2-Dimension from the topological viewpoint

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Abstract

In this paper we study the 2-dimension of a finite poset from the topological point of view. We use homotopy theory of finite topological spaces and the concept of a *beat point* to improve the classical results on 2-dimension, giving a more complete answer to the problem of all possible 2-dimensions of an n-point poset.

2000 Mathematics Subject Classification. 06A06, 06A07, 54A10, 54H99.
Key words and phrases. Posets, Finite Topological Spaces, 2-dimension.

1 Introduction

It is well known that finite topological spaces and finite preorders are intimately related. More explicitly, given a finite set X, there exists a $1-1$ correspondence between topologies and preorders on X [1]. Moreover, T_0-topologies correspond to orders.

In this way, one can consider finite posets as finite T_0-spaces and vice versa. Combinatorial techniques based on finite posets together with topological properties can result in stronger theorems [2, 6, 7, 8, 10, 14].

A basic result in topology says that a topological space X is T_0 if and only if it is a subspace of a product of copies of $\mathfrak{S}$, the Sierpinski space. Furthermore, if X is in addition finite, it is a subspace of a product of finitely many copies of $\mathfrak{S}$. Using the correspondence between T_0-spaces and posets, this result can be expressed as follows: A finite preorder X is a poset if and only if it is a subpreorder of $\mathbf{2}^n$ for some n.

From this result it seems natural to define the 2-dimension of a finite T_0-space as the minimum $n \in \mathbb{N}_0$ such that X is a subspace of $\mathfrak{S}^n$. This coincides with the classical definition of the 2-dimension of the poset X.

In 1963, Novák [11] introduced the notion of the k-dimension of a finite poset X (for any integer $k \geq 2$), extending the definition of dimension of posets given by Dushnik and Miller [4]. One of the first studies on 2-dimension is the foundational paper [15] by Trotter. In that paper he introduces the notion of the n-cube Q_n and defines the 2-dimension of a finite poset X as the smallest positive integer n such that X can be embedded as a subposet of Q_n. He also proves in [15] and [16] the classical formulas and bounds for the 2-dimension.

In the last 30 years the theory of the 2-dimension was studied by many mathematicians and computer scientists. New results and improvements of known results were obtained, but of course there is still much to investigate [5, 16, 17].

In this paper we will show how the classical bounds for the 2-dimension of a poset of cardinality n can be obtained from the topological point of view. Moreover, we will use homotopy theory of finite spaces to improve the classical results on 2-dimension, giving a more complete answer to the problem of all possible 2-dimensions of n-point posets.

The concept of a *beat point* of a finite T_0-space (poset) introduced by Stong [14] plays an essential role in our results. Explicitly, we prove below the following proposition.

Proposition 4.4. *Let X be a finite T_0-space (poset) and let $x \in X$ be a beat point. Then*

$$d(X) - 1 \leq d(X \smallsetminus \{x\}) \leq d(X).$$

Here $d(X)$ denotes the 2-dimension of X. This result improves (in the case of beat points) the *continuity* property of the 2-dimension. As an immediate consequence we have:

Corollary 4.5. *Let X be a finite contractible T_0-space. Then $d(X) \leq |X| - 1$.*

Using these results and the notion of *non-Hausdorff suspension* of a topological space [10], we deduce our main theorem:

Theorem 4.10. *Given $n \geq 2$ and m such that $\lceil \log_2 n \rceil \leq m \leq n$, there exists a T_0-space (poset) X of cardinality n with $d(X) = m$. Moreover, if $m \neq n$, X can be taken contractible.*

2 Preliminaries: topologies, preorders and initial maps

We start by recalling the basic correspondence between topologies and preorders on a finite set.

The first mathematician who related finite topological spaces with preorders was Alexandroff [1]. Many years later, Stong [14] and McCord [10] continued Alexandroff's ideas. Recently, a paper by Osaki [12] and a beautiful series of notes by Peter May [6, 7, 8] captured the attention of algebraic topologists. In [2] we used combinatorial techniques based on finite spaces and posets to solve topological problems concerning the homotopy groups of the spheres.

