ON QUILLEN'S THEOREM A FOR POSETS

JONATHAN ARIEL BARMAK †

Abstract. A theorem of McCord of 1966 and Quillen's Theorem A of 1973 provide sufficient conditions for a map between two posets to be a homotopy equivalence at the level of complexes. We give an alternative elementary proof of this result and we deduce also a stronger statement: under the hypotheses of the theorem, the map is not only a homotopy equivalence but a simple homotopy equivalence. This leads then to stronger formulations of the simplicial version of Quillen's Theorem A, the Nerve Lemma and other known results. In particular we establish a conjecture of Kozlov on the simple homotopy type of the crosscut complex and we improve a well-known result of Cohen on contractible mappings.

1. Introduction

In his seminal paper [23] McCord gives a criterion for recognizing weak homotopy equivalences between topological spaces (i.e. maps which induce isomorphisms in all the homotopy groups). Roughly speaking, his theorem [23, Theorem 6] says that a map is a weak homotopy equivalence if it is locally a weak homotopy equivalence. This result allowed him to establish the relationship between the homotopy theory of finite topological spaces and finite complexes.

Given a finite poset X, the *associated complex* (also called *order complex*) $\mathcal{K}(X)$ is the simplicial complex whose simplices are the non-empty chains of X. An order preserving map $f : X \to Y$ between finite posets induces a simplicial map $\mathcal{K}(f) : \mathcal{K}(X) \to \mathcal{K}(Y)$ which coincides with f on vertices. A finite poset X can be considered as a finite topological space and it can be deduced from McCord's Theorem that there is a weak homotopy equivalence $\mathcal{K}(X) \to X$.

The celebrated Theorem A of Quillen [25] provides a sufficient condition for a functor between two categories to induce a homotopy equivalence between the classifying spaces.

Although these powerful and general results apply in very different contexts, they have a particular common application, which is without discussion one of the most useful known tools to study the relationship between posets and homotopy theory. The McCord-Quillen Theorem 1.1, many times referred to as "Quillen's Fiber Lemma", is on one hand McCord's Theorem applied to finite spaces and the covers given by the minimal bases, and on the other hand Quillen's Theorem A applied to finite posets.

Theorem 1.1 (McCord, Quillen). *Let $f : X \to Y$ be an order preserving map between two finite posets. Suppose that for every $y \in Y$, the complex $\mathcal{K}(f^{-1}(U_y))$ is contractible. Then $\mathcal{K}(f)$ is a homotopy equivalence.*

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Here, $U_y \subseteq Y$ denotes the subset of elements which are smaller than or equal to y. Quillen's statement is implicit in [25] and explicit in [26, Proposition 1.6]. Theorem 1.1 has shown to be indispensable in the study of the topology of order complexes of posets. Some important consequences are, for example, the simplicial version of Theorem A, the so called Nerve Lemma, Dowker's Theorem on complexes associated to a relation and the Crosscut Theorem. Other versions of Theorem 1.1 can be found in [2, 9, 10].

Both McCord's Theorem and Quillen's Theorem A have technical nontrivial proofs. In [27] Walker gives an elementary proof of Theorem 1.1 using a homotopy version of the Acyclic Carrier Theorem. In this article we give a different proof of Theorem 1.1. Our proof is also very basic but the most important consequence is that it can be easily improved to obtain a stronger statement of the theorem.

Whitehead's simple homotopy theory aimed to give a combinatorial description of homotopy types of simplicial complexes. This theory is of great importance for its applications to combinatorial group theory [19], differential topology and piecewise-linear topology. One of its most crucial applications is the s-cobordism Theorem which is used to prove the Poincaré Conjecture in dimensions greater than or equal to 5 and which is a fundamental part of the surgery program [22].

The concepts of simplicial collapse and expansion give rise to the notions of simple homotopy types and simple homotopy equivalences. Simple homotopy equivalent complexes are homotopy equivalent and simple homotopy equivalences are homotopy equivalences, but these implications are strict. These notions coincide for instance in the case of simply connected complexes. We will show that under the same hypotheses as in Theorem 1.1, the simplicial map $\mathcal{K}(f)$ is not only a homotopy equivalence but a simple homotopy equivalence.

