ is the tree satisfying $P(T + \mathfrak{o}) = P(T) \cup \{\pi(\mathfrak{o})\}$, if such a tree exists (if it does, it is necessarily unique). Finally, we write

$$T \pm \mathfrak{o} = \begin{cases} T - \mathfrak{o} & \text{if } \pi(\mathfrak{o}) \in P(T), \\ T + \mathfrak{o} & \text{otherwise.} \end{cases}$$

Let $\underline{\Pi}_n^G$ be the subposet of $\Pi_n^G \setminus \{\hat{0}\}$ induced by the partitions with trivial zero block (i.e. the partitions in which 0 is a singleton block).

Lemma 3.8. *The complexes* $\mathcal{T}_n^G$ *and* $\Delta(\underline{\Pi}_n^G)$ *are homotopy equivalent.*

Proof. Consider a tree $T \in \mathcal{T}_n^G$. Removing the root and all its incident edges from T, we are left with a collection of disjoint trees. If we add a singleton block consisting of 0, the partition of the leaves induced by this collection is an element $\tau(T) \in \Pi_n^G$. Clearly, this gives an order-preserving map $\tau : \mathcal{T}_n^G \setminus \{\hat{0}\} \to \Pi_n^G$ whose image is $\underline{\Pi}_n^G$.

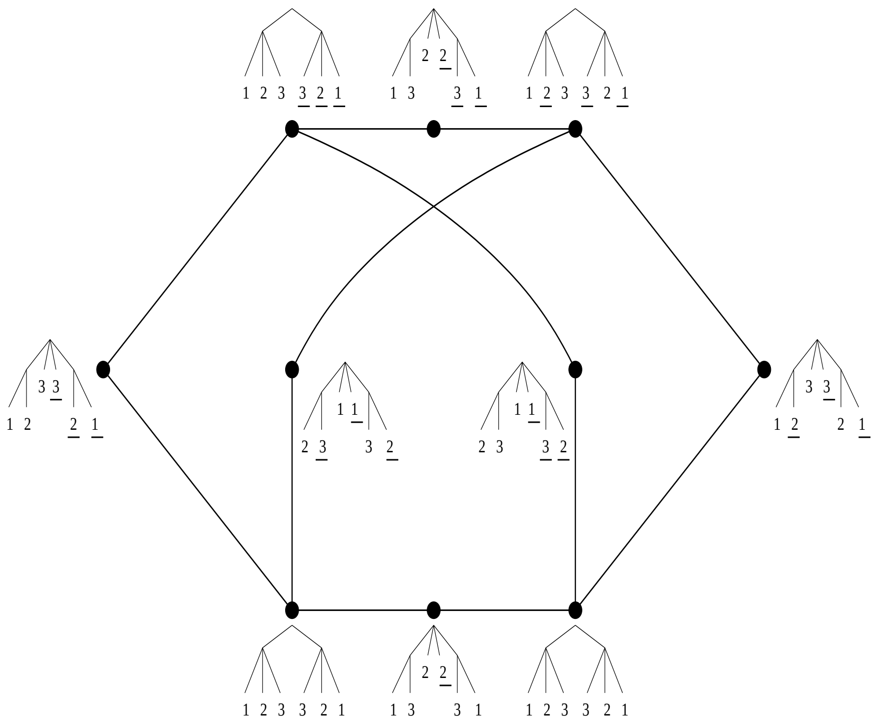

FIGURE 2. A picture of $\mathcal{T}_3^{\mathbb{Z}_2}$ viewed as a simplicial complex with vertex labels indicated. The two rightmost edges coincide with the maximal elements that generate the order ideal in Figure 1.

Let $\sigma \in \underline{\Pi}_n^G$, and suppose σ_1 is a non-singleton block in σ. Observe that if $\tau(T) \leq \sigma$, then $\tau(T \pm \mathfrak{o})$ is well-defined and $\tau(T \pm \mathfrak{o}) \leq \sigma$, too, where $\mathfrak{o} \in V_n^G$ satisfies that the only non-zero block orbit in $\pi(\mathfrak{o})$ is the orbit of σ_1. This shows that the subposet of $\mathcal{T}_n^G$ induced by

$$\{T \in \mathcal{T}_n^G \setminus \{\hat{0}\} \mid \tau(T) \leq \sigma\}$$

is a cone with apex $\mathfrak{o}$; in particular it is contractible. Theorem 2.1 now implies that $\Delta(\underline{\Pi}_n^G)$ and $\Delta(\mathcal{T}_n^G \setminus \{\hat{0}\})$ are homotopy equivalent. The latter complex is the barycentric subdivision of $\mathcal{T}_n^G$, and the lemma follows. □

Remark 3.9. Delucchi [5] has extended Lemma 3.8 by showing that $\Delta(\underline{\Pi}_n^G)$ can be obtained from $\mathcal{T}_n^G$ by a sequence of stellar subdivisions. In particular, the two complexes are homeomorphic.