Given a finite topological space (X, τ) and $x \in X$, we define the *minimal open set* U_x of x as the intersection of all the open sets containing x. The preorder on X associated to the topology τ is defined as follows: $x \leq y$ if $x \in U_y$.

Conversely, given a preorder $\leq$ on X, we define for each $x \in X$ the sets

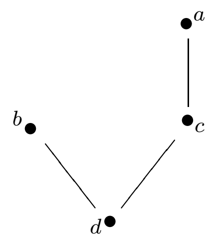
$$U_x = \{y \in X \mid y \leq x\}.$$

It is easy to see that these sets form a basis for a topology, which will be the topology associated to $\leq$.

These applications are mutually inverse. Moreover, the order relations on X are in correspondence with the T_0-topologies on X. Recall that a space X is said to be T_0 if for every pair of points in X there exists some open set that contains one and only one of those points.

From now on, we will identify finite T_0-spaces with finite posets.

Example 2.1. Let $X = \{a, b, c, d\}$ the 4-point space whose open subsets are $\emptyset, \{a, c, d\}$, $\{b, d\}$, $\{c, d\}$, $\{d\}$, $\{a, b, c, d\}$, $\{b, c, d\}$. Then X is a T_0-space with Hasse diagram



Note that there exists a bijection between the topology of a T_0-space X and the antichains of the poset X that assigns to each open subset of X the antichain of its maximal elements.

Remark 2.2. The correspondence between topologies and preorders takes products to products, disjoint unions to disjoint unions and subspaces to subpreorders.

It is not hard to prove that a function between finite spaces is continuous if and only if it is order preserving.

The concept of initial map or initial topology is related to the notions of subspace and product [3].

A map $f : X \to Y$ is initial if the topology on X is induced by f. More precisely

Definition 2.3. A function $f : X \to Y$ between topological spaces is an *initial map* (or X has the *initial topology* with respect to f) if the topology of X is the coarsest such that f is continuous. Explicitly, $U \subseteq X$ is open if and only if there exists an open set $V \subseteq Y$ such that $U = f^{-1}(V)$.

More generally, we say that a family $\{f_\lambda : X \to X_\lambda\}_{\lambda\in\Lambda}$ of functions between topological spaces is an *initial family* (or that X has the *initial topology* with respect to $\{f_\lambda\}_{\lambda\in\Lambda}$) if the topology of X is the coarsest such that f_λ is continuous for every $\lambda \in \Lambda$.

Proposition 2.4. *Let $\{f_\lambda : X \to X_\lambda\}_{\lambda\in\Lambda}$ be a family of functions between topological spaces. The following are equivalent*

(*i*) $\{f_\lambda\}_{\lambda\in\Lambda}$ *is initial.*

(*ii*) $\{f_\lambda^{-1}(U) |\ \lambda \in \Lambda,\ U \subseteq X_\lambda$ *is open*$\}$ *is a subbase of the topology of* X.

(*iii*) *For every space Z and every function $g : Z \to X$, g is continuous if and only if $f_\lambda g$ is continuous for every $\lambda \in \Lambda$.*

Example 2.5. Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of spaces. Then the family of projections $\{\prod\limits_{\lambda\in\Lambda} X_\lambda \overset{p_\gamma}{\to} X_\gamma\}_{\gamma\in\Lambda}$ is an initial family.

It is easy to see that a family $\{f_\lambda : X \to X_\lambda\}_{\lambda\in\Lambda}$ is initial if and only if the induced function $f : X \to \prod\limits_{\lambda\in\Lambda} X_\lambda$ is an initial map.

If $A \subseteq Y$ is a subspace, then the inclusion $i : A \hookrightarrow Y$ is initial. One can generalize the concept of subspace as follows.

Definition 2.6. A function $i : X \to Y$ is a *subspace map* if it is initial and injective.

If $i : X \to Y$ is a subspace map, i becomes a homeomorphism between X and its image $i(X)$ viewed as a subspace of Y. We simply denote $X \subseteq Y$.

We denote $\mathfrak{S} = \{0, 1\}$ the Sierpinski space, whose unique proper open set is $\{0\}$. Note that $\mathfrak{S} = \mathbf{2}$ is the finite T_0-space (order), with $0 < 1$.