Theorem 1.2. *Let $f : X \to Y$ be an order preserving map between two finite posets. Suppose that for every $y \in Y$, the complex $\mathcal{K}(f^{-1}(U_y))$ is contractible. Then $\mathcal{K}(f)$ is a simple homotopy equivalence.*

Theorem 1.2 originally appears in the author's Thesis [3, Proposition 6.2.9] formulated in the setting of finite spaces. In Section 4 we present a self contained proof of the simplicial statement which is more transparent than the one of [3]. From this result we immediately obtain stronger formulations of the simplicial version, the Nerve Lemma and Dowker's Theorem. We also deduce a simple homotopy version of the Crosscut Theorem, settling in this way a conjecture by Kozlov [21, Conjecture 5.6]. Theorem 1.2 is used to provide an alternative proof of a well-known result by Cohen on contractible mappings. Moreover, we will show that our simplicial variant of Theorem A, Theorem 4.2, is a stronger version of Cohen's result.

The key point of our approach is the so called *non-Hausdorff mapping cylinder* of a map between posets introduced by Barmak and Minian in [5] where it is used to establish the relationship between finite topological spaces (finite posets) and simple homotopy theory of polyhedra.

In the last section of the paper we adapt the method used in the proof of Theorem 1.1 to give a short proof of one extension of this result for n-equivalences and of its homological version.

2. Preliminaries

The star $\mathrm{st}_K(v)$ of a vertex v in a simplicial complex K is the subcomplex of simplices $\sigma \in K$ such that $\sigma \cup \{v\} \in K$. The link $\mathrm{lk}_K(v)$ is the subcomplex of $\mathrm{st}_K(v)$ of simplices which do not contain v. The join of two (disjoint) simplicial complexes K and L is the simplicial complex $K * L$ whose simplices are those of K, those of L and unions of a simplex of K with a simplex of L. If two complexes are homotopy equivalent, their joins with a third complex are also homotopy equivalent. In particular, the join of a contractible complex with another complex is contractible. For simplicity we will identify a simplicial complex with its geometric realization.

The following result [26, 1.3] is a particular case of the well known fact that natural transformations induce homotopies in the classifying spaces. We include a simple proof for completeness which appears in the author's Thesis [3, Proposition 2.1.2] and in [7].

Lemma 2.1. *Let $f, g : X \to Y$ be two order preserving maps between finite posets. Suppose that $f(x) \leq g(x)$ for every $x \in X$. Then $\mathcal{K}(f)$ and $\mathcal{K}(g)$ are homotopic.*

Proof. We proceed by induction on the number of order preserving maps $h : X \to Y$ such that $f(x) \leq h(x) \leq g(x)$ for every $x \in X$. Suppose that $f \neq g$. Let $x \in X$ be a maximal point with the property that $f(x) \neq g(x)$. Let $y \in Y$ be an element covering $f(x)$ and such that $y \leq g(x)$. Consider the map $h : X \to Y$ which coincides with f in every point different from x and such that $h(x) = y$. By the maximality of x, h is order preserving. The simplicial maps $\mathcal{K}(f)$ and $\mathcal{K}(h)$ are contiguous (i.e. $\mathcal{K}(f)(\sigma) \cup \mathcal{K}(h)(\sigma) \in \mathcal{K}(Y)$ for every simplex $\sigma \in \mathcal{K}(X)$) and in particular the linear homotopy between them is well defined and continuous. By induction $\mathcal{K}(h) \simeq \mathcal{K}(g)$ and therefore $\mathcal{K}(f) \simeq \mathcal{K}(g)$. □

3. An alternative proof of the McCord-Quillen Theorem 1.1

The idea of our approach is to prove the theorem in some very particular cases in which the map is just an inclusion of a poset into another poset with only one more point. The general case will follow taking compositions of these basic maps and homotopy inverses.

The next result follows immediately from Theorem 1.1 as it is observed in [27, Proposition 6.1] (see also [5]) but here we use a different idea (cf. [6, Proposition 3.10]) since we will need it in the proof of the theorem.

Given a finite poset X, we will denote

$$U_x^X = \{x' \in X \mid x' \leq x\},\ F_x^X = \{x' \in X \mid x' \geq x\},$$

$$\hat{U}_x^X = \{x' \in X \mid x' < x\},\ \hat{F}_x^X = \{x' \in X \mid x' > x\}.$$

When there is no risk of confusion we will just write $U_x, F_x, \hat{U}_x$ and $\hat{F}_x$.