Let $\Gamma = (V, E)$ be a simple graph on $V = [n]$. Given an edge-labelling $\phi : E \to G$ and $e = \{x, y\} \in E$, let

$$\epsilon(x,y) = \begin{cases} \phi(e) & \text{if } x < y, \\ \phi(e)^{-1} & \text{if } x > y. \end{cases}$$

The labelling ϕ is called *consistent* if given any $x, y \in V$, and any pair of paths $x = v_0, v_1, \ldots, v_s = y$ and $x = v'_0, v'_1, \ldots, v'_t = y$, we have $\epsilon(v_0, v_1)\ldots\epsilon(v_{s-1}, v_s) = \epsilon(v'_0, v'_1)\ldots\epsilon(v'_{t-1}, v'_t)$.

Example 3.10. With $G = \mathbb{Z}_2$, we may think of ϕ as colouring the edges blue or red. This labelling is then consistent if for all $x, y \in V$, any two paths from x to y have the same number of red edges modulo 2.

Clearly, removing edges from a consistently labelled graph yields a new graph with this property. Thus, we have a simplicial complex Λ_n^G whose vertices are all possible labelled edges (i.e. $\binom{n}{2}|G|$ vertices in all) and whose simplices are the consistently labelled graphs on $[n]$.

Lemma 3.11. *The complexes $\Delta(\underline{\Pi}_n^G)$ and Λ_n^G are homotopy equivalent.*

Proof. The poset $\underline{\Pi}_n^G \cup \{\hat{0}\}$ is an order ideal in the lattice Π_n^G. Therefore, $\underline{\Pi}_n^G \cup \{\hat{0}, \hat{1}\}$ is also a lattice. By Theorem 2.2, $\underline{\Pi}_n^G$ is homotopy equivalent to the simplicial complex on the atoms of $\underline{\Pi}_n^G \cup \{\hat{0}, \hat{1}\}$ whose simplices are the atom sets with join in $\underline{\Pi}_n^G$. Each atom has exactly one block of the form $\{(i, \mathrm{id}), (j, g)\}$ for $1 \le i < j \le n$, $g \in G$ (this block is a representative of the only block orbit which is not comprised of singletons). We identify such an atom with the vertex of Λ_n^G which consists of the edge $\{i, j\}$ with label g. Clearly, this gives a bijection between the vertices of Λ_n^G and the atoms of $\underline{\Pi}_n^G \cup \{\hat{0}, \hat{1}\}$. Now observe that a (nonempty) set of atoms has its join in $\underline{\Pi}_n^G$ if and only if the corresponding graph is consistently labelled. This proves the lemma. □

Let $\Gamma = (V, E)$ be a graph with $v_1, v_2 \in V$, and set $e = \{v_1, v_2\}$. Below, we will employ the following notation:

$$\Gamma \pm e = \begin{cases} (V, E \cup \{e\}) & \text{if } e \notin E, \\ (V, E \setminus \{e\}) & \text{otherwise.} \end{cases}$$

Theorem 3.12. *The complex Λ_n^G is homotopy equivalent to a wedge of spheres of dimension $n-2$. The number of spheres is*

$$(|G| - 1)(2|G| - 1)\ldots((n-1)|G| - 1).$$

Proof. We will give an acyclic matching on Λ_n^G with the prescribed number and dimensions of the critical cells. It proceeds in two steps.

Matching 1: Consider a graph $\Gamma \in \Lambda_n^G$. Let $w = w(\Gamma) \in [n]$ be the maximal vertex from which at least two edges to smaller vertices emanate, if such a vertex exists. If w does not exist, Γ is a critical cell of the first matching. Otherwise, let $v > u$ be the largest and the second largest vertices among those neighbours of w that are smaller than w. Let e be an edge between u and v with label $\phi(e) = \phi(\{u, w\})\phi(\{w, v\})^{-1}$. We then let Γ be matched with $M(\Gamma) = \Gamma \pm e$. Observe that adding e to Γ preserves the consistency of the edge-labelling, and removing e trivially does. Furthermore, $w(\Gamma) = w(\Gamma \pm e)$. Thus, M is a well-defined matching on Λ_n^G.