As we pointed out in the introduction, it is a basic topological fact that a space is T_0 if and only if it is a subspace of a product of copies of $\mathfrak{S}$ (cf. [3]). Explicitly

Proposition 2.7. *Let X be T_0-space. Then the function*

$$i : X \to \prod_{\substack{h:X\to\mathfrak{S} \\ continuous}} \mathfrak{S}$$

defined by $i(x) = (h(x))_h$ is a subspace map.

We will use this result to define the 2-dimension of a finite T_0-space.

3 Initial maps and subposets

We give a characterization of initial maps between finite spaces in terms of preorders.

Proposition 3.1. *Let $i : X \to Y$ be a function between finite spaces. The following are equivalent:*

(i) i is initial.

(ii) For every $x, x' \in X$ it holds that $x \le x'$ if and only if $i(x) \le i(x')$.

Proof. If i is an initial map, then it is continuous and therefore order preserving. Suppose that $x, x' \in X$ are such that $i(x) \le i(x')$, and suppose that $i^{-1}(U)$ is an open set of X that contains x'. Then $i(x') \in U$, and therefore $i(x) \in U_{i(x')} \subseteq U$ which implies that $x \in i^{-1}(U)$. It follows that $x \in U_{x'}$ and hence $x \le x'$.

Conversely, if condition (ii) holds, i is order preserving and then continuous. We want to show that the topology τ of X is the coarsest that makes i continuous. Suppose that τ' is another topology which induces the preorder $\preceq$ and makes i continuous. If $x \preceq x'$, then $i(x) \le i(x')$ and, by (ii), $x \le x'$. Since $id : (X, \tau') \to (X, \tau)$ is order preserving, it is continuous, and then $\tau \subseteq \tau'$. □

Suppose now that $i : X \to Y$ is initial and x, x' are such that $i(x) = i(x')$. Since $i(x) \le i(x')$ and $i(x) \ge i(x')$, it follows by the previous proposition that $x \le x'$ and $x \ge x'$. Therefore, if the preorder on X is antisymmetric, i becomes injective. Then we obtain the following

Corollary 3.2. *Let X, Y be finite spaces such that X is T_0 and suppose that $i : X \to Y$ is an initial map. Then i is a subspace map.*

Remark 3.3. In fact, the previous result works for infinite spaces as well. But, as we have seen, the proof is very simple in the finite space case by the tractability of posets.

By 2.7 every finite T_0-space is a subspace of a product of finitely many copies of $\mathfrak{S}$. Then, the following definition makes sense.

Definition 3.4. Let X be a finite T_0-space. We define the 2-*dimension* of X as the minimum $n \in \mathbb{N}_0$ such that X is a subspace of a product of n copies of $\mathfrak{S}$.

Note that the 2-dimension is well defined by 2.7 and that $2^{|X|}$ is an upper bound. Here $|X|$ denotes the cardinality of X. The 2-dimension of X will be denoted $d(X)$.

Remark 3.5. By 2.2 it is clear that X is a subspace of a product of n copies of $\mathfrak{S}$ if and only if X is a subposet of $\mathbf{2}^n$. From 2.7 one can deduce that every finite poset is a subposet of a finite boolean algebra. It is easy to see that our definition of 2-dimension coincides with the classical definition in terms of posets.

A topological interpretation of the monotonicity of the 2-dimension could be the following. If X is a subposet of Y, then X is a subspace of Y, which is a subspace of $\mathfrak{S}^{d(Y)}$. Therefore $d(X) \leq d(Y)$.

Now we prove the well-known result on the bounds of the 2-dimension of an n-point T_0-space, in terms of topology.

Proposition 3.6. *Let X be a finite non-empty T_0-space. Then*

$$\lceil \log_2 |X| \rceil \leq d(X) \leq |X|.$$

Proof. The first inequality is trivial because $X \subseteq \mathfrak{S}^{d(X)}$ and then $|X| \leq 2^{d(X)}$. In order to prove the second inequality let us consider the function $h : X \to \prod_{x \in X} \mathfrak{S} = \mathfrak{S}^{|X|}$ defined by $h(y) = (\chi_{U_x^c}(y))_{x \in X}$, where $\chi_{U_x^c}$ is the characteristic function of $U_x^c = X \smallsetminus U_x$ (i.e. $\chi_{U_x^c}(y) = 1$ if $y \notin U_x$ and $\chi_{U_x^c}(y) = 0$ in other case).