Lemma 3.1. *Let X be a finite poset and let $x \in X$ be such that $\mathcal{K}(\hat{U}_x)$ or $\mathcal{K}(\hat{F}_x)$ is contractible. Then $\mathcal{K}(X \smallsetminus \{x\}) \hookrightarrow \mathcal{K}(X)$ is a homotopy equivalence.*

Proof. By hypothesis, $\mathrm{lk}_{\mathcal{K}(X)}(x) = \mathcal{K}(\hat{U}_x) * \mathcal{K}(\hat{F}_x)$ is contractible. Therefore the inclusion $\mathrm{lk}_{\mathcal{K}(X)}(x) = \mathrm{st}_{\mathcal{K}(X)}(x) \cap \mathcal{K}(X \smallsetminus \{x\}) \hookrightarrow \mathrm{st}_{\mathcal{K}(X)}(x)$ is a homotopy equivalence. Moreover, being an inclusion of complexes, it is a strong deformation retract (see [18, Proposition 0.16, Corollary 0.20]). Then, $\mathcal{K}(X \smallsetminus \{x\})$ is a strong deformation retract of $\mathrm{st}_{\mathcal{K}(X)}(x) \cup \mathcal{K}(X \smallsetminus \{x\}) = \mathcal{K}(X)$. □

Definition 3.2. Let $f : X \to Y$ be an order preserving map between finite posets. The *non-Hausdorff mapping cylinder* $B(f)$ of f is a poset whose underlying set is the disjoint union $X \sqcup Y$. The given ordering within X and Y is kept and for $x \in X$, $y \in Y$ one has $x \leq y$ in $B(f)$ if $f(x) \leq y$ in Y. Therefore, the cover relations in $B(f)$ are the cover relations of X, those of Y and of the form $x < f(x)$ for $x \in X$. The canonical inclusions of X and Y into the non-Hausdorff mapping cylinder will be denoted by $i : X \hookrightarrow B(f)$ and $j : Y \hookrightarrow B(f)$.

Proposition 3.3. *Let $f : X \to Y$ be an order preserving map between finite posets. Then $\mathcal{K}(j) : \mathcal{K}(Y) \to \mathcal{K}(B(f))$ is a homotopy equivalence.*

Proof. Let $x_1, x_2, \ldots, x_n$ be a linear extension of X (i.e. an ordering of the elements of X such that $x_r \leq x_s$ implies $r \leq s$) and denote $Y_r = Y \cup \{x_1, x_2, \ldots, x_r\} \subseteq B(f)$ for each $0 \leq r \leq n$. Then

$$\hat{F}_{x_r}^{Y_r} = \{y \in Y \mid y \geq f(x_r)\} = F_{f(x_r)}^{Y}.$$

Therefore $\mathcal{K}(\hat{F}_{x_r}^{Y_r}) = \mathcal{K}(F_{f(x_r)}^{Y})$ is a cone (with apex $f(x_r)$) and in particular, contractible. By Lemma 3.1, $\mathcal{K}(Y_{r-1}) \hookrightarrow \mathcal{K}(Y_r)$ is a homotopy equivalence and then the inclusion $\mathcal{K}(j) : \mathcal{K}(Y) = \mathcal{K}(Y_0) \hookrightarrow \mathcal{K}(Y_n) = \mathcal{K}(B(f))$ is also a homotopy equivalence. □

Another way to see that $\mathcal{K}(j)$ is a homotopy equivalence, is by defining a retraction $r : B(f) \to Y$ with $r(x) = f(x)$ for every $x \in X$. Since $jr \geq 1_{B(f)}$, by Lemma 2.1 $\mathcal{K}(j)$ is a homotopy equivalence with homotopy inverse $\mathcal{K}(r)$. The idea shown in our proof of Proposition 3.3 will be used in the proof of Theorem 1.1 and in its simple version.

Proof of Theorem 1.1. Consider the non-Hausdorff mapping cylinder $B(f)$. Following the idea of the proof of Proposition 3.3 we show that $\mathcal{K}(i) : \mathcal{K}(X) \to \mathcal{K}(B(f))$ is a homotopy equivalence. Let $y_1, y_2, \ldots, y_m$ be a linear extension of Y and let $X_r = X \cup \{y_{r+1}, y_{r+2}, \ldots, y_m\} \subseteq B(f)$ for every $0 \leq r \leq m$. Then

$$\hat{U}_{y_r}^{X_{r-1}} = \{x \in X \mid f(x) \leq y_r\} = f^{-1}(U_{y_r}^{Y}).$$

By hypothesis, $\mathcal{K}(\hat{U}_{y_r}^{X_{r-1}})$ is contractible. By Lemma 3.1, $\mathcal{K}(X_r) \hookrightarrow \mathcal{K}(X_{r-1})$ is a homotopy equivalence and then so is $\mathcal{K}(i) : \mathcal{K}(X) = \mathcal{K}(X_m) \hookrightarrow \mathcal{K}(X_0) = \mathcal{K}(B(f))$.