We now show acyclicity of M. Suppose $x \lhd M(x) \rhd y \lhd M(y)$ for some $x, y \in \Lambda_n^G$, where the coverings are in the face poset of Λ_n^G. By construction, $w(x) \ge w(y)$. If $w(x) = w(y)$ but $x \ne y$, y must be constructed from $M(x)$ by removing an edge incident to $w(x)$; otherwise we would have $y \rhd M(y)$. In particular, $w(x)$ has strictly fewer neighbours in y than in x. Thus, whenever we have a sequence

$$x_0 \lhd M(x_0) \rhd x_1 \lhd M(x_1) \rhd \cdots \lhd M(x_{t-1}) \rhd x_t$$

in the face poset of Λ_n^G, then either $w(x_0) > w(x_t)$ or $w(x_0) = w(x_t)$ and the number of neighbours of this vertex is strictly smaller in x_t than in x_0. In particular, we always have $x_0 \neq x_t$. This proves that M is acyclic.

A graph $\Gamma \in \Lambda_n^G$ is a critical cell of M iff every vertex in Γ has at most one edge in common with a smaller vertex. This happens iff Γ is a forest with the following property: in every connected component, the unique shortest path from the smallest element to any other element is increasing. (We may think of the smallest elements as forming the roots of the connected components. Then, downward paths from a root must increase.) Clearly, these forests form a subcomplex C of Λ_n^G. By Theorem 2.4, C is homotopy equivalent to Λ_n^G.

Matching 2: We now define a matching M_2 on C. Given $\Gamma \in C$, consider the set $K(\Gamma) \subseteq \{2, 3, \ldots, n\}$ defined by $v \in K(\Gamma)$ iff v is the smallest vertex in a connected component of Γ or if $\{1, v\}$ is an edge of Γ labelled with the identity element (i.e. $\phi(\{1, v\}) = \mathrm{id}$). If $K(\Gamma) \neq \emptyset$, we set $M_2(\Gamma) = \Gamma \pm e$, where e is the edge $e = \{1, \min(K(\Gamma))\}$ with label $\phi(e) = \mathrm{id}$. It is easily seen that $M_2(\Gamma) \in C$ and $K(\Gamma) = K(M_2(\Gamma))$. Hence, $M_2(M_2(\Gamma)) = \Gamma$.

To prove that M_2 is acyclic, assume $x \lhd M(x) \rhd y$ for $x, y \in C$ (coverings in the face poset of C). Observe that $K(x) \subseteq K(y)$. If $K(x) = K(y)$, then either $x = y$ or $M(y) \lhd y$. Hence, the existence of a sequence

$$x_0 \lhd M(x_0) \rhd x_1 \lhd M(x_1) \rhd \cdots \lhd M(x_{t-1}) \rhd x_t$$

with $x_0 \neq x_1$ implies that $K(x_0) \neq K(x_t)$. In particular, $x_0 \neq x_t$ and M_2 is acyclic.

The critical cells under M_2 are the $\Gamma \in C$ with $K(\Gamma) = \emptyset$. These are the trees on $[n]$ with G-labelled edges in which the unique shortest path from 1 to any other element is increasing, and every edge containing 1 has a label different from the identity element in G. Let T be the set of such trees. Every element in T of course has $n-1$ edges. By Theorem 2.3, Λ_n^G is homotopy equivalent to a wedge of $|T|$ spheres of dimension $n-2$.

A tree in T is uniquely determined by the choice for each $v \in \{2, \ldots, n\}$ of the neighbour w of v which is on the unique shortest path from v to 1 together with the choice of the label $\phi(\{v, w\})$. If $w = 1$, we have $|G| - 1$ choices for this label (since the identity is not allowed). Otherwise, we have $|G|$ possibilities. Thus,

$$|T| = \prod_{i=2}^{n} ((i-1)|G| - 1).$$

□

Combining this theorem with Lemma 3.8 and Lemma 3.11 yields

Corollary 3.13. *The complexes $\mathcal{T}_n^G$, $\Delta(\underline{\Pi}_n^G)$ and Λ_n^G are all homotopy equivalent to a wedge of $(|G|-1)(2|G|-1)\ldots((n-1)|G|-1)$ spheres of dimension $n-2$.*

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