This is a continuous map because $p_x h = \chi_{U_x^c}$ is continuous for every $x \in X$.

If $U \subseteq X$ is open,

$$U = \bigcup_{x \in U} U_x = \bigcup_{x \in U} \chi_{U_x^c}^{-1}(\{0\}) = \bigcup_{x \in U} h^{-1}(p_x^{-1}(\{0\})) = h^{-1}(\bigcup_{x \in U} p_x^{-1}(\{0\})).$$

Since $p_x^{-1}(\{0\}) \subseteq \mathfrak{S}^{|X|}$ is open for every $x \in X$, U is an open set in the initial topology. It follows that h is initial, and by 3.2, h is subspace. □

4 Homotopy and 2-dimension

In 1966 Stong classified finite spaces by their homotopy types [14]. His ideas turned out to be very illuminating to improve the well known result of 3.6.

First we recall some definitions from [14], although this formulation is from [6].

Definition 4.1. Let X be a finite T_0-space. We say that $x \in X$ is an *up-beat point* if $\{y \in X | \ y > x\}$ has a minimum. Analogously, we say that x is a *down-beat point* if $\{y \in X | \ y < x\}$ has a maximum. In either of these cases we say that x is a *beat point.*

We say that a finite space is a *minimal finite space* if it is T_0 and it has no beat points.

We recall that a subspace X of a topological space Y is a strong deformation retract of Y if there exists a continuous retraction $r : Y \to X$ of the inclusion $i : X \hookrightarrow Y$, such that ir and the identity of Y are homotopic with a homotopy that is stationary on $i(X)$.

Note that if X is a strong deformation retract of Y, X and Y have the same homotopy type.

Stong proves that if x is a beat point of a finite T_0-space X, then $X \smallsetminus \{x\}$ is a strong deformation retract of X.

Definition 4.2. Let X be a finite space. A subspace $Y \subseteq X$ is a *core* of X if it is a minimal finite space which is a strong deformation retract of X.

Every finite space X has a core, the core X_c is unique up to isomorphism and it is the smallest space which is homotopy equivalent to X. Moreover, if X is T_0, there exists a sequence $X = X_0 \supset X_1 \supset X_2 \supset \ldots \supset X_n = X_c$ where X_{i+1} is obtained from X_i by removing a beat point.

Remark 4.3. If X is a finite T_0-space with maximum m, a maximal point x of $X \smallsetminus \{m\}$ is an up-beat point. Now $X \smallsetminus \{x\}$ is homotopy equivalent to X and has maximum m. By an inductive argument, it follows that X is homotopy equivalent to a point, i.e. contractible. A topological proof of this fact can be found in [6, 10].

If X is a finite space, we can define the space X^{op} whose open sets are the closed sets of X. It is easy to prove that the induced preorder in X^{op} is the opposite (or dual) of X.

It is well known that if X is a finite poset and $x \in X$, then $d(X) - 2 \leq d(X \smallsetminus \{x\}) \leq d(X)$. We improve this result in the case that x is a beat point.

Proposition 4.4. *Let X be a finite T_0-space (poset) and let $x \in X$ be a beat point. Then*

$$d(X) - 1 \leq d(X \smallsetminus \{x\}) \leq d(X).$$

Proof. The second inequality is clear by the monotonicity of d.

Now suppose that $x \in X$ is an up-beat point. Then there exists $y > x$ such that $z > x$ implies $z \geq y$.

We regard the elements of $\mathfrak{S}^n$ as n-tuples of 0's and 1's. If $t \in \mathfrak{S}^{d(X \smallsetminus \{x\})}$, we denote by $t0 \in \mathfrak{S}^{d(X \smallsetminus \{x\})+1}$ the $d(X \smallsetminus \{x\})$-tuple t followed by a 0. Similarly, $t1$ denotes the $d(X \smallsetminus \{x\})$-tuple t followed by a 1.