Since $i(x) \leq jf(x)$ for every $x \in X$, by Lemma 2.1, $\mathcal{K}(i) \simeq \mathcal{K}(jf) = \mathcal{K}(j)\mathcal{K}(f)$. Since $\mathcal{K}(i)$ is a homotopy equivalence and $\mathcal{K}(j)$ is a homotopy equivalence by Proposition 3.3, then so is $\mathcal{K}(f)$. □

4. A simple stronger statement

We will show that the proof of Theorem 1.1 can be easily modified to obtain the stronger Theorem 1.2.

If K is a finite simplicial complex with a simplex τ which is a proper face of a unique simplex σ, we say that there is an elementary collapse from K to the subcomplex $L \subset K$ which is obtained from K by removing the simplices σ and τ. If there is a sequence of elementary collapses from a complex K to a subcomplex L, we say that K collapses to L. Two complexes have the same simple homotopy type if it is possible to obtain one from the other by performing collapses and their inverses (expansions). A class of maps $\mathcal{C}$ between topological spaces is said to satisfy the 2-out-of-3 property if whenever there are three maps f, g, h such that the composition fg is well defined, $fg \simeq h$ and two of the three maps are in $\mathcal{C}$, then so is the third. The class of simple homotopy equivalences is the smallest class satisfying the 2-out-of-3 property and containing all the inclusions $L \hookrightarrow K$ where K is a complex and L is a subcomplex which expands to K (see [20, p. 118]). Note that this definition is not exactly the same as it appears in [20] but it is easily shown to be equivalent to that one. For basic properties on simple homotopy theory we encourage the readers to consult [13].

Theorem (20.1) of [13] states that if L is a subcomplex of a complex K, the inclusion $L \hookrightarrow K$ is a homotopy equivalence and every connected component of the space $K \smallsetminus L$ is simply connected, then $L \hookrightarrow K$ is a simple homotopy equivalence.

From this result and Lemma 3.1 we obtain a refined statement of Lemma 3.1. Note that if X is a finite poset and $x \in X$, then the space $\mathcal{K}(X) \smallsetminus \mathcal{K}(X \smallsetminus \{x\})$ is the open star of x in $\mathcal{K}(X)$ (i.e. the union of the interiors of the simplices containing x) which is contractible.

Lemma 4.1. *Let X be a finite poset and let $x \in X$ be such that $\mathcal{K}(\hat{U}_x)$ or $\mathcal{K}(\hat{F}_x)$ is contractible. Then $\mathcal{K}(X \smallsetminus \{x\}) \hookrightarrow \mathcal{K}(X)$ is a simple homotopy equivalence.*

Proof of Theorem 1.2. The proof is essentially the same as before, but in the proofs of Proposition 3.3 and Theorem 1.1 we use Lemma 4.1 instead of Lemma 3.1. Note that the inclusions $\mathcal{K}(Y_{r-1}) \hookrightarrow \mathcal{K}(Y_r)$ in the proof of Proposition 3.3 are simple homotopy equivalences by Lemma 4.1. Therefore their composition $\mathcal{K}(j) : \mathcal{K}(Y) \to \mathcal{K}(B(f))$ is a simple homotopy equivalence (in fact $\mathcal{K}(B(f))$ collapses to $\mathcal{K}(Y)$ [5]). Analogously, the inclusions $\mathcal{K}(X_r) \hookrightarrow \mathcal{K}(X_{r-1})$ in the proof of Theorem 1.1 are simple homotopy equivalences and then so is $\mathcal{K}(i) : \mathcal{K}(X) \to \mathcal{K}(B(f))$. Now, since $\mathcal{K}(i) \simeq \mathcal{K}(j)\mathcal{K}(f)$, by the 2-out-of-3 property, $\mathcal{K}(f)$ is a simple homotopy equivalence. $\square$

Given a finite simplicial complex K, its *associated poset* $\mathcal{X}(K)$ (also known as face poset) is the poset of simplices of K ordered by containment. A simplicial map $\varphi : K \to L$ also has associated an order preserving map $\mathcal{X}(\varphi) : \mathcal{X}(K) \to \mathcal{X}(L)$ defined by $\mathcal{X}(\varphi)(\sigma) = \varphi(\sigma)$. Note that $\mathcal{K}(\mathcal{X}(K))$ coincides with the barycentric subdivision K'. It is a standard fact that a simplicial complex and its barycentric subdivision are simple homotopy equivalent. Moreover, the linear map $s_K : K' \to K$ that maps a simplex of K into its barycenter is a simple homotopy equivalence (see [13, (25.1)]). Given a simplicial map $\varphi : K \to L$, we denote by $\varphi' = \mathcal{K}(\mathcal{X}(\varphi)) : K' \to L'$ the map induced in the barycentric subdivisions. Since the maps $s_L\varphi'$ and φs_K are homotopic for any simplicial map $\varphi : K \to L$, from the 2-out-of-3 property we deduce that φ is a simple homotopy equivalence if and only if φ' is a simple homotopy equivalence.

We prove the following result which is a stronger version of the simplicial statement of Quillen's Theorem A [25]. It also sharpens Theorem 4.3.14 of [3] which requires a more restrictive hypothesis on the map φ.

Theorem 4.2. *Let $\varphi : K \to L$ be a simplicial map between two finite complexes. Suppose that the preimage of each closed simplex of L is contractible. Then φ is a simple homotopy equivalence.*

Proof. We show that the associated map $\mathcal{X}(\varphi)$ satisfies the hypotheses of Theorem 1.2. Given $\sigma \in \mathcal{X}(L)$,
$$\mathcal{K}(\mathcal{X}(\varphi)^{-1}(U_\sigma)) = \mathcal{K}(\mathcal{X}(\varphi^{-1}(\sigma)))$$
is the barycentric subdivision of $\varphi^{-1}(\sigma)$ which is contractible by hypothesis. Thus, φ' is a simple homotopy equivalence and then so is φ. $\square$

The original result of Quillen concludes under the same hypotheses that φ is a homotopy equivalence.

Another consequence of Theorem 1.2 is the following improvement of the Nerve Lemma proved by Borsuk (see [8, Theorem 10.6]). Recall that the *nerve* of a family $\mathcal{U} = \{U_i\}_{i\in I}$ of subsets of a set is the simplicial complex $\mathcal{N}(\mathcal{U})$ whose simplices are the finite subsets J of I such that $\bigcap\limits_{i\in J} U_i \neq \emptyset$. Given a poset X, we denote by X^{op} the poset with the reversed order. We follow the proof of [8, Theorem 10.6].

Theorem 4.3. *Let K be a finite simplicial complex and let $\mathcal{U} = \{L_i\}_{i\in I}$ be a finite family of subcomplexes of K such that $\bigcup\limits_{i\in I} L_i = K$ and such that every intersection of elements of $\mathcal{U}$ is empty or contractible. Then K has the same simple homotopy type as $\mathcal{N}(\mathcal{U})$.*

Proof. The map $\mathcal{X}(K) \to \mathcal{X}(\mathcal{N}(\mathcal{U}))^{op}$ that maps a simplex $\sigma \in K$ into $\{i \in I \mid \sigma \in L_i\}$ satisfies the hypotheses of Theorem 1.2. Therefore there is a simple homotopy equivalence from K' to $\mathcal{N}(\mathcal{U})'$. $\square$

We deduce then a stronger version of Dowker's Theorem [15]. The proof is the same as in [8, Theorem 10.9] but using Theorem 4.3.

Theorem 4.4. *Let X and Y be two finite sets and let $R \subseteq X \times Y$ be a relation. Consider the simplicial complex K whose simplices are the subsets σ of X for which exists an element of Y that is related to all the elements of σ. Symmetrically, the simplices of the complex L are subsets σ of Y for which there is an element of X related to every element of σ. Then K and L have the same simple homotopy type.*

Recall that a *crosscut* in a poset X is an antichain $A \subseteq X$ which satisfies the following two properties: (1) For every chain $C \subseteq X$ there exists $a \in A$ comparable with every element in C and (2) Every bounded subset B of A (i.e. with an upper or lower bound in X) has a supremum or an infimum in X. The *crosscut complex* $\Gamma(X, A)$ is the simplicial complex whose simplices are the non-empty bounded subsets of A. Rota's Crosscut Theorem asserts that for any crosscut A in a finite poset X, $\Gamma(X, A)$ and $\mathcal{K}(X)$ have the same homotopy type. A proof can be obtained from a direct application of the Nerve Lemma (see [8, Theorem 10.8]). In [21, Theorem 5.2], Kozlov proves that if the poset X is a *reduced lattice* (i.e. a finite lattice with its maximum and minimum removed), and A is the set of minimal elements, then the crosscut complex and $\mathcal{K}(X)$ have the same simple homotopy type. He conjectures that this happens in general for any crosscut in a reduced lattice [21, Conjecture 5.6]. As a corollary of our simple homotopy version of the Nerve Lemma, Theorem 4.3, we establish this conjecture of Kozlov, which holds even when X is not a reduced lattice.