Let $i : X \smallsetminus \{x\} \hookrightarrow \mathfrak{S}^{d(X \smallsetminus \{x\})}$ be a subspace map. In this case we define $i' : X \to \mathfrak{S}^{d(X \smallsetminus \{x\})+1}$ in the following way

$$i'(z) = \begin{cases} i(y)0 & \text{if } z = x \\ i(z)0 & \text{if } z < x \\ i(z)1 & \text{in other case.} \end{cases}$$

If we prove that i' is an initial map, then by 3.2, i' is subspace and $d(X) \leq d(X \smallsetminus \{x\}) + 1$.

We will use proposition 3.1 to show that i' is initial. We first show that i' is order preserving. Suppose $z < z'$

- If $z = x$, $z' > x$ and then $z' \geq y$. Since i is order preserving, $i(z') \geq i(y)$. Therefore $i(z')1 \geq i(y)0$ and $i'(z') \geq i'(z)$.
- If $z' = x$, $z < x < y$, then $i(z) \leq i(y)$ and therefore $i(z)0 \leq i(y)0$. It follows that $i'(z) \leq i'(z')$.
- The case $z \neq x \neq z'$ is clear.

Now suppose that $z \neq z'$ are such that $i'(z) \leq i'(z')$.

- If $z = x$, $i(y)0 \leq i'(z')$ and then $i(y) \leq i(z')$. By 3.1 $z = x < y \leq z'$.
- If $z' = x$, $i'(z')$ ends in 0 and then, so does $i'(z)$. Therefore $z \leq x = z'$.
- In other case, $i'(z) \leq i'(z')$ implies $i(z) \leq i(z')$ and since i is initial, $z \leq z'$.

Again by 3.1, it follows that i' is initial.

If $x \in X$ is a down-beat point, the result follows from the up-beat point case, considering X^{op}. □

Corollary 4.5. *Let X be a finite contractible T_0-space. Then $d(X) \leq |X| - 1$.*

Proof. Since X is contractible, its core X_c consists of only one point. If $X = X_c = *$ the result is trivial. Otherwise, X is not minimal and it has a beat point x. The space $X \smallsetminus \{x\}$ is a strong deformation retract of X and then contractible. By an inductive argument

$$d(X \smallsetminus \{x\}) \leq |X \smallsetminus \{x\}| - 1 = |X| - 2.$$

Now the result follows from the previous proposition. □

There are two standard constructions in Topology, more precisely in Homotopy Theory: the cone and the suspension of a topological space. These constructions are not useful for finite spaces, but in the finite case there are two analogous constructions introduced in [10], namely, the *non-Hausdorff cone* and the *non-Hausdorff suspension.*

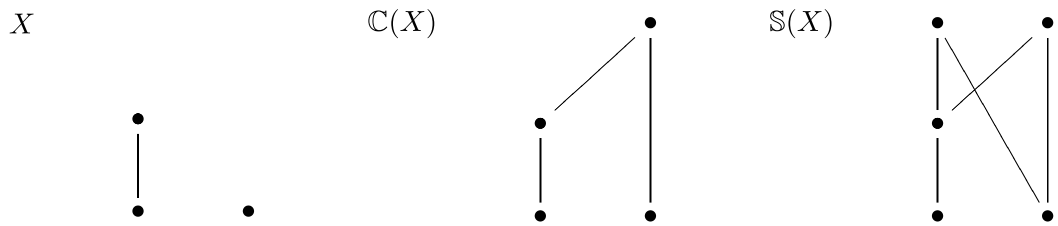
Definition 4.6. (McCord) We define $\mathbb{C}(X)$ the *non-Hausdorff cone* of a space X as the space $X \cup \{*\}$, whose open sets are the open sets of X together with $X \cup \{*\}$.

We define $\mathbb{S}(X)$ the *non-Hausdorff suspension* of X as the space $X \cup \{+, -\}$, whose open sets are those of X, together with $X \cup \{+\}$, $X \cup \{-\}$ and $X \cup \{+, -\}$. We define recursively the *n-fold non-Hausdorff suspension* of X by $\mathbb{S}^n(X) = \mathbb{S}(\mathbb{S}^{n-1}(X))$.