Theorem 4.5. *Let A be a crosscut in a finite poset X. Then $\Gamma(X, A)$ and $\mathcal{K}(X)$ are simple homotopy equivalent.*

One of the main results in [21] (Theorem 7.2) says that the *neighborhood complex* $N(G)$ of a finite graph G and the polyhedral complex $\mathrm{Hom}(K_2, G)$ have the same simple homotopy type (see [21] for definitions). A shorter alternative proof can be found in [14, Theorem 8, Theorem 9]. However, this result follows directly from the well-known proof that these complexes are homotopy equivalent [1, Proposition 4.2] and our Theorem 1.2, since the map $\mathcal{X}(\mathrm{Hom}(K_2, G)) \to \mathcal{X}(N(G))$ defined in the proof of [1, Proposition 4.2] satisfies the hypothesis of the Fiber Lemma.

5. Relationship with Cohen's Theorem on contractible mappings

Theorem 4.2 is closely related to a result of Cohen on the so called *contractible mappings.* A simplicial map $\varphi : K \to L$ is called a contractible mapping if the preimage $\varphi^{-1}(y)$ of each point y in the underlying space of L is contractible. Cohen's result [12, Theorem 11.1] is the following

Theorem 5.1 (Cohen)**.** *Let $\varphi : K \to L$ be a contractible mapping between two finite simplicial complexes. Then φ is a simple homotopy equivalence.*

Contractible mappings are related to the Poincaré Conjecture (see the discussion in [12, p. 243]). They are fundamental in the development of Hatcher's theory in [17]. It is not hard to prove that compositions of contractible mappings are again contractible and that two simplicial complexes have the same simple homotopy type if and only if there is a chain of contractible mappings connecting them. We provide here a short and simple proof of Theorem 5.1 using our simple homotopy version of the McCord-Quillen Theorem.

Remark 5.2. If $\varphi : K \to L$ is a simplicial map, the preimage of each subcomplex of L is a subcomplex of K. Although the barycenter $b(\sigma)$ of a simplex $\sigma \in L$ is a vertex of L', the preimage $\varphi^{-1}(b(\sigma))$ need not be a subcomplex of K' because $\varphi : K' \to L'$ is not simplicial in general. Here we consider the vertices of K' and L' as the barycenters of the simplices of K and L, and we identify the underlying spaces of K' and K and of L' and L. If instead of K' we take a suitable derived subdivision δK of K choosing points different from the barycenters, we can make $\varphi : \delta K \to L'$ simplicial. Specifically, for each $\tau \in K$ let a_τ be a point in $\overset{\circ}{\tau} \cap \varphi^{-1}(b(\varphi(\tau)))$ (here $\overset{\circ}{\tau}$ denotes the interior of τ), for instance the convex combination

$$\sum_{v \in \tau} \frac{v}{\#\varphi(\tau)\#(\varphi^{-1}(\varphi(v)) \cap \tau)}.$$

Now, the derived subdivision δK is a simplicial complex whose vertices are the points a_τ and whose simplices are the sets $\{a_{\tau_0}, a_{\tau_1}, \ldots, a_{\tau_n}\}$ where $\tau_0 \subsetneq \tau_1 \subsetneq \ldots \subsetneq \tau_n$. Clearly δK is isomorphic to K'. The complex δK is a subdivision of K and therefore we identify both underlying spaces. For more information about derived subdivisions see [16, Chapter I, B].

For this subdivision, the map $\varphi : \delta K \to L'$ is simplicial since $\varphi(a_\tau) = b(\varphi(\tau))$. Thus, the preimage of $b(\sigma) \in L'$ is a subcomplex of δK. Concretely, it is the full subcomplex spanned by the vertices a_τ such that $\varphi(\tau) = \sigma$. This is isomorphic to the full subcomplex M of K' spanned by the vertices $b(\tau) \in K'$ (or equivalently $\tau \in K$) such that $\varphi(\tau) = \sigma$. This idea appears in [12, Remark 1, p. 225].

Proof of Theorem 5.1. Consider the map $\mathcal{X}(\varphi) : \mathcal{X}(K) \to \mathcal{X}(L)$ between the associated posets. Let σ be a simplex of L. Then σ is an element of $\mathcal{X}(L)$ and F_σ is a subset of $\mathcal{X}(L)$. Let $i : \mathcal{X}(\varphi)^{-1}(\sigma) \hookrightarrow \mathcal{X}(\varphi)^{-1}(F_\sigma)$ be the inclusion and define a map $r : \mathcal{X}(\varphi)^{-1}(F_\sigma) \to \mathcal{X}(\varphi)^{-1}(\sigma)$ by $r(\tau) = \tau \cap \varphi^{-1}(\sigma)$. This map is well-defined, order preserving and it is a retraction of i. Moreover $ir(\tau) \leq \tau$ and then by Lemma 2.1, $\mathcal{K}(i)\mathcal{K}(r) \simeq 1_{\mathcal{K}(\mathcal{X}(\varphi)^{-1}(F_\sigma))}$. Therefore $\mathcal{K}(\mathcal{X}(\varphi)^{-1}(\sigma))$ and $\mathcal{K}(\mathcal{X}(\varphi)^{-1}(F_\sigma))$ are homotopy equivalent.

On the other hand, $\mathcal{K}(\mathcal{X}(\varphi)^{-1}(\sigma))$ is exactly the subcomplex $M \subseteq K'$ of Remark 5.2, homeomorphic to the preimage $\varphi^{-1}(b(\sigma))$ of the barycenter of σ, which is contractible by hypothesis. Therefore, $\mathcal{K}(\mathcal{X}(\varphi)^{-1}(F_\sigma))$ is contractible.

We can then apply Theorem 1.2 to the map $\mathcal{X}(\varphi)^{op} : \mathcal{X}(K)^{op} \to \mathcal{X}(L)^{op}$ to conclude that $\varphi' : K' \to L'$ is a simple homotopy equivalence, and hence, so is φ. $\square$

Contractible mappings are closely related to the maps considered in Theorem 4.2. In fact we will see that any contractible mapping satisfies the hypothesis of the simplicial version of Quillen's Theorem A. Furthermore, we will show that Theorem 4.2 improves Cohen's result since it applies also to maps which are not necessarily contractible (see Example 5.4).

Proposition 5.3. *Let $\varphi : K \to L$ be a simplicial map between two finite simplicial complexes. If φ is a contractible mapping, then the preimage $\varphi^{-1}(\sigma)$ of each simplex σ of L is a contractible space.*

Proof. Let $\sigma \in L$. Then $\varphi|_{\varphi^{-1}(\sigma)} : \varphi^{-1}(\sigma) \to \sigma$ is a contractible mapping. By Theorem 5.1, $\varphi|_{\varphi^{-1}(\sigma)}$ is a homotopy equivalence and therefore $\varphi^{-1}(\sigma)$ is contractible. □

Example 5.4. Let K by an acyclic and non-contractible finite simplicial complex. Let L be a contractible finite simplicial complex containing K as a subcomplex (for instance we can take L as the simplex with the same vertices as K). Let v be a vertex not in L and consider the simplicial cone vK. Let $M = vK \cup L$. Let $\sigma = \{w_0, w_1\}$ be a 1-dimensional simplex and let $\varphi : M \to \sigma$ be the simplicial map that maps v into w_0 and all L into w_1.

Since M is covered by two contractible subcomplexes vK and L, it is homotopy equivalent to the suspension $\Sigma(vK \cap L) = \Sigma(K)$ of their intersection (see [4, Lemma 3.3] for instance). Since K is acyclic, the suspension $\Sigma(K)$ is acyclic and simply connected, and then contractible by Hurewicz and Whitehead's Theorems. Therefore, M is contractible. Then, the preimages v, L, M of the simplices of σ are contractible. However, the preimage $\varphi^{-1}(b(\sigma))$ of the barycenter of σ is the middle section of the cone vK parallel to the base, which is homeomorphic to the non-contractible complex K.

6. Two more applications

Other versions of the McCord-Quillen Theorem can be obtained by modifying the hypotheses on the subcomplexes $\mathcal{K}(f^{-1}(U_y))$. The following result was proved by Björner [9, Theorem 2] using the homotopy version of the Acyclic Carrier Theorem. We exhibit here an alternative proof using our approach to Theorem 1.1. Recall that a continuous map $f : X \to Y$ between two topological spaces is said to be an n-equivalence if for every $x \in X$, the induced map $\pi_i(X, x) \to \pi_i(Y, f(x))$ is an isomorphism for $i < n$ and an epimorphism for $i = n$.