Note that if X is T_0 and finite, the induced order on $\mathbb{C}(X)$ is the one that results when we add a maximum to X, and $\mathbb{S}(X)$ results when we add to X two incomparable points $+$ and $-$ which are greater than all the points of X.

In terms of the join operation $\oplus$ of posets, it is easy to see that $\mathbb{C}(X) = X \oplus *$ and $\mathbb{S}(X) = X \oplus S^0$, where $*$ is the 1-point poset and $S^0 = D_2$ denotes the 2-point antichain (2-point discrete space).

Example 4.7. Let $X = \mathfrak{S} \sqcup *$, then the Hasse diagrams of X, $\mathbb{C}(X)$ and $\mathbb{S}(X)$ are the following



The construction of $\mathbb{S}(X)$ becomes important when we work with finite models of spheres (finite spaces with the same homotopy groups of the spheres) [2, 10]. However we give here a completely different use of this construction.

We deduce from Theorem 2.3 of [15] that for every finite T_0-space X,

$$d(\mathbb{S}(X)) = d(X) + d(S^0) = d(X) + 2.$$

Inductively one proves that $d(\mathbb{S}^n(S^0)) = 2n + 2 = |\mathbb{S}^n(S^0)|$ and $d(\mathbb{S}^n(D_3)) = 2n + 3 = |\mathbb{S}^n(D_3)|$ for every $n \geq 0$. Therefore we have the following

Lemma 4.8. *Given $n \geq 2$, there exists a T_0-space (poset) with n points and whose 2-dimension is n.*

The next result follows from 4.3 and 4.5.

Lemma 4.9. *Let $n \in \mathbb{N}$ and $X = [\![1, n]\!]$ the T_0-space with the usual order. Then $d(X) = n - 1$.*

Now we state the main result of this paper.

Theorem 4.10. *Given $n \geq 2$ and m such that $\lceil \log_2 n \rceil \leq m \leq n$, there exists a T_0-space (poset) X of cardinality n with $d(X) = m$. Moreover, if $m \neq n$, X can be taken contractible.*

Proof. The case $m = n$ follows from 4.8, so it remains to analize the case $\lceil \log_2 n \rceil \leq m \leq n - 1$.

Let $X_1 = \{u_1, \ldots, u_n\}$ be an n-point subspace of $\mathfrak{S}^{\lceil \log_2 n \rceil}$ such that u_1 is the maximum of $\mathfrak{S}^{\lceil \log_2 n \rceil}$. By 3.6, it follows that

$$d(X_1) = \lceil \log_2 n \rceil. \tag{1}$$

We define recursively $X_2, \dots, X_n$ by $X_{i+1} = \mathbb{C}(X_i \smallsetminus \{u_{i+1}\})$ and we denote by u'_i the maximum of X_i. This means that the spaces are constructed from the previous ones by removing a point and putting it in the top.

For every $2 \leq i \leq n$, u'_i is a down-beat point of X_i because $X_i \smallsetminus \{u'_i\}$ has maximum. By 4.4, if $1 \leq j < n$, $d(X_{j+1} \smallsetminus \{u'_{j+1}\}) \geq d(X_{j+1}) - 1$. However, $X_{j+1} \smallsetminus \{u'_{j+1}\} = X_j \smallsetminus \{u_{j+1}\}$ is a subspace of X_j, and then $d(X_j) \geq d(X_{j+1} \smallsetminus \{u'_{j+1}\}) \geq d(X_{j+1}) - 1$. Therefore

$$d(X_{j+1}) \leq d(X_j) + 1. \tag{2}$$

Every space in this sequence has maximum, and by 4.3 is contractible. Moreover, X_n is linear, therefore, by 4.9

$$d(X_n) = n - 1. \tag{3}$$

From (1), (2) and (3) it follows that the sequence $d(X_i)$, with $1 \leq i \leq n$, takes all the values between $\lceil \log_2 n \rceil$ and $n - 1$. This completes the proof. □

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