Theorem 6.1 (Björner)**.** *Let $f : X \to Y$ be an order preserving map between two finite posets and let n be a nonnegative integer. Suppose that for every $y \in Y$, the complex $\mathcal{K}(f^{-1}(U_y))$ is n-connected. Then $\mathcal{K}(f)$ is an $(n+1)$-equivalence.*

We need a third version of Lemma 3.1.

Lemma 6.2. *Let $n \geq 0$, let X be a finite poset and let $x \in X$ be such that $\mathcal{K}(\hat{U}_x)$ is n-connected. Then $\mathcal{K}(X \smallsetminus \{x\}) \hookrightarrow \mathcal{K}(X)$ is an $(n+1)$-equivalence.*

Proof. The link $\mathrm{lk}_{\mathcal{K}(X)}(x) = \mathcal{K}(\hat{U}_x) * \mathcal{K}(\hat{F}_x)$ is also n-connected by [24, Lemma 2.3] and therefore the pair $(\mathrm{st}_{\mathcal{K}(X)}(x), \mathrm{lk}_{\mathcal{K}(X)}(x))$ is $(n+1)$-connected by the long exact sequence for relative homotopy groups. We can assume that $\mathcal{K}(X)$ is connected and therefore, $(\mathcal{K}(X \smallsetminus \{x\}), \mathrm{lk}_{\mathcal{K}(X)}(x))$ is 0-connected. By the Excision Theorem for homotopy groups [18, Theorem 4.23], the map $\pi_i(\mathrm{st}_{\mathcal{K}(X)}(x), \mathrm{lk}_{\mathcal{K}(X)}(x)) \to (\mathcal{K}(X), \mathcal{K}(X \smallsetminus \{x\}))$ induced by the inclusion is an isomorphism for $i < n+1$ and an epimorphism for $i = n+1$. Thus, $(\mathcal{K}(X), \mathcal{K}(X \smallsetminus \{x\}))$ is $(n+1)$-connected and the lemma follows. □

Proof of Theorem 6.1. Following the idea of the proof of Theorem 1.1, $\mathcal{K}(i)$ is a composition of $(n+1)$-equivalences by Lemma 6.2, and then it also is an $(n+1)$-equivalence. Since $\mathcal{K}(j)$ is a homotopy equivalence by Proposition 3.3 and $\mathcal{K}(i) \simeq \mathcal{K}(j)\mathcal{K}(f)$, $\mathcal{K}(f)$ is an $(n+1)$-equivalence. □

Before proving the homological analog of Theorem 6.1 due to Quillen [26], we state a fourth version of Lemma 3.1.

Lemma 6.3. *Let $n \geq 0$, let X be a finite poset and let $x \in X$ be such that the reduced (integral) homology groups $\widetilde{H}_i(\mathcal{K}(\hat{U}_x))$ are trivial for $i \leq n$. Then the map $\widetilde{H}_i(\mathcal{K}(X \smallsetminus \{x\})) \hookrightarrow \widetilde{H}_i(\mathcal{K}(X))$ induced by the inclusion is an isomorphism for $i \leq n$ and an epimorphism for $i = n+1$.*

Proof. The groups $\widetilde{H}_i(\mathrm{lk}_{\mathcal{K}(X)}(x)) = \widetilde{H}_i(\mathcal{K}(\hat{U}_x) * \mathcal{K}(\hat{F}_x))$ are trivial for $i \leq n$ by [24, Lemma 2.1]. The result then follows from the Mayer-Vietoris sequence for the decomposition $\mathcal{K}(X) = \mathcal{K}(X \smallsetminus x) \cup \mathrm{st}_{\mathcal{K}(X)}(x)$. □

Using again the ideas of the proof of Theorem 1.1 we deduce the following result.

Theorem 6.4 (Quillen)**.** *Let $f : X \to Y$ be an order preserving map between two finite posets and let n be a nonnegative integer. Suppose that for every $y \in Y$, the reduced homology groups $\widetilde{H}_i(\mathcal{K}(f^{-1}(U_y)))$ are trivial for $i \leq n$. Then $\mathcal{K}(f)_* : \widetilde{H}_i(\mathcal{K}(X)) \to \widetilde{H}_i(\mathcal{K}(Y))$ is an isomorphism for $i \leq n$ and an epimorphism for $i = n+1$.*

Corollary 6.5. *Let $f : X \to Y$ be an order preserving map between two finite posets. If $\mathcal{K}(f^{-1}(U_y))$ is acyclic for every $y \in Y$, then $\mathcal{K}(f)$ induces isomorphisms in all the homology groups.*

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Mathematics Department, Kungliga Tekniska högskolan, Stockholm, Sweden
E-mail address: jbarmak@kth.se